

Conservative dissipation shapes symmetry breaking in nonequilibrium currents

Matteo Sireci^{1,2,*}, Luca Peliti^{3,†} and Daniel Maria Busiello^{4,5,‡}¹*Departamento de Electromagnetismo y Física de la Materia and Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, E-18071 Granada, Spain*²*Laboratoire de Physique de l'École Normale Supérieure, CNRS, Paris, France*³*Santa Marinella Research Institute, 00058 Santa Marinella, Italy*⁴*Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany*⁵*Department of Physics and Astronomy "G. Galilei", University of Padova, Padova, Italy*

(Received 12 September 2025; accepted 26 June 2026; published 30 July 2026)

Most natural thermodynamic systems operate far from equilibrium, developing persistent currents and organizing into nonequilibrium stationary states (NESSs). Yet, the general principles governing how nonequilibrium driving reshapes probability distributions and breaks equilibrium symmetries remain poorly understood. Here, we establish a geometric framework connecting dissipation, current organization, and symmetry breaking in mesoscopic stochastic systems governed by Langevin dynamics. Using the inverse diffusion matrix as a metric, we derive the general geometric properties of stationary current velocities, employing a small-noise expansion to obtain further insights on their link to symmetry breaking. We then introduce a nonorthogonal decomposition of the velocity field into conservative and nonconservative components. The nonconservative contribution quantifies the violation of detailed balance induced by external driving, while the conservative one measures how the system reorganizes its internal state space relative to a reference equilibrium distribution. This decomposition naturally induces a splitting of entropy production and leads to an exact balance relation together with bounds defining the accessible thermodynamic region of a NESS. We show that the conservative contribution determines how nonequilibrium conditions amplify or generate symmetry-breaking phenomena, yielding a geometric classification of NESSs. We apply the framework to systems ranging from molecular machines to coupled oscillators and Brownian gyrators, and extend it to multiplicative-noise dynamics, where additional symmetry breaking emerges from curvature induced by space-dependent diffusion. Our results uncover a general connection between geometry, dissipation, and emergent organization in nonequilibrium thermodynamics.

DOI: [10.1103/4rsq-ybmk](https://doi.org/10.1103/4rsq-ybmk)

I. INTRODUCTION

In the natural world, thermodynamic systems are generally away from equilibrium. This particular state might be caused by environmental conditions, such as the presence of multiple thermal baths [1], nonconservative forces [2], and odd diffusion [3], or by internal features, such as energy consuming many-body interactions [4–6] and chemical dissipation [7]. Under such external conditions, thermodynamic systems depart from equilibrium and reorganize into a nonequilibrium stationary state (NESS). This departure typically involves two intertwined processes: on the one hand, the emergence of sustained currents (whether of mass, heat, electricity, or

molecules) and, on the other, a modification of the system's internal state.

Nonequilibrium stationary states can exhibit structural properties that qualitatively differ from their equilibrium counterparts. In particular, the presence of currents often leads to symmetry breaking and the emergence of geometric and topological order. A simple example is the appearance of vortices [8] or a preferred direction for heat flow between reservoirs at different temperatures [9,10]. More broadly, symmetry breaking has long been associated with dissipation and self-organization in nonequilibrium systems, following Prigogine's concept of dissipative structures [11–13]. Despite these insights, identifying universal symmetry-breaking principles governing NESSs remains an open problem.

The last decades have witnessed the discovery of fluctuation theorems [14–18], universal relations holding arbitrarily far from equilibrium. The core message of these results is that nonequilibrium systems break time-reversal symmetry while preserving a weaker version of it that quantifies dissipation. It is natural to think that breaking of the time symmetry would induce symmetry breaking in other aspects of the system, hinting at the existence of additional universal features.

In this work, we make a step toward the understanding of these open questions by providing a unified geometric

*Contact author: matteo.sireci@phys.ens.fr†Contact author: luca@peliti.org‡Contact author: danielmaria.busiello@unipd.it

framework that links the emergence of currents to symmetry breaking in NESSs. Through our approach, we identify the features of external nonequilibrium conditions that lead to dissipation and those that contribute to a change in the system's internal organization.

A. Outline of the work and summary of the results

We focus on continuous-space mesoscopic systems described by stochastic thermodynamics [19], a framework encompassing well-known, experimentally realizable examples from molecular machines [2] to chemical reaction networks [7] and colloidal particles [20].

In Sec. II, we present a general framework based on Langevin dynamics with additive noise. In particular, in Secs. II A and II B, by adopting the (inverse) diffusion matrix as a natural metric in configuration space, we formulate thermodynamics in terms of the current velocity and a generalized free energy, also named *self-information*, revealing their geometric and thermodynamic roles. In Sec. II C, we derive general geometric properties on both stationary velocity and self-information. To do so, we employ a weak-noise expansion of the self-information showing that, at the lowest order, the stationary velocity is divergence-free. At the next noise order, an additional gradient term appears and breaks the divergence-free condition, thereby modifying the topology of the current.

To characterize the connection between geometric properties of currents and thermodynamics, in Sec. III, we propose a natural decomposition of the velocity into a gradient and a residual part. Such decomposition is not unique since it relies on the choice of an equilibrium *reference* state. We remark that this gauge freedom reflects the fact that different physical mechanisms might drive the system into the same NESS. However, independently of this choice, the residual (*non-conservative*) component measures the contribution breaking detailed balance, while the gradient (*conservative*) one reflects the system's self-organization, connected to the relative entropy with respect to the equilibrium reference state. Furthermore, these two contributions are not perpendicular, but statistically oriented one against the other. These results lead to a decomposition of the entropy production, implying that, at stationarity, its value equals the nonconservative dissipation *minus* the conservative one. From these results, we derive an upper bound for the steady-state dissipation and interpret the conservative contribution as the energy absorbed from the environment to reshape the equilibrium state into the NESS.

In Secs. IV–VI, we apply this geometric decomposition to classify NESS thermodynamics. Specifically, we define in *local equilibrium* a system that satisfies generalized Maxwell-Boltzmann stationary distributions, in which the conservative velocity is harmonic and the self-information minimum coincides with the equilibrium one. If the two minima do not coincide but the mentioned geometric constraints are satisfied, we call the system *locally balanced*. Further away from equilibrium, we name *locally closed* a system characterized by closed nonconservative fluxes in the trajectory space. In this last case, we show that the geometry and topology of the velocity are dependent on the noise strength: the conservative

term can develop thermodynamic fluxes with nonzero divergence, creating repellers and increasing dissipation.

In Sec. V, we extend the framework to multiplicative noise, where variable-dependent diffusion induces nonzero curvature in state space, adding “thermophoreticlike” terms to the velocity and stabilizing high-energy states. In Sec. V C, we study the underdamped case in a temperature gradient, emphasizing the role of reversible and irreversible velocity components in the full phase space.

Finally, Sec. VI shows how our framework could be used to quantify the thermodynamics of nonequilibrium phase transitions. We analyze many-body coupled molecular oscillators undergoing a synchronization phase transition, revealing that nonconservative dissipation links to control parameters while the conservative component arises from the order parameter, shaping a complex velocity geometry reflecting free-energy transduction.

II. SYMMETRY BREAKING OF NONEQUILIBRIUM CURRENTS

A. Thermodynamics of self-information and current velocity

In this section, we set the mathematical stage by formulating stochastic thermodynamics in a geometric perspective using the inverse diffusion matrix as a metric [21]. Then, we introduce the main actors: the self-information (or nonequilibrium free energy) and the current velocity [22–25].

We consider a system in d dimensions, whose state is described by the variable $\mathbf{x} = (x^i)$, which evolves according to the Langevin equation in configuration space:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \boldsymbol{\xi}(t), \quad (1)$$

where $\mathbf{F}$ is an external drift, which may be conservative (i.e., stemming from a potential energy) or nonconservative, and $\boldsymbol{\xi}(t) = (\xi^i(t))$ is a zero-mean Gaussian white-noise process that satisfies

$$\langle \xi^i(t) \xi^j(t') \rangle_{\xi} = 2D^{ij} \delta(t - t'), \quad i, j = 1, \dots, d, \quad (2)$$

with a given, symmetric and positive-definite diffusion matrix $\mathbf{D} = (D^{ij})$. We write the indices as exponents for further convenience and denote by $\langle \cdot \cdot \rangle_{\xi}$ the average taken with respect to the noise. In the present case, the diffusion matrix is independent of $\mathbf{x}$; hence, the manifold of the configuration space is flat. We define the *diffusive* scalar product $\mathbf{a} * \mathbf{b}$ between the two vectors $\mathbf{a} = (a^i)$ and $\mathbf{b} = (b^i)$ by

$$\mathbf{a} * \mathbf{b} = a^i D_{ij} b^j, \quad (3)$$

where $D = (D_{ij})$ is the matrix inverse of $\mathbf{D} = (D^{ij})$. Throughout the whole article, we use the Einstein convention for summing over repeated indices, one upper and one lower. Thus, $\mathbf{D}^{-1} = D$ acts as the metrics in the state space, as already noted in Refs. [21,26]. This geometric structure can be viewed as a generalization of the factor $\beta = 1/k_B T$, with the inverse diffusivities entering naturally in the diffusive scalar product; i.e., the effective length scale along each direction is inversely proportional to its diffusivity. Then, the derivative of any scalar field $\phi(\mathbf{x})$ is straightforwardly defined as $\partial_i \phi$. Likewise, the *covariant* derivative of a vector field $\mathbf{w} = (w^i)$ is given by $\partial_j w^i$. On the other hand, the *contravariant* derivative

of the scalar field ϕ is given by

$$(\nabla\phi)^i = D^{ij}\partial_j\phi, \quad (4)$$

which is often called the gradient of ϕ . Finally, we consider the divergence of a vector field as the covariant derivative summed with respect to repeated indices, i.e., $\nabla * \mathbf{w} = \partial_i w^i$, which coincides with the scalar product of the gradient and the field. With this formalism, the Langevin equation (1) corresponds to the following Fokker-Planck equation for the probability P of finding the system at time t in the state $\mathbf{x}$:

$$\partial_t P = -\nabla * (\mathbf{F} P - \nabla P) = -\nabla * \mathbf{J}, \quad (5)$$

where the probability current $\mathbf{J} = (J^i)$ is given by

$$J^i = F^i P - D^{ij}\partial_j P = F^i - (\nabla P)^i. \quad (6)$$

Since we do not consider boundary-driven systems for the time being, we take boundary conditions for which $P \rightarrow 0$ for $|\mathbf{x}| \rightarrow \infty$ or for $\mathbf{x}$ approaching the boundary of the allowed state space.

Given a time-independent drift $\mathbf{F}(\mathbf{x})$, the system eventually reaches a stationary distribution, P_{st} , which can be characterized in terms of a nonequilibrium free energy $\mathcal{F}(\mathbf{x})$ (hereafter called self-information [25]):

$$P_{\text{st}}(\mathbf{x}) \propto e^{-\mathcal{F}(\mathbf{x})} \quad (7)$$

and which satisfies $\partial_t P = 0$. Then, we define the current velocity $\mathbf{v}(\mathbf{x}, t)$ via the relation

$$\mathbf{v} = \frac{\mathbf{J}}{P}, \quad (8)$$

which becomes equal to $\mathbf{v}_{\text{st}}(\mathbf{x})$ at stationarity. In terms of this steady current velocity, the stationarity condition takes the form

$$\nabla * \mathbf{v}_{\text{st}} = \mathbf{v}_{\text{st}} * \nabla \mathcal{F} = \nabla_{\mathbf{v}_{\text{st}}} \mathcal{F}. \quad (9)$$

The left-hand side represents the divergence of the stationary velocity, while the right-hand side represents the derivative of the self-information with respect to the local velocity, i.e., the rate of change of $\mathcal{F}$ along the current.

Following a standard approach in thermodynamics, we evaluate the time derivative of the average of a generic thermodynamic quantity $A(\mathbf{x}, t)$ by

$$d_t \langle A \rangle = \langle \mathbf{v} * \nabla A \rangle + \langle \dot{A} \rangle, \quad (10)$$

where $\langle \dots \rangle$ denotes the average over the probability distribution:

$$\langle \dots \rangle = \int d\mathbf{x} \dots P(\mathbf{x}). \quad (11)$$

The change induced by the explicit time dependence of A is due to driving mechanisms, while the one along the velocity defines the flux of A and emerges by averaging the change along a single trajectory over the noise [27,28]:

$$A_{\text{flux}} = \langle \mathbf{v} * \nabla A(\mathbf{x}) \rangle = \langle \dot{\mathbf{x}} * \nabla A(\mathbf{x}) \rangle_{\xi}. \quad (12)$$

Note that a positive sign indicates a flux from the system to the environment, and vice versa for a negative sign. As a particular case, if we consider the system Gibbs-Shannon entropy $S_{\text{sys}} = -\int d\mathbf{x} P \ln P$, its time derivative decomposes

into the total and environmental entropy productions that, in our notation, take the following form [24]:

$$\begin{aligned} \dot{S}_{\text{sys}} &= \dot{S}_{\text{tot}} - \dot{S}_{\text{env}}, \\ \dot{S}_{\text{tot}} &= \int d\mathbf{x} P v^i D_{ij} v^j = \langle \mathbf{v} * \mathbf{v} \rangle \geq 0, \\ \dot{S}_{\text{env}} &= \int d\mathbf{x} P(x) v^i D_{ij} F^j = \langle \mathbf{v} * \mathbf{F} \rangle. \end{aligned} \quad (13)$$

B. Orthogonality relations for the velocity

Equation (13) suggests that the velocity plays a more fundamental, yet underappreciated, role than the current $\mathbf{J}$. In particular, Eq. (9) implies that, while the stationary current is necessarily divergence-free, the stationary velocity is not. Therefore, we expect $\mathbf{v}_{\text{st}}$ to exhibit a richer geometric structure with nontrivial interplay with dissipative thermodynamic quantities that characterize a nonequilibrium steady state. Specifically, here we show two orthogonality conditions for the velocity field that represent the first steps of this study. Following Hatano and Sasa [29,30], we start by decomposing the velocity into two contributions, one reflecting the steady nonequilibrium condition (i.e., *adiabatic*, $\mathbf{v}_a$) and the other the time-dependent drivings or relaxation (i.e., *nonadiabatic*, $\mathbf{v}_{\text{na}}$):

$$\mathbf{v} = \mathbf{v}_a + \mathbf{v}_{\text{na}}. \quad (14)$$

Given the absence of time-dependent driving, these two contributions are given by

$$\mathbf{v}_a^i = \mathbf{v}_{\text{st}}^i = F - D^{ij}\partial_j \ln P_{\text{st}} = F + D^{ij}\partial_j \mathcal{F}, \quad (15)$$

$$\mathbf{v}_{\text{na}}^i = -D^{ij}\partial_j \ln \frac{P}{P_{\text{st}}} = D^{ij}\partial_j \mathcal{F}_{\text{ex}}(\mathbf{x}, t), \quad (16)$$

where we have defined the *excess self-information*, $\mathcal{F}_{\text{ex}} = \mathcal{F} - \mathcal{F}_{\text{st}}$, with $\mathcal{F}_{\text{st}} = -\ln P_{\text{st}}$. Thanks to these definitions, we can rewrite the Fokker-Planck equation solely in terms of velocity and time-dependent self-information, $\mathcal{F}(t) = -\ln P(\mathbf{x}, t)$:

$$\partial_t \mathcal{F}(t) + \mathbf{v} * \nabla \mathcal{F}(t) = \nabla * \mathbf{v}. \quad (17)$$

By taking the average of Eq. (17), we are able to connect the change in system entropy to the average velocity divergence:

$$\dot{S}_{\text{sys}} = d_t \langle \mathcal{F}(t) \rangle = \langle \mathbf{v} * \nabla \mathcal{F}(t) \rangle = \langle \nabla * \mathbf{v} \rangle. \quad (18)$$

Equations (9), (17), and (18) show that the divergence of the velocity field carries a thermodynamic meaning: a positive divergence corresponds to entropy-increasing expansions, while a negative divergence corresponds to compressions that store work in the system.

Using the definitions of adiabatic and nonadiabatic velocities [Eqs. (15) and (16)] directly from the stationarity condition [Eq. (9)], we obtain that $\mathbf{v}_a$ and $\mathbf{v}_{\text{na}}$ are on average orthogonal to each other at any time:

$$\langle \mathbf{v}_a * \mathbf{v}_{\text{na}} \rangle = \int d\mathbf{x} P v_a^i D_{ij} v_{\text{na}}^j = 0, \quad \forall t. \quad (19)$$

Equation (19) implies the standard decomposition of the entropy production into adiabatic and nonadiabatic contributions

as well:

$$\dot{S}_{\text{tot}} = \langle \mathbf{v} * \mathbf{v} \rangle = \dot{S}_a + \dot{S}_{\text{na}}, \quad (20)$$

where

$$\dot{S}_a = \langle \mathbf{v} * \mathbf{v}_a \rangle = \langle \mathbf{v}_a * \mathbf{v}_a \rangle \geq 0, \quad (21)$$

$$\dot{S}_{\text{na}} = \langle \mathbf{v} * \mathbf{v}_{\text{na}} \rangle = \langle \mathbf{v}_{\text{na}} * \mathbf{v}_{\text{na}} \rangle \geq 0. \quad (22)$$

At the stationary state, the nonadiabatic entropy production vanishes, and thus the adiabatic one yields the total entropy production:

$$\dot{S}_{\text{na}}^{\text{st}} = 0, \quad \dot{S}_{\text{tot}}^{\text{st}} = \dot{S}_a^{\text{st}}. \quad (23)$$

To derive the second orthogonality condition, let $\phi(\mathbf{x})$ be an arbitrary time-independent scalar field. From Eq. (10), we obtain that the derivative of its average is given by its flux:

$$\frac{d}{dt} \langle \phi \rangle = \langle \mathbf{v} * \nabla \phi \rangle = \phi_{\text{flux}}. \quad (24)$$

Since its average is constant at stationarity, we obtain

$$\phi_{\text{flux}}^{\text{st}} = \langle \mathbf{v}_a * \nabla \phi \rangle^{\text{st}} = 0. \quad (25)$$

This holds for any ϕ and in particular for the special choice $\phi = \mathcal{F}$. Therefore, Eq. (25) implies that, at the NESS, the adiabatic velocity is orthogonal on average to the contour lines of the self-information:

$$\frac{d}{dt} \langle \mathcal{F} \rangle^{\text{st}} = \mathcal{F}_{\text{flux}}^{\text{st}} = \langle \mathbf{v}_a * \nabla \mathcal{F} \rangle^{\text{st}} = 0. \quad (26)$$

Finally, from Eqs. (15) and (26), the drift $\mathbf{F}$ can be naturally decomposed into two (on average) orthogonal parts, one dissipative, $\mathbf{v}_a$, and a potential one, $\nabla \mathcal{F}$:

$$\mathbf{F} = \mathbf{v}_a - \nabla \mathcal{F}. \quad (27)$$

C. Characterizing the velocity geometry with a small-noise expansion

To advance in the general characterization of the geometry of the velocity field, we need to assume some minimal properties of the self-information. In the following, we show that an expansion in the noise strength is enough to find general properties of the divergence of the adiabatic velocity field.

1. Weak-noise limit

Following the pioneering works of Graham and Tél [22,31,32], together with recent developments in Macroscopic Fluctuation Theory [6,33,34], we propose here that there exists a single noise amplitude σ that controls all the elements of the diffusion matrix, so that in the weak-noise regime, the stationary probability distribution exhibits the following limiting behavior:

$$P_{\text{st}} \asymp e^{-\mathcal{F}^{(0)}/\sigma}, \quad (28)$$

where $\mathcal{F}^{(0)}$ is the rescaled zeroth-order self-information, which also constitutes the large deviation or rate function of the system. Furthermore, in this context, $\mathcal{F}^{(0)}$ is a quasipotential that can be rigorously defined and estimated via path-integral methods [6,33]. From Eq. (28), the adiabatic velocity is then composed by a contribution from the zeroth order, i.e., $\mathbf{v}_a = \mathbf{F} + \nabla \mathcal{F}^{(0)}$, where, with abuse of notation,

we identify $\nabla^i = D^{ij} \partial_j$, without the factor σ that rescales the diffusion matrix. By taking advantage of this general form for the stationary distribution, we uncover the geometrical properties of the velocity $\mathbf{v}_a$ in the limit of weak noise, i.e., at the leading order in σ . Plugging Eq. (28) into the Fokker-Planck equation, expanding up to the zeroth order in σ , and equating terms of the same order, we get two consistency equations at stationarity (see Appendix A for details):

$$\mathbf{v}_a * \nabla \mathcal{F}^{(0)} = 0, \quad \nabla * \mathbf{v}_a = 0. \quad (29)$$

The first relation states that $\mathbf{v}_a$ must be tangent to the quasipotential contour lines, here identified with $\mathcal{F}^{(0)}$, while the second one dictates that it also has to be a divergence-free field. These conditions imply that, at the minimum of the self-information, the current is zero.

2. Beyond weak noise: An additional gradient term in the velocity

When pushing the system beyond the weak-noise regime, the properties of the velocities are expected to change. This scenario is particularly relevant for mesoscopic thermodynamic systems in which σ can be identified with the temperature of the bath. Hence, starting from the rate function in Eq. (28), we consider an additional term in the expansion of the availability:

$$\mathcal{F} \approx \frac{1}{\sigma} \mathcal{F}^{(0)} + \mathcal{F}^{(1)}. \quad (30)$$

From the decomposition in Eq. (27), in addition to the zeroth-order term, we obtain a first-order contribution in the form of a gradient (see Appendix A for details):

$$\mathbf{v}_a = \mathbf{v}_a^{(0)} + \sigma \mathbf{v}_a^{(1)}, \quad (31)$$

$$\mathbf{v}_a^{(0)} = \mathbf{F} + \nabla \mathcal{F}^{(0)}, \quad (32)$$

$$\mathbf{v}_a^{(1)} = \nabla \mathcal{F}^{(1)} = \nabla \mathcal{F} - \nabla \mathcal{F}^{(0)}/\sigma. \quad (33)$$

By solving perturbatively for the stationary solution of Fokker-Planck equation and equating the terms of order σ^{-1} , we discover that the zeroth-order velocity is still orthogonal to the gradient of the quasipotential:

$$\mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(0)} = 0. \quad (34)$$

From the terms at following order, σ^0 , we see that its divergence is not zero:

$$\nabla * \mathbf{v}_a^{(0)} = \mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(1)} + \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(0)}. \quad (35)$$

Finally, the conditions emerging at order σ determine that the first-order velocity has a positive divergence:

$$\nabla * \mathbf{v}_a^{(1)} = \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(1)} = \mathbf{v}_a^{(1)} * \mathbf{v}_a^{(1)} \geq 0, \quad (36)$$

thus implying that this term will generate repellers.

By inspecting the average fluxes of self-information, it is immediate to derive additional geometric properties of the adiabatic velocity that hold on average (see Appendix A). First, we find an orthogonality condition for the first-order velocity:

$$\langle \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(0)} \rangle_{\text{st}} = 0, \quad (37)$$

implying that it is on average perpendicular to the contour lines of the quasipotential. Second, we can characterize the average

sign of the divergencies of both velocities:

$$\langle \nabla * \mathbf{v}_a^{(0)} \rangle = -\sigma \langle \nabla * \mathbf{v}_a^{(1)} \rangle \leq 0, \quad (38)$$

implying that the first-order self-information decreases along the zero-order velocity:

$$\langle \mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(1)} \rangle \leq 0. \quad (39)$$

III. GEOMETRIC DECOMPOSITION OF VELOCITY AND ENTROPY PRODUCTION

Our aim is to understand how the geometric properties of the velocity derived in Secs. II B and II C shape the thermodynamics of nonequilibrium steady states and induce symmetry breaking with respect to the equilibrium scenario. To this end, we introduce here an additional decomposition of the velocity field into a nonconservative component, responsible for breaking detailed balance and creating dissipative currents, and a conservative one, measured as the distance from a reference equilibrium distribution, which dictates how nonequilibrium shapes internal organization and symmetry-breaking phenomena.

A. Geometry identifies conservative and nonconservative velocities

A necessary condition for equilibrium is detailed balance or, equivalently, a vanishing adiabatic current, $\mathbf{J}_a = 0$. From Eq. (27), this implies that the force derives from a scalar potential, $\mathbf{F} = \nabla \ln P_{\text{eq}}$. In the presence of a single current and assuming Einstein's relation, this reduces to the integrability condition

$$\partial_i F_j = \partial_j F_i, \quad (40)$$

where $F_i = (D^{-1}\mathbf{F})_i$.

Away from equilibrium, the orthogonality conditions in Eq. (25), together with the divergence properties derived in Sec. II C, suggest a natural generalization of detailed balance. Specifically, the velocity can be decomposed into two components: one that breaks detailed balance and another that can be written as a gradient of a scalar potential and, as such, can be “reabsorbed” into a reference equilibrium distribution P_{eq}^* . By doing so, we find a decomposition for the adiabatic velocity into a conservative part (indicated by $\mathbf{v}_c$), which is purely gradient, and a residual part that we call nonconservative (and indicate by $\mathbf{v}_{\text{nc}}$):

$$\mathbf{v}_a = \mathbf{v}_{\text{nc}} + \mathbf{v}_c = \mathbf{v}_{\text{nc}} + \nabla \mathcal{A}, \quad (41)$$

with

$$\mathbf{v}_c^i = -D^{ij} \partial_j \ln \frac{P_{\text{st}}}{P_{\text{eq}}^*} \equiv D^{ij} \partial_j \mathcal{A}, \quad (42)$$

$$\partial_i (D_{jk} v_c^k) = \partial_j (D_{ik} v_c^k). \quad (43)$$

The scalar field $\mathcal{A}$, which we call the *availability* [35], quantifies the deviation from the reference equilibrium state. This decomposition is not unique in general, as it depends on the choice of the equilibrium reference distribution P_{eq}^* . However, as we show in the following, physical considerations select a natural choice—analogue to a gauge fixing—such that $\mathcal{A}$ acquires the meaning of the available work that can be extracted

from the system if the nonconservative constraints encoded in $\mathbf{v}_{\text{nc}}$ are removed and converges to P_{eq}^* :

$$S_{\text{rel}} = - \int dx P_{\text{st}}(x) \ln \frac{P_{\text{st}}(x)}{P_{\text{eq}}^*(x)} = \langle \mathcal{A}(t) \rangle. \quad (44)$$

In Appendix B, we explicitly connect the nonconservative velocity to the tendency to circulate, thereby breaking a parity symmetry at the trajectory level. We additionally show that the conservative component plays a role in selecting open trajectory over their time reversal in a NESS. These observations provide a link between symmetry-breaking phenomena in the trajectory space and the proposed geometric decomposition.

B. Geometric decomposition of dissipation encodes external conditions and internal organization

To show how the above decomposition naturally partitions dissipation, we consider the dynamics leading to the NESS. By combining the adiabatic decomposition [Eqs. (15) and (16)] with the geometric one [Eqs. (41) and (42)], we notice that the nonadiabatic component can be naturally included into the conservative one:

$$\mathbf{v} = \mathbf{v}_{\text{nc}} + \mathbf{v}_c + \mathbf{v}_{\text{na}}(t) \quad (45)$$

$$= \mathbf{v}_{\text{nc}} + \mathbf{v}_c(t) = \mathbf{v}_{\text{nc}} + \nabla \mathcal{A}(t), \quad (46)$$

with

$$\mathbf{v}_c(t) = \nabla \mathcal{A}(t), \quad (47)$$

$$\mathcal{A}(t) = \mathcal{A} + \mathcal{F}_{\text{ex}}(t), \quad (48)$$

where $\mathcal{A}(t)$ includes both the availability and the excess self-information and is named *time-dependent availability*. The evolution of its average reads

$$\begin{aligned} \frac{d}{dt} \langle \mathcal{A}(t) \rangle &= \langle \mathbf{v} * \nabla \mathcal{A}(t) \rangle + \langle \partial_t \mathcal{A}(t) \rangle \\ &= \langle \mathbf{v}_c(t) * \mathbf{v}_c(t) \rangle + \langle \mathbf{v}_{\text{nc}} * \nabla \mathcal{A}(t) \rangle + \langle \partial_t \mathcal{A}(t) \rangle, \end{aligned} \quad (49)$$

by using Eq. (47). The first term on the right-hand side is non-negative and defines the *conservative entropy production*:

$$\dot{S}_c(t) = \langle \mathbf{v}_c(t) * \mathbf{v}_c(t) \rangle \geq 0. \quad (50)$$

The second term represents the availability flux along the nonconservative velocity.

Using Eq. (49) together with the orthogonality condition in Eq. (25), at stationarity we obtain

$$\dot{S}_c^{\text{st}} = -\langle \mathbf{v}_{\text{nc}} * \nabla \mathcal{A} \rangle^{\text{st}} = -\langle \mathbf{v}_{\text{nc}} * \mathbf{v}_c \rangle^{\text{st}} \geq 0. \quad (51)$$

Similarly, we define the *nonconservative entropy production* as

$$\dot{S}_{\text{nc}} = \langle \mathbf{v}_{\text{nc}} * \mathbf{v}_{\text{nc}} \rangle \geq 0. \quad (52)$$

The total entropy production can then be written as

$$\dot{S}_{\text{tot}} = \dot{S}_{\text{nc}} + \dot{S}_c + 2\langle \mathbf{v}_{\text{nc}} * \nabla \mathcal{A} \rangle. \quad (53)$$

Combining this expression with Eq. (51), we obtain our first main result:

$$\dot{S}_{\text{tot}}^{\text{st}} = \dot{S}_{\text{nc}}^{\text{st}} - \dot{S}_c^{\text{st}}, \quad (54)$$

which immediately implies the bounds

$$\dot{S}_{\text{tot}}^{\text{st}} \leq \dot{S}_{\text{nc}}^{\text{st}}, \quad \dot{S}_{\text{c}}^{\text{st}} \leq \dot{S}_{\text{nc}}^{\text{st}}. \quad (55)$$

This decomposition admits a clear thermodynamic interpretation. When driven out of equilibrium, the system sustains dissipative currents through the nonconservative velocity $\mathbf{v}_{\text{nc}}$, which contribute to the energy released into the environment. At the same time, the system reorganizes its internal state, modifying its availability with respect to the reference equilibrium distribution. This reorganization is encoded in the conservative velocity and gives rise to the term $\dot{S}_{\text{c}}$. If the nonconservative component vanished, the dynamics would reduce to a pure gradient flow and the system would relax to an effective equilibrium state. Conversely, a nonzero $\mathbf{v}_{\text{nc}}$ maintains the system out of equilibrium, while the conservative dissipation $\dot{S}_{\text{c}}^{\text{st}}$ quantifies which part of the injected energy coming from the nonequilibrium driving is being redistributed into the internal degrees of freedom to sustain organization. Equation (51) expresses this accumulation of injected energy and geometrically constrains the conservative response to be statistically oriented against the nonconservative driving. This energetic balance is exactly captured by Eq. (54) that states that the net dissipation is given by the nonconservative contribution minus the part stored in reshaping the stationary distribution, for instance, by populating higher-energy states compared to the reference equilibrium.

The conservative dissipation $\dot{S}_{\text{c}}^{\text{st}}$ can be identified with the minimum energetic cost $\dot{S}_{\text{min}}$ required to maintain a given NESS using nonequilibrium controls, provided that the uncontrolled equilibrium distribution coincides with the reference distribution P_{eq}^* [36]. Despite this correspondence, $\dot{S}_{\text{min}}$ was introduced in a framework where the dynamics is explicitly decomposed into an uncontrolled part and a set of driven transitions, so that the total entropy production can be resolved into contributions associated with these two processes. In contrast, our decomposition does not rely, in general, on such a separation. As discussed at the end of the paper, extending our framework to Markov jump processes may help clarify the conditions under which the setting of Ref. [36] is naturally recovered and whether the combination of the two approaches yields tighter bounds on the nonconservative entropy production.

The idea of geometrically decomposing current velocities and entropy production is not new in stochastic thermodynamics. In particular, Dechant and collaborators showed that several excess/housekeeping decompositions, including the Hatano-Sasa [29] and Maes-Netočný [37] constructions, can be understood by splitting the velocity field into a gradient contribution and an orthogonal residual [38–40]. In the Hatano-Sasa framework, the stationary state is taken as reference: the steady velocity defines the housekeeping contribution, while the excess part is identified with the gradient of the distance between the instantaneous distribution and the stationary one (in this study, this corresponds to the adiabatic/nonadiabatic decomposition). Conversely, the Maes-Netočný construction identifies as excess the minimum dissipative conservative driving reproducing the dynamics, while the remaining orthogonal component defines the housekeeping contribution.

Our decomposition shares the same geometric spirit, but differs conceptually in two important ways. First, instead of relying on orthogonal contributions, it separates the dynamics into a nonconservative component sustaining irreversible currents and a conservative one quantifying the reorganization of the stationary distribution with respect to a reference equilibrium state. Equation (51) shows that the projection between the two contributions measures the extractable work required to maintain the NESS away from equilibrium. Second, it is not unique. Thermodynamics has been shown to be a gauge theory, with a prior distribution playing the role of the gauge choice [41,42]. In this view, thermodynamic potentials are not uniquely defined. However, physically measurable quantities, such as the entropy production, are indeed gauge invariant. Hence, as in other physical theories, such as electromagnetism and hydrodynamics, in our framework, one can fix a gauge to make different symmetry-breaking mechanisms transparent and highlight how dissipative processes contribute to organize the deviations from different reference equilibrium states.

IV. GEOMETRIC DECOMPOSITION SHAPES AND CLASSIFIES SYMMETRY BREAKING

We are finally ready to combine the general geometric properties of a NESS derived in Sec. II and the decomposition introduced in Sec. III to derive a NESS characterization scheme that reveals how currents break equilibrium symmetries, potentially leading to internal organization. We introduce and discuss the concepts of locally closed, locally balanced, and local equilibrium systems. In Table I, we report a brief summary of this classification.

Let us start from Eq. (9) that dictates the relationship between the adiabatic (stationary) velocity $\mathbf{v}_{\text{a}}$ and $\mathcal{F}$ in a generic NESS. In the most general case, we would expect the divergence of $\mathbf{v}_{\text{a}}$ not to vanish:

$$\nabla * \mathbf{v}_{\text{a}} = \mathbf{v}_{\text{a}} * \nabla \mathcal{F} \neq 0. \quad (56)$$

As a consequence, the velocity field would not have a simple geometry, eventually admitting the presence of sinks and sources. We call *open* those systems that exhibit such complex properties, and their geometry and thermodynamics will be studied in Sec. IV B.

A relevant class of nonequilibrium systems, however, satisfies a stringent condition on the velocity:

$$\nabla * \mathbf{v}_{\text{a}} = 0 = \mathbf{v}_{\text{a}} * \nabla \mathcal{F}, \quad (57)$$

where the last equality comes from Eq. (9). Therefore, the adiabatic velocity must be tangent to the contour lines of the self-information, so that the NESS is characterized by closed fluxes in the configuration space. We name these systems *locally closed*, because the stationary probability does not undergo local expansion or contraction along the velocity field, in direct analogy with incompressible flow in fluid dynamics [see Eq. (17)]. From the geometric results in Sec. II, we can conclude that any nonequilibrium system with additive noise is locally closed in the weak-noise limit, as also remarked by the last line of Table I. In the weak-noise regime, by employing the geometric decomposition

$$\mathbf{v}_{\text{a}}^{(0)} = \mathbf{v}_{\text{nc}} + \mathbf{v}_{\text{c}}^{(0)}, \quad (58)$$

TABLE I. NESS characterization. Classes toward the right require increasingly stringent conditions. The first three rows determine the geometric characterization discussed in Sec. IV, while the last row indicates noise expansion properties (Sec. IIC).

	Open	Locally closed	Locally balanced	Local equilibrium
Self-information	$\nabla * \mathbf{v}_a = \mathbf{v}_a * \nabla \mathcal{F}$	$\mathbf{v}_a * \nabla \mathcal{F} = 0$	$\nabla^2 \mathcal{A} = 0$	Same as locally balanced
Velocity	$\mathbf{v}_a = \mathbf{v}_{nc} + \mathbf{v}_c$	$\nabla * \mathbf{v}_{nc} = -\nabla * \mathbf{v}_c$	$\nabla * \mathbf{v}_{nc} = 0$	Same as locally balanced
Extreme points	$\nabla \mathcal{F}^* = 0$	$\mathbf{v}_{nc}^* = -\mathbf{v}_c^* = \beta \nabla U^*$	Same as locally closed	$\nabla \mathcal{F}^* = \beta \nabla U^* = 0$
Noise expansion	$P_{st} \sim e^{-\mathcal{F}^{(0)}/\sigma - \mathcal{F}^{(1)}}$	$P_{st} \asymp e^{-\mathcal{F}^{(0)}/\sigma}$	Same as locally closed	$P_{st} = Z^{-1} e^{-\mathcal{F}^{(0)}/\sigma}$

$$\mathbf{v}_c^{(0)} = \nabla \mathcal{A} = \nabla \mathcal{F}^{(0)} - \beta \nabla U, \quad (59)$$

we see that there is no *a priori* constraint on the divergences of conservative and nonconservative contributions, but they need to cancel out:

$$\nabla * \mathbf{v}_{nc} = -\nabla * \mathbf{v}_c. \quad (60)$$

Moreover, again from Eq. (57), we obtain that the adiabatic velocity must be zero at the extreme points of the self-information, i.e., where P_{st} is maximized:

$$\nabla \mathcal{F}(\mathbf{x}^*) \equiv \nabla \mathcal{F}^* = 0 \rightarrow \mathbf{v}_a(\mathbf{x}^*) = 0, \quad (61)$$

with the short-hand notation $X^* = X(\mathbf{x}^*)$. Hence, for locally closed systems, both the conservative and nonconservative velocities and their fluxes need to balance each other at $\mathbf{x}^*$, implying that the nonconservative contribution is also equal to the local change in the self-information of the reference equilibrium distribution:

$$-\mathbf{v}_{nc}^* = \mathbf{v}_c^* = \nabla \mathcal{A}^* = \nabla \ln P_{eq}^* = \beta \nabla U^*. \quad (62)$$

The conditions in Eqs. (60) and (62) are very general and state that any source for the nonconservative flow must be a sink for the conservative one, and vice versa, together with the fact that they must have common center/saddles. Furthermore, let us note that, by using Eq. (41), Eq. (57) is equivalent to a Poisson equation for the availability where the divergence of the nonconservative velocity acts as the source term [43]:

$$\nabla^2 \mathcal{A} = -\nabla * \mathbf{v}_{nc}. \quad (63)$$

To interpret Eq. (63), we recall that, intuitively, the Laplacian measures the local concavity of field, or, more rigorously, the difference between the field at that point and its average in a local neighborhood [44]. Hence, how much the availability concavity deviates from equilibrium is directly influenced by the nonconservative velocity that reflects the external nonequilibrium constrains. Furthermore, Eq. (63) gives information on the topology of the velocity field and could in principle be expanded in multipole moments.

Nevertheless, the condition defining a locally closed system contains the great majority of thermodynamic models. To narrow down the space of possible geometric properties, a natural simplification from Eq. (63) is to consider the additional constraint:

$$\nabla^2 \mathcal{A} = 0 = \nabla * \mathbf{v}_{nc}. \quad (64)$$

This condition amounts to assuming that the availability is harmonic in the entire phase space. As a consequence, the equilibrium and nonequilibrium free energies have the same

Laplacian:

$$\nabla^2 \mathcal{A} = 0 \rightarrow \nabla^2 \mathcal{F} = \beta \nabla^2 U. \quad (65)$$

This condition also strongly constrains the topology of non-conservative velocity field to have no monopole contribution. We call this class of systems locally balanced. Finally, as a minimal way of constructing a NESS, we consider an even more stringent requirement that we name local equilibrium. In addition to local balance, we impose that the equilibrium and nonequilibrium probabilities have the same extreme point, $\mathbf{x}^*$ [45,46]:

$$\nabla \mathcal{F}^* = \beta \nabla U^* = 0. \quad (66)$$

It is trivial to show that these conditions are valid for any linear system, i.e., with $\mathbf{F} = \mathbf{A}\mathbf{x}$. From Eq. (66), it follows that both geometric velocity components need to vanish at the extreme points:

$$\mathbf{v}_c^* = \mathbf{v}_{nc}^* = 0. \quad (67)$$

The combination of Eqs. (64) and (66) implies that the mean value of the availability in any configuration space region containing an extreme point must be zero. In synthesis, in local equilibrium systems, the geometry of the velocities is fully captured by a solenoidal nonconservative and a harmonic conservative contribution:

$$(\mathbf{v}_{nc})^i = \partial_j B^{ij}, \quad B^{ij} = -B^{ji}, \quad (68)$$

$$(\mathbf{v}_c)^i = D^{ij} \partial_j \psi, \quad \partial_i D^{ij} \partial_j \psi = 0, \quad (69)$$

where B is a skew-symmetric tensor and ψ a scalar potential. Equations (66), (68), and (69) together signal that local equilibrium conditions support the existence of a current, induced by nonequilibrium external conditions, that breaks rotational symmetry, while not allowing the system to stably occupy excited states at stationarity. Indeed, as we show in Sec. IV A with two examples, local equilibrium systems can either occupy the same state space as in equilibrium, i.e., $S_{rel}^{eq} = 0$, which we hence call them *pseudoequilibrium* systems, or explore excited states via nonequilibrium fluctuations while having the same minimum as in equilibrium, i.e., $S_{rel}^{eq} > 0$.

A. Examples in local equilibrium

Here, we present two pedagogical examples in which the connection between nonconservative dissipation and parity symmetry breaking (see also Appendix B) is easy to visualize, and the possibility to explore excited states can be appreciated through the exact computation of the availability.

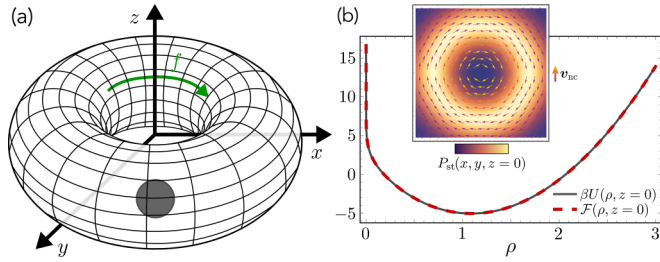


FIG. 1. Parity symmetry breaking for a driven particle in a circular potential. (a) A sketch of the model. (b) In the NESS, the stationary self-information (red dashed curve) coincides with the equilibrium potential (gray curve). Hence, the total stationary entropy production coincides with the nonconservative one, $\dot{S}_{\text{tot}}^{\text{st}} = \dot{S}_{\text{nc}}^{\text{st}}$. The inset shows the steady-state distribution projected in the (x, y) plane (in color scale) with the velocity vector field on top (a bigger arrow corresponds to a stronger field). It clearly shows the tendency to rotate in a preferential direction, therefore breaking parity symmetry. Model parameters are $\rho^* = 1$, $f = 1$, $\gamma = 1$, and $D = 0.1$.

1. Driven particle in a circular potential

Consider a 3D Brownian particle confined on a circle of radius ρ^* by a (quadratic) potential and driven along the circle itself by a constant nonconservative force f [see Fig. 1 (a)]. It is instructive to write the Langevin equations describing this system in cylindrical coordinates (ρ, ϕ, z) :

$$\begin{aligned}\dot{\rho} &= -\gamma(\rho - \rho^*) + \frac{D}{\rho} + \sqrt{2D} \xi_{\rho}(t), \\ \dot{\phi} &= \frac{f}{\rho} + \frac{\sqrt{2D}}{\rho} \xi_{\phi}(t), \\ \dot{z} &= -\gamma z + \sqrt{2D} \xi_z(t),\end{aligned}\quad (70)$$

with the equilibrium potential $U = (\gamma/2)(\rho - \rho^*)^2 + (\gamma/2)z^2$, the nonconservative force f only acting in the direction of increasing the angle ϕ , and D/ρ appearing due to the Ito's formula for the change of variables. The stationary solution is found to support a probability flux only along the angle variable ϕ but, surprisingly, the steady-state self-information coincides with the equilibrium potential [see Fig. 1(b)]. Hence, the availability is zero and the stationary velocity $\mathbf{v}_a = \mathbf{v}_{\text{nc}} = -(f/\rho)\hat{\mathbf{v}}_{\phi}$, where $\hat{\mathbf{v}}_{\phi}$ is the versor associated with ϕ . As expected, $\mathbf{v}_{\text{nc}}$ is a divergence-free field and indicates that the adiabatic dissipation originates from cyclic trajectories running across the entire circle in a preferential direction, i.e., dragging the particle in the direction of increasing ϕ , therefore breaking parity symmetry (see Appendix F). Yet, the nonequilibrium driving does not affect the steady-state distribution with respect to equilibrium, thereby the system does not explore excited states either stably or through fluctuations.

2. The Brownian gyration

As a second paradigmatic example, we study a two-dimensional (2D) Brownian gyration, i.e., a diffusive particle in a confining quadratic potential coupled to two reservoirs at different temperatures, T_x and T_y , each acting along one direction [see Fig. 2(a)] [47]. The Langevin equations describing

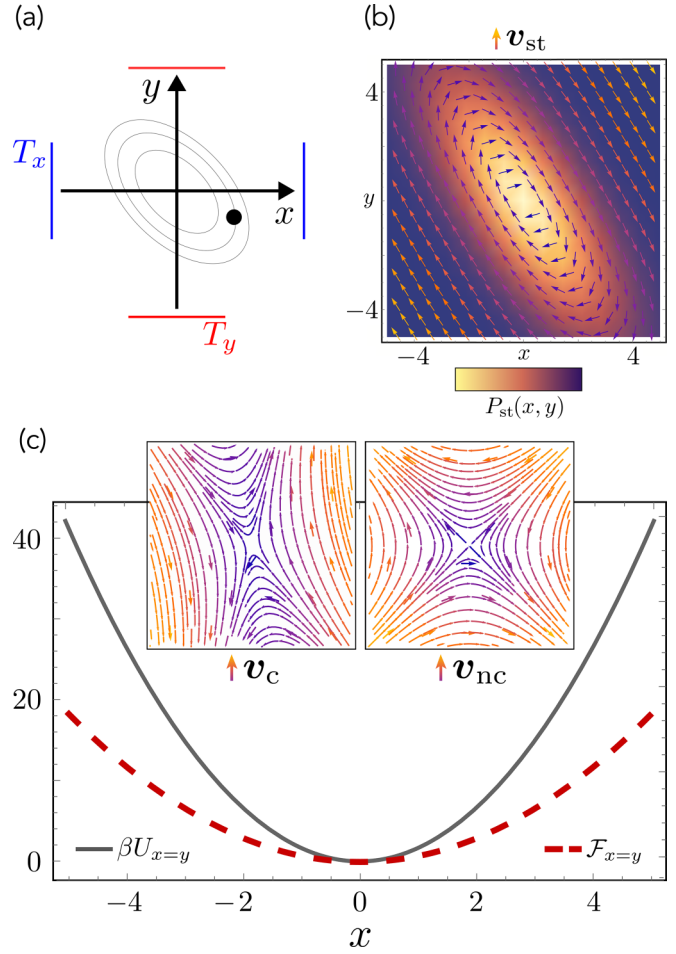


FIG. 2. Parity symmetry breaking in a Brownian gyration. (a) A sketch of the model. (b) Steady-state distribution (in color code) with the adiabatic velocity field on top (larger brighter arrows indicate a stronger field). (c) Self-information (red dashed curve) and potential (gray curve) along the line $x = y$ show that excited states can be explored solely due to nonequilibrium fluctuations. In the inset, conservative and nonconservative velocities at steady state are presented in color code and as a stream plot. Model parameters are $u = 0.7$, $T_x = 1$, and $T_y = 5$.

the system are

$$\begin{aligned}\dot{x} &= -\partial_x U(x, y) + \sqrt{2T_x} \xi_x(t), \\ \dot{y} &= -\partial_y U(x, y) + \sqrt{2T_y} \xi_y(t),\end{aligned}\quad (71)$$

with ξ a Gaussian white noise with zero mean and unit variance, and $T_y = T_x(1 + \delta)$. Following Appendix E, we isolate the interaction part of the potential, U_{int} , from the confining contributions, U_x and U_y :

$$\begin{aligned}U &= U_{\text{int}} + U_x + U_y, \\ U_{\text{int}} &= uxy, \quad U_{\alpha} = \frac{\alpha^2}{2}, \quad \alpha = x, y,\end{aligned}\quad (72)$$

where u quantifies the asymmetry of the elliptic potential, with $|u| < 1$ to ensure stability. As long as $T_x \neq T_y$, i.e., $\delta \neq 0$, since the mobilities along x and y are the same, the fluctuation-dissipation relation cannot be satisfied and the system breaks

detailed balance [48]. The stationary distribution can be explicitly obtained in the following exact form [49]:

$$P_{\text{st}} = \frac{e^{-\mathcal{F}}}{Z}, \quad \mathcal{F} = \beta_{\text{eff}} \left(\frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{2} y^2 + u \gamma_3 xy \right),$$

where Z is a normalization factor, $\beta_{\text{eff}} = (T\eta)^{-1}$ an effective inverse temperature, and

$$\eta = 1 + \delta + \frac{u^2}{4} \delta^2, \quad \gamma_1 = 2 - \frac{u^2}{2} \delta, \\ \gamma_2 = 1 + \frac{u^2}{2} \delta, \quad \gamma_3 = 1 + \frac{\delta}{2}.$$

The adiabatic velocity is a solenoidal field vanishing in the minimum of the self-information, making it apparent that the system exhibits parity symmetry breaking [see Fig. 2(b)]. The adiabatic vorticity $w_{\text{st}} = \partial_x(v_a)_y - \partial_y(v_a)_x$, i.e., the 2D curl of the velocity, is connected to the tendency of performing closed trajectories and reads (see Appendix F)

$$w_{\text{st}} = \frac{4T_x u}{4T_x T_y + (T_x - T_y)^2 u^2} (T_x^2 - T_y^2). \quad (73)$$

Following the geometric decomposition presented in Eq. (E2), we have

$$v_{\text{nc}}^i = -\partial_i U_{\text{int}}, \quad v_{\text{c}}^i = \nabla^i \mathcal{A}, \quad (74)$$

with the availability

$$\mathcal{A} = \beta_{\text{eff}} \gamma_3 \left((\gamma_1 - 2) \frac{x^2}{2} + (\gamma_2 - 1) \frac{y^2}{2(1 + \delta)} + u xy \right).$$

This system fulfills the local equilibrium condition: both velocities are divergence-free, vanish at the self-information minimum, and flow in opposite directions [see Fig. 2(c)]. As a result, excited states cannot be stably occupied, although transient explorations occur during fluctuations. The deviation from equilibrium arises because the conservative velocity induces compensating expansions and contractions along different directions, even though the total divergence is zero. The sum of v_{nc} and v_{c} defines the adiabatic velocity v_a , which is a solenoidal field: its circulation does not stem from a nonconservative force, as in the previous example, but from the interplay between the potential concavity and the temperature anisotropy. We remark once again that this geometric decomposition is not unique and considers the external nonequilibrium constraints as stemming from the interaction potential U_{int} .

B. Geometric and thermodynamic properties beyond weak noise

The most general case of open system might be more challenging to analyze through simple examples. However, from the analysis in Sec. II C 2 beyond the weak-noise limit, it is clear that, in this regime, systems can be open, given that the divergence of the adiabatic velocity is not constrained to vanish. In the previous examples, we have seen that the conservative velocity represents how the system compensates the action of the nonequilibrium driving, without breaking additional symmetries. Here, we show that, when the system is open, the conservative contribution encodes the interactions between the system and the environment, and quantifies

how phenomena breaking additional symmetries emerge by increasing the noise strength. Starting from Eq. (30), we can identify $\mathcal{F}^{(1)}$ as the self-information contribution that vanishes in the weak-noise limit. Therefore, the velocity stemming from its gradient can be identified as an additional conservative contribution:

$$v_a = v_a^{(0)} + \sigma v_a^{(1)}, \quad (75)$$

$$v_a^{(0)} = F + \nabla \mathcal{F}^{(0)} = v_{\text{nc}} + v_{\text{c}}^{(0)}, \quad (76)$$

$$v_{\text{c}}^{(0)} = \nabla \mathcal{A}^{(0)}, \quad (77)$$

$$v_a^{(1)} = \nabla \mathcal{F}^{(1)} = \nabla \mathcal{A}^{(1)} = v_{\text{c}}^{(1)}. \quad (78)$$

As a first observation, we notice that the divergence of the conservative velocity gets modified by the effect of the noise encoded in $\mathcal{F}^{(1)}$:

$$\nabla * v_{\text{c}} = \nabla * v_{\text{c}}^{(0)} + \sigma v_{\text{c}}^{(1)} * v_{\text{c}}^{(1)}, \quad (79)$$

while the nonconservative contribution stays unchanged because of Eq. (78). Furthermore, let us remark that, from Eqs. (9) and (17), the stationary probability can now locally expand and contract; i.e., the flow is compressible, and hence the self-information can increase or decrease along the velocity lines. As a result, single trajectories can travel between different self-information valleys. This substantiates the fact that, beyond the weak-noise limit, the system is no longer locally closed since Eq. (57) cannot be satisfied at any order in σ . By means of the geometric decomposition, the geometric relations derived in Sec. II C acquire a thermodynamic meaning. The conservative entropy production exhibits additional terms:

$$\dot{S}_{\text{c}} = \dot{S}_{\text{c}}^{(0)} + \dot{S}_{\text{c}}^{(1)} + 2\langle v_{\text{c}}^{(0)} * v_{\text{c}}^{(1)} \rangle, \quad (80)$$

where we introduced

$$\dot{S}_{\text{c}}^{(0)} = \frac{1}{\sigma} \langle v_a^{(0)} * v_{\text{c}}^{(0)} \rangle, \quad \dot{S}_{\text{c}}^{(1)} = \sigma \langle v_{\text{c}}^{(1)} * v_{\text{c}}^{(1)} \rangle.$$

By using Eqs. (36) and (38), it is immediate to realize that the first-order conservative entropy production measures the flow of its corresponding velocity:

$$\dot{S}_{\text{c}}^{(1),\text{st}} = \sigma \langle \nabla * v_{\text{c}}^{(1)} \rangle_{\text{st}} \geq 0. \quad (81)$$

Equation (81) expresses the fact that the interactions with the reservoir create local dissipative probability expansions that tend to increase the self-information. From Eq. (A20), expressing the divergence of $v^{(1)}$ from Eq. (A19), and then using the geometric decomposition for the zeroth-order velocity, we obtain

$$\langle \nabla^2 \mathcal{A}^{(0)} \rangle_{\text{st}} = -\langle \nabla * v_{\text{nc}} \rangle_{\text{st}} - \dot{S}_{\text{c}}^{(1),\text{st}}. \quad (82)$$

This equation specializes Eq. (63) for open systems at the first order in σ . It clearly shows that the deviation of availability concavity from equilibrium is dictated not only by the nonconservative velocity, reflecting external nonequilibrium constraints, but also by an additional conservative entropy production term stemming from the system-reservoir thermodynamic coupling.

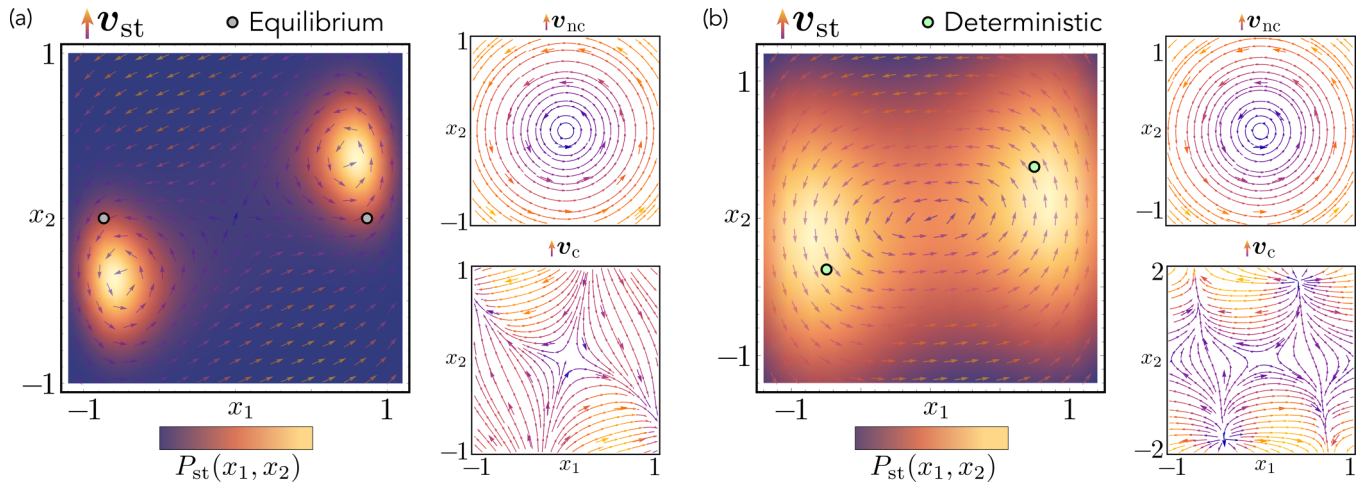


FIG. 3. Beyond weak noise for a Brownian particle in bistable potential and shear flow. (a) Contour plot of the steady-state distribution in the weak-noise limit with the adiabatic velocity as a vector field in color code (brighter arrow corresponds to stronger field). The minima of the equilibrium potential are shown: they do not coincide with the peaks of the self-information, leading to a nonzero availability. We report both the corresponding conservative velocity as a stream plot in color code and the nonconservative component that coincides with the shear flow. (b) Same as in panel (a), but for the strong-noise scenario. Notice that the self-information minima do not coincide with the deterministic solution, and the conservative velocity shows the presence of repellers at $(x_1, x_2) \approx \pm(0.5, 1.8)$. However, the nonconservative component is unchanged, since it only depends on external nonequilibrium conditions. Parameters are $a = 1$, $b = 1.5$, $\gamma = 1$, and $T = 0.1$ (weak noise) or 1 (strong noise).

1. Example: Brownian particle in a bistable potential and shear flow

As an example, we consider a Brownian particle in 2D immersed in a bath at temperature T under the effect of a double well potential $U(\mathbf{x}) = ax_1^4 - bx_1^2 + x_2^2$ and a shear flow $\mathbf{f} = \gamma(-x_2, x_1)$ (by components):

$$\dot{x}_i = -\partial_i U + f_i + \sqrt{2T}\xi_i. \quad (83)$$

In the absence of flow ($\gamma = 0$), the system reaches the equilibrium distribution $P_{\text{eq}} \sim \exp(-U/T)$. Conversely, in the presence of driving, a large-deviation function in the form of Eq. (28) exists and can be numerically evaluated, as shown in Ref. [50]. Even if $\mathcal{F}^{(0)}$ is not accessible analytically, we study the deterministic dynamics through a fixed-point analysis and perform numeric integration of the Fokker-Planck equation. In the strong driving limit, i.e., $\gamma^2 > 4b$, the self-information will have a single minimum in $\mathbf{x} = 0$, while by reducing γ , $\mathbf{x} = 0$ becomes a maximum and two minima appear:

$$x_1^* = \pm \frac{1}{2} \sqrt{\frac{4\tilde{b} - \gamma^2}{2a}}, \quad x_2^* = \gamma \frac{x_1^*}{2}, \quad (84)$$

where $\tilde{b} = b - 6a\text{Var}(x_1)$ indicates how the parameters are modified by the effect of noise through the variance of x_1 (the correction is of order T for small temperatures; see Ref. [50] for details). By studying the system in the weak-noise limit, we see that the nonequilibrium conditions, i.e., the presence of the shear flow, modifies the landscape and the minima of $\mathcal{F}$ become equal to those in the deterministic limit [Eq. (84)] [24]. They do not coincide with those of the potential U , so that the system is not in local equilibrium [see Fig. 3(a)]. As shown in Appendix D, $\mathbf{v}_{\text{nc}}$ coincides with the shear flow $\mathbf{f}$; i.e., it is divergence-less and, as a consequence, the system falls in the locally balanced scenario. In

Fig. 3(a), we show the stationary distribution and the conservative/nonconservative velocities in the weak-noise regime. However, when we increase the temperature that controls the noise amplitude, phenomena appear. The system is now locally open and exchanges energy with the bath. As such, $\mathbf{v}_c$ is not necessarily harmonic, repellers appear in the system, and the self-information minima do not coincide with the deterministic solution [see Fig. 3(b)]. However, the divergence-less nature of the nonconservative velocity ensures that the two conservative contributions balance each other on average; i.e., while the system is locally open, it remains globally closed:

$$\langle \nabla * \mathbf{v}_c^{(0)} \rangle_{\text{st}} = -\dot{S}_c^{(1),\text{st}} = -\sigma \langle \nabla * \mathbf{v}_c^{(1)} \rangle \leq 0, \quad (85)$$

where the conservative entropy production measures the energy flow from the bath to the system. These equalities have been derived by using Eqs. (79) and (82). Therefore, an increase in temperature generates dissipative expansions of probability and the system is doing work at the single trajectory level by populating excited states. In summary, the nontrivial interplay between temperature bath, equilibrium potential, and energy flow has the effect of rising the system's typical energies and increasing the conservative dissipation.

V. GENERALIZATION TO MULTIPLICATIVE FLUCTUATIONS

A. A velocity stemming from curved geometry

The proposed framework naturally extends beyond the case of additive noise to the challenging scenario of multiplicative fluctuations. In general, state-dependent diffusion coefficients may emerge in a variety of systems, from Brownian motion in inhomogeneous media or constrained on curved surfaces [51–53] to chemical reaction networks affected by thermal gradients [54–56] and fluctuating environments [57].

Moreover, finite-size fluctuations are inherently multiplicative and important in fluctuating hydrodynamics [58], field theories [59,60], and population dynamics [61,62].

We start from the following general Langevin equation:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + G(\mathbf{x})\boldsymbol{\xi}(t), \quad (86)$$

with $\boldsymbol{\xi}$ a vector of Gaussian white noises with zero mean and correlation matrix $2\sigma\mathbb{1}$ (the identity matrix), where σ controls noise amplitude. Here, G is a state-dependent matrix; hence, the total diffusion matrix is $D = \sigma G^T G$. When G is diagonal, this system reduces to the additive noise case. This scenario extends and generalizes our approach by considering again the inverse of the diffusion matrix to be the metric of our space, which can now be curved because of the space dependence of D (that we assume to be invertible). Following the classic works by Graham [21,63] and a more recent deep analysis by Poletini [26], we use standard tools of differential geometry to recast the Fokker-Planck equation in a fully covariant form (i.e., coordinate independent; see Appendix C for details). First, we need to upgrade the standard derivatives to covariant ones:

$$\nabla_j u^i = \partial_j u^i + \Gamma_{jk}^i u^k, \quad (87)$$

where Γ is the Levi-Civita connection that encodes the effect of the curvature on the derivative, while the first contribution measures the transport in the tangent space (that in a flat manifold coincides with the usual configuration space). Recall that the covariant derivative of a scalar field coincides with the partial derivative. The contravariant derivative, or gradient, is obtained from the covariant derivative: $\nabla^i v^k = D^{ij} \nabla_j v^k$. As a consequence, the divergence now depends on the diffusion matrix determinant $|D|$:

$$\nabla * \mathbf{v} = \sqrt{|D|} \partial_i \left(\frac{v^i}{\sqrt{|D|}} \right) = \partial_i v^i - \Gamma_i v^i, \quad (88)$$

with $\Gamma_i = \Gamma_{ik}^k = -\partial_i \ln \sqrt{|D|}$. By defining the invariant probability Q and the invariant measure $d\Omega$, i.e., $Q = P\sqrt{|D|}$ and $d\Omega = dx/\sqrt{|D|}$, one can rewrite the Fokker-Planck equation in a fully covariant form, interpreting the Langevin equation with the Stratonovich ($\alpha = 0$) or Ito ($\alpha = 1$) prescription:

$$\partial_t Q = -\nabla_i J^i, \quad J^i = \tilde{F}^i Q - \sigma \nabla^i Q. \quad (89)$$

In this expression, the force is modified by a term $\mathbf{v}_D$ that we call the diffusive velocity:

$$\tilde{F}^i = F^i + v_D^i, \quad v_D^i = D^{ij} \omega_j - \alpha \partial_j G_l^i G_k^j \delta^{lk}, \quad (90)$$

composed by a vector field $\boldsymbol{\omega}$ that measures the breaking of detailed balance condition on the diffusive force G :

$$\omega_j = G_k^l (\partial_j G_l^k - \partial_l G_j^k), \quad (91)$$

and by the Ito phoretic term, $G^T \partial G$. The vector field $\boldsymbol{\omega}$ stems directly from the configuration space geometry: it vanishes when G can be written as the derivative of a collection of vector fields, $G_j^i = -\partial_j \psi^i$, that corresponds to the condition of zero curvature of the metric induced by the diffusion matrix (see Ref. [21] and Appendix C 1). This modification of the force field directly implies also a change in the velocity decomposition:

$$\mathbf{v}_a = \tilde{\mathbf{F}} - \nabla Q = \mathbf{F} - \nabla Q + \mathbf{v}_D, \quad (92)$$

where the diffusive (geometry-driven) velocity emerges.

B. Geometric and thermodynamic properties of multiplicative fluctuations

In the case of multiplicative fluctuations, equilibrium cannot generally be reached, making the identification of a potential energy less clear. Nevertheless, the zero current condition implies the following detailed balance relationship for the modified force in Eq. (90):

$$\partial_i \tilde{F}_j = \partial_j \tilde{F}_i. \quad (93)$$

It is then trivial to see that, once again, any gradient field does not break this condition. Therefore, it can be collected into the conservative velocity and interpreted as a contribution to a pseudoequilibrium distribution, i.e., a modification of the internal organization of system's states. Yet, the diffusive velocity has a complex structure that makes challenging the general identification of the conservative and nonconservative terms, along with their geometric properties. This is mainly due to the presence of $\boldsymbol{\omega}$ that introduces cross dependencies into the fluctuations.

Let us investigate two different scenarios. In the first one, the diffusion matrix is space dependent, but the diffusive forces satisfy the potential condition $G_k^i = -\partial_k \psi^i$. As a result, $\boldsymbol{\omega}$ vanishes, the geometry is flat, and it is always possible to find a set of coordinates $\mathbf{x}' = G^{-1} \mathbf{x}$ (called *harmonic*) where the system has additive noise [21,26]. Focusing on the Ito prescription ($\alpha = 1$), in this simple setting the diffusive velocity turns out to be a gradient, and thus conservative:

$$v_D^i = \frac{1}{2} \partial^i (\partial_j \psi_k \partial_z \psi_l \delta^{kl} \delta^{jz}) = v_{c,D}^i. \quad (94)$$

Hence, it can be absorbed into the conservative component of the velocity field and does not contribute to the breaking of detailed balance [Eq. (93)]. An intuitive understanding comes from considering a system subjected to a thermal gradient along each direction, $T_i(x_i)$ [51,64]. If the gradients cause independent fluctuations on the system, the diffusion matrix is diagonal $D = G^T G = \text{diag}(T_i(x_i))$ and $G_{ii} = \partial_i \psi(x_i)$. Therefore, the velocity is a “phoretic” term:

$$v_D^i = -\nabla^i \ln \sqrt{|D|} = -\delta^{ik} \partial_k T_i(x_i) = \Gamma^i, \quad (95)$$

stemming directly from the connection Γ and contributing to the covariant divergence [Eq. (88)]. By considering the noise expansion for the self-information, we obtain a covariant version of the equations in Sec. II C 2 with this phoretic term contributing to the first-order velocity $\tilde{\mathbf{v}}_a^{(1)} = \nabla \mathcal{F}^{(1)} + \mathbf{v}_D$ (see Appendix C 1). As a result, the conservative dissipation up to the first order in noise amplitude is given by the velocity divergence in the tangent space:

$$\dot{\mathcal{S}}_{\text{ex},a}^{(1)} = \sigma \langle \tilde{\mathbf{v}}_a^{(1)} * \tilde{\mathbf{v}}_a^{(1)} \rangle_{\text{st}} = \sigma \langle \partial * \tilde{\mathbf{v}}_a^{(1)} \rangle_{\text{st}} \geq 0. \quad (96)$$

We can conclude that the system is open and the phoretic velocity will drive the system to stably occupy “excited” states. It is important to remark that, in this case, even though the curvature vanishes, the Levi-Civita connection does not vanish and directly generates the diffusive velocity. Therefore, $\mathbf{v}_D$ plays the role of a fictitious force, arising solely from the use of a coordinate system in which the metric (and thus the

diffusion tensor) varies in space. While in Newtonian frames fictitious forces originate from accelerated or curvilinear coordinates, here they emerge from spatially varying diffusion, and the effective drift reflects how the local diffusive motion changes across space.

Let us now briefly discuss the more general case in which $\omega \neq 0$. This is generally expected to happen when the coordinates exhibit coupled diffusivity or when the diffusion tensor is nondiagonal, encompassing scenarios such as Brownian particles in magnetic fields [65], diffusion on curved surfaces [52], and interacting active particle systems [66,67]. In this scenario, $\mathbf{v}_D$ arises intrinsically from the curvature and cannot be eliminated by a change of coordinates. This is analogous to a static gravitational field likewise producing thermal gradients according to the Ehrenfest-Tolman effect [68]. The drift in this $\omega \neq 0$ case contributes simultaneously to conservative and nonconservative velocities, since the distinction between gradient and rotational components becomes entangled with curvature. A precise separation is nontrivial, and we leave this interesting but challenging problem for future work.

C. Example: underdamped particle in a temperature gradient

In this section, we extend our approach to the case of an underdamped system in which the multiplicative noise originates from a temperature gradient. For simplicity, we consider the following one-dimensional underdamped dynamics of a Brownian particle in the position-velocity phase space (x, v) in the presence of a temperature gradient $T(x)$:

$$\dot{x} = v, \quad (97)$$

$$\dot{v} = -\frac{\gamma}{m}v - \frac{1}{m}\partial_x U + f + \frac{1}{m}\sqrt{2\gamma T(x)}\xi(t), \quad (98)$$

where γ represents the friction, m the mass, f a nonconservative force even under time reversal, the potential $U = ux^2$ for simplicity, and the diffusion coefficient satisfies the Einstein's relation $D(x) = \gamma T(x)$ at each point in space. We can define the timescale of the friction $\tau = m/\gamma$ so that the overdamped limit is attained for $\tau \rightarrow 0$, when velocities equilibrate infinitely fast.

The current velocity is now a two-dimensional vector in the phase space and can be split into a reversible $\mathbf{v}_{\text{rev}}$ and an irreversible component $\mathbf{v}_{\text{irr}}$, according to the behavior they exhibit under time-reversal operation [69]:

$$\begin{aligned} \mathbf{v}_{\text{rev}} &= \left\{ v, -\frac{1}{m}\partial_x U(x) + \frac{f}{m} \right\}, \\ \mathbf{v}_{\text{irr}} &= \left\{ 0, -\frac{1}{\tau}v - \frac{1}{m\tau}T(x)\partial_v \underbrace{\ln(P(x, v, t))}_{\mathcal{F}} \right\}. \end{aligned} \quad (99)$$

At equilibrium, when $T(x) \equiv T$ and $f = 0$, the system cannot support any dissipative flux. Solving the Fokker-Planck equation associated with the Langevin dynamics without a temperature gradient, we obtain the equilibrium distribution

$$P_{\text{eq}}(x, v) \sim e^{-\frac{mv^2}{2T} - \frac{U(x)}{2T}} = e^{-\beta E}, \quad (100)$$

where $\beta = 1/k_B T$, with $k_B = 1$, and E is the total energy of the system, as expected.

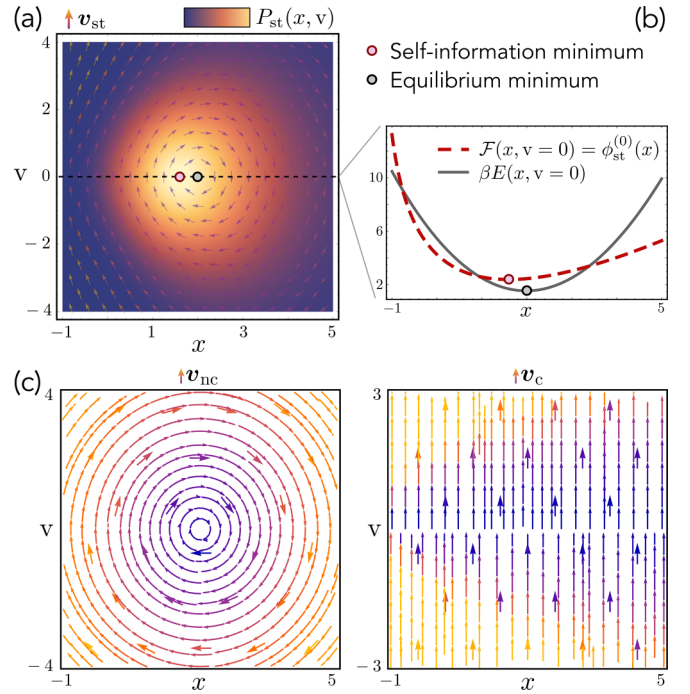


FIG. 4. Geometry of velocities for underdamped dynamics with thermal gradient. (a) The stationary solution of the underdamped Fokker-Planck equation up to the first order in τ is shown [Eq. (102)], for a linear thermal gradient $T(x) = T_0(1 + \alpha x)$ and a quadratic potential $U = u(x - \mu)^2$. Arrows represent the stationary velocity in color code (brighter arrows indicate stronger field). (b) Comparison between self-information and equilibrium distribution for $v = 0$. Notice that the self-information along this line coincides with $\phi_{\text{st}}^{(0)}(x)$ [Eq. (102)]. The self-information minimum does not coincide with the equilibrium one. (c) Stream plot in color code of nonconservative (reversible) and conservative (irreversible) velocities. In this figure, the spatial domain is $x \in [-1/\alpha, +\infty]$ and we set all parameters so that the probability distribution and its derivative vanish at the boundaries. The model parameters are $m = 1$, $\gamma = 0.01$, $\mu = u = 2$, $T_0 = 1$, and $\alpha = 0.8$.

In analogy to the two bath scenario discussed in Appendix E, we define the conservative velocity by fixing P_{eq} as reference distribution for every point in space, i.e., $P_{\text{eq}}|_{\beta \rightarrow \beta(x)}$. The resulting decomposition is

$$\mathbf{v}_{\text{nc}} = \mathbf{v}_{\text{rev}}, \quad \mathbf{v}_{\text{c}} = \nabla(\mathcal{F} - \beta(x)E) = \mathbf{v}_{\text{irr}}. \quad (101)$$

A crucial observation to obtain this correspondence is that only v has a metric term given by the diffusion, as the system is effectively one dimensional. Therefore, the gradient only acts on the v space. As said above, this decomposition is arbitrary, but it is informed (and substantiated) by the physical decomposition of the velocity into reversible and irreversible components. As a consequence, the conservative part accounts for the deviation of the self-information from the equilibrium condition. It is nonzero and, as such, it allows the system to stably explore excited states due to the presence of a thermal gradient. Conversely, the nonconservative velocity is divergence-free, contains the nonconservative force pushing the system out of equilibrium, and also reflects the presence of fluxes in the full phase space that vanish in the overdamped

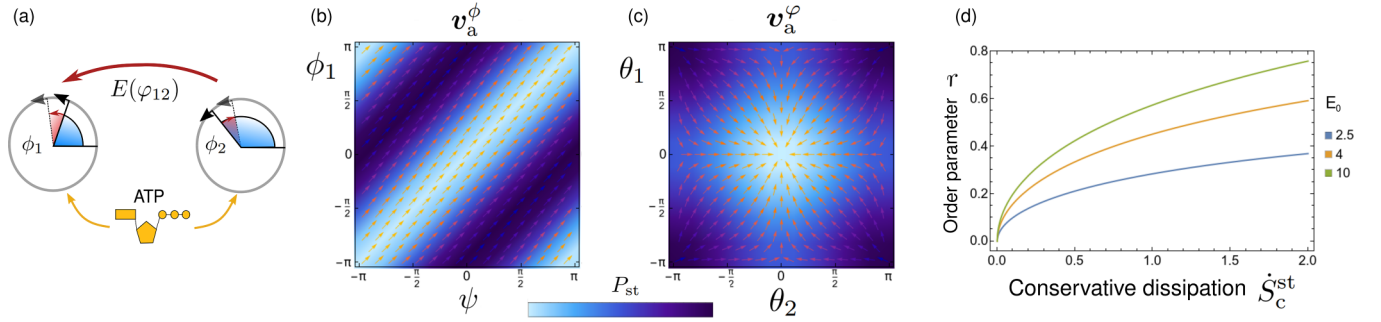


FIG. 5. The geometry of current velocities for coupled molecular oscillators. (a) Pictorial sketch of two coupled molecular biochemical oscillators. Each oscillator is represented by a phase $\phi_i \in [0, 2\pi]$ (in shaded blue color) that grows with a constant velocity ke_g (solid black arrows) fueled by adenosine triphosphate (ATP) consumption. Furthermore, oscillators can interact with a phase exchange potential $E_{12}(\phi_{12})$ depending on their phase difference $\phi_{12} = \phi_1 - \phi_2$ (in shaded red), eventually leading to anticorrelated phase changes (small red arrows). (b) Stream plot of v_a^ϕ for a typical oscillator with phase ϕ_1 as a function of the collective phase ψ with the heat map of $P_{st}(\theta_1)$, $\theta_1 = \phi_1 - \psi$ in the mean-field limit in the synchronized phase. Here, v_a^ϕ drives the synchronized oscillations (yellow arrows and light-blue regions, high probability density) but also indicates that interactions tend to destabilize them (dark blue regions of extremely low probability). (c) Stream plot of v_a^ϕ for two typical oscillators with the heat map of $P_{st}(\theta_1, \theta_2)$, with ψ the mean-field collective phase. v_a^ϕ tends to reduce the oscillators phase differences ϕ , stabilizing the dynamics and promoting synchronization. (d) Plot of the synchronization order parameter r as a function of the conservative entropy production for different values of E_0 . Model parameters are $k = 0.5$, $e_g = 4\pi$, $E_0 = 8$, and $\Omega = 1$.

limit. Indeed, v acts as a nonequilibrium driving, since collapsing the system into the x space (i.e., fast relaxation of v leading to the overdamped regime) will result into an effective equilibrium distribution. When the system is pushed out of equilibrium by a temperature gradient, the Fokker-Planck equation can be solved at any time by expanding in powers of τ , i.e., close to the overdamped regime. Up to the first order, the time-dependent nonequilibrium solution reads (see Appendix F for the derivation):

$$P^{(\tau)}(x, v, t) = e^{-\frac{mv^2}{2T}} \left\{ \Phi^{(0)} - \tau \left[v \partial_x \Phi^{(0)} + \left(\frac{mv^3}{6T^2} + \frac{v}{T} \right) \Phi^{(0)} \partial_x T + \frac{v}{T} \Phi^{(0)} \partial_x U \right] \right\}, \quad (102)$$

where $\Phi^{(0)}(x, t) = \phi^{(0)}(x, t) \sqrt{2\pi T(x)/m}$, with $\phi^{(0)}$ satisfying the zero-flux overdamped Fokker-Planck equation with Ito prescription [see Figs. 4(a) and 4(b)], as dictated by the timescale separation employed [56,70]. First, we notice that the form of the solution in Eq. (102) goes beyond the weak-noise regime, thereby showing a peak that is different from the equilibrium one [see Fig. 4(b)]. In Fig. 4(c), we show the components of the stationary velocity stemming from this solution.

VI. GENERALIZATION TO MANY-BODY SYSTEMS: GEOMETRY OF FREE-ENERGY TRANSDUCTION IN COUPLED MOLECULAR OSCILLATORS

Our framework is able to provide physical insights in even more general settings. To present a concrete example, we now employ it to analyze a many-body system, i.e., a model of N coupled biochemical oscillators introduced in Ref. [4]. Consider N oscillators, each one with a phase $\phi_i \in [0, 2\pi]$ and undergoing a stochastic dynamics that advances its phase. The resulting phase precession rate is equal to ke_g , where e_g is a typical thermodynamic force driving the system out of

equilibrium, while the fluctuation strength is proportional to k , a typical microscopic reaction rate. The bath temperature has been put to 1 for simplicity. Then, we include the possibility that two oscillators can exchange a phase through the interaction potential, $E_{ij} = E(\phi_i - \phi_j) = -E_0 \cos(\phi_i - \phi_j)$, with E_0 the coupling energy [see Fig. 5(a) for a pictorial illustration]. This system can hence be described by N coupled Langevin equations:

$$\dot{\phi}_i = ke_g + \eta_i - \frac{\Omega}{N} \sum_{i < j} (E'_{ij} + \xi_{ij}(t)), \quad (103)$$

where Ω is a typical frequency. Here, η_i and ξ_{ij} are two different zero-mean Gaussian noises, respectively, stemming from single-oscillator dynamics and generated by interactions between oscillators, with respective variances:

$$\begin{aligned} \langle \eta_i(t) \eta_j(t') \rangle &= 2k \delta_{ij} \delta(t - t'), \\ \langle \xi_{ij}(t) \xi_{kl}(t') \rangle &= \frac{2\Omega}{N} [\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}] \delta(t - t'). \end{aligned} \quad (104)$$

For the detailed derivation of this model from a microscopic description, along with its biological motivation, we refer to Ref. [4]. Let us emphasize that this example goes beyond the settings presented so far because of the presence of two different currents (and hence two noise parameters) and we do not have a natural reference equilibrium distribution. Nevertheless, we now show that our framework is still valid and indeed illuminates the system phase transition. One can calculate the exact stationary solution of this system and identify the adiabatic availability as

$$P_{st} = \frac{e^{-\beta_{\text{eff}} \mathcal{F}}}{Z}, \quad \mathcal{F} = N^{-1} \sum_{i < j} E^{ij}, \quad (105)$$

with $\beta_{\text{eff}} = 1 + k/\Omega$ (see also Appendix F for its derivation) and the total energy plays the role of the self-information. Given the many-body nature of the problem, a very particular thermodynamics emerges. First, we notice that Eq. (F16) is equivalent to N equations describing phase changes, $\dot{\phi}_i$,

coupled to N^2 equations capturing phase differences, $\varphi_{ij} = \phi_i - \phi_j$. With this splitting, the whole system can be described using the formalism introduced in Sec. II, but in two different spaces. First, we identify nonconservative and conservative velocities associated with the single-oscillator phase change:

$$\begin{aligned} \mathbf{v}_a^\phi &= \mathbf{v}_{nc}^\phi + \mathbf{v}_c^\phi, \quad \mathbf{v}_{nc}^\phi = k\mathbf{e}_g, \\ (\mathbf{v}_c^\phi)^i &= \nabla_\phi^i \mathcal{F} = \frac{k\beta_{\text{eff}}}{N} \partial_j E^{ij}, \end{aligned} \quad (106)$$

where the gradient is defined as $\nabla_\phi^i = k\delta^{ij}\partial_j$ and the scalar product is denoted by $a \cdot b = k^{-1}a_i b^i$. Second, in the space of the phase difference, indicating the gradient as $\nabla_\varphi^{ij} = (\Omega/N)(\partial^i - \partial^j)$, we obtain the velocities

$$\begin{aligned} \mathbf{v}_a^\varphi &= \mathbf{v}_{nc}^\varphi + \mathbf{v}_c^\varphi, \quad (\mathbf{v}_{nc}^\varphi)^{ij} = -\frac{\Omega}{N} E^{ij}, \\ (\mathbf{v}_c^\varphi)^{ij} &= \nabla_\varphi^{ij} \mathcal{F} = \frac{\Omega\beta_{\text{eff}}}{N^2} \partial_k (E^{ik} - E^{jk}), \end{aligned} \quad (107)$$

and the scalar product is $a \cdot b = (\Omega/N)^{-1} \sum_{i,j} a^{ij} b^{ij}$. The geometrical picture is very rich and generalizes our previous results. In each space, there is a nonconservative part that represents the injected energy, i.e., the adenosine triphosphate (ATP) for single phases, where $\mathbf{v}_{nc}$ is harmonic and $\nabla_\phi \cdot \mathbf{v}_{nc}^\phi = 0$, and the interaction potential for phase differences, where $\nabla_\varphi \cdot \mathbf{v}_{nc}^\varphi \neq 0$. The conservative parts are given by the derivatives of the self-information in each space, with the one associated with φ representing the energy difference between oscillators. In this case, the reference distribution for the phases is the uniform one, and hence $\mathcal{F}$ and $\mathcal{A}$ coincide. Finally, both adiabatic velocities have nonzero divergences, and their values balance:

$$\nabla_\phi \cdot \mathbf{v}_a^\phi = -\nabla_\varphi \cdot \mathbf{v}_a^\varphi = k\beta_{\text{eff}} \nabla_\phi^2 \mathcal{F}. \quad (108)$$

By performing a mean-field limit, valid for an infinite number of oscillators ($N \rightarrow \infty$), one discovers that the system can undergo a transition from an asynchronous to a coherent phase by varying the coupling frequency Ω (see the original work [4] for more details). A plot of the system velocities for two mean-field oscillators in the synchronized phase is presented in Fig. 5 (see Appendix F for the explicit expressions). It shows intuitively that while $\mathbf{v}^\phi$ drives the phase of each single oscillator to complete a cycle (nonzero vorticity but not divergence-free), $\mathbf{v}^\varphi$ tends to reduce the phase differences to zero and hence to synchronize the system (i.e., a multiple dipole structure emerges akin to a group of charges in an electric field, zero vorticity). In such a mean-field limit, we can also compute the average velocity divergence,

$$\langle \nabla \cdot \mathbf{v}_a^\varphi \rangle_{\text{st}} = k\beta_{\text{eff}} E_0 r = k\beta_{\text{eff}} \langle \nabla_\phi^2 \mathcal{F} \rangle_{\text{st}} \geq 0, \quad (109)$$

and the various entropy production terms,

$$\begin{aligned} \dot{S}_{nc}^{\text{st}} &= k\mathbf{e}_g^2 + \frac{\Omega E_0}{\beta_{\text{eff}}} \left(1 - \frac{1}{\beta_{\text{eff}} E_0}\right), \\ \dot{S}_c^{\text{st}} &= \Omega E_0 r^2 = -\mathcal{F}_{nc}^{\text{flux}} \geq 0, \end{aligned} \quad (110)$$

which contribute to the total dissipation, as for Eq. (54). Equations (109) and (110) give a complete thermodynamic characterization of the synchronization phase transition: the order parameter $r \in [0, 1]$ stems from the velocity divergence

(i.e., the availability Laplacian), so that the nonconservative dissipation correctly quantifies energy inputs (depending only on the control parameters), as anticipated above, while the conservative one quantifies exactly the resulting emergent behavior, as it originates from the transduction work done by the oscillators and, as such, is proportional to the square of the order parameter. This suggests that, in many-body systems, the nontrivial current geometry, i.e., the divergence of $\mathbf{v}$ and conservative dissipation, reflects the transduction of free energy from one components to the others, similarly to the view developed by Hill [71,72].

VII. CONCLUSIONS AND PERSPECTIVES

A. Summary

How order emerges far from thermodynamic equilibrium remains a far-reaching unsolved problem. In this article, we shed light on how geometrical and symmetry-breaking properties of nonequilibrium steady states emerge from the violation of detailed balance. By focusing on velocities, rather than currents, we introduce a geometric decomposition that allows us to isolate the conditions responsible for breaking the detailed balance (i.e., nonconservative components) from those associated with system's internal response to them (i.e., conservative components). This decomposition at the level of velocities reflects into a decomposition of thermodynamic dissipation as well. By combining this framework with a small-noise expansion, we propose a classification of NESSs and show how geometric and thermodynamic properties are shaped by the external noise. Furthermore, extending these results to systems with multiplicative fluctuations—a largely uncharted area—reveals that more complex scenarios can emerge. In particular, we show that the system breaks both parity and translational symmetry in trajectory space due to the presence of an additional phoretic drift. Finally, applying our ideas to many-body systems with coupled molecular oscillations reveals the geometric properties of free-energy transduction and nonequilibrium phase transitions when multiple currents are simultaneously at play.

B. Toward a thermodynamics of selection and symmetry breaking

An intuitive understanding of our result might come from interpreting the onset of these geometrical structures as a mechanism to cope with the external constraints that keep the system out of equilibrium. Indeed, the system selectively dissipates energy through these “large-scale” complex structures, which are also responsible for the breaking of parity and translational symmetry (in the most general case). Following inspiration by Zia and Schmittmann [73], the geometric decomposition of dissipation can be used to classify NESSs, as recapitulated in Table I and in Fig. 6. In this work, we verified our results mainly in cases where it is possible to access the analytical form of the stationary probability distribution, at least in an approximate sense. To advance further, we plan to investigate the problem numerically, using methodologies capable of reducing the error in velocity estimation. On the analytical side, the properties of velocity fields could eventually be studied using renormalization group techniques

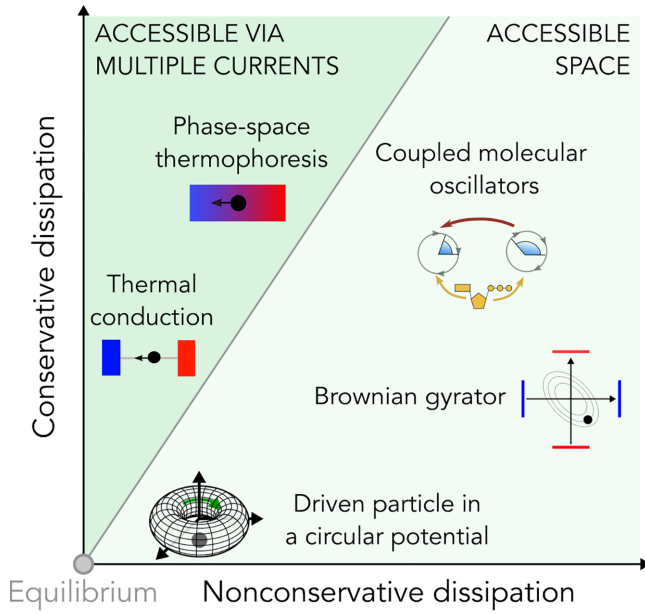


FIG. 6. NESS classification based on conservative and nonconservative dissipation. We report the examples studied along this article in the diagram, with their position qualitatively representing the thermodynamic classification on the basis of Table I. Local equilibrium system such as the driven particle in a circular potential or the Brownian gyrator has zero or modest conservative dissipation, and the velocity geometry is dominated by the nonconservative contribution breaking parity. They all have to live in the accessible space where $\dot{S}_{nc} \geq \dot{S}_c$, since they are characterized by a single physical current. For all these systems, the gray line and the gray dot indicate equilibrium conditions. However, open systems like a particle connected to many heat baths or an underdamped particle in a temperature gradient sustain the emergence of multiple currents that might enable the access to the whole space, even exhibiting a purely conservative dissipation associated with a translational symmetry breaking (see Sec. VC). Finally, a many-body open system like a biochemical oscillator exhibits an increasing conservative dissipation when approaching a synchronization phase transition.

[74,75], large-deviation theory [76], or by focusing on topological properties. Finally, on a more conceptual level, our approach calls for a deeper understanding of nonequilibrium thermodynamics through the lens of gauge transformations, geometry, and information theory [23,41,57,77]. Beyond these fundamental questions, our framework shows great promise as a versatile tool for addressing different interdisciplinary challenges, as already shown by flux/landscape approaches [24]. Active matter represents a promising field of application for the presented framework. In the simplest case of an active Brownian particle [78], for example, a solution can be readily obtained at all times and closely resembles that of the 2D Brownian gyrator. The conceptual leap, in this context, is that emergent dissipative cycles arise in the space describing both the particle and the bath, making physical interpretation more challenging [79,80]. Future work may further explore this direction, together with the role of nonreciprocal interactions [81]. More generally, following the many-body example presented here, our framework could pave the way for constructing a thermodynamics of

nonequilibrium phase transitions, where individual components break symmetry due to energy injection but coordinate to spontaneously induce macroscopic order [6,82]. Along the same lines, the idea that a form of selection can naturally occur out of equilibrium due to dissipation-driven processes is gaining momentum in the fields of physical and biological/metabolic chemistry [54,55,83–85]. Living matter, such as cells and ecological systems [86–89], is believed to operate at an optimal thermodynamic or metabolic state. A promising way to understand the selection of such optimality is to study the thermodynamic landscape of alternative states relative to the optimal one by introducing a particular external forcing/perturbation, as in recent experimental studies on cell cytoskeleton [90,91]. Hence, our framework offers a promising step toward a general theory of thermodynamic selection, suggesting a possible conceptual bridge between thermodynamics and evolutionary dynamics [62,92–94]. To further investigate this analogy, we leave for future investigation the extension of the framework to discrete-state dynamics (e.g., master equations), in line with existing exploratory efforts in this direction [40,74,75,77,95,96]. In summary, our work characterizes the geometric properties of nonequilibrium currents and how they signal the onset of symmetry breaking. Owing to its generality, it paves the way for future theoretical investigations and offers a unified approach to understanding selection phenomena across different contexts as a consequence of symmetry-breaking processes driven by nonequilibrium dissipation.

ACKNOWLEDGMENTS

M.S. acknowledges G. Jona-Lasinio for early inspiration in studying symmetry breaking in nonequilibrium statistical physics and M. Muñoz for introducing us to the quasipotential formalism. D.M.B. thanks M. Ciarchi (Retoño) for illuminating discussions on symmetry breaking. We are extremely grateful to M.A. Muñoz, E. Roldán, A. Maritan, P.I. Hurtado, P. Garrido, C. Maes, M. Murrell, and A. Goyal for important discussions and insights. M.S. has been supported by Grant No. PID2023-149174NB-I00 financed by the Spanish Ministry and Agencia Estatal de Investigación MICIU/AEI/10.13039/501100011033 and EDRF funds, by Daniel R. Amor (ENS-PSL) through Grant No. ANR-22-CPJ2-0064-01, and by the Human Frontier Science Program (HFSP) Postdoctoral Fellowship LT0022/2025-L. D.M.B. was funded by the program STARS@UNIPD with the project “ActiveInfo.”

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: WEAK-NOISE EXPANSION FOR ADDITIVE NOISE

If we assume a weak-noise ansatz for the stationary probability,

$$P_{st} \asymp e^{-\mathcal{F}^{(0)}/\sigma}, \quad (\text{A1})$$

the stationarity condition reads

$$\begin{aligned} \partial_t P_{\text{st}} &= \sigma^{-1} P_{\text{st}} (F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \partial_i \mathcal{F}^{(0)} \\ &\quad - (\partial_i F^i + D^{ij} \partial_i \partial_j \mathcal{F}^{(0)}) P_{\text{st}} = 0. \end{aligned} \quad (\text{A2})$$

Following various approaches [22,33,34], we perturbatively solve Eq. (A2) by equating to zero each order in σ . The first contribution is of order $1/\sigma$ and reads

$$(F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \partial_i \mathcal{F}^{(0)} = 0, \quad (\text{A3})$$

while the second (and last) contribution is of order σ^0 :

$$\partial_i F^i + D^{ij} \partial_i \partial_j \mathcal{F}^{(0)} = 0. \quad (\text{A4})$$

By using the definition of gradient, divergence, and adiabatic velocity,

$$\mathbf{v}_a = \mathbf{F} + \nabla \mathcal{F}^{(0)}. \quad (\text{A5})$$

Equations (A3) and (A4) in vector form read

$$\mathbf{v}_a * \nabla \mathcal{F}^{(0)} = 0, \quad \nabla * \mathbf{v}_a = 0. \quad (\text{A6})$$

If we now consider an additional order in the self-information expansion, we have

$$P_{\text{st}} \approx e^{-\mathcal{F}^{(0)}/\sigma - \mathcal{F}^{(1)}}, \quad (\text{A7})$$

so that the velocity can be split into two parts:

$$\mathbf{v}_a^{(0)} = \mathbf{F} + \nabla \mathcal{F}^{(0)}, \quad \mathbf{v}_a^{(1)} = \nabla \mathcal{F}^{(1)}. \quad (\text{A8})$$

The stationarity condition reads

$$\begin{aligned} \partial_t P_{\text{st}} &= \sigma^{-1} P_{\text{st}} (F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \partial_i \mathcal{F}^{(0)} \\ &\quad + (\partial_i (F^i + D^{ij} \partial_j \mathcal{F}^{(0)})) \\ &\quad - (F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \partial_i \mathcal{F}^{(1)} - D^{ij} \partial_j \mathcal{F}^{(1)} \partial_i \mathcal{F}^{(0)} P_{\text{st}} \\ &\quad + \sigma (-D^{ij} \partial_i \partial_j \mathcal{F}^{(1)} + D^{ij} \partial_i \mathcal{F}^{(1)} \partial_j \mathcal{F}^{(1)}) P_{\text{st}} = 0. \end{aligned} \quad (\text{A9})$$

The dominant order σ^{-1} is not modified with respect to Eq. (A3):

$$\mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(0)} = 0. \quad (\text{A10})$$

On the other hand, the order σ^0 is modified to

$$\begin{aligned} &-(F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \partial_i \mathcal{F}^{(1)} + \partial_i (F^i + D^{ij} \partial_j \mathcal{F}^{(0)}) \\ &- D^{ij} \partial_j \mathcal{F}^{(1)} \partial_i \mathcal{F}^{(0)} = 0. \end{aligned} \quad (\text{A11})$$

In vector forms, it reads

$$\nabla * \mathbf{v}_a^{(0)} = \mathbf{v}_a^{(0)} * \mathbf{v}_a^{(1)} + \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(0)}. \quad (\text{A12})$$

Finally, the order σ reads

$$-D^{ij} \partial_i \partial_j \mathcal{F}^{(1)} + D^{ij} \partial_i \mathcal{F}^{(1)} \partial_j \mathcal{F}^{(1)} = 0, \quad (\text{A13})$$

$$\nabla * \mathbf{v}_a^{(1)} = \mathbf{v}_a^{(1)} * \mathbf{v}_a^{(1)}. \quad (\text{A14})$$

To continue with geometric characterization, let us study the average flux of the different contributions to the self-information $\mathcal{F}^{(0)}$, $\mathcal{F}^{(1)}$. At stationarity, the time derivative of the average of the quasipotential part of the

self-information has to vanish,

$$\mathbf{d}_t \langle \mathcal{F}^{(0)} \rangle_{\text{st}} = \langle \mathbf{v}_a * \nabla \mathcal{F}^{(0)} \rangle_{\text{st}} = 0, \quad (\text{A15})$$

therefore

$$\langle \mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(0)} \rangle_{\text{st}} = -\sigma \langle \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(0)} \rangle_{\text{st}} \quad (\text{A16})$$

and, by using Eq. (34), we obtain the orthogonality condition:

$$\langle \mathbf{v}_a^{(1)} * \nabla \mathcal{F}^{(0)} \rangle_{\text{st}} = 0. \quad (\text{A17})$$

This relation implies that the first-order velocity, which consists of only a conservative term, is on average orthogonal to the contour lines of the self-information. Similarly, by looking at the first-order self-information $\mathcal{F}^{(1)}$, we find that its average change along the zeroth-order velocity balances the first-order velocity divergence. In fact,

$$\mathbf{d}_t \langle \mathcal{F}^{(1)} \rangle_{\text{st}} = \langle \mathbf{v}_a * \nabla \mathcal{F}^{(1)} \rangle_{\text{st}} = 0 \quad (\text{A18})$$

implies that

$$\begin{aligned} \langle \mathbf{v}_a^{(0)} * \nabla \mathcal{F}^{(1)} \rangle_{\text{st}} &= -\sigma \langle \mathbf{v}_a^{(1)} * \mathbf{v}_a^{(1)} \rangle_{\text{st}} \\ &= -\sigma \langle \nabla * \mathbf{v}_a^{(1)} \rangle_{\text{st}}, \end{aligned} \quad (\text{A19})$$

where we used the fact that $\mathbf{v}_a^{(1)} = \nabla \mathcal{F}^{(1)}$ and Eq. (36) for the last equality. Notice that, by using Eqs. (A18) and (35), the relation appearing in Eq. (A19) is equivalent to

$$\langle \nabla * \mathbf{v}_a^{(0)} \rangle_{\text{st}} = -\sigma \langle \nabla * \mathbf{v}_a^{(1)} \rangle_{\text{st}} \leq 0, \quad (\text{A20})$$

which can be obtained by directly averaging the divergence of the total velocity and remembering that $\langle \nabla * \mathbf{v}_a \rangle_{\text{st}} = 0$ at stationarity. These conditions, taken together, imply that the average fluxes of $\mathbf{v}_a^{(0)}$ and $\mathbf{v}_a^{(1)}$ must be related to each other.

APPENDIX B: SYMMETRY BREAKING IN THE TRAJECTORY SPACE

In our framework, we can improve the physical interpretation of the nonconservative and conservative dissipations by studying symmetry breaking at the trajectory level. Here, we show that the first one it accounts for a tendency to circulate, breaking a parity symmetry holding at equilibrium and leading to a stationary dissipation into the environment. Considering a system governed by Eq. (1), the probability of observing a trajectory Γ —starting in $\mathbf{x}_0$ at $t = 0$ and arriving in $\mathbf{x}_f$ —is given by

$$P(\Gamma) = \underbrace{e^{-\mathcal{A}(\Gamma)/2}}_{P(\Gamma|\mathbf{x}_0)} P(\mathbf{x}_0), \quad (\text{B1})$$

where $\mathcal{A}(\Gamma)$ is the Onsager-Machlup action [97] that, using the Stratonovich prescription, reads

$$\mathcal{A}(\Gamma) = \int_0^t d\tau \left[\frac{1}{2} (\dot{\mathbf{x}} - \mathbf{F}) * (\dot{\mathbf{x}} - \mathbf{F}) + \nabla * \mathbf{F} \right], \quad (\text{B2})$$

in which all quantities have to be evaluated along Γ . Equation (B2) can be rewritten as the time integral of a lagrangian A , and it can be divided into a time-symmetric part A_s , stemming from an “energy-stress tensor,” and a

time-antisymmetric part A_a :

$$\mathcal{A}(\Gamma) = \int_0^t d\tau A, \quad A = D_{ij}A_s^{ij} + D_{ij}A_a^{ij}, \quad (\text{B3})$$

$$A_s^{ij} = \frac{1}{2}\dot{x}^i\dot{x}^j + \frac{1}{2}F^iF^j + \partial_k D^{ik}F^j, \quad (\text{B4})$$

$$A_a^{ij} = -\dot{x}^iF^j. \quad (\text{B5})$$

The detailed balance condition in Eq. (40) can thus be rewritten as a condition for the stress-energy tensor to be symmetric with respect to indices:

$$\begin{aligned} w_{ij} &= (A_s)_{ij} - (A_s)_{ji} = \partial_i F_j - \partial_j F_i \\ &= \partial_i (v_{nc})_j - \partial_j (v_{nc})_i, \end{aligned} \quad (\text{B6})$$

where w_{ij} is the “torque” tensor, i.e., the antisymmetric part of the stress-energy tensor. This field encodes the nonconservation of the angular momentum, hence parity symmetry breaking, and coincides with the vorticity field $\mathbf{w}$ in two and three dimensions. The dissipation along a trajectory Γ , $s(\Gamma)$, customarily quantified by the ratio between the probability of Γ and its time reversal $\tilde{\Gamma}$ [2], is associated with the antisymmetric part of the action. To deepen our understanding, consider that in a closed trajectory $\Gamma_\circ$, i.e., $x_0 = x_f$, we have

$$s(\Gamma_\circ) = \ln \frac{P(\Gamma_\circ)}{\tilde{P}(\tilde{\Gamma}_\circ)} = \int_{\Gamma_\circ} d\mathbf{x} * \mathbf{v}_{nc} \equiv s_{nc}(\Gamma_\circ). \quad (\text{B7})$$

In two and three dimensions, we can further apply Stokes’ theorem and thus rewrite the dissipation as the flux of the vorticity field $\mathbf{w}$ across the surface $\Sigma(\Gamma_\circ)$ defined by the closed trajectory:

$$s_{nc}(\Gamma_\circ) = \int_{\Sigma(\Gamma_\circ)} d\Sigma \mathbf{w}. \quad (\text{B8})$$

These results establish a clear connection between the propensity of performing closed trajectories in a preferential direction, i.e., the breaking of the equilibrium parity symmetry and the nonconservative velocity that breaks the space symmetries in the stress-energy tensor. Furthermore, these observations reinforce the validity of our identification of the nonconservative contribution as stemming from the nonequilibrium conditions, and the availability $\mathcal{A}$ as capturing pseudoequilibrium modifications. Indeed, we notice that $\mathcal{A}$ does not contribute to the dissipation along any closed trajectory $\Gamma_\circ$.

Furthermore, if we look at the dissipation along an open trajectory, we note that an additional contribution proportional to the availability can arise if the system is in a NESS:

$$\ln \frac{P_{eq}(\mathbf{x}_0)P(\Gamma|x_0)}{P_{eq}(\mathbf{x}_f)\tilde{P}(\tilde{\Gamma}|x_f)} = s_{nc}(\Gamma), \quad (\text{B9})$$

$$\ln \frac{P_{st}(\mathbf{x}_0)P(\Gamma|x_0)}{P_{st}(\mathbf{x}_f)\tilde{P}(\tilde{\Gamma}|x_f)} = s_{nc}(\Gamma) + \Delta\mathcal{A}, \quad (\text{B10})$$

where $\Delta\mathcal{A} = \mathcal{A}(\mathbf{x}_f) - \mathcal{A}(\mathbf{x}_0)$ and $\mathcal{A} = \mathcal{F} - \beta U$. Equations (B9) and (B10) explain the role on single trajectories of the geometric components of the velocity field: The nonconservative velocity selects the probability of producing a fluctuation from the equilibrium state over its time reversal, while the conservative contribution tends to select a trajectory

over its time reversal in a NESS if it leads to an increase in availability.

APPENDIX C: COVARIANT FORMULATION OF THE FOKKER-PLANCK EQUATION WITH MULTIPLICATIVE NOISE

We first consider the covariant formulation of the Fokker-Planck equation in the Ito scheme and second in the Stratonovich case. Our exposition follows various works by Graham [21,63].

In the Ito case, the Fokker-Planck equation reads

$$\begin{aligned} \partial_t P &= -\partial_i (F^i P - \partial_j D^{ij} P) \\ &= -\partial_i ((F^i - \partial_j D^{ij}) P - D^{ij} \partial_j P). \end{aligned} \quad (\text{C1})$$

As in Sec. II, we take the inverse of the diffusion matrix as a metric to introduce the covariant derivative:

$$\nabla_j u^i = \partial_j u^i + \Gamma_{jk}^i u^k, \quad (\text{C2})$$

where Γ is the affine connection. By imposing the parallel transport of the metric,

$$\nabla_j D^{ij} = 0, \quad (\text{C3})$$

the connection is determined to be

$$\Gamma_{\beta\kappa}^\alpha = \frac{1}{2} D^{\alpha\mu} (\partial_\beta D_{\mu,\kappa} + \partial_\kappa D_{\mu\alpha} - \partial_\mu D_{\alpha\beta}). \quad (\text{C4})$$

The covariant divergence is defined as

$$\begin{aligned} \nabla * \mathbf{v} &= \nabla_i v^i = \sqrt{|D|} \partial_i \left(\frac{v^i}{\sqrt{|D|}} \right) \\ &= \partial_i v^i - \Gamma_i v_i, \end{aligned} \quad (\text{C5})$$

where $|D|$ is the diffusion matrix determinant and $\Gamma_i = \Gamma_{ik}^k$. The notation in terms of the diffusive scalar product follows from

$$\begin{aligned} \nabla * \mathbf{v} &= \nabla^i D_{ij} v^j = D^{ik} \nabla_k D_{ij} v^j \\ &= \nabla_j v^j + D^{ik} v^j \nabla_k D_{ij} = \nabla_j v^j + D_i^k v_j \nabla_k D^{ij} \\ &= \nabla_j v^j + v_j \nabla_i D^{ij} = \nabla_j v^j. \end{aligned} \quad (\text{C6})$$

The Laplace-Beltrami operator, i.e., the divergence of a gradient of scalar function, is defined as

$$\nabla * \nabla \phi = \sqrt{|D|} \partial_i \left(\frac{D^{ij} \partial_j \phi}{\sqrt{|D|}} \right). \quad (\text{C7})$$

Now, by defining the invariant probability Q and the invariant measure $d\Omega$,

$$Q = P\sqrt{|D|}, \quad d\Omega = \frac{d\mathbf{x}}{\sqrt{|D|}}, \quad (\text{C8})$$

one can rewrite the Fokker-Planck equation in a fully covariant form. First, we need to set the position of the diffusion matrix in a compatible way with the Laplace-Beltrami operator. This will produce an inhomogeneity in the force field. In

the Ito prescription, this reads

$$\begin{aligned} \sqrt{|D|} \partial_i \partial_j D^{ij} P &= \sqrt{|D|} \partial_i \partial_j \left(\frac{D^{ij} Q}{\sqrt{|D|}} \right) = \sqrt{|D|} \partial_i \left(\frac{D^{ij} \partial_j Q}{\sqrt{|D|}} \right) \\ &+ \sqrt{|D|} \partial_i \left(Q \partial_j \left(\frac{D^{ij}}{\sqrt{|D|}} \right) \right). \end{aligned} \quad (\text{C9})$$

By rewriting everything now using Eqs. (C5), (C7), and (C8), we arrive at

$$\begin{aligned} \partial_i Q &= -\sqrt{|D|} \partial_i \left(\frac{\tilde{F}^i Q}{\sqrt{|D|}} - \frac{D^{ij} \partial_j Q}{\sqrt{|D|}} \right) \\ &= -\nabla * J, \end{aligned} \quad (\text{C10})$$

where

$$\begin{aligned} J^i &= \tilde{F}^i Q - D^{ij} \partial_j Q, \\ \tilde{F}^i &= F^i - \sqrt{|D|} \partial_j \left(\frac{D^{ij}}{\sqrt{|D|}} \right) = F^i + v_D^i. \end{aligned} \quad (\text{C11})$$

This additional term appearing in the force, which we name v_D , is the covariant divergence of the diffusion matrix. By applying the definition of the covariant divergence [Eq. (88)] it can be rewritten as

$$\begin{aligned} v_D^i &= \partial_j D^{ij} - \Gamma^i, \\ \Gamma^i &= \Gamma_{jk}^i D^{jk} = -\nabla^i \ln \sqrt{|D|}. \end{aligned} \quad (\text{C12})$$

Finally, by using Eq. (91) for the first term and after some manipulations, it can be decomposed as the sum of a first contribution stemming from the vector ω and a second one from the derivative of the diffusive forces G :

$$v_D^i = D^{ij} \omega_j - \partial_j G_i^j G_k^{jk} \delta^{lk}. \quad (\text{C13})$$

On the other hand, in the case of the Stratonovich prescription, we first have to rewrite the Fokker-Planck to make manifest the presence of the diffusive forces (as we did in the Ito case):

$$\begin{aligned} \partial_t P &= -\partial_i (F P - G_b^i \partial_j (G_a^j \delta^{ab} P)) \\ &= -\partial_i ((F^i + \partial_j G_k^i G_k^j) P - \partial_j D^{ij} P), \end{aligned} \quad (\text{C14})$$

and then apply Eq. (C9). It is immediate to see that the additional drift term in the force in Eq. (C14) counterbalances the second term in Eq. (C13), leading to the following expression for the diffusive velocity:

$$v_D^i = D^{ij} \omega_j. \quad (\text{C15})$$

The combination of Eqs. (C13) and (C15), with the introduction of a parameter $\alpha = 0, 1$, gives Eq. (90). Now, let us comment on the curvature and its relation with the diffusive velocity. As usual, the curvature can be quantified by the Ricci scalar in terms of the affine connection:

$$R = D^{ij} (\partial_l \Gamma_{ij}^l - \partial_j \Gamma_{il}^l + \Gamma_{ij}^k \Gamma_{lk}^l - \Gamma_{il}^k \Gamma_{jk}^l). \quad (\text{C16})$$

The diffusive forces G are the metric frames, and intuitively they tell how the variable feels the noise “kicks” at a certain configuration [26]. The vector ω is called anholonomy vector and measures how G behaves when parallel transported along a closed loop, i.e., the anisotropy of fluctuations. This is equivalent to measuring the curvature of the associated manifold.

Hence, one defines the anholonomy tensor as

$$\omega_{ab}^c = G_a^i G_b^j (\partial_i G_j^c - \partial_j G_i^c), \quad (\text{C17})$$

which quantifies how a motion along the directions a and b generates anisotropy in c . The corresponding anholonomy vector capturing the net twist only along the direction a is thus obtained by contraction. Then, one can rewrite the scalar curvature as

$$R = \frac{1}{4} \omega_{abc} \omega^{abc} - \frac{1}{2} \omega_{abc} \omega^{bca} - \omega_a^{ab} \omega^c_{cb}. \quad (\text{C18})$$

1. Perturbative solution of the covariant Fokker-Planck equation

We consider a noise expansion for the invariant stationary probability:

$$Q_{\text{st}} \approx e^{-\mathcal{F}^{(0)}/\sigma - \mathcal{F}^{(1)}} \quad (\text{C19})$$

and solve order by order in sigma as in Appendix A. The results are equivalent to the previous expansions but stated in curved geometry, and with the addition of the “phoretic” term in the first-order velocity,

$$\tilde{v}_a^{(1)} = \nabla \mathcal{F}^{(1)} + v_D. \quad (\text{C20})$$

The dominant order is still given by Eq. (A10), while the following orders read:

$$\begin{aligned} \nabla * v_a^{(0)} &= v_a^{(0)} * \nabla \mathcal{F}^{(1)} + \tilde{v}^{(1)} * \nabla \mathcal{F}^{(0)}, \\ \nabla * \tilde{v}_a^{(1)} &= \tilde{v}_a^{(1)} * \nabla \mathcal{F}^{(1)} = \tilde{v}_a^{(1)} * (\tilde{v}_a^{(1)} - v_D). \end{aligned}$$

By taking the average, the last equation leads to the first-order entropy production:

$$\begin{aligned} \dot{S}^{(1)} &= \sigma \langle \tilde{v}_a^{(1)} * \tilde{v}_a^{(1)} \rangle \\ &= \sigma \langle \nabla * \tilde{v}_a^{(1)} \rangle + \sigma \langle v_D * \tilde{v}_a^{(1)} \rangle \geq 0. \end{aligned} \quad (\text{C21})$$

In the case of vanishing curvature, $\omega = 0$, and it is immediate to show that the Ito diffusive velocity [Eq. (C13)] reduces to the contracted connection [see Eq. (95)]. Hence, by applying the definition of covariant divergence [Eq. (88)] in Eq. (C21), one obtains

$$\dot{S}_a^{(1)} = \sigma \langle \partial * \tilde{v}_a^{(1)} \rangle, \quad (\text{C22})$$

coherently with the fact that the manifold is flat.

APPENDIX D: GEOMETRIC DECOMPOSITION FROM NONCONSERVATIVE FORCES

A more transparent physical interpretation of the decomposition stems by the study of the simple case in which nonequilibrium conditions originate from the presence of *a priori* known nonconservative forces, f (e.g., external fields, shear flows). In this scenario, one has

$$F^i = f^i - \partial^i U, \quad D^{ij} = \delta^{ij} T, \quad \nabla^i = T \partial^i, \quad (\text{D1})$$

where $\partial^i = \delta^{ij} \partial_j$, U is the potential acting on the system, and T the temperature of the external bath. We also employ the Einstein relation to link diffusion and temperature to ensure that, when $f = 0$, the system relaxes to the Boltzmann distribution. Here, $\beta = T^{-1}$ with the Boltzmann constant set to 1

for simplicity. From the force decomposition in Eq. (27), the adiabatic velocity can be expressed as follows:

$$\mathbf{v}_a = \mathbf{f} - \partial U + \nabla \mathcal{F} = \mathbf{f} + \nabla (\mathcal{F} - \beta U), \quad (\text{D2})$$

where all gradient terms have been isolated and can be identified with $\nabla \mathcal{F}_{\text{ex},a}$. The first contribution coincides with the nonconservative force—the sole source of nonequilibrium in this scenario—and corresponds to $\mathbf{v}_{\text{nc}}$. Thereby, in this simple scenario, we obtain a decomposition of the velocity:

$$\mathbf{v}_{\text{nc}} = \mathbf{f}, \quad \mathbf{v}_c = \nabla (\mathcal{F} - \beta U) = \nabla \mathcal{A}. \quad (\text{D3})$$

Out of equilibrium, the particle could visit excited states with nonzero availability. The adiabatic entropy production is thus given by

$$\dot{S}_a = \langle \mathbf{v}_a * \mathbf{v}_a \rangle = \langle \mathbf{v} * \mathbf{f} \rangle + \langle \mathbf{v} * \nabla \mathcal{A} \rangle, \quad (\text{D4})$$

where the first term corresponds to the heat dissipated by the nonconservative force and the second one to the extractable work (and duly vanishes in the stationary state). In the steady state, we have

$$\dot{S}_a^{\text{st}} = \langle \mathbf{v}_a * \mathbf{f} \rangle_{\text{st}} = \beta \left(\underbrace{\langle \mathbf{f}^2 \rangle_{\text{st}}}_{\dot{Q}_{\text{out}}} - \underbrace{\langle \nabla \mathcal{A}^2 \rangle_{\text{st}}}_{-\dot{Q}_{\text{in}}} \right) \geq 0. \quad (\text{D5})$$

The first term multiplying the inverse temperature can be interpreted as the heat due to the nonconservative force released into the environment, while the second one as the heat absorbed from the nonequilibrium conditions that modifies the system self-information.

APPENDIX E: GEOMETRIC DECOMPOSITION FROM MULTIPLE BATHS

Another paradigmatic yet more intricate scenario to induce nonequilibrium conditions involves systems coupled to multiple baths, such as the Brownian gyrator and a Brownian particle in contact with two thermal reservoirs. In these cases, the identification of $\mathcal{A}$ becomes more subtle since there are multiple possible given equilibrium distributions, i.e., one for each bath, $P_{\text{eq}}^{T_i} \propto \exp(-U/T_i)$, where U is the potential. However, interaction terms in U between different components might result into an interaction between different baths. Indeed, in general $U = \sum_i U_i + U_{\text{int}}$, where U_i depends only on the i th component. For simplicity, we assume that each degree of freedom i is coupled to a different bath at temperature T_i , i.e., $D^{ij} = \delta^{ij} \sqrt{T_i T_j}$. The stationary velocity for each component can be readily obtained as before:

$$\mathbf{v}_{\text{st}}^i = -\partial^i U + \nabla^i \mathcal{F} = -\partial^i U_{\text{int}} + \nabla^i \left(\mathcal{F} - \frac{U_i}{T_i} \right). \quad (\text{E1})$$

The second term is a gradient and, as such, can be identified with the contribution from the availability. Indeed, it accounts for the difference between $\mathcal{F}$ and the equilibrium case, where no interaction exists between components and each one relaxes to the Boltzmann distribution associated with its respective bath. As a consequence, the first term coincides with the nonconservative velocity and, once again, only contains external contributions triggering nonequilibrium conditions.

Thus,

$$\mathbf{v}_{\text{nc}} = -\partial U_{\text{int}}, \quad \mathbf{v}_c = \nabla \left(\underbrace{\mathcal{F} - \sum_i \frac{U_i}{T_i}}_{\mathcal{A}} \right). \quad (\text{E2})$$

In this case, it is manifest how the choice of the reference equilibrium distribution changes the identification of the geometric contributions. Another possibility would have been to compare the NESS with the equilibrium distribution where $T_i = T$ for all baths, in the presence of an interaction potential. In such a case, the component of the velocity encoding the external nonequilibrium conditions would have encoded the fact that baths have different temperatures. To conclude this case, we note that the metric is flat but anisotropic, with $D^{-1} = \text{diag}(1/T_1, 1/T_2, \dots, 1/T_d)$. It can be mapped to the standard Euclidean metric by rescaling each degree of freedom by $1/\sqrt{T_i}$. Thermodynamically, this implies that in this rescaled space, each direction is effectively “measured” by a different temperature; i.e., the length scale along direction i is inversely proportional to its temperature T_i . We also generalize our framework to include multiple thermal baths acting simultaneously on the same degrees of freedom and generating different currents. In this case, since there is not only one physical current at play, Eq. (40) does not coincide with the equilibrium condition; hence, the reference equilibrium distribution does not naturally appear in the stationary velocity and has to be set *a priori*. As a concrete example, and for simplicity of notations, we consider a one-dimensional Brownian particle attached to two baths at different temperatures and subject to two corresponding potentials [30,98]:

$$\dot{x} = -\partial_x U_1 + \sqrt{2T_1} \xi_1(t) - \partial_x U_2 + \sqrt{2T_2} \xi_2(t). \quad (\text{E3})$$

The system reaches a generalized Boltzmann distribution with an effective temperature:

$$P_{\text{st}} \propto e^{-\mathcal{F}}, \quad \mathcal{F} = \frac{U_1 + U_2}{T_1 + T_2}. \quad (\text{E4})$$

In this particular condition, the total current is zero, as the contributions from each bath balance each other. Nevertheless, as we shall see shortly, there is an ongoing entropy production. Moreover, the absence of interactions (or any other external nonequilibrium source) implies that $\mathbf{v}_a$ stems purely from conservative terms from a geometrical perspective:

$$\begin{aligned} v_a &= v_{1,c} + v_{2,c} = 0, \\ v_{\alpha,c} &= -\partial_x U_\alpha + T_\alpha \partial_x \mathcal{F}, \quad \alpha = 1, 2. \end{aligned} \quad (\text{E5})$$

However, when we set a reference equilibrium distribution, say $P_1 \sim \exp(-U_1/T_1)$, the presence of the second bath can be identified as the nonequilibrium driving of the system and, as such, it has to reflect into the nonconservative part of velocity:

$$v_c = v_{1,c}, \quad v_{\text{nc}} = v_{2,c}. \quad (\text{E6})$$

Hence, the total dissipation is given by the conservative contribution associated with thermal conduction:

$$\dot{S}_a = \dot{S}_c = \sum_{i=1}^2 \left\langle \frac{v_{i,c}^2}{T_i} \right\rangle = \left\langle \frac{v_c^2}{T_1} \right\rangle + \left\langle \frac{v_{\text{nc}}^2}{T_2} \right\rangle. \quad (\text{E7})$$

We emphasize here that a purely geometrical interpretation of v would give only conservative contributions. However, a physically informed interpretation of our results in the presence of multiple currents, based on Eq. (44), leads to one of the possible correct decompositions (depending on the equilibrium prior).

Here, we report additional mathematical details and derivations on the examples we presented in the main text.

APPENDIX F: EXAMPLES

1. Driven particle in a circular potential

Consider a Brownian particle confined on a circle of radius ρ^* by a quadratic potential and driven along the circle itself by a constant nonconservative force f , as described in the main text. The presence of f breaks detailed balance, and the stationary flux along the circle will induce a parity symmetry breaking in the system. Let us start with the description of this Brownian motion:

$$\begin{aligned}\dot{x} &= f_x(x, y) - \partial_x U(x, y, z) + \sqrt{2D} \xi_x(t), \\ \dot{y} &= f_y(x, y) - \partial_y U(x, y, z) + \sqrt{2D} \xi_y(t), \\ \dot{z} &= f_z(x, y) - \partial_z U(x, y, z) + \sqrt{2D} \xi_z(t),\end{aligned}\quad (\text{F1})$$

where f_i is the component of the driving force along i and $\xi_i(t)$ is a Gaussian white noise with unit variance. These equations can be rewritten in cylindrical coordinates (ρ, ϕ, z) , with $\rho = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. Employing the Ito's formula for the change of variable, we obtain

$$\begin{aligned}\dot{\rho} &= -\gamma(\rho - \rho^*) + \frac{D}{\rho} + \sqrt{2D} \xi_\rho(t), \\ \dot{\phi} &= \frac{f}{\rho} + \frac{\sqrt{2D}}{\rho} \xi_\phi(t), \\ \dot{z} &= -\gamma z + \sqrt{2D} \xi_z(t),\end{aligned}\quad (\text{F2})$$

with $U = (\gamma/2)(\rho - \rho^*)^2 + (\gamma/2)z^2$, f only acts along ϕ , and D/ρ is the usual Ito term. The resulting Fokker-Planck equation describing the system is

$$\partial_t P = (\mathcal{L}_\rho + \mathcal{L}_\phi + \mathcal{L}_z)P, \quad (\text{F3})$$

where we introduced the following operators:

$$\begin{aligned}\mathcal{L}_\rho &= \partial_\rho \left(\gamma(\rho - \rho^*) - \frac{D}{\rho} \right) + D \partial_{\rho\rho}, \\ \mathcal{L}_\phi &= \partial_\phi \left(-\frac{f}{\rho} \right) + \frac{D}{\rho^2} \partial_{\phi\phi}, \\ \mathcal{L}_z &= \partial_z (\gamma z) + D \partial_{zz}.\end{aligned}\quad (\text{F4})$$

The motion along z is decoupled, while the other two equations can be solved imposing that the flux only flows along ϕ . Therefore, the steady-state probability distribution is

$$P_{\text{st}} = Z^{-1} \exp \left(-\frac{\rho(\rho - 2\rho^*)\gamma + \gamma z^2 - 2D \ln \rho}{2D} \right), \quad (\text{F5})$$

with Z the normalization factor. By computing the flux, we have

$$\mathbf{J}_{\text{st}} = Z^{-1} \exp \left(-\frac{\rho(\rho - 2\rho^*)\gamma + \gamma z^2}{2D} \right) f \mathbf{i}_\phi, \quad (\text{F6})$$

where $\mathbf{i}_\phi$ is the versor indicating that it flows along the coordinate ϕ . Hence, the stationary velocity reads

$$\mathbf{v}_{\text{st}} = -\frac{f}{\rho} \mathbf{i}_\phi. \quad (\text{F7})$$

We also remark that, in the small-noise limit, we have

$$P_{\text{st}} \simeq Z^{-1} \exp \left(-\frac{\rho(\rho - 2\rho^*)\gamma + \gamma z^2}{2D} \right), \quad (\text{F8})$$

which coincides with the form outlined in the main text stemming from the noise expansion.

The parameters employed to obtain Fig. 1 are $f = 1$, $\gamma = 1$, $D = 0.1$, and $\rho^* = 2$.

2. The Brownian gyrator

As a second example, consider the 2D motion in the x - y plane of a particle under the effect of a quadratic potential U and coupled to two baths, each acting along one direction and at different temperatures, $T_x = T$ and $T_y = T(1 + \delta)$. The dynamics is described by

$$\begin{aligned}\dot{x} &= -\partial_x U(x, y) + \xi_x(t), \\ \dot{y} &= -\partial_y U(x, y) + \xi_y(t), \\ \langle \xi_i(t) \rangle &= 0, \\ \langle \xi_i(t) \xi_j(t') \rangle &= 2T_i \delta_{ij} \delta(t - t'),\end{aligned}\quad (\text{F9})$$

where the potential is $U(x, y) = \frac{x^2}{2} + \frac{y^2}{2} + uxy$, with $|u| < 1$ for stability. The system approaches the following stationary solution [49]:

$$P_{\text{st}}(x, y) = Z^{-1} e^{-\mathcal{F}}, \quad (\text{F10})$$

$$\mathcal{F}(x, y) = \beta_{\text{eff}} \left(\frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{2} y^2 + u\gamma_3 xy \right), \quad (\text{F11})$$

where Z is the normalization factor and $\beta_{\text{eff}} = (T\eta)^{-1}$. To have a compact notation, we introduced

$$\begin{aligned}\eta &= 1 + \delta + \frac{u^2}{4} \delta^2, \\ \gamma_1 &= 2 - \frac{u^2}{2} \delta, \\ \gamma_2 &= 1 + \frac{u^2}{2} \delta, \\ \gamma_3 &= 1 + \frac{\delta}{2}.\end{aligned}$$

Note that when $\delta = 0$, the self-information $\mathcal{F}$ reduces to U , as expected. Note that the stationary solution respects the form resulting from the small-noise expansion, where T is the parameter regulating the noise, while δ and u affect the detailed balance condition that reads as follows:

$$T_x \partial_x F_y = T_y \partial_y F_x \rightarrow u\delta = 0. \quad (\text{F12})$$

Due to the anisotropy of the temperatures, detailed balance is broken and a current emerges with a stationary velocity:

$$v_{st,x} = \partial_x U - T \partial_x \mathcal{F}, \quad (\text{F13})$$

$$v_{st,y} = \partial_y U - T(1 + \delta) \partial_y \mathcal{F}. \quad (\text{F14})$$

It is straightforward to show that its divergence vanishes, and its vorticity reads

$$w = \partial_x (v_a)_y - \partial_y (v_a)_x = -\frac{4Tu\delta}{4(1 + \delta) + (u\delta)^2} (2 + \delta), \quad (\text{F15})$$

which coincides with the expression reported in the main text. Hence, the interplay between particle interaction and temperature difference is accountable for the emergence of a parity symmetry breaking in the NESS.

The parameters employed in Fig. 2 are $u = 0.7$, $T = 1$, and $\delta = 4$.

3. Coupled molecular oscillators

a. Model introduction and stationary solution

In Ref. [4], Tu and co-workers formulate and study a nonequilibrium model of coupled molecular oscillators chemically fueled by ATP and interacting between each other. Notably, the thermodynamics of such a complex many-body problem can be easily constructed. Here, we rapidly review this work and study it using our framework generalized to the presence of multiple currents.

At the individual level, a single oscillator is thought as a “molecular clock” with phase ϕ_i , $i \in [1, N]$, formed by a cycle of n asymmetric reactions fueled by the ATP consumption. Each reaction has a direct Poissonian rate k and a reversed one γk , where γ is related to the energy consumption. Furthermore, we consider a coupled totality of N oscillators that can also interact and exchange a small phase with reaction potential $E(\phi_i - \phi_j) = E_{ij} = -E_0 \cos(\phi_i - \phi_j)$, with E_0 the coupling strength. In the limit of many individual internal reactions $n \rightarrow \infty$, the model of coupled oscillators can be directly studied as a system of N Langevin equations with two types of noise:

$$\dot{\phi}_i = ke_g + \eta_i - \frac{\Omega}{N} \sum_{j \neq i} (2E'_{ij} + \xi_{ij}(t)), \quad (\text{F16})$$

$$\langle \eta_i(t) \eta_j(t') \rangle = 2k\delta_{ij}\delta(t - t'), \quad (\text{F17})$$

$$\langle \xi_{ij}(t) \xi_{kl}(t') \rangle = \frac{2\Omega}{N} [\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}] \delta(t - t'), \quad (\text{F18})$$

where η_i ($T = 1$) is the noise stemming by the internal individual processes, while η_{ij} is generated by the interactions. Therefore, it is needed to evaluate separately the two types of currents. In the original article, this is done by considering the microscopic master equation and approximating it. Here, we propose an equivalent analysis directly starting with the Langevin equations that generalize our results exposed in Sec. II. First, we separate the dynamics [Eq. (F16)] into the change in the single phase, $\dot{\phi}_i^0$, due to internal processes, and the change in phase differences $\phi_{ij} = \phi_i - \phi_j$, $\dot{\phi}_{ij}$, driven by

phase exchange. The resulting system of equations is

$$\dot{\phi}_i = \dot{\phi}_i^0 + \sum_j \dot{\phi}_{ij}, \quad \dot{\phi}_i^0 = ke_g + \eta_i, \quad \dot{\phi}_{ij} = -\frac{2\Omega}{N} E'_{ij} + \xi_{ij}. \quad (\text{F19})$$

In the same way, we also decompose the Fokker-Planck equation and its current as

$$\partial_t P = - \sum_{i=1}^N \partial_i J_i^\phi - \sum_{i=1}^N \sum_{j>i}^N \frac{\partial}{\partial \phi_{ij}} J_{ij}^\phi, \quad (\text{F20})$$

with $\partial/\partial \phi_{ij} = \partial_i - \partial_j$. The whole system can be described using the geometric formalism introduced in the main text, but in two different geometric spaces.

The first term accounts for the single-oscillator current where the single phase gradient is defined as $\nabla_\phi^i = k\delta^{ij}\partial_j$:

$$(J^\phi)^i = ke_g P - \nabla_\phi^i P = k[e_g - \delta^{ij}\partial_j]P, \quad (\text{F21})$$

while the second one accounts for phase differences due to the phase exchange interactions, with gradient $\nabla_\phi^{ij} = \frac{\Omega}{N}(\partial^i - \partial^j)$:

$$(J^\phi)^{ij} = \frac{\Omega}{N} E'^{ij} - \nabla_\phi^{ij} P = \frac{\Omega}{N} [E'^{ij} - (\partial^i - \partial^j)]P. \quad (\text{F22})$$

In resume, the Fokker-Planck equation can be rewritten as

$$\partial_t P = -\nabla_\phi \cdot \mathbf{J}_\phi - \nabla_\phi * \mathbf{J}_\phi. \quad (\text{F23})$$

We search for a stationary solution P_{st} by imposing $\partial_t P_{st} = 0$, i.e., the sum of the two divergences to zero:

$$ke_g \sum_i \partial_i P - \sum_i \partial_i \frac{\Omega}{N} \sum_{j \neq i} 2E'_{ij} P - \sum_i (k + \Omega(N-1)/N^{-1}) \partial_i^2 P + 2 \frac{\Omega}{N} \sum_{j \neq i} \partial_j \sum_i \partial_i P = 0.$$

Equivalently, the stationary condition reads in terms of velocities:

$$\nabla_\phi \cdot \mathbf{v}_\phi + \nabla_\phi * \mathbf{v}_\phi = (\mathbf{v}_\phi \cdot \nabla_\phi + \mathbf{v}_\phi * \nabla_\phi) \mathcal{F}. \quad (\text{F24})$$

Given that the stationary solution needs to be rotationally symmetric, i.e., $\sum_i \partial_i P_{st} = 0$, the stationary condition [Eq. (F24)] simplifies to an *effective* equilibrium one:

$$\sum_i \partial_i \left[\frac{\Omega}{N} \sum_{j \neq i} 2E'_{ij} P_{st} + (k + \Omega) \partial_i P_{st} \right] = \sum_i \partial_i J_i^{\text{eff}} = 0, \quad (\text{F25})$$

allowing us to consider $J_i^{\text{eff}} = 0$ and leading to an *exact* solution in the form of a self-information with an effective temperature:

$$P_{st}(\boldsymbol{\phi}) = Z^{-1} \exp \left(-N^{-1} \beta \sum_{i<j} E(\phi_i - \phi_j) \right),$$

$$\beta^{-1} = T_{\text{eff}} = 1 + \frac{k}{\Omega}. \quad (\text{F26})$$

Let us note that here the reference distribution can be thought as the uniform one obtained in absence of interactions. Hence,

the potential in the stationary probability can be directly identified as the availability:

$$\mathcal{A} = \frac{1}{N} \sum_{i < j} E_{ij}, \quad P_{\text{st}} = \frac{e^{-\beta_{\text{eff}} \mathcal{A}}}{Z}. \quad (\text{F27})$$

b. Geometrical and thermodynamic properties of the velocities

With the splitting introduced in the precedent section, the whole system can be described using the geometric formalism introduced in the main text, but in two different spaces. First, we identify conservative and nonconservative velocities associated with the single-oscillator phase change:

$$\begin{aligned} \mathbf{v}_a^\phi &= \mathbf{v}_{\text{nc}}^\phi + \mathbf{v}_c^\phi, & \mathbf{v}_{\text{nc}}^\phi &= k e_g, \\ (\mathbf{v}_c^\phi)^i &= \beta_{\text{eff}} \nabla_\phi^i \mathcal{A} = \frac{k \beta_{\text{eff}}}{N} \partial_j E^{ij}, \end{aligned} \quad (\text{F28})$$

where the gradient is defined as $\nabla_\phi^i = k \delta^{ij} \partial_j$ and scalar product is indicated as $a \cdot b = k^{-1} a_i b^i$. The nonconservative part is harmonic, while the conservative has a nonzero divergence, leading to

$$\nabla_\phi \cdot \mathbf{v}_a^\phi = \frac{k \beta_{\text{eff}}}{N} \sum_{ij} E''_{ij} = \beta_{\text{eff}} \nabla_\phi^2 \mathcal{A} = -\frac{k \beta_{\text{eff}}}{N} \sum_{ij} E_{ij}. \quad (\text{F29})$$

Second, in the space of the phase difference, indicating the gradient as $\nabla_\varphi^{ij} = \frac{\Omega}{N} (\partial^i - \partial^j)$, we obtain the velocities

$$\begin{aligned} \mathbf{v}_a^\varphi &= \mathbf{v}_{\text{nc}}^\varphi + \mathbf{v}_c^\varphi, & (\mathbf{v}_{\text{nc}}^\varphi)^{ij} &= -2 \frac{\Omega}{N} E'^{ij}, \\ (\mathbf{v}_c^\varphi)^{ij} &= \nabla_\varphi^{ij} \mathcal{F} = 2 \frac{\Omega \beta_{\text{eff}}}{N^2} \partial_k (E^{ik} - E^{jk}), \end{aligned} \quad (\text{F30})$$

and the scalar product is $a \cdot b = (\Omega/N)^{-1} a_i b^i$. Here, both velocity contribution have a nonzero divergence:

$$\nabla_\varphi \cdot \mathbf{v}_a^\varphi = \nabla_\varphi \cdot \mathbf{v}_{\text{nc}}^\varphi + \nabla_\varphi \cdot \mathbf{v}_c^\varphi, \quad (\text{F31})$$

$$\nabla_\varphi \cdot \mathbf{v}_{\text{nc}}^\varphi = -\frac{\Omega}{N} \sum_{i < j} (\partial_i - \partial_j) \partial_i E^{ij} = 2 \frac{\Omega}{N} \sum_{i < j} E^{ij}, \quad (\text{F32})$$

$$\nabla_\varphi \cdot \mathbf{v}_c^\varphi = -\frac{\beta \Omega}{N} \sum_{ij, i > j} (E''_{ij} - E''_{ji}) = -2 \frac{\beta \Omega}{N} \sum_{ij} E_{ij}. \quad (\text{F33})$$

By comparing Eq. (F31) with Eq. (F29), we see that they balance each other (in the synchronized phase):

$$\begin{aligned} \nabla_\varphi \cdot \mathbf{v}_a^\varphi &= \frac{\Omega}{N} (1 - \beta) \sum_{ij} E_{ij} = k \frac{\beta}{N} \sum_{ij} E_{ij} \\ &= -\nabla_\phi \cdot \mathbf{v}_a^\phi = -k \beta_{\text{eff}} \nabla_\phi^2 \mathcal{F}. \end{aligned} \quad (\text{F34})$$

By using the definitions above and the stationary solution, we can easily identify the different dissipation functions but not explicitly calculate all of them. Let us start with the single-oscillator one:

$$\begin{aligned} \dot{S}_{\text{nc}}^{\text{st}, \phi} &= \langle \mathbf{v}_{\text{nc}}^\phi \cdot \mathbf{v}_{\text{nc}}^\phi \rangle_{\text{st}} = k e_g^2, & \dot{S}_c^{\text{st}, \phi} &= \langle \mathbf{v}_c^\phi \cdot \mathbf{v}_c^\phi \rangle_{\text{st}} \\ &= k \beta_{\text{eff}}^2 E_0^2 \left\langle \sum_i \left(\frac{\sum_j \sin(\phi_i - \phi_j)}{N} \right)^2 \right\rangle_{\text{st}}, \end{aligned} \quad (\text{F35})$$

Due to the symmetry of the energy function, we have that

$$\langle \mathbf{v}_{\text{nc}}^\phi \cdot \mathbf{v}_c^\phi \rangle_{\text{st}} = 0, \quad (\text{F36})$$

leading to

$$\dot{S}_a^{\text{st}, \phi} = \dot{S}_{\text{nc}}^{\text{st}, \phi} + \dot{S}_c^{\text{st}, \phi}. \quad (\text{F37})$$

Similarly, in the phase-difference space we have that

$$\dot{S}_{\text{nc}}^{\text{st}, \varphi} = \langle \mathbf{v}_{\text{nc}}^\varphi \cdot \mathbf{v}_{\text{nc}}^\varphi \rangle_{\text{st}} = \Omega E_0^2 \left\langle \sum_{ij, i < j} \frac{\sin^2 \varphi_{ij}}{N^2} \right\rangle_{\text{st}}, \quad (\text{F38})$$

$$\begin{aligned} \dot{S}_c^{\text{st}, \varphi} &= \langle \mathbf{v}_c^\varphi \cdot \mathbf{v}_c^\varphi \rangle_{\text{st}} \\ &= \frac{\Omega}{N} \frac{E_0 \beta_{\text{eff}}^2}{N^2} \left\langle \sum_{ij, i > j} \left(\sum_k (\sin \varphi_{ik} - \sin \varphi_{jk}) \right)^2 \right\rangle_{\text{st}}, \quad (\text{F39}) \\ \langle \mathbf{v}_{\text{nc}}^\varphi \cdot \mathbf{v}_c^\varphi \rangle_{\text{st}} &= -\frac{\Omega E_0^2}{N} \frac{\beta}{N} \\ &\times \sum_{ij, i > j} \left\langle \sin \varphi_{ij} \left(\sum_k (\sin \varphi_{ik} - \sin \varphi_{jk}) \right) \right\rangle_{\text{st}}. \end{aligned} \quad (\text{F40})$$

By looking at the self-information balance at stationary state, we can generalize the orthogonality condition in the main text to the case where multiple currents are present:

$$\langle \partial_i \mathcal{F} \rangle = \langle \mathbf{v}^\phi \cdot \nabla_\phi \mathcal{F} \rangle + \langle \mathbf{v}^\varphi \cdot \nabla_\varphi \mathcal{F} \rangle, \quad (\text{F41})$$

which leads to

$$\langle \mathbf{v}^\phi \cdot \nabla_\phi \mathcal{F} \rangle_{\text{st}} = -\langle \mathbf{v}^\varphi \cdot \nabla_\varphi \mathcal{F} \rangle_{\text{st}}. \quad (\text{F42})$$

By using the conservative/nonconservative decomposition into Eq. (F42), we have that

$$\dot{S}_c^{\text{st}, \phi} + \dot{S}_c^{\text{st}, \varphi} = -\langle \mathbf{v}_{\text{nc}}^\varphi \cdot \nabla_\varphi \mathcal{F} \rangle \geq 0, \quad (\text{F43})$$

which finally leads to the total entropy production balance:

$$\dot{S}_{\text{tot}} = \dot{S}_{\text{nc}}^{\text{st}, \phi} + \dot{S}_{\text{nc}}^{\text{st}, \varphi} - \dot{S}_c^{\text{st}, \phi} - \dot{S}_c^{\text{st}, \varphi}. \quad (\text{F44})$$

More precisely, we can show a stronger relation for the φ dissipation using Eqs. (F36) and (F43):

$$\begin{aligned} \dot{S}_{\text{tot}}^{\text{st}, \varphi} &= \dot{S}_{\text{nc}}^{\text{st}, \varphi} + \dot{S}_c^{\text{st}, \varphi} + 2 \langle \mathbf{v}_{\text{nc}}^\varphi \cdot \nabla_\varphi \mathcal{F} \rangle \\ &= \dot{S}_{\text{nc}}^{\text{st}, \varphi} - \dot{S}_c^{\text{st}, \varphi} - 2 \dot{S}_c^{\text{st}, \phi} \geq 0, \end{aligned} \quad (\text{F45})$$

from which, thanks to the positivity of the conservative contribution,

$$\dot{S}_{\text{nc}}^{\text{st}, \varphi} \geq \dot{S}_c^{\text{st}} = \dot{S}_c^{\text{st}, \phi} + \dot{S}_c^{\text{st}, \varphi}. \quad (\text{F46})$$

This relation confirms that the conservative contribution encodes the dissipation cost due to collective organization, which needs to be smaller than the dissipation due to the interaction reactions.

c. Mean-field approximation

To go further into the analytics, we follow the original reference and introduce a mean-field approximation in the limit of large system size $N \rightarrow \infty$:

$$P(\phi) = \Pi_{i=1}^N \rho(\phi_i). \quad (\text{F47})$$

To characterize the collective behavior of the system, the standard order parameter $r \in [0, 1]$ is introduced:

$$r e^{i\psi} = \langle e^{i\phi} \rangle, \quad (\text{F48})$$

where ψ is the collective phase and $\theta = \phi - \psi$ the difference of a single phase with respect to ψ . By marginalizing the original Fokker-Planck equation and applying the mean-field approximation, one obtains an effective one-body mean-field equation:

$$\partial_t \rho(\phi, t) = \partial_\phi \left[-ke_g + \Omega E_0 \int d\phi' \sin(\phi - \phi') \rho(\phi') \right. \\ \left. + (\Omega + k) \partial_\phi \right] \rho(\phi, t), \quad (\text{F49})$$

using the fact that $\langle \sin(\phi - \phi') \rangle_{\phi'} = r \sin \theta$, the stationary state has the following solution:

$$\rho_{\text{st}}(\theta) = \frac{1}{Z} \exp(r\beta E_0 \cos \theta) \quad (\text{F50})$$

[remember that, due to rotational symmetry, $\rho_{\text{st}}(\phi) = \rho_{\text{st}}(\phi - \psi) = \rho_{\text{st}}(\theta)$]. Following the original reference, it is possible to use this solution to calculate the critical (minimal) exchange frequency for synchronized oscillations to emerge and perform an approximation for $\beta E_0 \rightarrow \infty$ of the order parameter:

$$\Omega_c = \frac{2k}{E_0 - 2}, \quad r = 1 - \frac{1}{2\beta E_0} - \frac{3}{8(\beta E_0)^2}. \quad (\text{F51})$$

By inspecting the Fokker-Planck equation, we identify the mean-field value of the various velocity components:

$$\bar{v}_{\text{nc}}^\phi = ke_g, \quad \bar{v}_{\text{c}}^\phi = -k\partial_k \ln \rho = krE_0\beta_{\text{eff}} \sin \theta \quad (\text{F52})$$

$$\bar{v}_{\text{nc}}^\varphi = \Omega E_0 r \sin \theta, \quad \bar{v}_{\text{c}}^\varphi = -\Omega \partial \ln \rho = \Omega \beta_{\text{eff}} E_0 r \sin \theta. \quad (\text{F53})$$

The geometrical properties of the system are well represented by these mean-field approximations, given that the geometries have nonzero divergences, they match the original geometric relation in Eq. (F34), and their average is related to the order parameter:

$$\langle \nabla_\phi \cdot \bar{v}_{\text{a}}^\phi \rangle_{\text{st}} = -\langle \nabla_\theta \cdot \bar{v}_{\text{a}}^\varphi \rangle_{\text{st}} = k\beta_{\text{eff}} \langle (\nabla_\phi)^2 \mathcal{A} \rangle = kE_0\beta r \geq 0. \quad (\text{F54})$$

In Fig. 5, we illustrate the velocity geometry: In panel (b), plot the mean-field phase velocities for two oscillators in the case $\phi_2 = \psi$ in the (ϕ_1, ψ) plane; hence, $(\bar{v}_1^\phi, \bar{v}_2^\phi) = (ke_g + krE_0\beta_{\text{eff}} \sin(\phi_1 - \psi), ke_g + krE_0\beta_{\text{eff}})$. In panel (c), we consider the φ velocity and we do not need to fix the collective phase to plot them in the (θ_1, θ_2) plane; hence, $(\bar{v}_1^\varphi, \bar{v}_2^\varphi) = (-krE_0\beta_{\text{eff}} \sin(\theta_1), -krE_0\beta_{\text{eff}} \sin(\theta_2))$.

Now, we put our attention on calculating the mean-field dissipations functions in Eq. (F44). To do so, one needs to apply the mean-field approximation ($N \rightarrow \infty$) to Eqs. (F35) and (F38), and normalize by N . In the mean-field approximation, the conservative part of the single phase velocity of each oscillator reads using that $\langle \cos \theta \rangle = r$ (see Ref. [4] for more

details on the calculation):

$$\dot{S}_{\text{c}}^{\text{st},\phi} = \frac{1}{N} \dot{S}_{\text{c}}^{\text{st},\phi} \approx k\beta_{\text{eff}}^2 E_0^2 \int d\theta_1 d\theta_2 d\theta_3 \\ \times \sin(\theta_1 - \theta_3) \sin(\theta_1 - \theta_2) \rho(\theta_1) \rho(\theta_2) \rho(\theta_3) \\ = k\beta_{\text{eff}}^2 E_0^2 \langle \sin(\theta_1 - \theta_2) \sin(\theta_1 - \theta_3) \rangle = k\beta_{\text{eff}} E_0 r^2, \quad (\text{F55})$$

where we have used that $\langle \sin^2 \theta \rangle = (\beta_{\text{eff}} E_0)^{-1}$ and $\langle \cos \theta \rangle = r$. Passing to φ , for the nonconservative component we have

$$\dot{S}_{\text{nc}}^{\text{st},\varphi} = \frac{1}{N} \dot{S}_{\text{nc}}^{\text{st},\varphi} \approx \frac{1}{2} \Omega E_0^2 \langle \sin^2(\theta_1 - \theta_2) \rangle_{1,2} \\ = \frac{\Omega E_0}{\beta_{\text{eff}}} \left(1 - \frac{1}{\beta_{\text{eff}} E_0} \right), \quad (\text{F56})$$

and finally its conservative part reads

$$\dot{S}_{\text{c}}^{\text{st},\varphi} = \frac{1}{N} \dot{S}_{\text{c}}^{\text{st},\varphi} \approx \Omega \frac{E_0^2 \beta_{\text{eff}}^2}{2} (2 \langle \sin(\theta_1 - \theta_2) \sin(\theta_1 - \theta_3) \rangle \\ - 2 \langle \sin \theta \rangle^2) = \Omega E_0 \beta_{\text{eff}} r^2, \quad (\text{F57})$$

where we have used that $\langle \sin \theta \rangle = 0$. By putting everything together, we arrive to

$$\dot{S}_{\text{tot}}^{\text{st}} = \dot{S}_{\text{nc}}^{\text{st}} - \dot{S}_{\text{c}}^{\text{st}}, \quad \dot{S}_{\text{nc}}^{\text{st}} = ke_g^2 + \frac{\Omega E_0}{\beta_{\text{eff}}} \left(1 - \frac{1}{\beta_{\text{eff}} E_0} \right), \\ \dot{S}_{\text{c}}^{\text{st}} = \Omega E_0 r^2, \quad (\text{F58})$$

with $\dot{S}_{\text{tot}}^{\text{st}} = ke_g^2$ directly from Eq. (F52).

4. Underdamped particle in a temperature gradient

The motion of a particle, subjected to a confining potential U , has to be described in the entire phase space $(x, \mathbf{v})$. Hence, a 1D dynamics along x only turns into a 2D one when velocities and positions are considered at the same time. In this case, including also the presence of a temperature gradient, the Langevin equations describing the system are

$$\dot{x} = v, \quad (\text{F59})$$

$$\dot{\mathbf{v}} = -\frac{\gamma}{m} \mathbf{v} - \frac{1}{m} \partial_x U + \frac{1}{m} \sqrt{2\gamma T(x)} \xi(t), \quad (\text{F60})$$

where γ represents the friction, m the mass, $\partial_x U$ a (confining) potential, and the diffusion coefficient satisfies the Einstein's relation $D(x) = \gamma T(x)$ at each point in space. Notice that the temperature gradient translates into a multiplicative noise on $\mathbf{v}$ that solely depends on x . The associated Fokker-Planck equation is

$$\partial_t P + \mathbf{v} \partial_x P = \frac{1}{\tau} \partial_v \left(\mathbf{v} P + \frac{\partial_x U}{\gamma} P + \frac{T(x)}{m} \partial_v P \right), \quad (\text{F61})$$

where $\tau = m/\gamma$ is the characteristic friction time. When $\tau \rightarrow 0$, velocities equilibrate infinitely faster, the system reaches the overdamped limit, and thus can be described only through its dynamics in the position space.

To investigate the system behavior beyond the overdamped limit, we can employ a timescale separation approach by expanding in the small parameter τ up to the first order. Since

the definition of τ involves both m and γ , we first need to perform a change of variable in the system to express Eq. (F61) solely in terms of τ . To this aim, we introduce $\tilde{x} = x\sqrt{\gamma}$ and $\tilde{v} = v\sqrt{m}$ so that the rescaled Fokker-Planck is

$$\partial_t P + \frac{1}{\sqrt{\tau}} \tilde{v} \partial_{\tilde{v}} P = \frac{1}{\tau} \partial_{\tilde{v}} \left(\tilde{v} P + \sqrt{\tau} \frac{\partial_{\tilde{x}} U}{\gamma} P + T(\tilde{x}) \partial_{\tilde{v}} P \right). \quad (\text{F62})$$

Multiplying both sides by τ , we obtain an equation where both τ and $\sqrt{\tau}$ appear. Thus, we can propose a solution in powers of $\sqrt{\tau}$ in the following form:

$$P^{(\tau)}(\tilde{x}, \tilde{v}, t) = P^{(0)}(\tilde{x}, \tilde{v}, t) + \sqrt{\tau} P^{(1)}(\tilde{x}, \tilde{v}, t). \quad (\text{F63})$$

Inserting this expression into the dynamical equation, we can solve order by order the resulting relationships. The leading term is of order τ^0 and reads

$$\partial_{\tilde{v}}(\tilde{v} P^{(0)} + T(\tilde{x}) \partial_{\tilde{v}} P^{(0)}) = 0, \quad (\text{F64})$$

whose solution can be immediately obtained in the following form:

$$P^{(0)} = e^{-\frac{1}{2}\tilde{v}^2} \phi(\tilde{x}, \tilde{v}). \quad (\text{F65})$$

Proceeding with the expansion, we have to solve the dynamics of order $\sqrt{\tau}$ to determine the first-order correction:

$$\tilde{v} \partial_{\tilde{x}} P^{(0)} = \partial_{\tilde{v}}(\tilde{v} P^{(1)} + (\partial_{\tilde{x}} U) P^{(0)} + T(\tilde{x}) \partial_{\tilde{v}} P). \quad (\text{F66})$$

After simple manipulations, the first order can be expressed as follows:

$$P^{(1)} = -e^{-\frac{1}{2}\tilde{v}^2} \left(\tilde{v} \partial_{\tilde{x}} \phi + \left(\frac{\tilde{v}^3}{6T^2} + \frac{\tilde{v}}{T} \right) \phi \partial_{\tilde{x}} T + \frac{\tilde{v}}{T} \phi \partial_{\tilde{x}} U \right). \quad (\text{F67})$$

Finally, using both $P^{(0)}$ and $P^{(1)}$ to determine the evolution equation of order τ , and integrating over $\tilde{v}$, we reconstruct the overdamped equation in Ito form, as expected (additional details on this step can be found in Ref. [56]). The solution shown in the main text is the one shown here, where we moved back to the original variables, x and v .

Regarding the identification of the mixed term appearing in the nonadiabatic functional, from the decomposition into reversible and irreversible parts of the velocity, we have

$$\begin{aligned} \langle v_{\text{rev}} * v_{\text{irr}} \rangle &= \int dx dv \frac{m^2}{\gamma T} \left(\frac{\gamma}{m^2} v \partial_x U + \frac{\gamma T}{m^3} \frac{\partial_v P}{P} \partial_x U \right. \\ &\quad \left. - \frac{\gamma}{m^2} v f - \frac{\gamma T}{m^3} f \frac{\partial_v P}{P} \right) P \\ &= - \int dx dv \frac{1}{T} v (-\partial_x U + f) = - \left\langle \frac{vF}{T} \right\rangle = \left\langle \frac{\dot{W}}{T} \right\rangle \end{aligned} \quad (\text{F68})$$

due to vanishing boundary conditions on P , where F is the total force acting on the system, and hence $\dot{W}$ is the usual definition for the rate of stochastic work performed on the system.

-
- [1] D. M. Busiello, D. Gupta, and A. Maritan, Coarse-grained entropy production with multiple reservoirs: Unraveling the role of time scales and detailed balance in biology-inspired systems, *Phys. Rev. Res.* **2**, 043257 (2020).
 - [2] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, *Rep. Prog. Phys.* **75**, 126001 (2012).
 - [3] C. Hargus, J. M. Epstein, and K. K. Mandadapu, Odd diffusivity of chiral random motion, *Phys. Rev. Lett.* **127**, 178001 (2021).
 - [4] D. Zhang, Y. Cao, O. Qi, and Y. Tu, The energy cost and optimal design for synchronization of coupled molecular oscillators, *Nat. Phys.* **16**, 95 (2019).
 - [5] C. W. Lynn, C. M. Holmes, W. Bialek, and D. J. Schwab, Decomposing the local arrow of time in interacting systems, *Phys. Rev. Lett.* **129**, 118101 (2022).
 - [6] G. Falasco and M. Esposito, Macroscopic stochastic thermodynamics, *Rev. Mod. Phys.* **97**, 015002 (2025).
 - [7] R. Rao and M. Esposito, Nonequilibrium thermodynamics of chemical reaction networks: Wisdom from stochastic thermodynamics, *Phys. Rev. X* **6**, 041064 (2016).
 - [8] T. Bodineau, B. Derrida, and J. L. Lebowitz, Vortices in the two-dimensional simple exclusion process, *J. Stat. Phys.* **131**, 821 (2008).
 - [9] G. Jona-Lasinio, Spontaneous symmetry breaking: Variations on a theme, *Prog. Theor. Phys.* **124**, 731 (2010).
 - [10] Y. Baek, Y. Kafri, and V. Lecomte, Dynamical symmetry breaking and phase transitions in driven diffusive systems, *Phys. Rev. Lett.* **118**, 030604 (2017).
 - [11] I. Prigogine and G. Nicolis, On symmetry-breaking instabilities in dissipative systems, *J. Chem. Phys.* **46**, 3542 (1967).
 - [12] P. W. Anderson and D. L. Stein, Broken symmetry, emergent properties, dissipative structures, life: Are they related, in *Basic Notions of Condensed Matter Physics* (CRC Press, Boca Raton, 2018), pp. 263–277.
 - [13] P. W. Anderson, Can broken symmetry occur in driven systems, in *Equilibrium and Nonequilibrium Statistical Mechanics* (Wiley, New York, 1981), pp. 289–297.
 - [14] G. Gallavotti and E. G. D. Cohen, Dynamical ensembles in nonequilibrium statistical mechanics, *Phys. Rev. Lett.* **74**, 2694 (1995).
 - [15] C. Jarzynski, Nonequilibrium equality for free energy differences, *Phys. Rev. Lett.* **78**, 2690 (1997).
 - [16] J. Kurchan, Fluctuation theorem for stochastic dynamics, *J. Phys. A: Math. Gen.* **31**, 3719 (1998).
 - [17] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, *Phys. Rev. E* **60**, 2721 (1999).
 - [18] J. L. Lebowitz and H. Spohn, A Gallavotti–Cohen-type symmetry in the large deviation functional for stochastic dynamics, *J. Stat. Phys.* **95**, 333 (1999).
 - [19] L. Peliti and S. Pigolotti, *Stochastic Thermodynamics: An Introduction* (Princeton University Press, Princeton, 2021).
 - [20] S. Ciliberto, Experiments in stochastic thermodynamics: Short history and perspectives, *Phys. Rev. X* **7**, 021051 (2017).

- [21] R. Graham, Covariant formulation of non-equilibrium statistical thermodynamics, *Z. Phys. B Condens. Matter* **26**, 397 (1977).
- [22] R. Graham and T. Tél, On the weak-noise limit of Fokker-Planck models, *J. Stat. Phys.* **35**, 729 (1984).
- [23] H. Feng and J. Wang, Potential and flux decomposition for dynamical systems and non-equilibrium thermodynamics: Curvature, gauge field, and generalized fluctuation-dissipation theorem, *J. Chem. Phys.* **135**, 234511 (2011).
- [24] X. Fang, K. Kruse, T. Lu, and J. Wang, Nonequilibrium physics in biology, *Rev. Mod. Phys.* **91**, 045004 (2019).
- [25] J. N. Freitas and M. Esposito, Emergent second law for non-equilibrium steady states, *Nat. Commun.* **13**, 5084 (2022).
- [26] M. Polettini, Generally covariant state-dependent diffusion, *J. Stat. Mech.* (2013) P07005.
- [27] U. Seifert, Entropy production along a stochastic trajectory and an integral fluctuation theorem, *Phys. Rev. Lett.* **95**, 040602 (2005).
- [28] D. M. Busiello and S. Pigolotti, Hyperaccurate currents in stochastic thermodynamics, *Phys. Rev. E* **100**, 060102(R) (2019).
- [29] T. Hatano and S.-i. Sasa, Steady-state thermodynamics of Langevin systems, *Phys. Rev. Lett.* **86**, 3463 (2001).
- [30] C. Van den Broeck and M. Esposito, Three faces of the second law. II. Fokker-Planck formulation, *Phys. Rev. E* **82**, 011144 (2010).
- [31] R. Graham and T. Tél, Existence of a potential for dissipative dynamical systems, *Phys. Rev. Lett.* **52**, 9 (1984).
- [32] R. Graham and T. Tél, Weak-noise limit of Fokker-Planck models and nondifferentiable potentials for dissipative dynamical systems, *Phys. Rev. A* **31**, 1109 (1985).
- [33] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, *Rev. Mod. Phys.* **87**, 593 (2015).
- [34] P. L. Garrido, Quasipotentials in the nonequilibrium stationary states or a method to get explicit solutions of Hamilton–Jacobi equations, *J. Stat. Mech.* (2021) 113206.
- [35] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (Wiley, New York, NY, 1985).
- [36] J. M. Horowitz, K. Zhou, and J. L. England, Minimum energetic cost to maintain a target nonequilibrium state, *Phys. Rev. E* **95**, 042102 (2017).
- [37] C. Maes and K. Netočný, A nonequilibrium extension of the Clausius heat theorem, *J. Stat. Phys.* **154**, 188 (2014).
- [38] A. Dechant, S.-I. Sasa, and S. Ito, Geometric decomposition of entropy production in out-of-equilibrium systems, *Phys. Rev. Res.* **4**, L012034 (2022).
- [39] A. Dechant, S.-I. Sasa, and S. Ito, Geometric decomposition of entropy production into excess, housekeeping, and coupling parts, *Phys. Rev. E* **106**, 024125 (2022).
- [40] K. Yoshimura, A. Kolchinsky, A. Dechant, and S. Ito, Housekeeping and excess entropy production for general nonlinear dynamics, *Phys. Rev. Res.* **5**, 013017 (2023).
- [41] M. Polettini, Nonequilibrium thermodynamics as a gauge theory, *Europhys. Lett.* **97**, 30003 (2012).
- [42] M. Polettini, Of dice and men, Subjective priors, gauge invariance, and nonequilibrium thermodynamics, *arXiv:1307.2057*.
- [43] F. Khodabandehlou, C. Maes, and K. Netočný, On the Poisson equation for nonreversible Markov jump processes, *J. Math. Phys.* **65**, 043301 (2024).
- [44] J. S. Ovall, The Laplacian and mean and extreme values, *Am. Math. Mon.* **123**, 287 (2016).
- [45] H. Qian, Thermodynamics of the general diffusion process: Equilibrium supercurrent and nonequilibrium driven circulation with dissipation, *Eur. Phys. J.: Spec. Top.* **224**, 781 (2015).
- [46] M. Mendler and B. Drossel, Predicting properties of the stationary probability currents for two-species reaction systems without solving the Fokker-Planck equation, *Phys. Rev. E* **102**, 022208 (2020).
- [47] R. Exartier and L. Peliti, A simple system with two temperatures, *Phys. Lett. A* **261**, 94 (1999).
- [48] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.* **29**, 255 (1966).
- [49] S. Cerasoli, V. Dotsenko, G. Oshanin, and L. Rondoni, Asymmetry relations and effective temperatures for biased Brownian gyrators, *Phys. Rev. E* **98**, 042149 (2018).
- [50] D. Santolin, N. Freitas, M. Esposito, and G. Falasco, Bridging Freidlin-Wentzell large deviations theory and stochastic thermodynamics, *Phys. Rev. E* **111**, 024106 (2025).
- [51] N. Van Kampen, Diffusion in inhomogeneous media, *J. Phys. Chem. Solids* **49**, 673 (1988).
- [52] N. Van Kampen, Brownian motion on a manifold, *J. Stat. Phys.* **44**, 1 (1986).
- [53] R. Landauer, Motion out of noisy states, *J. Stat. Phys.* **53**, 233 (1988).
- [54] D. M. Busiello, S. Liang, F. Piazza, and P. De Los Rios, Dissipation-driven selection of states in non-equilibrium chemical networks, *Commun. Chem.* **4**, 16 (2021).
- [55] A. V. Dass, T. Georgelin, F. Westall, F. Foucher, P. De Los Rios, D. M. Busiello, S. Liang, and F. Piazza, Equilibrium and non-equilibrium furanose selection in the ribose isomerisation network, *Nat. Commun.* **12**, 2749 (2021).
- [56] S. Liang, D. M. Busiello, and P. De Los Rios, Emergent thermophoretic behavior in chemical reaction systems, *New J. Phys.* **24**, 123006 (2022).
- [57] G. Nicoletti and D. M. Busiello, Mutual information disentangles interactions from changing environments, *Phys. Rev. Lett.* **127**, 228301 (2021).
- [58] P. L. Garrido, Notes about the macroscopic fluctuating theory, *J. Stat. Mech.* (2021) 024001.
- [59] M. te Vrugt, H. Löwen, and R. Wittkowski, Classical dynamical density functional theory: From fundamentals to applications, *Adv. Phys.* **69**, 121 (2020).
- [60] F. Cornalba, T. Shardlow, and J. Zimmer, A regularized Dean–Kawasaki model: Derivation and analysis, *SIAM J. Math. Anal.* **51**, 1137 (2019).
- [61] S. Azale, S. Pigolotti, J. R. Banavar, and A. Maritan, Dynamical evolution of ecosystems, *Nature (London)* **444**, 926 (2006).
- [62] M. Sireci and M. A. Muñoz, Statistical mechanics of phenotypic eco-evolution: From adaptive dynamics to complex diversification, *Phys. Rev. Res.* **6**, 023070 (2024).
- [63] R. Graham, Covariant stochastic calculus in the sense of Itô, *Phys. Lett. A* **109**, 209 (1985).
- [64] A. Celani, S. Bo, R. Eichhorn, and E. Aurell, Anomalous thermodynamics at the microscale, *Phys. Rev. Lett.* **109**, 260603 (2012).
- [65] R. Czipnik and P. Garbaczewski, Brownian motion in a magnetic field, *Phys. Rev. E* **63**, 021105 (2001).
- [66] M. E. Cates and J. Tailleur, Motility-induced phase separation, *Annu. Rev. Condens. Matter Phys.* **6**, 219 (2015).

- [67] E. D. Mackay, G. Janzen, D. A. Matoz-Fernandez, and R. Sknepnek, Emergent dynamics of active systems on curved environments, *Phys. Rev. Res.* **8**, 013172 (2026).
- [68] C. Rovelli and M. Smerlak, Thermal time and Tolman–Ehrenfest effect: ‘Temperature as the speed of time’, *Class. Quantum Grav.* **28**, 075007 (2011).
- [69] A. Dechant and S.-i. Sasa, Entropic bounds on currents in Langevin systems, *Phys. Rev. E* **97**, 062101 (2018).
- [70] R. Kupferman, G. A. Pavliotis, and A. M. Stuart, Itô versus Stratonovich white-noise limits for systems with inertia and colored multiplicative noise, *Phys. Rev. E* **70** (3), 036120 (2004).
- [71] T. L. Hill, *Free Energy Transduction and Biochemical Cycle Kinetics* (Springer, New York, 1989).
- [72] A. Wachtel, R. Rao, and M. Esposito, Free-energy transduction in chemical reaction networks: From enzymes to metabolism, *J. Chem. Phys.* **157** (2), 024109 (2022).
- [73] R. K. P. Zia and B. Schmittmann, A possible classification of nonequilibrium steady states, *J. Phys. A: Math. Gen.* **39**, L407 (2006).
- [74] Q. Yu, D. Zhang, and Y. Tu, Inverse power law scaling of energy dissipation rate in nonequilibrium reaction networks, *Phys. Rev. Lett.* **126**, 080601 (2021).
- [75] Q. Yu and Y. Tu, State-space renormalization group theory of nonequilibrium reaction networks: Exact solutions for hypercubic lattices in arbitrary dimensions, *Phys. Rev. E* **105**, 044140 (2022).
- [76] R. Gutiérrez and C. Pérez-Espigares, Dynamical phase transition to localized states in the two-dimensional random walk conditioned on partial currents, *Phys. Rev. E* **104**, 044134 (2021).
- [77] S. Ito, Information geometry, trade-off relations, and generalized Glansdorff–Prigogine criterion for stability, *J. Phys. A: Math. Theor.* **55**, 054001 (2022).
- [78] L. Dabelow, S. Bo, and R. Eichhorn, Irreversibility in active matter systems: Fluctuation theorem and mutual information, *Phys. Rev. X* **9**, 021009 (2019).
- [79] A. Datta, P. Pietzonka, and A. C. Barato, Second law for active heat engines, *Phys. Rev. X* **12**, 031034 (2022).
- [80] J. O’Byrne, Nonequilibrium currents in stochastic field theories: A geometric insight, *Phys. Rev. E* **107**, 054105 (2023).
- [81] M. Fruchart, R. Hanai, P. B. Littlewood, and V. Vitelli, Non-reciprocal phase transitions, *Nature (London)* **592**, 363 (2021).
- [82] Q. Yu and Y. Tu, Energy cost for flocking of active spins: The cusped dissipation maximum at the flocking transition, *Phys. Rev. Lett.* **129**, 278001 (2022).
- [83] K. Liu, A. Blokhuis, C. van Ewijk, A. Kiani, J. Wu, W. Roos, and S. Otto, Light-driven eco-evolutionary dynamics in a synthetic replicator system, *Nat. Chem.* **16**, 79 (2024).
- [84] S. Ameta, A. Blokhuis, C. Jeancolas, and P. Nghe, Toward evolution in chemical reaction networks, in *Prebiotic Chemistry and Life’s Origin* (The Royal Society of Chemistry, Cambridge, 2022), Vol. 6.
- [85] A. Goyal, A. I. Flamholz, A. P. Petroff, and A. Murugan, Closed ecosystems extract energy through self-organized nutrient cycles, *Proc. Natl. Acad. Sci. USA* **120**, e2309387120 (2023).
- [86] J. R. Banavar, A. Maritan, and A. Rinaldo, Size and form in efficient transportation networks, *Nature (London)* **399**, 130 (1999).
- [87] J. Hidalgo, J. Grilli, S. Suweis, M. A. Munoz, J. R. Banavar, and A. Maritan, Information-based fitness and the emergence of criticality in living systems, *Proc. Natl. Acad. Sci. USA* **111**, 10095 (2014).
- [88] A. Despons, Y. De Decker, and D. Lacoste, Structural constraints limit the regime of optimal flux in autocatalytic reaction networks, *Commun. Phys.* **7**, 224 (2024).
- [89] M. A. Munoz, Colloquium: Criticality and dynamical scaling in living systems, *Rev. Mod. Phys.* **90**, 031001 (2018).
- [90] S. Chen, D. S. Seara, A. Michaud, S. Kim, W. M. Bement, and M. P. Murrell, Energy partitioning in the cell cortex, *Nat. Phys.* **20**, 1824 (2024).
- [91] Z. G. Sun, N. Zimmerberg, P. Kelly, C. Floyd, G. Papoian, and M. Murrell, Feedback between F-actin organization and active stress governs criticality and energy localization in the cell cytoskeleton, *Nat. Phys.* **21**, 1290 (2025).
- [92] G. R. Price, The nature of selection, *J. Theor. Biol.* **175**, 389 (1995).
- [93] R. Rao and S. Leibler, Evolutionary dynamics, evolutionary forces, and robustness: A nonequilibrium statistical mechanics perspective, *Proc. Natl. Acad. Sci. USA* **119**, e2112083119 (2022).
- [94] C. D. Kocher and K. A. Dill, Darwinian evolution as a dynamical principle, *Proc. Natl. Acad. Sci. USA* **120** (11), e2218390120 (2023).
- [95] S. Dal Cengio, V. Lecomte, and M. Polettni, Geometry of nonequilibrium reaction networks, *Phys. Rev. X* **13**, 021040 (2023).
- [96] S. Liang, P. De Los Rios, and D. M. Busiello, Thermodynamic bounds on symmetry breaking in linear and catalytic biochemical systems, *Phys. Rev. Lett.* **132**, 228402 (2024).
- [97] L. F. Cugliandolo and V. Lecomte, Rules of calculus in the path integral representation of white noise Langevin equations: The Onsager–Machlup approach, *J. Phys. A: Math. Theor.* **50**, 345001 (2017).
- [98] J. M. R. Parrondo and P. Español, Criticism of Feynman’s analysis of the ratchet as an engine, *Am. J. Phys.* **64**, 1125 (1996).