

Quantum reflection time and the Goos-Hänchen effect

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We explore the analogy between following the motion of a reflected wave packet and a quantum measurement of the spatial delay imposed on the particle by the scattering potential. It is shown that converting such delays into temporal durations can lead to “negative times,” and best be avoided. It is also demonstrated that a “soft” potential can be replaced by a superposition of hard walls. This representation is used for calculating the Goos-Hänchen shift of a Gaussian beam incident on a potential step or a barrier at an oblique angle.

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I. INTRODUCTION

The Goos-Hänchen (GH) effect, first observed in 1947 for electromagnetic waves [1], consists in lateral displacement of a beam reflected off a plane surface. The effect is not unique to electromagnetism and can occur in a variety of wave phenomena, including quantum scattering (see, for example, Refs. [2–9]). The displacement is often related to the time delay, associated with the scattering process [5], and can be seen as a way of measuring it. However, there is still little consensus about the definition of the quantum mechanical time delay. Much attention has been paid to the “tunneling time problem” [10–14] of defining the duration a tunneling particle is supposed to spend in the barrier region. Recently, we argued that the answer is provided by the measurement theory, in a uniquely quantum way [15–17]. According to the uncertainty principle [18], to know the duration of interest one needs to destroy the interference, and that destroys tunneling. Conversely, one can preserve tunneling, but with interference intact, the duration of interest must remain indeterminate [15,16]. The same should be true in quantum reflection, where one would expect similar nonclassical behavior for a particle repulsed by a potential well, or reflected above a potential barrier.

In this work, we revisit the quantum Goos-Hänchen effect in order to establish whether determination of the GH shift amounts to measuring the time spent by the particle in the scattering potential. We will answer the question by the

negative, and proceed to determine what is being measured, and to what accuracy. The paper is organized as follows. In Sec. II, we will rewrite the reflected wave packet (WP) in terms of the freely propagating counterpart, and a number of its mirror images. In Sec. III, we exploit the analogy with a von Neumann quantum measurement performed on a pre- and postselected quantum system, and identify the measured quantity. In Sec. IV, we show that in the classical limit, one does determine the duration spent by the classical trajectory in the scattering potential. In Sec. V, we attempt to extend the idea to the fully quantum case, and fail. In Sec. VI, we obtain the reflection “phase time” and discuss its properties. Sections VII and VIII offer two simple examples of what has been said so far. In Sec. IX, we abandon the attempts to define a meaningful “reflection time”, and replace instead a soft potential barrier by a superposition of virtual hard walls. In Sec. X, we use this representation to reconstruct the reflected beam and evaluate the GH shift in terms of the spatial shifts experienced by the reflected particle. Section XI contains our conclusions.

II. REFLECTION IN THE MOMENTUM AND COORDINATE REPRESENTATIONS

We consider a potential barrier $V(x)$, contained in the region $0 < x < d$, and a wave packet state $|\Psi(0)\rangle$, $\langle x|\Psi(0)\rangle \equiv 0$ for $x > 0$. A convenient choice is a Gaussian WP moving toward the barrier (see below). The initial state at $t = 0$ can be expanded in plane waves (we use $\hbar = 1$):

$$\Psi(x, 0) = \int_{-\infty}^{\infty} dk \langle x|k\rangle \langle k|\Psi(0)\rangle, \quad (1)$$

where $\langle x|k\rangle = (2\pi)^{-1/2} \exp(ikx)$. The time evolution of the WP [see Fig. 1(a)] is found by expanding it in the complete set of scattering states, incident on the barrier from the left and from the right (see Appendix A). Thus, for $t > 0$, the wave

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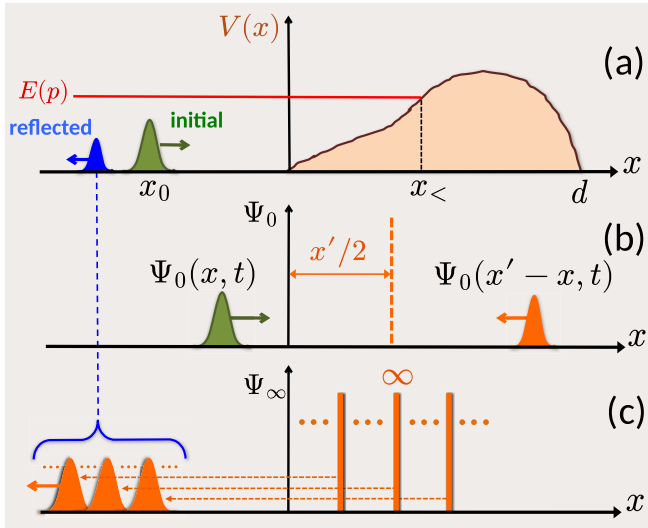


FIG. 1. (a) A wave packet with mean momentum p , incident on a potential barrier $V(x)$, is partly reflected. Its mean energy is lower than the barrier height, and $x_<$ denotes the classical turning point $V(x_<) = E(p)$. (b) Freely propagating WP and its reflection in a mirror placed at $x = x'/2$. (c) The reflected WP shown in panel (a) is a superposition of the WPs reflected by a variety of infinite walls. A detailed discussion of this representation is given in Sec. IX.

function in the left half-space $x < 0$ is given by

$$\Psi(x, t) = \Psi_0(x, t) + \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} R(k) \langle k | \Psi(0) \rangle \exp[-ikx - iE(k)t], \quad (2)$$

where

$$\Psi_0(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \langle k | \Psi(0) \rangle \exp[ikx - iE(k)t] \quad (3)$$

is the freely evolving initial state and $R(k)$ is the reflection amplitude for a particle with momentum k , incident on the potential from the left. The function $R(k)$ is known to be analytic in the complex k plane, with poles in its lower half, and also on the positive imaginary axis, if $V(x)$ supports bound states [19]. Note that Eq. (2) is valid even when the initial state (1) contains negative momenta, in which case one uses $R(-|k|) = R^*(|k|)$ (see Appendix A).

One can transform Eq. (2) to the coordinate representation by inserting the identity

$$R(k) = \int_{-\infty}^{\infty} dx' \exp(ikx') \xi(x'), \quad (4)$$

$$\xi(x') = (2\pi)^{-1} \int_{-\infty}^{\infty} dk' \exp(-ik'x') R(k') \quad (5)$$

into Eq. (2). This yields

$$\Psi(x, t) = \Psi_0(x, t) + \int_{-\infty}^{\infty} dx' \Psi_0(x' - x, t) \xi(x'), \quad (6)$$

where $\Psi_0(x' - x, t)$ is the image of the freely propagated WP (3) in a mirror placed at $x = x'/2$ [see Fig. 1(b)]. Each image enters the superposition (6) with a weight $\xi(x')$, the Fourier

transform of the reflection amplitude with respect to particle's momentum. For a potential $V(x)$, such that $R(k) \rightarrow 0$ as $|k| \rightarrow \infty$, $\xi(x')$ is a smooth function of x' , and, in the absence of bound states, one has

$$\xi(x') \equiv 0 \quad \text{for } x' < 0. \quad (7)$$

The evolution of the wave function in the left half-space can now be visualized as follows. At first, the WP localized far to the left moves freely toward the scatterer, and the wave function is given by the first term in Eq. (2), $\Psi(x, t) \approx \Psi_0(x, t)$. For a while, its mirror images $\Psi_0(x' - x, t)$, moving in the opposite direction, remain “off screen” in the right half-space $x > 0$. As the free WP crosses the origin and disappears from the screen, its mirror images enter the region $x < 0$, replacing there $\Psi_0(x, t)$. Scattering is finished when the left half-space contains only the reflected (R) part of the wave function,

$$\Psi(x, t) \xrightarrow{t \rightarrow \infty} \int_{-\infty}^{\infty} dx' \xi(x') \Psi_0(x' - x, t) \equiv \Psi_R(x, t). \quad (8)$$

Needless to say, the representation (6) has no physical significance in the right region $x > 0$, but is exact for any $x < 0$. We will give a further interpretation of Eq. (8) in Sec. IX.

III. ANALOGY WITH A QUANTUM MEASUREMENT

Let the initial state of a particle of mass m be a Gaussian WP with a mean momentum $p = mv > 0$, and coordinate width Δx , placed at $x = x_0 < 0$. Throughout this paper, we will neglect the WP's spreading, so the free state in Eq. (3) propagates without distortion (see Appendix B), and we have

$$\Psi_0(x, t) \approx G_0(x - x_0 - vt) \exp[ip(x - x_0) - iE(p)t], \quad (9)$$

where

$$G_0(x) = \left(\frac{2}{\pi \Delta x^2} \right)^{1/4} \exp\left(-\frac{x^2}{\Delta x^2}\right). \quad (10)$$

Reflecting $\Psi_0(x, t)$ about the point $x'/2$ requires changing $x \rightarrow x' - x$, and for the reflected state (8), we find

$$\begin{aligned} \Psi_R(x, t) &\approx \exp[-ip(x + x_0) - iE(p)t] \\ &\times \int_{-\infty}^{\infty} dx' G_0(x' - X(t)) \\ &\times \exp(ipx') \xi(x') \quad (x < 0), \end{aligned} \quad (11)$$

where we introduced a shorthand

$$X(t) \equiv x + x_0 + vt. \quad (12)$$

Similarity with a quantum measurement made on a pre- and postselected quantum system is now evident. The amplitude of finding the particle at x , $\Psi_R(x, t)$, matches that of a von Neumann pointer [20] of a mass m and momentum $-p$, set up to determine the manner in which a postselected system reaches its final state [see Eq. (D7) of Appendix D]. As with the von Neumann pointer, the amplitude $\Psi_R(x, t)$ can be seen as a result of interference between various alternatives, each labeled by x' . The experimenter, who finds the reflected particle in x at t , knows that x' (whatever it may be) lies in the interval $X(t) - \Delta x \lesssim x' \lesssim X(t) + \Delta x$, since only the values within this range contribute to the corresponding probability

$|\Psi_R(x, t)|^2$. Destruction of interference between the alternatives, and therefore, the accuracy with which x' is determined, is the greater the smaller is the WP's width Δx .

The peculiarity of the situation is that, unlike in the von Neumann example, no other degree of freedom is involved, so the particle appears to “measure itself.” But what precisely is being measured? Rewriting the reflected plane wave as

$$R(p) \frac{\exp(-ipx)}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}} \exp[-ip(x-x')]\xi(x'), \quad (13)$$

we note that x' represents a *spatial displacement*, or *shift* with which the reflected particle leaves the scatterer, with $\xi(x')$ giving the corresponding probability amplitude.

The usual dilemma is now evident. In the von Neumann example of Appendix D, interaction with the pointer alters the likelihood of the postselection. Similarly, the reflection probability $P_R(t) \equiv \int dx |\Psi_R(x, t)|^2$ depends on the WP's width Δx . An accurate measurement, $\Delta x \rightarrow 0$, destroys reflection, $P_R(t) \rightarrow 0$, since most of the WP, being broad in the momentum space, is now transmitted. Conversely, the choice of a nearly monochromatic WP, $\Delta x \rightarrow \infty$, preserves $P_R(t) \rightarrow |R(p)|^2$ but fails to distinguish between different shifts x' . A similar claim that, in a double-slit experiment, one cannot both know the slit chosen by the particle and retain the interference pattern on the screen is often used to illustrate the uncertainty principle [18].

It is instructive to see how the notion of reflection time occurs in the classical limit of the just described “measurement.”

IV. (SEMI)CLASSICAL REFLECTION TIME

Consider the semiclassical limit of Eq. (11) in the case of the barrier shown in Fig. 1(a). For energies less than the barrier height, the reflection amplitude is given by [21]

$$R(k) \approx -i \exp \left[2i \int_0^{x_<(k)} q(x, k) dx \right], \quad (14)$$

where $q(x, k) = \sqrt{k^2 - 2mV(x)}$ is the local momentum and $x_<$ is the turning point, where $q(x_<(k), k) = 0$ [see Fig. 1(a)]. The integral over k in Eq. (2) contains rapidly oscillating exponentials and can be evaluated by the stationary phase method [22]. Integral over dx' in Eq. (8) is also oscillatory and, as is easy to check, has a stationary point at $x'_s(p)$,

$$\frac{mx'_s(p)}{p} = 2 \int_0^{x_<(p)} \frac{mdx}{q(x, p)}. \quad (15)$$

As a result, a single envelope is selected in Eq. (11),

$$\begin{aligned} \Psi_R(x, t) &\approx \exp[-ip(x+x_0) - iE(p)t] \\ &\times G_0(x'_s(p) - X(t)). \end{aligned} \quad (16)$$

We note that Eq. (15) equates the time it takes the free particle to travel a distance x' to the actual time of travel from the origin $x = 0$ to the turning point $x_<$ and back. The classical time measurement now looks like this. The experimenter prepares a classical particle with momentum p at $x_0 < 0$ to the left of the barrier, and waits until it bounces off it. Then, he determines its position x at time t , calculates $x'_s = X(t) = x + x_0 + vt$,

and finally finds the duration spent in the potential,

$$\tau = \frac{mx'_s}{p} \equiv \frac{x'_s}{v}. \quad (17)$$

For a semiclassical particle, prepared in a Gaussian state, a convenient reference point is the center of mass of the reflected WP (16),

$$\langle x \rangle \equiv \frac{\int_{-\infty}^{\infty} dx x |\Psi_R(x, t)|^2}{\int_{-\infty}^{\infty} dx |\Psi_R(x, t)|^2}, \quad (18)$$

from which one obtains

$$\langle \tau \rangle \equiv \frac{\langle x \rangle}{v} + \frac{x_0}{v} + t. \quad (19)$$

The classical experiment cannot, however, determine such a duration for a reflected particle with energy higher than the barrier top, or for a particle reflected by a potential well. Such events are, however, possible in the quantum case, which we will consider next.

V. QUANTUM “REFLECTION TIME”

One way to extend the classical result (17) to a fully quantum case is to associate a duration $\tau = x'/v$ with each spatial shift x' in Eq. (11). Now, having found the reflected particle in x at t , one can assume that it had spent in the potential a duration $X(t)/v - \Delta x/v \lesssim \tau \lesssim X(t)/v + \Delta x/v$. The danger, as will be shown shortly, is that one will encounter negative durations whenever $V(x)$ provides negative shifts, $\xi(x' < 0) \neq 0$. We will proceed nonetheless.

For an accurate measurement, one requires $\Delta x \rightarrow 0$, and since $\xi(\tau)$ is a smooth function, one finds ($x < 0$)

$$|\Psi_R(x, t)|^2 \approx \left| \int_{-\infty}^{\infty} dx' G_0(x') \right|^2 |\xi(x + vt + x_0)|^2. \quad (20)$$

With interference between the alternatives destroyed, a single duration $\tau(x) = t + \frac{x}{v} + \frac{x_0}{v}$ can be ascribed to a particle found in x . Averaging over all positions yields the mean “reflection time” (19)

$$\langle \tau \rangle \xrightarrow{\Delta x \rightarrow 0} \frac{1}{v} \frac{\int_{-\infty}^{\infty} dx x |\xi(x + vt + x_0)|^2}{\int_{-\infty}^{\infty} dx |\xi(x + vt + x_0)|^2}. \quad (21)$$

The danger here, we repeat, is that, since $\xi(x')$ may not vanish for $x' < 0$, the average (21) can, in principle, turn negative, and no longer represent a meaningful duration.

We note also that since $G_0(x) \xrightarrow{\Delta x \rightarrow 0} (2\pi \Delta x^2)^{1/4} \delta(x)$, the first integral in Eq. (20) behaves as $\sim \Delta x^{1/2}$, and $|\Psi_R(x, t)|^2 \rightarrow 0$, because most of the momenta now go above the barrier (see also Sec. III).

Next, we consider the case of a highly inaccurate measurement, $\Delta x \rightarrow \infty$, where all the alternatives, be they spatial shifts x' or times x'/v , interfere to produce the probability $|\Psi_R(x, t)|^2$ of finding the reflected particle in x .

VI. “PHASE TIME” FOR REFLECTION

As the initial wave packet becomes broader, $\Delta x \rightarrow \infty$, $G_0(x' - X(t))$ in Eq. (8) varies little across the range of the x 's contained in $\xi_r(\tau)$, and can be approximated by the first

two terms of its Taylor series, $G_0(x' - X(t)) \approx G_0(X(t)) - x' \partial_x G_0(X(t))$. Inserting this into Eq. (11) yields

$$\begin{aligned} \Psi_R(x, t) &\approx \exp[-ip(x + x_0) - iE(p)t] \\ &\times G_0(X(t)) \int dx' \eta(p, x') - \partial_x G_0(X(t)) \\ &\times \int dx' x' \eta(p, x'), \end{aligned} \quad (22)$$

where we wrote $\eta(p, x') \equiv \exp(ipx') \xi(x')$. Inserting Eq. (22) into Eq. (18) and using Eq. (19) yields the “phase time”

$$\langle \tau \rangle \xrightarrow{\Delta x \rightarrow \infty} \frac{1}{v} \operatorname{Re} \left[\frac{\int_{-\infty}^{\infty} dx' x' \eta(p, x')}{\int_{-\infty}^{\infty} dx' \eta(p, x')} \right] = \frac{1}{v} \frac{\partial \phi_R(p)}{\partial k} \equiv \tau_{\text{ph}}, \quad (23)$$

where $\phi_R(p)$ is the phase of the reflection coefficient, $R(p) = |R(p)| \exp[i\phi_R(p)]$. The phase time (23) is often thought to represent the duration a particle with momentum p spends in the reflecting potential. [In quantum measurements, the complex quantity in the brackets is known as “weak value of x' ” [23,24]. It can be shown that, as in the case of transmission [17], its imaginary part determines the small change in the reflected particle’s mean momentum, which occurs because higher momenta are transmitted more easily, and are less likely to be reflected (see also Ref. [5]).]

However, one needs to be even more cautious. Since complex-valued $\eta(p, x')$ may change sign, τ_{ph} can be negative, even if all phase shifts in Eq. (13) are positive and $\xi(x')$ vanishes for $x' < 0$ (for example, see Ref. [25]).

Simple illustrations will be given next.

VII. ZERO-RANGE POTENTIALS: NEGATIVE “REFLECTION TIMES”

For a particle scattered by a zero-range potential $V(x) = \Omega \delta(x)$, where $\delta(x)$ is the Dirac delta, the reflection amplitude is well known to be

$$R(k) = \frac{-im\Omega}{k + im\Omega}. \quad (24)$$

Thus, we have

$$\xi(x') = m|\Omega| \exp(-m|\Omega x'|) \theta(\pm x'), \quad (25)$$

where the upper and lower signs are for a barrier, $\Omega > 0$, and a well, $\Omega < 0$, respectively.

For a δ barrier, both the average (21) and the phase time (23) are positive:

$$\langle \tau \rangle \xrightarrow{\Delta x \rightarrow 0} \frac{1}{2|\Omega|v} > 0, \quad \tau_{\text{ph}}(p) = v^{-1} \frac{m|\Omega|}{p^2 + m^2\Omega^2} > 0. \quad (26)$$

For a δ well [26], both $\langle \tau \rangle$ and τ_{ph} change sign and become negative,

$$\langle \tau \rangle \xrightarrow{\Delta x \rightarrow 0} \frac{-1}{2|\Omega|v} < 0, \quad \tau_{\text{ph}}(p) = v^{-1} \frac{-m|\Omega|}{p^2 + m^2\Omega^2} < 0. \quad (27)$$

The quantities in Eq. (27) cannot be interpreted as durations the particle spends in the right half-space. The fault is with

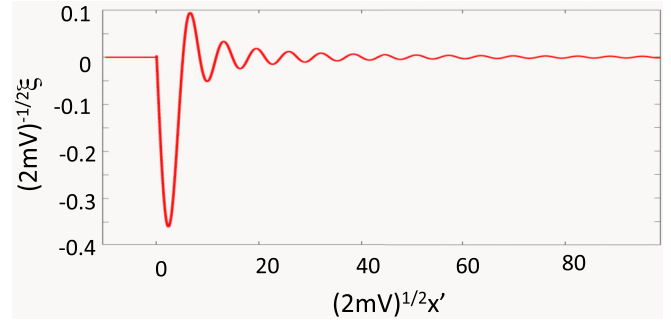


FIG. 2. The amplitude distribution $\xi(x')$ for a step potential (29), as given by Eq. (31).

conversion of the phase shifts in Eq. (13) into temporal intervals, which yields a negative time τ whenever $x' < 0$. Finally, the reflection probability

$$P_R \xrightarrow{\Delta x \rightarrow 0} \sqrt{2\pi} \Delta x m |\Omega| / 2 \quad (28)$$

vanishes, as expected, for a WP narrow in the coordinate space.

VIII. A STEP POTENTIAL: INFINITE PHASE TIME

Consider next a step potential

$$V(x) = V\theta(x), \quad (29)$$

for which the reflection amplitude is easily found to be

$$R(k, q(k)) = \frac{k - q}{k + q}, \quad (30)$$

where $q(k) = \sqrt{k^2 - 2mV}$. Evaluation of the integral (5) yields (for details see Appendix E)

$$\xi(x') = -\frac{2}{x'} J_2(\sqrt{2mV}x') \theta(x'), \quad (31)$$

where $J_2(z)$ is the Bessel function of the first kind of order 2 [27] (see Fig. 2). It is readily seen that $\int_0^\infty dx' \xi(x') = -1$, and as the height of the step increases, $\xi(x') \xrightarrow{V \rightarrow \infty} -\delta(x')$, the reflection amplitude tends to that of a hard wall,

$$R(p) = \int_0^\infty dx' \exp(ipx') \xi(x') \xrightarrow{V \rightarrow \infty} -1. \quad (32)$$

To understand the behavior of the phase time τ_{ph} , we note first that the integrand in the denominator of Eq. (23) contains the term $\exp[i(p - \sqrt{2mV})x']/x'^{3/2}$, and converges, as illustrated in Fig. 3. The integrand in the numerator of Eq. (23) falls off as $\sim 1/x'^{1/2}$, and

$$\tau_{\text{ph}}(p) = \begin{cases} \frac{2m}{p\sqrt{2mV-p^2}}, & \text{for } 0 \leq p \leq \sqrt{2mV}, \\ 0, & \text{for } p > \sqrt{2mV} \end{cases} \quad (33)$$

becomes infinite as $p \rightarrow \sqrt{2mV}$. The fault is with the approximation (22), without which both $\langle x \rangle$ in Eq. (18) and the $\langle \tau \rangle$ defined by Eq. (21) remain finite, as shown in Fig. 4 (see also Ref. [5]).

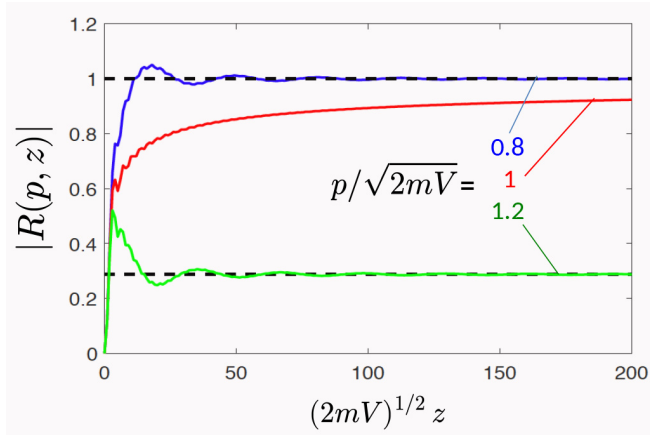


FIG. 3. Integral $R(p, z) \equiv \int_0^z dx' \exp(ipx') \xi(x')$ converges to the reflection amplitude $R(p)$ (dashed) as $z \rightarrow \infty$. Convergence is slower for energies close to the step's height, $p \approx \sqrt{2mV}$.

IX. SOFT BARRIER AS A SUPERPOSITION OF HARD WALLS

Having abandoned attempts to describe quantum reflection in terms of a “reflection time,” we look at the problem from a different angle. In Eq. (8), the free state $\Psi_0(x' - x, t)$ differs only by sign from the state $\Psi_R^\infty(x, t|a)$, reflected by an infinite hard wall placed at $a = x'/2$, and we can write

$$\Psi_R(x, t) = \int_0^\infty da \xi_a(a) \Psi_R^\infty(x, t|a) \quad (x < 0), \quad (34)$$

where $\xi_a(a) = -2 \xi(2a)$, and

$$\begin{aligned} \Psi_R^\infty(x, t|a) = & -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dk \langle k | \Psi(0) \rangle \\ & \times \exp[ik(2a - x) - iE(k)t]. \end{aligned} \quad (35)$$

As long as only the reflected part of the WP is of interest, one can replace the actual potential $V(x)$ by a superposition of hard walls, $V_\infty(x|a) = 0$ for $x < a$, and ∞ otherwise. The reflected WP is now given by the superposition of the states reflected by all infinite walls, with appropriate amplitude weights $\xi_a(a)$. Now, $x' = 0$ corresponds to reflection by a wall placed at the origin, in which case the particle does not enter the region $x > 0$. For a positive $x' > 0$, the wall is placed

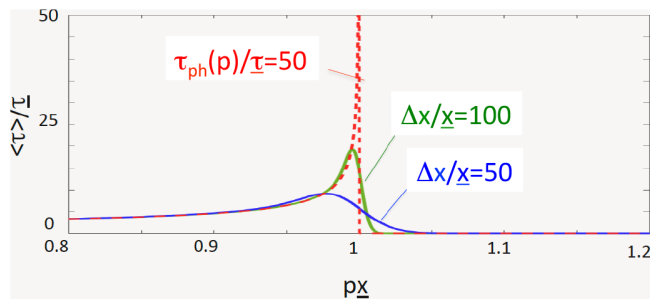


FIG. 4. The center-of-mass delay for a WP incident on the potential step (29) for two coordinate widths. Also shown by the dashed line is the phase time (23), which diverges when $E(p) = V$. The units are $\underline{x} = 1/\sqrt{2mV}$ and $\underline{t} = p^{-1}(2V/m)^{-1/2}$.

inside the $x > 0$ region, so the center of the WP, reflected from it, lags behind $\Psi_R^\infty(x, t|0)$. The opposite is true for $x' < 0$.

If the potential supports bound states, some of the walls will have to be placed in the $x < 0$ region, suggesting reflections occurring even before the particle impacts the potential [cf. Eq. (27)]. However, for a pure barrier, having no poles in the upper half of the complex k plane, we have

$$\xi_a(a) \equiv 0 \quad \text{for } a < 0, \quad (36)$$

so that all fictitious walls are found in the region $x > 0$ [see Fig. 1(c)]. Next, we use Eq. (34) to analyze reflection in two spatial dimensions.

X. QUANTUM GOOS-HÄNCHEN EFFECT

In two dimensions $[\mathbf{r} = (x, y), \mathbf{k} = (k_x, k_y)]$, consider a potential occupying the $x > 0$ half-space $V(x, y) = V(x)\theta(x)$. Now, if the particle's initial state, localized in the potential-free region $x < 0$, is given by

$$\Psi(\mathbf{r}, 0) = \int_{-\infty}^\infty d\mathbf{k} \langle \mathbf{r} | \mathbf{k} \rangle \langle \mathbf{k} | \Psi(0) \rangle, \quad \langle \mathbf{r} | \mathbf{k} \rangle = \frac{\exp(i\mathbf{k}\mathbf{r})}{2\pi}, \quad (37)$$

for the reflected WF one finds

$$\begin{aligned} \Psi_R(\mathbf{r}, t) = & \frac{1}{2\pi} \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y R(k_x) \\ & \times \exp[-ik_x x + ik_y y - iE(k)t] \langle \mathbf{k} | \Psi(0) \rangle, \end{aligned} \quad (38)$$

where $R(k)$ is the reflection coefficient for one-dimensional potential $V(x)$.

Acting as before, for a barrier not supporting bound states, we find

$$\Psi_R(\mathbf{r}, t) = \int_0^\infty da \xi_a(a) \Psi_R^\infty(\mathbf{r}, t|a) \quad (x < 0), \quad (39)$$

where $\Psi_R^\infty(\mathbf{r}, t|a)$ is the freely propagating initial state, reflected by a hard wall placed at $x = a$,

$$\Psi_R^\infty(x, y, t|a) = -\Psi_0(2a - x, y, t). \quad (40)$$

Let $\Psi_0(x, y, t)$ be a transverse Gaussian beam (or a WP broad in the direction of propagation, if a pulsed laser is used; see Refs. [4–7]), making an angle θ with the normal to the surface $x = 0$ [see Fig. 5(a)]. In the coordinate system (x_1, y_1) , obtained by rotating the axes (x, y) counterclockwise by θ , we have

$$\Psi_0(x_1, y_1, t) = G_0(y_1) \exp(ipx_1 - ip^2 t/2m), \quad (41)$$

where we neglect the beam's broadening by putting

$$G_0(y) = \left(\frac{2}{\pi \Delta y^2} \right)^{1/4} \exp\left(-\frac{y^2}{\Delta y^2} \right) \quad (42)$$

(see Appendix G). Returning to the spatial shifts (13), $x' = 2a$, after a simple exercise in trigonometry, we find ($x < 0$)

$$\begin{aligned} \Psi_R(x_2, y_2, t) \xrightarrow{x_2 \rightarrow \infty} & \exp[ipx_2 - iE(p)t] \\ & \times \int_0^\infty dx' G_0(y_2 - x' \sin \theta) \exp(ip_x x') \xi(x'), \end{aligned} \quad (43)$$

where the axes (x_2, y_2) are obtained by reflecting the axes (x_1, y_1) about the plane $x = 0$ [see Fig. 5(a)]. Again, we have a

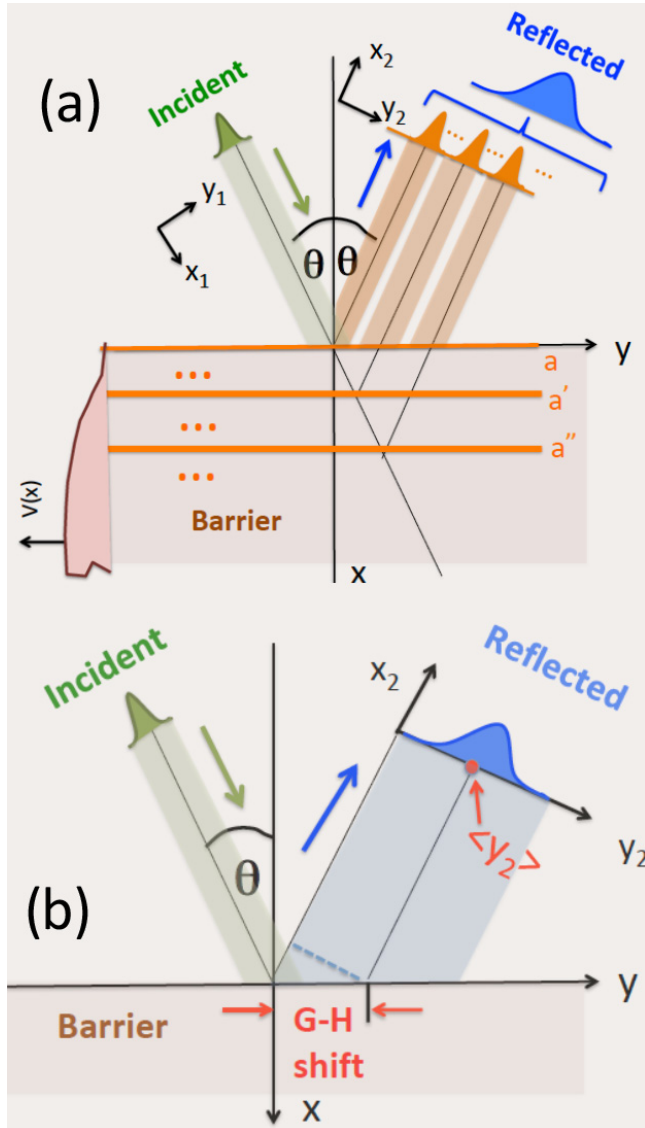


FIG. 5. (a) In two dimensions, a Gaussian beam is incident on a potential barrier $V(x, y) = V(x)\theta(x)$. The reflected beam is a superposition of all beams, reflected by a variety of infinite walls placed at $x > 0$. (b) The center of the reflected beam [cf. Eq. (44)] and the Goos-Hänchen shift (45).

kind of quantum measurement. Detecting the particle at some y_2 , one can deduce that it has experienced a spatial shift in the interval $(y_2 - \Delta y_2)/\sin \theta \lesssim x' \lesssim (y_2 + \Delta y_2)/\sin \theta$. The measurement of x' is the more accurate the smaller is the ratio between the width of the beam Δy_2 and the sine of the angle of incidence θ : $\Delta x' = \Delta y_2/\sin \theta$.

The quantity of interest is the Goos-Hänchen shift δy_{GH} , i.e., the distance by which the reflected beam is displaced relative to the same beam reflected by a hard wall placed at $x = 0$ [see Fig. 5(b)]. We note that superposition of the Gaussians in Eq. (43) does not have to be a Gaussian itself. However, one can always calculate the center of the beam, the way it was done in Eq. (19):

$$\langle y_2 \rangle = \frac{\int_0^\infty dy_2 y_2 |\Psi_R(x_2 \rightarrow \infty, y_2, t)|^2}{\int_0^\infty dy_2 |\Psi_R(x_2 \rightarrow \infty, y_2, t)|^2}, \quad (44)$$

and from it obtain

$$S_{\text{GH}} = \frac{\langle y_2 \rangle}{\cos(\theta)}. \quad (45)$$

In the limits of a narrow and a broad beam, we find expressions similar to Eqs. (21) and (23), respectively,

$$S_{\text{GH}} \xrightarrow{\Delta y/\sin \theta \rightarrow 0} \tan(\theta) \frac{\int_0^\infty dx' x' |\xi(x')|^2}{\int_0^\infty dx' |\xi(x')|^2} \quad (46)$$

and

$$S_{\text{GH}} \xrightarrow{\Delta y/\sin \theta \rightarrow \infty} \tan(\theta) \text{Re} \left[\frac{\int_0^\infty dx' x' \exp(ip_x x') \xi(x')}{\int_0^\infty dx' \exp(ip_x x') \xi(x')} \right] \\ = \tan(\theta) \left. \frac{\partial \phi_R(k)}{\partial k} \right|_{k=p_x = p \cos \theta}. \quad (47)$$

To calculate $\langle y_2 \rangle$ for any Δy , we write $G_0(y) = \int \tilde{G}_0(q) \exp(iyq) dq$ and use Eqs. (6) and (F4). The result is

$$\langle y_2 \rangle \equiv \text{Im} \left[\frac{\int_{-\infty}^\infty dq F^*(q, p_x, \theta) \partial_q F(q, p_x, \theta)}{\int_{-\infty}^\infty dq |F(q, p_x, \theta)|^2} \right], \quad (48)$$

$$F(q, p_x, \theta) \equiv R(p_x - q \sin \theta) \tilde{G}_0(q).$$

Finally, in the paraxial approximation of Appendix G, effects of beam's broadening can be included by replacing $G_0(y)$ and $\tilde{G}_0(q)$ in Eqs. (41) and (48) by their coordinate-dependent versions (G4) and (G6). This approach is also useful beyond the paraxial approximation. For example, consider a doubly Gaussian wave packet, reflected by a potential step (see Sec. VIII) in the regime of total reflection (i.e., when all energies in the wave packet are much lower than the step height). In this case, both the longitudinal delay and the lateral displacement can be directly related to the average depth at which the particle is reflected,

$$\langle x_2 \rangle = -x_0 - \frac{p_0}{m} t + \langle x \rangle \cos \theta, \quad \langle y_2 \rangle = \langle x \rangle \sin \theta. \quad (49)$$

The average depth at which the reflection occurs is related to the derivative of the phase [see Eq. (33)] of the reflection coefficient (30):

$$\langle x \rangle \approx \iint dk_{x_2} dk_{y_2} \frac{d\phi_R(k_x)}{dk_x} \Big|_{k_x=k_{x_2} \cos \theta - k_{y_2} \sin \theta} \\ \times \frac{2}{\pi} \frac{\exp(-2k_{y_2}^2/\Delta k_{y_2}^2)}{\Delta k_{y_2}} \frac{\exp[-2(k_{x_2} - k_0)^2/\Delta k_{x_2}^2]}{\Delta k_{x_2}}. \quad (50)$$

Here, both the wave packet widths Δk_{x_2} and Δk_{y_2} must be sufficiently small, so that adding them to the average momentum k_0 keeps the corresponding energies significantly lower than the barrier height.

XI. CONCLUSIONS

Above, we attempted to give a most detailed description of the quantum Goos-Hänchen effect that an elementary theory can offer. For a classical particle with a known momentum launched at an oblique angle at a repulsive potential, the

distance S_{GH} between the exit and entry points at the surface, where the potential vanishes, can be found by multiplying its lateral velocity $v_{\parallel}$ by the duration τ its trajectory spends in the potential: $S_{GH} = v_{\parallel}\tau$.

A straightforward extension of this analysis to the quantum case is, however, not possible. In one dimension, the quantum state of a particle with a known momentum $p = mv$ is a plane wave, and the moments when it enters and leaves the potential region simply cannot be defined. One notes, however, that the reflected state is a superposition (13) of multiple plane waves with phases corresponding to spatial displacements, or shifts, x' , induced by the potential. One notes also that in an experiment where one prepares the particle in a wave packet state with a mean momentum p , and then observes it at x , the shifts are restricted to the range, determined by the WP's coordinate width Δx . In the classical limit of this “quantum measurement,” the unique displacement x'_s , with which the reflected particle leaves the potential, corresponds to the stationary region where rapid oscillations of the amplitude distribution $\xi(x')$ slow down. This classical value allows one to deduce the duration τ that the particle's trajectory spends in the potential as x'_s/v .

There are two good reasons why one cannot continue to use the time argument beyond the classical limit, e.g., by converting each spatial shift into a duration using $\tau(x') = x'/v$. One is that some of these “durations” will have to be negative in the case of a potential well, or, indeed, of any potential supporting bound states. The idea that a particle that makes a U-turn before entering the potential spends in the scatterer a “negative duration” does not help explain the phenomenon in classical-like terms. (We are aware of the recently reported “experimental evidence of negative times” [22], and remain sceptical.) The notion that a particle's state can acquire a negative phase shift can, however, be made without causing a further controversy.

The other reason is that, as with any quantum measurement of this kind, one faces a dilemma rooted in the uncertainty principle [18]. One can determine the shift experienced by the particle accurately, but the reflection probability will not correspond to that of a particle with a momentum p . In fact, it will vanish. Or, one can keep the probability (almost) intact, but lose all the knowledge about the x 's to interference. In this case, it is still possible to find the displacement of the center of mass of a very broad reflected WP, and from it the “phase time” (23). However, both are related to the first moment of an oscillatory complex-valued distribution, and can be negative even for a potential barrier, where all shifts x' are positive (for an example, see Ref. [25]).

One can represent the scattering potential by a superposition of virtual hard walls, distributed in such a manner that the outgoing WP is a sum of all WPs reflected by each of the walls. Such a representation is not without its problems, since in the case of a well, some of the fictitious walls would need to be placed in the potential-free region, implying that the scattering occurs even before the particle reaches the scatterer. Still, the idea is somewhat less confusing than the one invoking negative durations, and can be used for illustrative purposes. For example, the hard-wall picture helps one to visualize what happens in the quantum Goos-Hänchen case [3]. The outgoing Gaussian beam is composed of the beams

reflected by each fictitious wall [see Fig. 5(a)]. The cross section of the beam's wave function is a superposition of Gaussians (we neglect the lateral spreading of the wave packet during propagation), similar to what one would obtain in a von Neumann quantum measurement. As the width of the incident beam increases (the measurement becomes less accurate), the Goos-Hänchen shift changes from Eq. (46) to (47), involving the “strong” and the “weak” values of the coordinate shift x' .

Finally, the above approach is likely to find an application in the study of soft-wall potentials, often employed to model realistic ballistic and mesoscopic systems. Such are the soft-wall billiards and related models used, for example, to study strong scarring and robust wave packet transport [29], or lateral shifts for ballistic electrons in semiconductor slabs [30].

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SCATTERING BY A ONE-DIMENSIONAL BARRIER

Consider scattering by a one-dimensional potential barrier $V(x)$, contained between $x = 0$ and $x = d$,

$$V(x) \equiv 0 \quad \text{for } x < 0 \quad \text{and } x > d. \quad (\text{A1})$$

The left (right) scattering states, for a particle incident from the left (right),

$$\begin{aligned} \langle x | \Phi_l(k) \rangle &= \frac{1}{\sqrt{2\pi}} [e^{ikx} + R_l(k)e^{-ikx}], \quad x < 0, \\ &= \frac{1}{\sqrt{2\pi}} T_l(k)e^{ikx}, \quad x > d, \end{aligned}$$

$$\begin{aligned}\langle x | \Phi_r(k) \rangle &= \frac{1}{\sqrt{2\pi}} [e^{-ikx} + R_r(k)e^{ikx}], \quad x > d, \\ &= \frac{1}{\sqrt{2\pi}} T_r(k)e^{-ikx}, \quad x < 0,\end{aligned}\quad (\text{A2})$$

form a complete orthonormal basis,

$$\int_0^\infty dk [|\Phi_l(k)\rangle\langle\Phi_l(k)| + |\Phi_r(k)\rangle\langle\Phi_r(k)|] = 1. \quad (\text{A3})$$

It follows from the Wronskian relations between the states (A2) and the fact that the complex conjugates $\Phi_{l,r}^*(k, x)$ are also solutions of the Schrödinger equation that

$$\begin{aligned}T_l(k) &= T_r(k) = T(k), \quad R_l(k)T^*(k) = -R_r^*(k)T(k), \\ T(-k^*) &= T(k)^*, \quad R_{l,r}(-k^*) = R_{l,r}^*(k).\end{aligned}\quad (\text{A4})$$

The initial state is a wave packet $|\Psi(x, 0)\rangle$ completely localized to the left of the barrier, which we expand in plane waves $|k\rangle$ as

$$\begin{aligned}|\Psi(x, 0)\rangle &= \int_{-\infty}^\infty dk |k\rangle\langle k | \Psi(0)\rangle \equiv \int_{-\infty}^\infty dk A(k) |k\rangle, \\ \langle x | k \rangle &= \frac{1}{\sqrt{2\pi}} e^{ikx}.\end{aligned}\quad (\text{A5})$$

To evolve the state in time, we expand it in the scattering states (A2),

$$\begin{aligned}\Psi(x, t) &= \int_0^\infty dk \langle \Phi_r(k) | \Psi(0) \rangle e^{-iE(k)t} \langle x | \Phi_r(k) \rangle \\ &+ \int_0^\infty dk \langle \Phi_l(k) | \Psi(0) \rangle e^{-iE(k)t} \langle x | \Phi_l(k) \rangle,\end{aligned}\quad (\text{A6})$$

where $E(k) = k^2/2m$ and m is the particle's mass. Using $\int_{-\infty}^\infty dk' \int_{-\infty}^\infty dx' |x'\rangle\langle x' | k'\rangle\langle k' | = 1$, we find

$$\begin{aligned}\langle \Phi_{l,r}(k) | \Psi(0) \rangle &= \int_{-\infty}^\infty dk' \int_{-\infty}^\infty dx' \\ &\times \langle \Phi_{l,r}(k) | x' \rangle \langle x' | k' \rangle \langle k' | \Psi(0) \rangle.\end{aligned}\quad (\text{A7})$$

Now, since the initial state lies to the left of the barrier, i.e., at $x < 0$, we can use the corresponding expressions for $\Phi_{l,r}^*(k, x)$ [cf. Eq. (A2)] and extend integration over x' to $(-\infty, \infty)$. With $(2\pi)^{-1} \int_{-\infty}^\infty \exp(ikx) dx = \delta(k)$, this gives

$$\begin{aligned}\langle \Phi_l(k) | \Psi(0) \rangle &= (2\pi)^{-1} \int_{-\infty}^\infty dk' \int_{-\infty}^\infty dx' \times [e^{-ikx'} + R_l^*(k)e^{ikx'}] e^{ik'x'} \langle k' | \Psi(0) \rangle = \langle k | \Psi(0) \rangle + R_l^*(k) \langle -k | \Psi(0) \rangle, \\ \langle \Phi_r(k) | \Psi(0) \rangle &= (2\pi)^{-1} \int_{-\infty}^\infty dk' \int_{-\infty}^\infty dx' \times T^*(k) e^{ikx'} e^{ik'x'} \langle k' | \Psi(0) \rangle = T^*(k) \langle -k | \Psi(0) \rangle.\end{aligned}\quad (\text{A8})$$

Inserting Eq. (A8) into Eq. (A6) for $x > d$ yields

$$\begin{aligned}\Psi(x, t) &= \frac{1}{\sqrt{2\pi}} \left\{ \int_0^\infty dk [\langle k | \Psi(0) \rangle + R_l^*(k) \langle -k | \Psi(0) \rangle] T(k) e^{ikx - iE(k)t} \right. \\ &\quad \left. + \int_0^\infty dk T^*(k) \langle -k | \Psi(0) \rangle [e^{-ikx} + R_r(k) e^{ikx}] e^{-iE(k)t} \right\},\end{aligned}\quad (\text{A9})$$

which, after taking into account (A4), and changing $k \rightarrow -k$, simplifies to

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dk T(k) \langle k | \Psi(0) \rangle e^{ikx - iE(k)t}. \quad (\text{A10})$$

Similarly, for $x < 0$ we have

$$\begin{aligned}\Psi(x, p, t) &= \int_0^\infty dk [\langle k | \Psi(0) \rangle + R_l^*(k) \langle -k | \Psi(0) \rangle] \times \frac{1}{\sqrt{2\pi}} [e^{ikx} + R_l(k) e^{-ikx}] e^{-iE(k)t} \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^\infty dk T^*(k) \langle -k | \Psi(0) \rangle T(k) e^{-ikx} e^{-iE(k)t}.\end{aligned}\quad (\text{A11})$$

Or, recalling that $|T(k)|^2 + |R_l(k)|^2 = 1$ and $R_l^*(k) = R_l(-k)$,

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dk \langle k | \Psi(0) \rangle e^{ikx - iE(k)t} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty R_l(k) dk \langle k | \Psi(0) \rangle e^{-ikx - iE(k)t}. \quad (\text{A12})$$

Validity of Eq. (A12) is easily proven. Define

$$I_1 = \int_0^\infty dk \langle k | \Psi(0) \rangle e^{ikx - iE(k)t}, \quad (\text{A13})$$

$$I_2 = \int_0^\infty dk R_l^*(k) \langle -k | \Psi(0) \rangle e^{ikx - iE(k)t} + \int_0^\infty dk R_l(k) \langle k | \Psi(0) \rangle e^{-ikx - iE(k)t} = \int_{-\infty}^\infty dk R_l(k) \langle k | \Psi(0) \rangle e^{-ikx - iE(k)t}, \quad (\text{A14})$$

$$I_3 = \int_0^\infty dk |R_l(k)|^2 \langle -k | \Psi(0) \rangle e^{-ikx - iE(k)t} + \int_0^\infty dk |T(k)|^2 \langle -k | \Psi(0) \rangle e^{-ikx - iE(k)t} = \int_0^\infty dk \langle -k | \Psi(0) \rangle e^{-ikx - iE(k)t} \\ = \int_{-\infty}^0 dk \langle k | \Psi(0) \rangle e^{ikx - iE(k)t}. \quad (\text{A15})$$

Then, adding $I_1 + I_2 + I_3$ yields $\int_{-\infty}^\infty dk \langle k | \Psi(0) \rangle e^{ikx - iE(k)t}$.

Equations (A10) and (A12) allow one to monitor the time evolution of the wave function in the regions $x < 0$ and $x > d$, but not inside the potential, $0 < x < d$, where one needs to know the behavior of $\Phi_{l,r}(k, x)$, not specified by Eq. (A2). For example, the initial WP located to the left of the potential is represented by the first term in Eq. (A12). As time progresses, it gradually “disappears from the screen” and is replaced by the second term in Eq. (A12) in the region $x < 0$, while the transmitted part (A10) appears in the $x > 0$ region. Note that Eqs. (A10) and (A11) are valid even when the WP expansion (B2) contains negative momenta, propagating away from the barrier, as we illustrate next.

APPENDIX B: GAUSSIAN WAVE PACKETS

It is common practice to study scattering of Gaussian states. Choosing

$$\langle k | \Psi(0) \rangle = (2\pi^{-1} \Delta k^2)^{-1/4} \exp \left[-\frac{(k-p)^2}{\Delta k^2} - ikx_0 \right], \quad (\text{B1})$$

with $x_0 < 0$ and $|x_0| \gg 2/\Delta k$, yields an initial Gaussian wave packet, set at a distance $|x_0|$ to the left of the barrier, and moving with velocity $v = p/m$ toward it or away from it, depending on the sign of p :

$$\langle x | \Psi(0) \rangle = \left(\frac{\pi \Delta x^2}{2} \right)^{-1/4} \exp \left[-\frac{(x-x_0)^2}{\Delta x^2} + ip(x-x_0) \right], \quad (\text{B2})$$

where $\Delta x = 2/\Delta k$. For $p < \Delta k$, the initial WP contains also negative momenta, even though it moves toward the barrier. Similarly, with $p > 0$, the WP moving now away from the barrier contains positive momenta. The first term in Eq. (A12), describing free motion of the initial wave packet, is now given by

$$\Psi(x, t) = \int_{-\infty}^\infty dk \langle k | \Psi(0) \rangle e^{ikx - iE(k)t} \quad (\text{B3}) \\ = e^{ip(x-x_0) - iE(p)t} G_0(x - pt/m, t),$$

where the envelope G , which contains the effects of spreading, is given by

$$G_0\left(x - \frac{pt}{m}, t\right) = \left(\frac{2\Delta x^2}{\pi \Delta x_t^4} \right)^{1/4} \exp \left[-\frac{(x - vt - x_0)^2}{\Delta x_t^2} \right], \quad (\text{B4}) \\ \Delta x_t \equiv \sqrt{\Delta x^2 + \frac{2it}{m}}.$$

If the spreading can be neglected, e.g., for $2t/m \ll \Delta x^2$, the envelope travels without distortion with a velocity $v = p/m$,

$$G_0\left(x - \frac{pt}{m}, t\right) \approx \left(\frac{2}{\pi \Delta x^2} \right)^{1/4} \\ \times \exp \left[-\frac{(x - vt - x_0)^2}{\Delta x^2} \right]. \quad (\text{B5})$$

Equations (B3) and (B5) are valid, we repeat, even if the initial Gaussian state (B) contains momenta in the direction opposite to the WP's velocity v .

APPENDIX C: FREE MOTION AND REFLECTION FROM AN INFINITE (HARD) WALL

To check the consistency of Eqs. (A10) and (A12), we consider two simple cases. For free motion, $V(x) \equiv 0$, $T(k) = 1$, $R(k) = 0$ for $-\infty < k < \infty$, Eq. (A10) recovers, as expected, the correct answer (B3) for $x > d$.

More interesting is the case of reflection by an infinite wall placed at some $a > 0$, $V(x) = 0$ for $x < a$, and $V(x) = \infty$ for $x \geq a$. Now, we have

$$R(k) = -\exp(2ika), \quad -\infty < k < \infty, \quad (\text{C1}) \\ \xi(x') = -\delta(x' - 2a),$$

where the negative sign ensures that the wave function vanishes at $x = a$. Now the reflected WP entering the region $x < 0$ is, apart from the change of the sign, a copy of the freely propagating initial WP, reflected about $x = a$:

$$\Psi_R(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dk R_l(k) \langle k | \Psi(0) \rangle e^{-ikx - iE(k)t} \\ = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dk \langle k | \Psi(0) \rangle e^{-ik(x-2a) - iE(k)t}. \quad (\text{C2})$$

This is also the correct result, which we will need in the following.

APPENDIX D: QUANTUM MEASUREMENTS

Consider, in one dimension, a particle (pointer) (P) with position f and momentum λ , briefly interacting with a quantum system (S),

$$\hat{H} = \hat{H}^S + \hat{H}^P + \hat{H}_{\text{int}}, \quad (\text{D1})$$

where

$$\hat{H}^P = \hat{\lambda}^2/2\mu, \quad \hat{H}_{\text{int}} = \hat{\lambda} \hat{B} \delta(t + \epsilon), \quad \epsilon \rightarrow 0, \quad (\text{D2})$$

where μ is the pointer's mass and $\hat{B} = \sum_i |b_i\rangle B_i \langle b_i|$ is the system's operator to be measured. The system is pre- and

postselected in states $|I\rangle$ and $|F\rangle$ at $t = 0$ and t , respectively. At $t = 0$, the pointer is prepared in the Gaussian state (A5) ($x \rightarrow f$, $p \rightarrow \lambda$, $\Delta x \rightarrow \Delta f$, $x_0 \rightarrow f_0 = 0$), so that the composite's initial state is given by $|\Phi(0)\rangle = |I\rangle|\Psi_0\rangle$. At time t , the observer accurately determines the pointer's position, hoping in this way to learn something about the value of $\hat{B}$ at the time the system interacted with the pointer. The pointer's state after the system was found in $|F\rangle$ at t is

$$\langle f|\langle F|\Phi(t)\rangle = \sum_i \Psi(f - B_i, t) \mathcal{A}^S(F \leftarrow b_i \leftarrow I), \quad (\text{D3})$$

where $\Psi(f - B_i, t)$ is given by Eqs. (B3) and (B4), and $\mathcal{A}_i^S$ is the system's transition amplitude,

$$\mathcal{A}^S(F \leftarrow b_i \leftarrow I) \equiv \langle F|\exp(-i\hat{H}^S t)|b_i\rangle\langle b_i|I\rangle. \quad (\text{D4})$$

It is convenient to introduce an amplitude distribution $\xi(f')$ (if required, the sum can be replaced by an integral):

$$\xi(f') \equiv \sum_i \delta(f' - B_i) \mathcal{A}^S(F \leftarrow b_i \leftarrow I), \quad (\text{D5})$$

so that Eq. (D3) can be rewritten as

$$\Phi(f, t) \equiv \langle f|\Phi(t)\rangle = \int df' \Psi(f - f', t) \xi(f'), \quad (\text{D6})$$

where $\xi(f')$ is the probability amplitude of the quantity, represented by the operator $\hat{B}$, having a value f' . Suppose we can neglect the spreading of the pointer's wave packet, prepared with mean momentum $\lambda = 0$, $\Psi(f - f', t) \approx (\frac{2}{\pi\Delta x^2})^{1/4} \exp[-\frac{(f-f')^2}{\Delta f^2}] \equiv G_0(f - f')$. Now only the values in the interval $f - \Delta f \lesssim f' \lesssim f + \Delta f$ contribute to $\Phi(f, t)$, and by determining f accurately, one measures $\hat{B}$ to a (quantum) accuracy Δf . Note that as $\Delta f \rightarrow \infty$, the probability of the postselection, $P(F)$, is no longer affected by the pointer, $P(F) \rightarrow |\langle F|\exp(-i\hat{H}^S t)|I\rangle|^2$.

If the spreading can still be neglected, but the pointer is prepared with nonzero mean momentum $\lambda \neq 0$,

$$\begin{aligned} \Phi(f, t) &\equiv \langle f|\Phi(t)\rangle = \exp[i\lambda f - iE(\lambda)t] \\ &\times \int df' G_0(f - f') \exp(-i\lambda f') \xi(f'), \end{aligned} \quad (\text{D7})$$

and finding the pointer at f one can still know that $\hat{B}$ had the value roughly in the interval $(f - \Delta f, f + \Delta f)$. However, now sending $\Delta f \rightarrow \infty$ does not fully decouple the pointer from the system, as $P(F) \rightarrow |\int df' \exp(-i\lambda f') \xi(f')|^2 \neq |\langle F|\exp(-i\hat{H}^S t)|I\rangle|^2$. The similarity between Eqs. (D6) and (8) provide the required analogy.

APPENDIX E: RECTANGULAR BARRIER AND POTENTIAL STEP

Consider a rectangular barrier $V(x)$ of height $V = \text{const}$, occupying the region $0 \leq x \leq d$. For a particle of mass m , the reflection amplitude is well known to be

$$R(k) = \frac{(k^2 - q^2) \sin qd}{2ikq \cos qd + (k^2 + q^2) \sin qd}, \quad (\text{E1})$$

where $q(k) \equiv \sqrt{k^2 - 2mV} \equiv \sqrt{k^2 - W^2}$. The amplitude is an entire function in the complex k plane, unchanged by replacement $q \rightarrow -q$ and satisfying, for real k , $R(-k) = R^*(k)$.

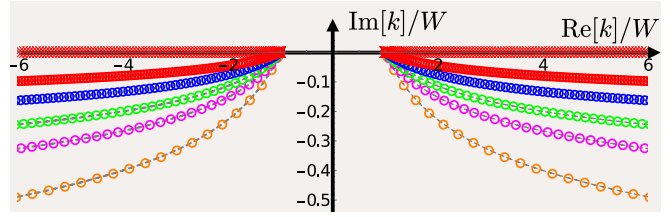


FIG. 6. The exact (numerical) poles of Eq. (E1) evaluated for $d = 10, 15, 20, 30, 50$ (circles) in orange, magenta, green, blue, and red, respectively. Also shown are the zeros of $R(k)$ for $d = 50$ (red crosses). The dashed gray lines show the approximations from Eq. (E2).

For finite d , the reflection coefficient has zeros in the real k axis and isolated poles in the lower half plane [complex zeros of the denominator in Eq. (E1)]. The zeros at $k_l = \pm\sqrt{(l\pi/d)^2 + W^2}$ correspond to energies at which the barrier is perfectly transparent. The poles, corresponding to the zeros of the denominator in Eq. (E1), are associated with the resonances (or quasistationary states) [19] at the energies above the barrier, $k^2/2m > V$.

For large d , their approximate position in the k plane is given by

$$\begin{aligned} k_n &\approx \chi_n(d) - \frac{1}{d} \frac{\beta_n(d)}{\chi_n(d)} \times \left(i \ln \left| \frac{\chi_n(d) + \beta_n(d)}{\chi_n(d) - \beta_n(d)} \right| \right. \\ &\quad \left. - \arg \left(\frac{\chi_n(d) + \beta_n(d)}{\chi_n(d) - \beta_n(d)} \right) \right) + \mathcal{O}(d^{-2}), \end{aligned} \quad (\text{E2})$$

where $\beta_n(d) \equiv n\pi/d$ and $\chi_n(d) \equiv \sqrt{\beta_n(d)^2 + W^2}$ (see Fig. 6). As $d \rightarrow \infty$, the poles approach the real axis and together with the zeros form cuts drawn from the branch points $k = \pm\sqrt{W}$ to $\pm\infty$. A similar behavior of densely coalescing poles was reported, e.g., in Ref. [28]. This branch cut corresponds to the one present in the reflection amplitude for the step potential found in Eq. (E3). Thus, to calculate $\xi(x') = (2\pi)^{-1} \int_{-\infty}^{\infty} dk \exp(-ikx') R(k)$, we may shift the contour of integration into the upper half plane by putting $k = k + i\epsilon$, $\epsilon \rightarrow 0$, so $\xi(x') = 0$ for $x' < 0$. Note that on the contour shown in Fig. 7, $q(-k) = q(k)$ for $|k| \leq W$, and $q(-k) = -q(k)$ for $|k| > W$.

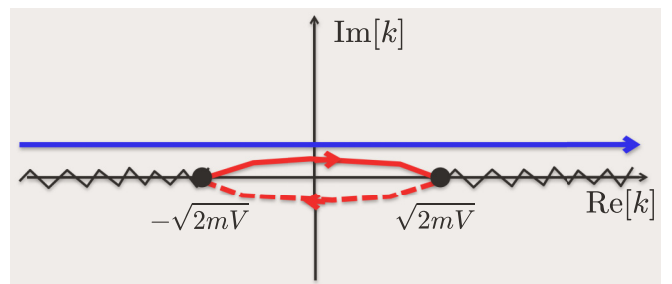


FIG. 7. For $x' > 0$, the contour of integration running just above the real k axis can be replaced by a loop around the branching points at $k = \pm\sqrt{2mV}$.

To obtain the reflection amplitude for a potential step, $V(x) = V\theta(x)$, we send $d \rightarrow \infty$ before taking the limit $\epsilon \rightarrow 0$, and retain only the growing exponentials. (For example, for $k > W$, we have $q(k + i\epsilon) \approx q(k) + ik\epsilon/q$ so that $\exp[-iq(k + i\epsilon)d] \rightarrow \infty$ as $d \rightarrow \infty$.) This gives the well-known expression

$$R(k, q(k)) = \frac{k - q}{k + q}, \quad (\text{E3})$$

which has no poles, but two branch cuts from $-\infty$ to $-W$, and from W to ∞ , formed by the coalescence of the resonance poles as d increases. To evaluate the amplitude distribution of the shifts, $\xi(x')$, we integrate along the contour passing just above the cuts (see Fig. 7). For $x' < 0$, the contour can be closed in the upper half plane where $R(k)$ has no singularities, so $\xi(x') \equiv 0$. For $x' > 0$, the contour can be transformed into a loop around the branch points as shown in Fig. 7, and we have

$$\begin{aligned} \xi(x' \geq 0) &= (2\pi)^{-1} \oint dk R(k) \exp(-ikx') \\ &= -\frac{4W}{\pi} \int_0^1 dt t \sqrt{1-t^2} \sin(Wx't), \\ &= \frac{2}{\sqrt{2mV}} \frac{d}{dx'} \left(\frac{J_1(\sqrt{2mV}x')}{x'} \right), \end{aligned} \quad (\text{E4})$$

and $\xi(x' < 0) \equiv 0$. In Eq. (E4), $J_1(z)$ is the Bessel function of the first order [27]. It is easy to check that $\int_0^\infty \xi(x') = R(p = 0) = -1$. Finally, using Eq. (9.1.30) of Ref. [27], one obtains

$$\xi(x' \geq 0) = -\frac{2}{x'} J_2(\sqrt{2mV}x'). \quad (\text{E5})$$

APPENDIX F: CENTRE OF MASS OF THE REFLECTED WAVE PACKET

Calculation of the average

$$\langle x \rangle \equiv \frac{\int_{-\infty}^{\infty} dx x |\Psi_R(x, t)|^2}{\int_{-\infty}^{\infty} dx |\Psi_R(x, t)|^2}, \quad (\text{F1})$$

where

$$\begin{aligned} \Psi_R(x, t) &= \int_{-\infty}^{\infty} dk F(k, t) \exp[-ikx], \\ F(k, t) &\equiv R_l(k) \langle k | \Psi(0) \rangle \exp[-iE(k)t], \end{aligned} \quad (\text{F2})$$

reduces to evaluation of simple quadratures. Insertion of the relations

$$\begin{aligned} \int dx \exp[i(k - k')x] &= 2\pi \delta(k - k'), \\ \int dx x \exp[i(k - k')x] &= i\pi (\partial_{k'} - \partial_k) \delta(k - k') \end{aligned} \quad (\text{F3})$$

into Eqs. (F1) and (F2) yields Eq. (48),

$$\langle x \rangle \equiv \frac{\text{Im} \left[\int_{-\infty}^{\infty} dk F^*(k, t) \partial_k F(k, t) \right]}{\int_{-\infty}^{\infty} dk |F(k, t)|^2}. \quad (\text{F4})$$

APPENDIX G: GAUSSIAN BEAM IN PARAXIAL APPROXIMATION

Consider a monoenergetic two-dimensional Gaussian beam, propagating freely along the x axis,

$$\begin{aligned} \Psi_0(x, y, t) &= \exp(-ip^2t/2m) \\ &\times \int dq \tilde{G}_0(q) \exp[iqy + i\sqrt{p^2 - q^2}(x - x_0)], \end{aligned} \quad (\text{G1})$$

where

$$\tilde{G}_0(q) = (2\pi^{-1} \Delta q^2)^{-1/4} \exp(-q^2/\Delta q^2). \quad (\text{G2})$$

If the spread of the transverse momentum Δq is small compared to p , for $x > x_0$ we have

$$\begin{aligned} \Psi_0(x, y, t) &= \exp(ip(x - x_0) - ip^2t/2m) \\ &\times \int dq \tilde{G}_0(q, T) \exp(iqy), \end{aligned} \quad (\text{G3})$$

where $T = T(x) = (x - x_0)/v$ is the time it takes a particle with velocity $v = p/m$ to travel a distance $x - x_0$, and

$$\tilde{G}_0(q, T) = (2\pi^3 \Delta q^2)^{-1/4} \exp\left(-\frac{q^2}{\Delta q^2} - i\frac{q^2 T}{2m}\right). \quad (\text{G4})$$

The integration over dq yields the paraxial approximation for the beam:

$$\Psi_0(x, y, t) = \exp(ipx - ip^2t/2m) G_0(y, T(x)), \quad (\text{G5})$$

where the envelope

$$\begin{aligned} G_0(y, T) &= \left(\frac{2\Delta y^2}{\pi \Delta y_T^4} \right)^{1/4} \exp\left[\frac{-y^2}{\Delta y_T^2} \right], \\ \Delta y_T &\equiv \sqrt{\Delta y^2 + \frac{2iT}{m}} \end{aligned} \quad (\text{G6})$$

now accounts for beam's broadening. Neglecting the broadening yields Eq. (41).

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Correction: Figure 1 was incorrectly formatted and has been replaced.