

Manifestations of flow topology in a quantum driven-dissipative system

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(Received 19 September 2025; accepted 26 February 2026; published 29 April 2026)

Driven-dissipative bosonic systems exhibit a rich variety of nonequilibrium steady states (NESS) due to the interplay between coherent drives, interactions, and dissipation. In the semiclassical limit, the *flow topology* of phase-space dynamics governs the stability and structure of these dynamical phases. Topological transitions occur when the number, chirality, and connectivity of NESS changes, reflecting a global reorganization of the dynamical phase-space landscape. Here, we study the impact of these topological signatures in a quantum driven-dissipative Kerr oscillator. Employing a Lindblad master equation and quantum-trajectory methods, we reveal that quantum dynamics retain key topological features of the underlying semiclassical flows, with clear signatures accessible via quantum state tomography and linear response. In this manner, we predict flow-topological phases whose signatures arise from a reorganization of fluctuation modes and are not captured by Liouvillian-gap closing alone. We, thus, generalize the conventional criteria for diagnosing phase transitions. Our findings position phase-space flow topology as a powerful tool to identify and control robust quantum phases, enabling advances in error correction and sensing.

DOI: [10.1103/4qh9-j88q](https://doi.org/10.1103/4qh9-j88q)

I. INTRODUCTION

Phase transitions involve abrupt macroscopic changes, typically characterized by order parameters that break an underlying symmetry. At zero temperature, Ginzburg-Landau mean-field theory describes these symmetry-broken phases via free-energy minima [1,2]. This framework underlies systems ranging from ferromagnets [3,4] and superconductors [5–7] to Bose-Einstein condensates [8]. Some phases, in contrast, lack local order parameters and are instead distinguished by global topological invariants. Topological insulators, topological superconductors, and non-Hermitian systems provide paradigmatic examples [9–12]. Topological phase transitions occur when these invariants change, often signaled by spectral-gap closing or an altered system response [10–18]. Independent of the underlying mechanism, thermal and quantum fluctuations may restore symmetry and ergodicity in the long-time limit [2].

Dissipative phase transitions (DPTs) extend this concept to open quantum systems [7,19–23]. They arise across diverse platforms, from Kerr oscillators [21,24–26] and Bose-Hubbard lattices [27,28] to exciton polaritons [29,30], superconducting arrays [31–36], driven spins [37–39], and nonlinear photonic lattices [22,40]. Instead of relaxing to

energy minima, open systems reach nonequilibrium steady states (NESS), where conservative and nonconservative forces balance [41,42]. Here, thermal and quantum fluctuations also enable the system to explore the manifold of NESS, often rendering phase transitions as smooth crossovers accompanied by long-lived metastable states and critical slowing down [43–45]. True nonanalytic behavior emerges only in the limit of large excitations [20,28]. A commonly used diagnostic is the closing of the Liouvillian gap [20,46], but relying solely on gap closure can miss transitions, motivating complementary indicators [47].

Recently, the phase-space vector flows associated with the semiclassical equations of motion of driven-dissipative resonators have been classified by a topological graph index [48]. This index captures how phase-space regions are dynamically connected [49] and provides a comprehensive account of the chirality of excitations around each NESS (see Appendix B for further details). Crucially, between regions associated with different graph indexes, a flow topology transition must occur, leading to a (global) change in the variety of NESS and their distribution in phase space [48]. Such flow transitions have been confirmed through full graph reconstruction in a classical resonator. Yet, whether such topological signatures carry over as observable effects in the quantum regime, where fluctuations erase the deterministic flow [50,51], remains an open question. More specifically, this raises a key question: Do flow-topological features leave observable imprints once quantum fluctuations dominate?

In this work, we show that flow topology leaves measurable signatures in a quantum driven-dissipative Kerr resonator, thus identifying phases beyond the Liouvillian-gap criteria. We begin by demonstrating the features of flow topology in

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the semiclassical system and show how averaged order parameters will average over the flow topology signatures. We, then, propose linear-response observables sensitive to the flow topology and demonstrate their persistence under strong quantum fluctuations. Our findings establish a framework for topologically characterizing quantum states in driven-dissipative systems, with implications for error correction [52,53], information processing [54], sensing [34], and control [55].

The paper is organized as follows. Section II introduces the driven-dissipative Kerr model and the Lindblad master equation. Section II A summarizes the semiclassical flow-topology classification and the associated graph invariant (Sec. II B). Section III presents the quantum steady states and compares them to the semiclassical flows. Section III A defines the frequency-resolved chirality spectrum. Section III B analyzes the topological properties in the deep-quantum regime. Additional technical details are provided in Appendixes.

II. MODEL AND THEORETICAL FRAMEWORK

We study a driven-dissipative Kerr resonator described by the effective Hamiltonian [Fig. 1(a)]

$$\hat{\mathcal{H}} = (-\Delta + U)\hat{b}^\dagger\hat{b} + \frac{U}{2}\hat{b}^{\dagger 2}\hat{b}^2 + \frac{G}{2}(\hat{b}^{\dagger 2} + \hat{b}^2) + F(\hat{b}^\dagger e^{-i\phi} + \hat{b}e^{i\phi}), \quad (1)$$

where $\hbar = 1$ and $\hat{b}$ is the bosonic annihilation operator for drive photons [56,57]. Model (1) is written in a frame rotating at frequency ω ; the detuning between the drive frequency ω and the resonator frequency ω_0 is $\Delta = (\omega^2 - \omega_0^2)/2\omega$. Here, U is the Kerr nonlinearity, G is the two-photon (parametric) drive amplitude at frequency $\omega_p = 2\omega$, and F is the amplitude of a single-photon coherent drive at frequency ω , with phase ϕ . To account for dissipation, we model single-photon losses using a Lindblad master equation in which the system density matrix $\hat{\rho}$ evolves under the Liouvillian $\mathcal{L}$ [58],

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}\hat{\rho} = -i[\hat{\mathcal{H}}, \hat{\rho}] + \kappa\left(\hat{b}\hat{\rho}\hat{b}^\dagger - \frac{1}{2}(\hat{b}^\dagger\hat{b}\hat{\rho} + \hat{\rho}\hat{b}^\dagger\hat{b})\right). \quad (2)$$

A. Semiclassical flow topology

In the semiclassical limit of large occupations, the driven-dissipative dynamics governed by Eq. (2) reduce to a deterministic evolution in phase space. The dynamics then obey the Gross-Pitaevskii equation (GPE), a mean-field description of Eq. (2) for the complex field $\beta = \text{Tr}(\hat{\rho}\hat{b})$,

$$i\frac{d\beta}{dt} = \left(-\Delta - i\frac{\kappa}{2} + U|\beta|^2\right)\beta + G\beta^* + F e^{-i\phi}, \quad (3)$$

which defines a two-dimensional (2D) vector flow in the rotating-frame phase space $\{X, Y\} \equiv \{\text{Re}(\beta), \text{Im}(\beta)\}$ [see Figs. 1(b) and 1(c) and Appendix A]. Stationary solutions correspond to the fixed points (FPs) β_0^k , found by setting $d\beta_0^k/dt = 0$ in Eq. (3) and solving the resulting polynomial equations [59,60]. The flows organize around these FPs: Stable ones are attractors (corresponding to NESS), with a local flow pattern that spirals clockwise (CW), counterclockwise (CCW), or not at all, while unstable ones are saddles. Separatrices crossing the saddles partition phase space into

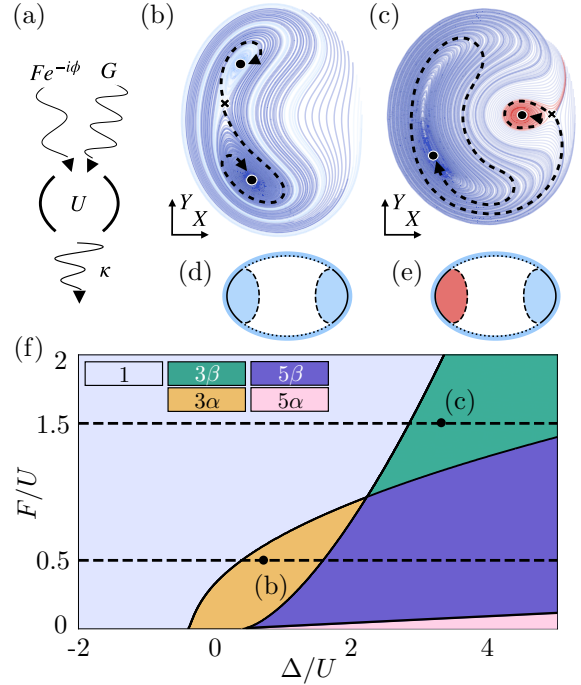


FIG. 1. Semiclassical phase-space topology. (a) Schematic of a driven-dissipative bosonic mode [cf. Eqs. (1)–(3)]. (b), (c) Phase-space flow from Eq. (3) at $(\Delta, F)/U = (0.7, 0.5)$ and $(\Delta, F)/U = (3.3, 1.5)$. Crosses (dots) mark saddles (attractors); dashed arrows delineate separatrices, simplified for visualisation purposes. (d), (e) Topological graph index corresponding to panels (b) and (c), capturing the structure of two attractors (regions encircled by solid and dashed lines) separated by a saddle (region encircled by dotted and dashed lines). The CW (CCW) chirality of the attractors associated with the particlelike (holelike excitations) character of local Bogoliubov excitations is marked by a blue (red) face color. (f) Flow-topological phase diagram, showing distinct NESS topologies (graph indices). Parameters: $\omega_0 = 10^2$, $U = 1$, $G = 0.4$, $\kappa = 0.1$, and $\phi = 0$.

different basins of attraction. By studying the eigenspectrum of the dynamical matrix associated with Bogoliubov excitations around each FP β_0^k , we can assess its stability, i.e., whether it is an attractor or a saddle, as well as its spiraling chirality [21,48,61]. We note that the vector flow defined by Eq. (3) contains no repellers [48,61].

B. Flow-topological graph invariant

The flow structure governed by Eq. (3) admits a compact topological representation through a graph invariant, obtained by extending Morse-Smale theory [49] to nonequilibrium flows [48] [see Figs. 1(d) and 1(e)]. This invariant encodes (1) the number and stability of FPs, (2) their connectivity, and (3) the local flow chirality around each NESS. The number of faces reflects (1), the network of edges between faces captures (2), and face colors encode (3) [48]. Changes to the invariant under smooth parameter variations indicate topological obstructions to deformation, signaling phase transitions. This construction yields a full topological phase diagram of structurally stable semiclassical flow patterns [see Fig. 1(f)].

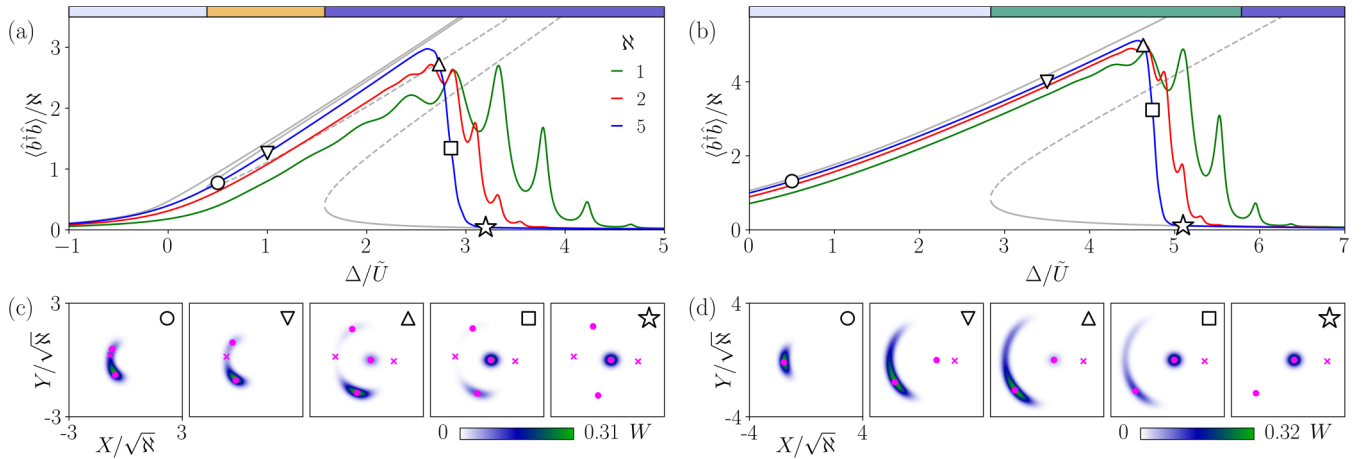


FIG. 2. Classical vs quantum stationary states across flow transitions. (a), (b) Rescaled steady-state population along $F = 0.5$ and $F = 1.5$ cuts of Fig. 1. Gray solid (dashed) lines mark the classical population, $|\beta_0^k|^2$ associated with the stable (unstable) FPs of the corresponding Eq. (3). Colored stripe indicates the corresponding classical phases as shown in Fig. 1(f). (c), (d) Wigner functions for $\aleph = 5$ associated, respectively, with panels (a) and (b) at representative detuning values (symbols). Magenta dots (crosses) mark stable (unstable) classical FPs.

The mean-field phase diagram in Fig. 1(f) illustrates this structure, mapping out regions where the invariant remains constant, thus defining a dynamical phase [48]. For example, the flow in region 1, with a single stable FP, cannot be smoothly deformed into regions 3α and 3β , each of which hosts three solutions: one saddle and two stable FPs, as shown in Figs. 1(b)–1(e). Phases 3α and 3β differ only in the chirality of the local flow around the stable FPs: 3α features CW rotation at both attractors, while 3β displays opposite chiralities [62,63]. They are separated by a saddle-node bifurcation at their phase boundary in Fig. 1(f). Similarly, the five-solution phases 5α and 5β each contain three stable FPs (two CW, one CCW) and two saddles, but differ in how the saddles are connected to the attractors [48].

III. QUANTUM MANIFESTATIONS OF FLOW TOPOLOGY

We now examine whether and how these classical topological features persist in the fully quantum nonequilibrium steady state, i.e., beyond the mean-field approximation. To address this, we numerically compute the stationary density matrix $\hat{\rho}_0$ by diagonalizing the Liouvillian of Eq. (2) in a truncated Hilbert space. To interpolate between the classical and quantum regimes, we rescale the complex field, the nonlinearity and drive amplitudes according to

$$\tilde{\beta} = \beta / \sqrt{\aleph}, \quad U = \tilde{U} / \aleph, \quad F = \tilde{F} \sqrt{\aleph}, \quad (4)$$

where $\aleph$ is the scaling parameter. This transformation leaves the GPE (3) invariant [64,65]. Increasing $\aleph$ reduces quantum fluctuations, pushing the system toward the classical regime. In Figs. 2(a) and 2(b), we show the rescaled steady-state population (RSSP), $\text{Tr}[\hat{b}^\dagger \hat{b} \hat{\rho}_0] / \aleph$, along cuts of Fig. 1(f), overlaid with the mean-field squared amplitudes $|\beta_0^k|^2$. The RSSP increases during a sweep in detuning until a first-order dissipative phase transition occurs (second order for $F = 0$) [19,24,66], inducing an abrupt drop in population. Importantly, the location of this DPT in parameter space does not align with the phase boundary predicted by the classical phase-space dynamics. Instead, it lies deep within the bistable

phases 3α and 3β , and for small $\aleph$, its position is further obscured by regions of multiphoton resonances. This discrepancy arises because the density matrix $\hat{\rho}_0$ is distributed across phase space due to ergodicity. The RSSP in Figs. 2(a) and 2(b) therefore reflects a weighted average over all semiclassical attractors. Such averaging reduces sensitivity to the features that define the semiclassical topological flows, and in the multiphoton regime these features may be entirely washed out. Since these structures determine the classical topological organization, one may conclude that quantum NESS have only limited resemblance to their classical counterparts.

To recover sensitivity to the classical flow features that are obscured in the RSSP, we examine the Wigner function $W(X, Y)$, a quasiprobability phase-space representation of $\hat{\rho}_0$ [67]. Here, the phase-space coordinates (X, Y) are identified with the field quadratures $\hat{X} = (\hat{b} + \hat{b}^\dagger) / \sqrt{2}$ and $\hat{Y} = (\hat{b} - \hat{b}^\dagger) / (i\sqrt{2})$, such that $X = \langle \hat{X} \rangle$ and $Y = \langle \hat{Y} \rangle$. In the classical limit ($\aleph \gg 1$), W becomes a sum over Gaussians centered at the stable FPs of Eq. (3). At moderate quantum fluctuations ($\aleph = 5$), W develops structured patterns around the FPs, as shown in Figs. 2(c) and 2(d). For example, at $\tilde{F} / \tilde{U} = 0.5$, a positive detuning sweep along $\Delta / \tilde{U} \geq 0$ through regions 1, 3α , and 5β in Fig. 1(a) transforms $\hat{\rho}_0$ from a low-amplitude coherent state into a squeezed state, then into a bimodal and finally into a trimodal distribution [cf. Fig. 2(c)]. In the final stage, the stationary distribution concentrates around a single low-amplitude FP, even though the classical mean-field flow supports three stable FPs. A similar sequence occurs in Fig. 2(d) for a cut at $\tilde{F} / \tilde{U} = 1.5$, crossing regions 1, 3β , and 5β in Fig. 1(a). Here, a bimodal distribution forms before quantum fluctuations eventually localize the state around the low-amplitude FP. Crucially, the phase-space structure of $W(X, Y)$ mirrors the classical flow topology of Eq. (3): the number of lobes corresponds to attractors, and their shapes and tails trace classical flow lines and larger structures, including basins of attraction and often extending into saddle regions. Thus, even in the presence of quantum fluctuations, Wigner quasiprobabilities largely retain semiclassical topological features.

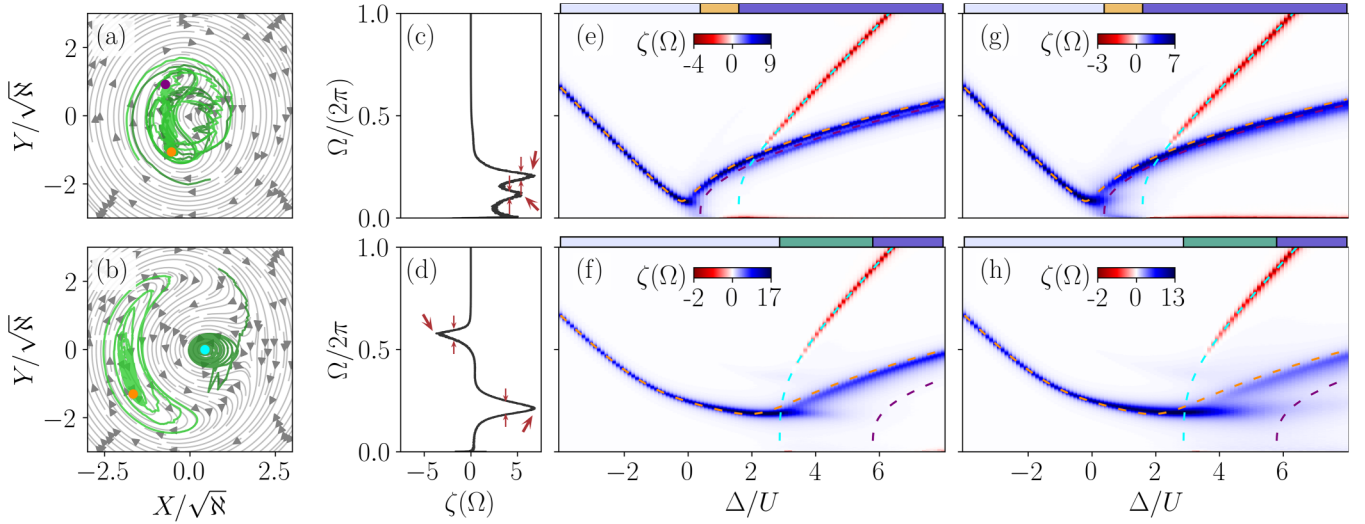


FIG. 3. Chirality signatures from quantum jump trajectories. (a) Representative quantum-jump trajectories (green) in region 3α in the semiclassical regime ($N = 20$), for $(F, \Delta)/U = (0.5, 1)$, overlaid on the corresponding classical flow (in gray) and attractors [dots colored as in the phase diagram in Fig. 1(e)]. (b) Same as PANEL (a) in region 3β for $(F, \Delta)/U = (1.5, 4)$. (c), (d) Chirality spectrum $\zeta(\Omega)$ [cf. Eq. (5)] for panels (a) and (b), respectively. Arrows mark the resonance peaks and their width. (e), (f) Chirality spectra for $N = 20$ along the detuning cuts in Fig. 1(f) (cf. Fig. 2). We sweep across the dynamical phase transitions $1 \rightarrow 3\beta \rightarrow 5\beta$ and $1 \rightarrow 3\alpha \rightarrow 5\beta$, respectively. Colored stripes as in Fig. 2. Dashed lines show the semiclassical response frequency around each attractor with corresponding color to PANELS (a) and (b). (g), (h) Same cuts as panels (e) and (f) for $N = 10$. Stronger quantum fluctuations introduce additional spectral features.

A. Chirality-resolved response function

While phase-space distributions obtained from the full density matrix capture many topological features, they average over trajectories and thus lose sensitivity to local chirality around each FP. Similarly, this insensitivity stems from the fact that the density matrix $\hat{\rho}_0$ represents an ensemble average over all quantum trajectories. As a result, the Wigner function reflects this averaging and smoothens out the local chirality experienced by individual trajectories exploring distinct regions of phase space. To capture these features in an observable, we define the *chirality spectrum* as the causal (retarded) response function

$$\zeta(\Omega) = \int_0^\infty e^{i\Omega\tau} \langle \hat{Y}(\tau)\hat{X}(0) - \hat{X}(\tau)\hat{Y}(0) \rangle_{ss} d\tau, \quad (5)$$

with steady-state expectation value $\langle \square \rangle_{ss} = \text{Tr}[\square \hat{\rho}_0]$. Inspired by Hamiltonian reconstruction [61,68], $\zeta(\Omega)$ captures how winding of trajectories around FPs distributes over fluctuation Fourier frequencies Ω . Away from bifurcations, each nondegenerate fluctuation mode produces a peak in $\zeta(\Omega)$, so the number of peaks coincides with the number of stable FPs. Saddles do not contribute, since their chirality is ill defined. Peak positions (linewidths) correspond to the imaginary (real) parts of the Bogoliubov eigenvalues near each FP, while their signs indicate CW (positive) or CCW (negative) quantum-trajectory rotation [61,68]. Each peak therefore reveals net winding at frequency Ω , with its center set by the oscillation frequency $\text{Im}(\lambda_k)$ of the dominant fluctuation mode and its width by the decay rate $\text{Re}(\lambda_k)$. Thus, $\zeta(\Omega)$ compactly encodes the essential ingredients of the graph invariant: The number of stable FPs equals the number of colored faces, and their chirality encodes the face colors.

To evaluate the chirality spectrum (5), we *unravel* the Lindblad master equation (2) into single-trajectory evolutions via the Monte Carlo wave-function (quantum-jump) method [50,51,69–71]. In this approach, each wave function $|\psi_r(t)\rangle$ evolves according to a non-Hermitian Schrödinger-like equation:

$$\frac{d}{dt} |\psi_r(t)\rangle = -i \left(\hat{H} - \frac{i\kappa}{2} \hat{b}^\dagger \hat{b} \right) |\psi_r(t)\rangle, \quad (6)$$

until a random photon-emission “jump” occurs with probability $dp = \kappa n(t) dt$, where $n(t) = \langle \psi_r(t) | \hat{b}^\dagger \hat{b} | \psi_r(t) \rangle$. The wave function is then projected onto $|\psi_r(t+dt)\rangle = \hat{b} |\psi_r(t)\rangle / \sqrt{n(t)}$. A single realization corresponds to the outcome of a continuously monitored experiment, reflecting real-time quantum jumps as detected, for example, in circuit-QED systems [72–76]. Averaging over many (N) such realizations with trajectory label r yields the density matrix $|\overline{\psi_r(t)}\rangle\langle\overline{\psi_r(t)}| = (1/N) \sum_{r=1}^N |\psi_r(t)\rangle\langle\psi_r(t)| = \hat{\rho}(t)$, matching the solution of Eq. (2).

Unlike the stationary state $\hat{\rho}_0$, individual quantum trajectories, with phase-space coordinates $(X_r(t), Y_r(t))$ given by the real and imaginary parts of $\tilde{\beta}_r(t) = \langle \psi_r(t) | \hat{b} | \psi_r(t) \rangle$, retain the full topological information about the underlying flow. In the semiclassical regime (large N), each quantum trajectory closely follows a classical streamline, drifting away from saddle points and fluctuating around stable FPs at long times [see Figs. 3(a) and 3(b)]. Therefore, the state $|\psi_r(t)\rangle$ witnesses flow topology in three ways: (1) The number and stability of FPs are encoded in the relative residence times within each basin, (2) the local chirality appears in the circulation of stochastic loops around each FP, and (3) the connectivity of FPs emerges from rare stochastic jumps that carry a trajectory between basins. From this trajectory

viewpoint, the chirality spectrum (5) can be written as

$$\zeta(\Omega) = -\frac{2}{\mathcal{N}} \sum_{r=1}^{\mathcal{N}} \text{Im}[\mathcal{F}(G_r(\tau))], \quad (7)$$

where on each trajectory we evaluate the time-delayed correlator $G_r(\tau) = \langle Y_r(t)X_r(t+\tau) - X_r(t)Y_r(t+\tau) \rangle_t$, with $\tau > 0$, and where $\langle \square \rangle_t$ denotes an average over all times t . Further details can be found in Appendix D. We then average over the imaginary part of the Fourier transform $\mathcal{F}$ of the correlator. Importantly, $\zeta(\Omega)$ can be tracked as a function of the control parameters (e.g., detuning Δ/U), where flow-topology transitions manifest as qualitative changes in the number, sign, and position of spectral peaks, providing a direct experimental diagnostic of the underlying topological phase structure [21,61].

We compute $\zeta(\Omega)$ along the detuning sweeps of Fig. 2 in the semiclassical regime ($\aleph = 20$), covering regions 1, 3α , and 3β [see Figs. 3(c)–3(f)]. In region 1, we observe a single CW mode that softens near the $1 \rightarrow 3\alpha$ transition. In region 3α , two nearly degenerate CW peaks emerge [see Fig. 3(e)]. In region 3β , one CW and one CCW peak of opposite sign appear [see Fig. 3(f)]. Finally, in region 5β , the spectrum exhibits two CW peaks and one CCW peak, reflecting the three distinct attractors of the system. Deeper in the quantum regime ($\aleph = 10$), see Figs. 3(g) and 3(h), the distinction between 3α and 3β persists away from phase boundaries, but their sharp transition smooths into a crossover. At the same time, multiphoton resonances involving higher eigenstates contribute (cf. Fig. 2 and Refs. [19,77]). Interestingly, the CW and CCW winding of the $\zeta(\Omega)$ peaks survives in the multiphoton regions. However, the connection to the underlying classical flow weakens, and mixed spectral features from both 3α and 3β can appear at fixed Δ/U .

B. Deep-quantum limit: Liouvillian analysis

In the deep-quantum limit ($\aleph = 1$), the quantum nature strongly modify the classical phase-space structure: Wigner peaks no longer match semiclassical FPs, and trajectories tunnel between multiple classical basins. In this limit, $\zeta(\Omega)$ can be obtained from the spectral decomposition of the Liouvillian [20],

$$\zeta(\Omega) = \sum_{k>0} \frac{w_k}{i\Omega - \lambda_k}, \quad (8)$$

where λ_k are complex Liouvillian eigenvalues ($\lambda_0 = 0$) and $w_k = \text{Tr}[\hat{Y} \hat{r}_k^\dagger \text{Tr}[\hat{\ell}_k^\dagger \hat{X} \hat{\rho}_0] - \text{Tr}[\hat{X} \hat{r}_k^\dagger \text{Tr}[\hat{\ell}_k^\dagger \hat{Y} \hat{\rho}_0]]$, where $\hat{\ell}_k$ ($\hat{r}_k$) are the left (right) Liouvillian eigenvectors. Equation (8) links each chirality peak to a Liouvillian mode with frequency $\text{Im}(\lambda_k)$, linewidth $\text{Re}(\lambda_k)$, and spectral weight set by overlap factors. Unraveling $\hat{\ell}_k, \hat{r}_k$ into quantum trajectories shows that $\zeta(\Omega)$ encodes in each peak the *net chirality* of the trajectory bundle forming it, which in the semiclassical limit corresponds to trajectories confined within their basins of attraction, where traces thereof remain even in the deep-quantum limit (see Fig. 4). Moreover, sweeping $\Delta/\tilde{U}$ across multiphoton resonances [Fig. 4(a)], the chirality spectrum $\zeta(\Omega)$ [colormap in Fig. 4(b)] reveals that chiralities can flip at avoided crossings ($\Delta/\tilde{U} = 2$) or persist ($\Delta/\tilde{U} = 3$), pointing

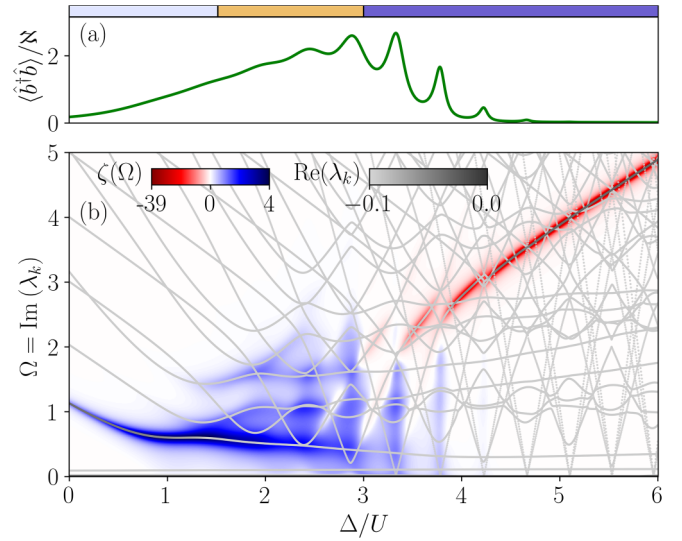


FIG. 4. Chirality spectrum in the multiphoton resonance regime. (a) RSSP vs detuning Δ/U for $F = 0.5$, $G = 0.4$ in the deep-quantum regime ($\aleph = 1$). The top stripe marks the classical phases. (b) Chirality spectrum $\zeta(\Omega)$ [Eq. (5)] shown as a colormap, overlaid with the imaginary parts $\text{Im}(\lambda_k)$ of Liouvillian eigenvalues; shading encodes decay rates $\text{Re}(\lambda_k)$ (darker = slower decay time). Even in this regime, spectral branches retain clear CW or CCW winding, making $\zeta(\Omega)$ a robust observable across multiphoton resonances.

to a reorganization of quantum trajectories in phase space. Importantly, all these transitions occur without the standard hallmark of dissipative phase transitions, namely, without a Liouvillian-gap closing [20]. This reflects that the steady state remains unique while the transition is encoded in the restructuring of subleading Liouvillian modes and their spectral weights rather than in a vanishing slowest decay rate.

Similar reorganizations also occur in the pure Kerr parametric and Kerr-driven oscillators, although in a more trivial form (see Appendixes). Finally, the amplitude asymmetry between $\zeta(+\Omega)$ and $\zeta(-\Omega)$ reflects unequal emission and absorption rates (see Appendixes); this arises from Kerr-induced squeezing of resonator fluctuations and is reinforced by drive- and dissipation-induced nonthermal occupations [78,79]. Our results establish $\zeta(\Omega)$ as a robust probe of dissipative phase structure deep in the quantum regime, beyond the conventional Liouvillian-gap criterion. A comprehensive analysis of the implications is left for future work.

IV. CONCLUSION AND OUTLOOK

We have shown that classical flow topology leaves distinct and experimentally accessible fingerprints in driven-dissipative quantum steady states. In a prototypical Kerr resonator with single- and two-photon drives, the Wigner function traces central semiclassical flow-topological features, including the number, arrangement, and stability of classical FPs. Yet, ensemble averaging washes out the local chirality contained in individual trajectories. The frequency-resolved chirality spectrum, $\zeta(\Omega)$, overcomes this limitation by extracting local winding signatures from fluctuation

modes. Remarkably, these spectral features remain robust deep in the quantum regime, even when quantum fluctuations smear out classical attractors.

All proposed diagnostics are directly accessible via heterodyne detection of the output field, providing a practical route to probe flow topology in current superconducting and photonic platforms [34,80,81]. In particular, $\zeta(\Omega)$ can be directly extracted from time-resolved quadrature correlations of the output field, making it experimentally measurable via standard heterodyne detection and frequency-resolved noise spectroscopy in circuit-QED and photonic platforms. Looking ahead, natural extensions include multimode generalizations, exploring the connection between flow topology and photon blockade [77], and leveraging topological signatures to develop bosonic codes tailored for fault-tolerant quantum computation [82,83]. More broadly, our results establish a spectral framework for identifying nonequilibrium transitions beyond the standard Liouvillian-gap paradigm.

ACKNOWLEDGMENTS

We acknowledge funding from the Deutsche Forschungsgemeinschaft (DFG) through Projects No. 449653034, No. 521530974, and No. 545605411, as well as via SFB 1432 (Project No. 425217212). We also acknowledge support from the Swiss National Science Foundation (SNSF) through the Sinergia Grant No. CRSII5_206008/1. J.d.P. acknowledges funding from MICIU/AEI/10.13039/501100011033 through the Ramón y Cajal program (RYC2023-043827-I, co-funded by FSE+), the “María de Maeztu” Programme (CEX2023-001316-M), and grant PID2024-158923NA-I00 (co-funded by FEDER, UE).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: MEAN-FIELD DYNAMICS

To analyze the semiclassical dynamics of the driven-dissipative system, we consider the evolution of the expectation value of the bosonic operator $\hat{b}$, defined as $\beta = \text{Tr}(\hat{\rho}\hat{b})$. Starting from the Lindblad master equation [Eqs. (1) and (2)], the time derivative of β is given by

$$\frac{d\beta}{dt} = \text{Tr}\left(\frac{d\hat{\rho}}{dt}\hat{b}\right) = \text{Tr}(\mathcal{L}\hat{\rho}\hat{b}). \quad (\text{A1})$$

Using the linear cyclic properties of the trace, this yields the evolution equation

$$\frac{d\beta}{dt} = \frac{1}{i\hbar}\text{Tr}([\hat{\mathcal{H}}, \hat{\rho}]\hat{b}) + \kappa\text{Tr}\left(\hat{b}^\dagger\hat{b}\hat{b}\hat{\rho} - \frac{1}{2}\hat{b}\hat{b}^\dagger\hat{b}\hat{\rho} - \frac{1}{2}\hat{b}^\dagger\hat{b}\hat{b}\hat{\rho}\right). \quad (\text{A2})$$

The closed equation for β is obtained by performing the classical limit. For this, we adopt a mean-field approximation where all higher-order moments are factorized, e.g., $\text{Tr}(\hat{b}^\dagger\hat{b}\hat{b}) \approx |\beta|^2\beta$ and $\text{Tr}(\hat{b}^2) \approx \beta^2$. Under this approximation, inserting the explicit form of the Hamiltonian [cf. Eq. (1)] (with the system parameters rewritten to make their dependence in $\hbar$ explicit), and taking the classical limit $\hbar \rightarrow 0$, we arrive at the nonlinear equation of motion

$$i\frac{d\beta}{dt} = \left(-\Delta - i\frac{\kappa}{2} + U|\beta|^2\right)\beta + G\beta^* + Fe^{-i\phi}, \quad (\text{A3})$$

which corresponds to a complex Gross-Pitaevskii equation for a driven Kerr resonator with both single- and two-photon driving and linear loss. Equation (A3) governs the dynamics in phase space in the large-occupation limit. We split real and imaginary parts in Eq. (A3). Using a shorthand notation, where $\text{Re}(\beta) \equiv \beta_r$ and $\text{Im}(\beta) \equiv \beta_i$, we obtain a system of coupled nonlinear ordinary differential equations:

$$\dot{\beta}_r = -\frac{\kappa}{2}\beta_r + U\beta_i^3 + U\beta_i\beta_r^2 - \Delta\beta_i - G\beta_i - F\sin(\phi), \quad (\text{A4})$$

$$\dot{\beta}_i = -\frac{\kappa}{2}\beta_i - U\beta_i^2\beta_r - U\beta_r^3 + \Delta\beta_r - F\cos(\phi) - G\beta_r. \quad (\text{A5})$$

Equations (A4) and (A5) define a 2D vector flow in phase space with coordinates (β_r, β_i) . We begin by analyzing its topology through the structure of fixed points, obtained by setting $\dot{\beta}_r = \dot{\beta}_i = 0$. This yields a coupled set of cubic polynomial equations, which we solve using homotopy continuation as implemented in HARMONICBALANCE.JL [59]. For a given set of parameters, the number of real solutions $\beta_0 = \{\beta_0^r, \beta_0^i\}$ of Eqs. (A4) and (A5) gives the total number of physical steady states N , including both stable and unstable states. In the phase diagram shown in the main text, with Δ and F as control parameters, this count provides a coarse classification into regions with distinct N .

Linear stability analysis

To assess the stability of each fixed point β_0 , we linearize Eqs. (A4) and (A5) around it. Introducing small deviations $\delta\beta = \{\delta\beta_r, \delta\beta_i\}$, the dynamics are governed by the Jacobian matrix J , defined as

$$J = \left(\begin{array}{cc} \partial_{\beta_r}\dot{\beta}_r & \partial_{\beta_i}\dot{\beta}_r \\ \partial_{\beta_r}\dot{\beta}_i & \partial_{\beta_i}\dot{\beta}_i \end{array}\right)_{\beta=\beta_0} = \left(\begin{array}{cc} -\frac{\kappa}{2} + 2U\beta_0^i\beta_0^r & 3U(\beta_0^i)^2 + U(\beta_0^r)^2 - \Delta - G \\ -3U(\beta_0^r)^2 - U(\beta_0^i)^2 + \Delta - G & -\frac{\kappa}{2} - 2U\beta_0^i\beta_0^r \end{array}\right). \quad (\text{A6})$$

The dynamical behavior of small fluctuations follows from the complex eigenvalues ϵ and eigenvectors of J , where $\text{Im}(\epsilon)$ sets the oscillation frequencies of the fluctuations, while $\text{Re}(\epsilon)$

determines whether the fluctuations grow or decay. In particular, $\text{Re}(\epsilon) < 0$ indicates that a perturbation away from the fixed point decays and thus the fixed point is locally stable,

whereas $\text{Re}(\epsilon) > 0$ signals an instability. Note that the Jacobian J equates the Bogoloubov–de Gennes Hamiltonian in a closed bosonic system ($\kappa = 0$), whose eigenvalues are the squeezed excitations denoted Bogoliubov modes [84].

APPENDIX B: VECTOR-FLOW TOPOLOGY

The standard approach to classifying phases in nonequilibrium systems is to count the number N of fixed points. Under such classification, two points A and B in parameter space, belong to the same (different) phase if $N_A = N_B$ ($N_A \neq N_B$). Phase transitions require a local bifurcation, where N changes abruptly, e.g., when a saddle collides with a stable node in the flow. This picture, however, misses transitions that do not involve a change in N . A finer classification comes from Morse-Smale theory, which captures not only the number and stability of fixed points but also their connectivity, all encoded in a topological invariant described later in this section [49]. Additionally, we observe regions split by phase transitions that do not fall under this classification: regions 3α and 3β in Figs. 1(b) and 1(d) both have $N = 3$, yet the steady-state distribution changes discontinuously between them: excitations switch from particlelike (confining potential) to holelike (inverted potential). This behavior, tied to local chirality around fixed points, is incorporated in our refined invariant [48]. The following outlines the key concepts of vector-flow topology relevant to semiclassical, out-of-equilibrium dynamics in 2D.

1. Morse-Smale theory and topological graph invariant

We begin with basic definitions. A smooth map $V : \mathcal{M} \rightarrow \mathbb{R}^2$ on a closed manifold $\mathcal{M}$ defines a two-dimensional *vector field* that assigns a vector to each point. The corresponding *vector flow* describes how points move over time and takes the form of two coupled differential equations, $\dot{X} = f(X, Y)$ and $\dot{Y} = g(X, Y)$. In our case, the flow follows from Eqs. (A4) and (A5), with an example shown in Fig. 5(a). Vector-flow topology classifies flows in 2D by their key features [49,85–87]:

(1) *Fixed points*, where $f(X, Y) = g(X, Y) = 0$, can be attractors, repellers, or saddles, depending on the local flow: Trajectories converge into attractors, diverge from repellers, and split (two in, two out) at saddles. In Fig. 5, black dots mark attractors and black crosses mark saddles; repellers would appear as white dots. However, the present flow does not feature repellers, as it results from the sum of a Hamiltonian (symplectic) flow and a dissipative term with positive damping rate ($\kappa > 0$), leading to contraction in phase space.

(2) *Separatrices* are special trajectories that connect distinct critical points of the flow, dividing the surface in regions with a given dynamics. They either originate from or terminate at saddle points. These trajectories describe the asymptotic behavior of the vector flow and are robust against small perturbations that maintain the inter-connectivity between critical points intact. In Fig. 5(b), separatrices are shown as black lines, with the dashing style indicating the type of connection: solid for source sink, dotted for source saddle, and dashed for saddle sink.

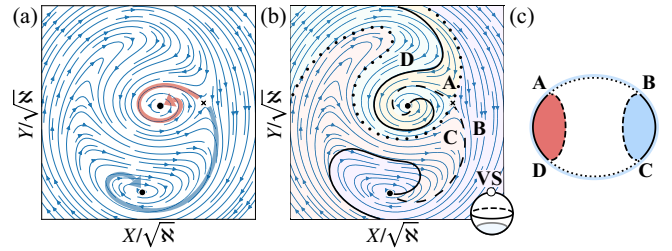


FIG. 5. Finding the Morse-Smale-based graph invariant of a 2D nonequilibrium flow. (a) Vector flow with $N = 3$, one saddle point and two (CW and CCW) attractors; see color-marked flow lines into the attractors. Note that this flow is topologically equivalent to that of Fig. 1(d). (b) Same flow as in panel (a), marking separatrices (flow lines) that connect the saddle to both attractors (dashed), and the flow coming from afar (i.e., from a virtual source) into the attractors (solid). The inset represents the compactification of the flow into the sphere, with the virtual source, here dubbed VS, placed at its north pole. The separatrices partition the flow into regions with topologically equivalent asymptotic behavior, each distinguished by a color and labeled by letters. The set of fixed points, flow lines, and faces is known as Morse-Smale complex. (c) Topological graph invariant of the flow. Nodes and edges correspond to the MSC regions and their shared boundaries, respectively. Each region is color coded by the chirality of the corresponding attractor. The outer boundary of the graph is colored to indicate the chirality of the distant flow. The lines drawn in panels (a) and (b) are simplified for visualisation purposes.

(3) Finally, *periodic orbits* correspond to closed, recurrent trajectories that are neither critical points nor separatrices. They are absent in the parameter regime considered here.

Morse-Smale theory provides a topological framework to classify flows on 2D closed manifolds and determine when they are *structurally stable*: A vector flow is structurally stable if the topological characteristics of the features described above (e.g., the separatrix connectivity) are robust against small perturbations. Formally, two structurally stable vector flows V and V' defined on the same manifold $\mathcal{M}$ are *topologically trajectory equivalent* if their flow lines can be smoothly deformed into each other via a homeomorphism $h : \mathcal{M} \rightarrow \mathcal{M}$. Note that, while the transformation h must preserve the orientation of flow lines (i.e., their asymptotics), though not necessarily the parametrization of the stream (i.e., their speed along trajectories).

Morse-Smale theory builds on Morse theory, which relates the topology of a manifold $\mathcal{M}$ to the critical points of a smooth Morse function $f : \mathcal{M} \rightarrow \mathbb{R}$ —isolated, nondegenerate maxima, minima, and saddles. For such an f , the gradient flow ∇f is *Morse-Smale* if

- (1) it has finitely many hyperbolic fixed points or periodic orbits, where a fixed point is *hyperbolic* if the Jacobian J has no eigenvalues with zero real part,
- (2) all trajectories end at one of them, and
- (3) stable and unstable manifolds intersect transversely (Smale condition).

These properties ensure *structural stability*: The flow's global pattern, including separatrices, is robust to small perturbations. In 2D, the Smale condition forbids direct saddle-to-saddle connections, whereas its violation signals instability.

For structurally stable 2D dynamical systems, Morse-Smale theory establishes a one-to-one correspondence between a vector flow V on a manifold $\mathcal{M}$ and a planar graph $\mathcal{G}$, known as the Oshemkov-Sharko three-colored graph [49]. This graph is a complete topological index for the classification of Morse-Smale vector flows, encapsulating all the information on the variety of critical points and their connectivity, as well as the topology of the underlying $\mathcal{M}$. Recall from Appendix B 1: “Two vector flows V and V' on the same manifold $\mathcal{M}$ are *topologically trajectory equivalent* if one can be smoothly deformed into the other.” This equivalently means: Two vector flows on 2D manifolds are trajectory equivalent if and only if their Oshemkov-Sharko graphs are isomorphic. Reference [49] provides a procedure to construct this topological invariant for Morse-Smale flows on compact 2D manifolds. We now summarize the key steps:

(1) *Construct the Morse-Smale complex.* For structurally stable flows, separatrices divide the manifold $\mathcal{M}$ into regions where all trajectories behave the same, i.e., they share origin, destination, and direction. There are three types: solid lines (source-to-sink), dashed (saddle to sink), and dotted (source to saddle). Dashed and dotted separatrices form bounded four-edged regions, defining the *Morse-Smale Complex* (MSC). Solid lines are then added in a step called *triangulation*, dividing the surface into three-edged regions—one edge of each dashing style—called crystals [colored areas in Fig. 5(b)]. The full set of nodes, edges, and crystals captures the structure of the flow at every point. For simplicity, we refer to this construction as MSC^* .

(2) *Compute the dual of the MSC.* The planar graph is obtained as the dual of MSC^* , as follows: (1) for each crystal, add a node in $\mathcal{G}$; (2) if two crystals share one (or more) side(s), connect the corresponding nodes in $\mathcal{G}$ with edges dashed as the shared side(s). When all nodes have been joined via the edges, this creates a pattern of adjacent regions: This is the Oshemkov-Sharko three-colored planar graph [black lines in Fig. 5(c)].

Note that the Oshemov invariant for 2D Morse-Smale flows [49] assumes that the flow “lives” on a compact (without a boundary) surface. The reason is twofold: First, compactness enforces global counting rules such as the Poincaré-Hopf theorem, which balances sources, sinks, and saddles with the Euler characteristic of the compact surface [88]; second, it guarantees that the invariant remains a finite graph, since trajectories cannot “leak out” to infinity. To use it in the phase plane, which is noncompact, we first need to compactify the space. This works because all trajectories in the plane ultimately “come from infinity,” so we can treat them as if they originate from a single source. Concretely, we map the plane onto a sphere of radius r via inverse stereographic projection, which sends infinity to a single point at the sphere’s north pole. In this way, the flow looks as though it emanates from that point [see inset of Fig. 5(b)]. This transformation reads

$$(X, Y) \mapsto \frac{(2Xr^2, 2Yr^2, r(A^2 - r^2))}{A^2 + r^2}, \quad (\text{B1})$$

with $A = \sqrt{X^2 + Y^2}$.

2. Extension of Morse-Smale theory to out-of-equilibrium dynamics

Morse-Smale theory, as traditionally formulated for gradient flows, classifies flows based on topological features like fixed points and separatrices. Out-of-equilibrium dynamical systems, such as those governed by Eqs. (A4) and (A5), include rotational (nongradient) components and allow for flow chirality, which standard Morse-Smale theory ignores (please refer to Ref. [48] for further information).

The key difference originates from the relevance of *chirality around fixed points*. In driven-dissipative systems, the dynamics lock to the drive frequency, which sets a phase reference. As a result, flow lines wind CW or CCW around each FP, depending on whether it is blue or red detuned. This winding directly reflects the particle- or holelike character of excitations [21]. Therefore, it impacts the physical properties of the system’s linear response around the attractors. The character of the response can be represented by ascribing, to the eigenstates of the Jacobian around a FP, Eq. (A6), a vector norm weighted with the symplectic metric $\mathcal{J}$,

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\text{B2})$$

the so-called symplectic norm ds^2 . Its possible values $\text{sgn}(ds^2) \in \{-1, 0, +1\}$ correspond to particlelike (+), holelike (−), and overdamped (0) excitations. Note that chirality is well defined by the sign of the nonlinearity U , which sets a background CCW (CW) winding for $U < 0$ ($U > 0$) in the distant flow, far from the attractors.

Changing the chirality of an NESS requires crossing exceptional points and is not a smooth deformation [21,48]. This imposes a key constraint: Flow deformations must preserve the symplectic norm (and thus chirality around each fixed point) to retain that two phases topologically are equivalent. Conversely, change in a chirality signals a phase transition and must alter the graph invariant. To encode the added symmetry constraint into the graph invariant, we apply a chirality-based coloring of the regions [Fig. 5(c)]. Blue (red) regions correspond to CW (CCW) fixed points, while the region is left white in case of an achiral fixed point (saddle or overdamped attractor). The chirality of the virtual source is represented by a colored halo encircling the graph. With this addition, the graph invariant provides a complete description of the vector flow for out-of-equilibrium dynamics. Two phases are topologically equivalent, if and only if their graphs are isomorphic, preserving face colors, nodes, and edge dashing.

APPENDIX C: THE TWO LIMITING CASES OF THE DUFFING AND THE PARAMETRON

In the main text, we discuss the competition between one-photon (F) and two-photon (G) drives in a nonlinear Kerr resonator. Because neither is a mere perturbation to the other, this results in the rise of both G - and F -dominated regions in the semiclassical phase diagram of Fig. 1(b). These two regions are separated by a multicritical point, as further discussed in Ref. [48]. Here, we briefly recap the analysis of the KDO (Kerr-driven oscillator) and KPO (Kerr parametric oscillator) semiclassical phases, and the quantum observables

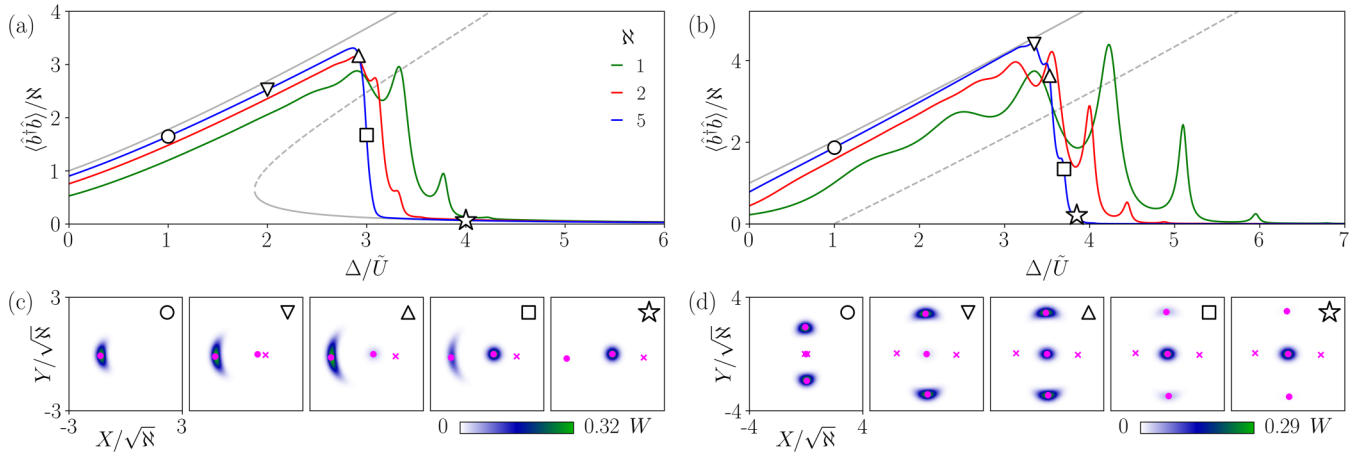


FIG. 6. (a), (b) Rescaled steady-state photon number $\langle \hat{b}^\dagger \hat{b} \rangle / \aleph$ vs detuning $\Delta / \tilde{U}$ for the two limiting cases: (a) the driven Kerr oscillator ($F/U = 1, G = 0$) and (b) the Kerr parametric oscillator (KPO; $G/U = 1, F = 0$). Colored lines correspond to the master equation results for different scaling parameter $\aleph$, while gray curves show semiclassical fixed-point solutions (solid = stable, dashed = unstable). (c), (d) Wigner functions of the steady state at representative detunings (symbols). Magenta crosses mark semiclassical fixed points, and magenta dots indicate the stable ones. In the KDO case (c), quantum fluctuations smear out the bistability predicted semiclassically, while in the KPO case (d), the quantum steady state appears as a catlike mixture of the two phase states.

for both such cases. For notation and the model parameters, we refer to Sec. II. Additional Wigner-function examples and parameter sweeps supporting the discussion below are shown in Fig. 6.

1. Kerr driven oscillator ($G = 0$) limit

When ($G = 0, F \neq 0$), the semiclassical phase space supports at most three fixed points ($N = 1$ or $N = 3$). At small detuning the system is in a single low-amplitude “gas” phase, which corresponds to the topologically trivial phase with $N = 1$. As the detuning is increased past a critical value, a saddle-node bifurcation creates a pair of new solutions: a high-amplitude “liquid” phase and an unstable saddle point [19]. In this bistable regime, the system could occupy either the low- or high-amplitude state hysteretically [24]. In the language of vector-field topology, the creation of a node-saddle pair marks a topological phase transition: The phase space splits into more basins, and the graph invariant gains two new faces, colored differently because the stable fixed points carry opposite chirality; see graph invariant discussion and Ref. [48]. In the full quantum model, the steady state remains unique: A finite Liouvillian gap enforces a single steady state and the semiclassical bistability appears only as metastable peaks in the Wigner function (see Fig. 6 for representative Wigner functions).

2. Kerr parametric oscillator ($F = 0$) limit

With only a two-photon (parametric) drive ($F = 0, G \neq 0$), the classical system has up to five fixed points. At low detuning, there is a single stable “normal” phase at zero field ($N = 1$). Upon increasing detuning past a threshold, a supercritical pitchfork bifurcation marks and spontaneous breaking of the $\mathbb{Z}_2$ symmetry ($X, Y \rightarrow -X, Y$). As a result, two new stable nonzero-amplitude solutions (phase states) related by $(X, Y) \rightarrow -(X, Y)$ emerge, plus an additional now-unstable fixed point at the origin (a total of $N = 3$ solutions). From

a flow-topology standpoint, at the pitchfork bifurcation, the flow undergoes a topological phase transition: the graph invariant acquires three faces, two of the same color because the stable fixed points share the same chirality [48]. This makes the bistable regime of the KPO and KDO topologically distinct. If detuning is increased further, a subcritical pitchfork adds two unstable symmetric solutions (for $N = 5$), corresponding to a coexistence region, where both the zero solution and the two phase states are classically stable. This regime is commonly referred to as the “coexistence” region of phase states. In the quantum steady state, the steady-state solution is a statistical mixture of opposite-phase Schrödinger cat states (even and odd parity) [89]. Accordingly, the Wigner function shows two peaks (the classical phase states), but the exact steady state preserves $\mathbb{Z}_2$ symmetry (see Fig. 6 for representative Wigner functions).

APPENDIX D: DETAILS ON THE TRAJECTORY-BASED CHIRALITY ANALYSIS

To quantify the local chirality of quantum fluctuations, we group trajectories by the basin of attraction they settle into at long times, assuming they remain confined there (i.e., without activation between attractors). Specifically, we assign trajectory r to the attractor k whose fixed-point phase-space coordinate β_0^k is closest to the initial complex amplitude $\beta_{r,\text{init}} = \langle \psi_r(0) | \hat{b} | \psi_r(0) \rangle$. For each cluster k , we record the trajectory coordinates $\{X_r^k(t), Y_r^k(t)\} = \{\text{Re}(\tilde{\beta}_r^k(t)), \text{Im}(\tilde{\beta}_r^k(t))\}$ and compute the cross correlation

$$G^k(\tau) = \overline{Y_r^k(0)X_r^k(\tau)} - \overline{X_r^k(0)Y_r^k(\tau)}, \quad (\text{D1})$$

where the overline denotes averaging over trajectories in cluster k . This retarded response $G^k(\tau)$ captures the tendency of trajectories to rotate around the attractor over a delay $\tau > 0$. The local chirality spectrum for attractor k then follow from

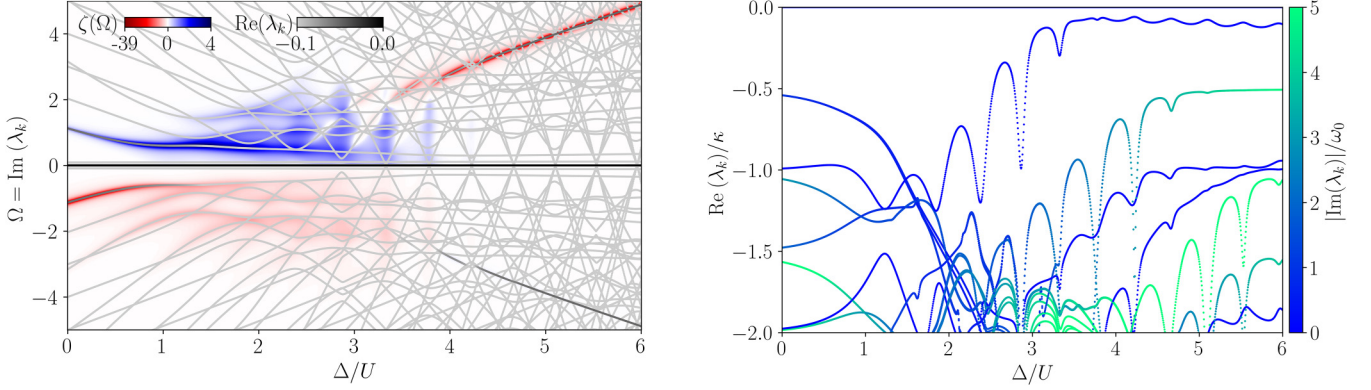


FIG. 7. Chirality spectrum and Liouvillian modes in the generic driven Kerr-parametric regime. Extension of Fig. 4(b) to negative frequencies. Left panel: The colormap shows $\zeta(\Omega)$, with overlaid lines marking the imaginary parts $\text{Im}(\lambda_k)$ of the slowest Liouvillian modes and shading indicating their decay rates $\text{Re}(\lambda_k)$. The unequal spectral weight at $\pm\Omega$ directly visualizes the positive/negative-frequency asymmetry discussed in Appendix E. Right panel: corresponding $\text{Re}(\lambda_k)$ (decay rates) for the same modes (color indicates $|\text{Im}(\lambda_k)|/\omega_0$). Parameters: $F/U = 0.5$, $G/U = 0.4$, $\kappa/U = 0.1$, and $\aleph = 1$. Positive (negative) values of $\zeta(\Omega)$ correspond to net CW (CCW) winding. Lines and shading are computed from the same Liouvillian used to produce ζ .

the Fourier transform of $G^k(\tau)$, $\mathcal{F}[G^k(\tau)]$ [21,90], as

$$\mathcal{C}^k(\Omega) = -2\text{Im}(\mathcal{F}(G^k(\tau))), \quad (\text{D2})$$

from τ to frequency Ω . Peaks in $\mathcal{C}^k(\Omega)$ indicate fluctuation modes around fixed point k that have a well-defined rotational sense in phase space. The sign of the peak at positive Ω reveals the local winding direction: a positive peak means clockwise (clockwise, CW) rotation, whereas a negative peak means counterclockwise (counterclockwise, CCW) rotation around β_0^k [21,45,48,68,91]. In this way, the chirality of each fixed point is read out from the quantum noise spectrum, arising from the trajectory cluster k , which could be experimentally retrieved via heterodyne detection of the output field [76].

For nondegenerate fluctuation modes, as in our main text cases (e.g., away from bifurcation points), the chirality spectrum $\sum_k \mathcal{C}^k(\Omega)$ averages over the chirality for quantum trajectories around all attractors into an average linear-response observable. In the quantum regime, trajectories may also switch or tunnel between attractors. These processes can be grouped under an additional label, s , so that $\mathcal{C}^s(\Omega)$ accounts for switching contributions. Extending the sum to include both k (fixed-attractor) and s (switching) classes, i.e.,

$$\zeta(\Omega) = \sum_k \mathcal{C}^k(\Omega) + \sum_s \mathcal{C}^s(\Omega) \quad (\text{D3})$$

makes $\zeta(\Omega)$ a fully averaged linear-response observable, measurable without the need to postselect quantum trajectories. Note that tunneling events are rare, so the switching term $\sum_s \mathcal{C}^s(\Omega)$ usually produces a negligible contribution.

APPENDIX E: SPECTRAL ASYMMETRY IN $\zeta(\Omega)$

By extending the chirality map of Fig. 4(b) to negative frequencies Ω , we observe a clear asymmetry between positive and negative frequencies in the chirality response $\zeta(\Omega)$, as shown in Fig. 7. This figure extends the deep-quantum chirality spectrum of Fig. 4(b) to negative frequencies and illustrates that peaks located at Liouvillian eigenvalues,

plotted as lines with brightness increasing with $|\text{Re}(\lambda_k)|$, have unequal weights for opposite signs of $\text{Im}(\lambda_k)$. The right panel displays the real parts of the Liouvillian spectrum, $\text{Re}(\lambda_k)$, highlighting the absence of a gap closing at $\text{Re}(\lambda_k) = 0$ across the chirality switching shown in the left panel of Fig. 7.

Using the Liouvillian spectral decomposition together with the quantum regression theorem, the chirality spectrum can be expressed in the form [cf., Eq. (8) in the main text]

$$\zeta(\Omega) = \sum_{k>0} \frac{w_k}{i\Omega - \lambda_k}, \quad (\text{E1})$$

where λ_k are nonzero Liouvillian eigenvalues and the complex weights are

$$w_k = \text{Tr}(\hat{Y} \hat{r}_k) \text{Tr}(\hat{\ell}_k^\dagger \hat{X} \hat{\rho}_0) - \text{Tr}(\hat{X} \hat{r}_k) \text{Tr}(\hat{\ell}_k^\dagger \hat{Y} \hat{\rho}_0). \quad (\text{E2})$$

Each mode contributes a Lorentzian-like feature centered at $\Omega = \text{Im}(\lambda_k)$ with linewidth $\text{Re}(\lambda_k)$, and amplitude set by w_k . Because the Liouvillian is non-Hermitian, its left and right eigenmodes $\hat{\ell}_k, \hat{r}_k$ are independent, so the weights w_k and $\tilde{w}_k$ of conjugate eigenvalues λ_k and λ_k^* need not match. Only with extra symmetries (e.g., particle-hole symmetry) does one recover $w_k = \tilde{w}_k$, see Ref. [20] for a comprehensive review of Liouvillian spectral theory. This non-Hermitian overlap structure is the primary, model-independent origin of amplitude asymmetry in $\zeta(\pm\Omega)$.

Physically, the asymmetry between positive-frequency peaks (CW modes) and negative-frequency peaks (CCW modes) reflects transitions that either emit or absorb excitations relative to the drive. These effects can be clarified by analyzing the limiting cases of the Kerr Driven Oscillator ($G = 0$) and the Kerr parametric oscillator ($F = 0$), shown in Figs. 8 and 9, respectively.

In the KDO limit, cf. Fig. 8, the asymmetry persists even without parametric drive, demonstrating the role of (1) Kerr anharmonicity and (2) broken detailed balance. The Kerr nonlinearity leads to photon-number-dependent level spacings, so that different transition pairs contribute with unequal matrix elements, producing asymmetric spectral weights at $\pm$

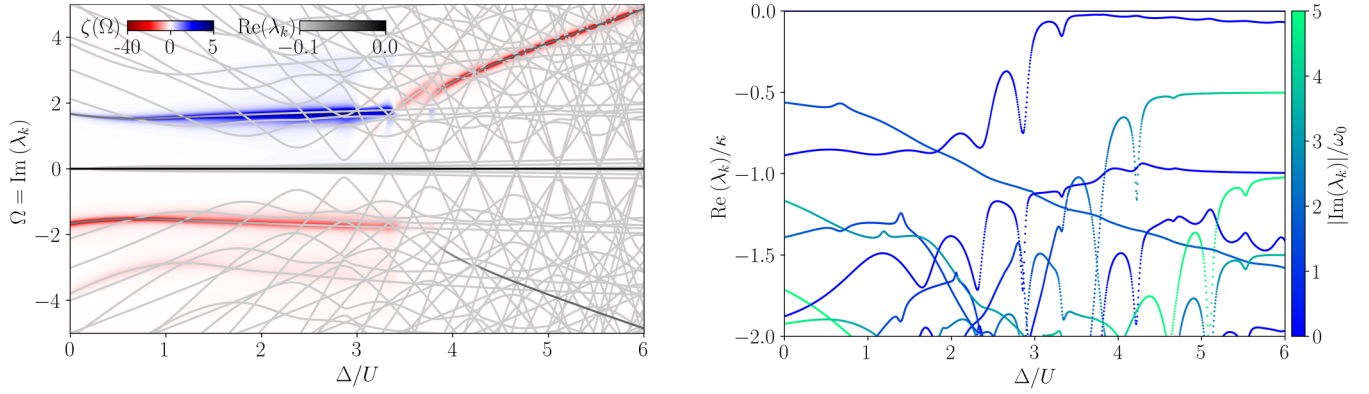


FIG. 8. Chirality spectrum and Liouvillian modes in the KDO limit. Same plotting conventions as Fig. 7, but for $F/U = 1$, $G = 0$. Even without parametric drive, the chirality spectrum shows pronounced imbalance between positive- and negative-frequency peaks (see text).

frequencies [92]. A similar effect occurs in classical Kerr mechanical oscillators, where quadrature squeezing yields asymmetric sidebands [78]. As for broken detailed balance, dissipation acts on eigenstates of the KPO that are not eigenstates of the loss operator, opening both decay and pumping channels. Unlike the purely monotonic decay of a harmonic oscillator in a zero-temperature bath, upward processes such as quantum heating occur, leading to a nonthermal steady state with finite absorption weight and transition-dependent linewidths [93]. A strong two-photon drive amplifies the imbalance through squeezing (cf. Fig 9). Together, these figures show that the asymmetry visible in Fig. 4(b) originates from a combination of anharmonicity, drive-dissipation imbalance, and squeezing, with their relative importance depending on the regime.

Liouvillian-gap closures versus chirality reorganizations

In the classical limit $\aleph \rightarrow \infty$, local bifurcations where the number of solutions changes signal topological phase transitions, that is, a breakdown of structural stability. From the density-matrix perspective, with $\hat{\rho}_0$ encoding the ensemble of driven-dissipative quantum trajectories, such transitions occur when the Liouvillian gap closes at bifurcations that alter the number of fixed points N . The spectral theory of dissipative phase transitions shows that gap closures correspond to

nonanalytic changes in $\hat{\rho}_0$ [20]. We emphasize that a “gap closure” here refers to the real part(s) of one or more nonzero Liouvillian eigenvalue(s) tending to zero (i.e., the emergence of additional slow modes). In particular, the saddle-node bifurcations of the KDO and the pitchfork bifurcations of the KPO coincide with such Liouvillian-gap closures. At finite $\aleph$, strict closures are avoided due to finite-size effects, but clear signatures of the semiclassical bifurcations appear near $\Delta/U \approx 3.5$ in Fig. 8 and $\Delta/U \approx 1$ in Fig. 9, where $\text{Re}(\lambda_k) \approx 0$.

At finite $\aleph$, we also encounter a different type of transition: sharp reorganizations of the chirality spectrum without gap closure, marked by abrupt changes in the number of CW and CCW peaks. Here, the residues w_k in Eq. (E1) vary abruptly while leaving the zero-eigenvalue (steady state) subspace unaffected. The eigenvalue $\text{Re}(\lambda_1)$, corresponding to the slowest-decaying eigenvector, remains finite with $\text{Re}(\lambda_1) < 0$ at the parameter values where the chirality spectrum reorganizes. This indicates that chirality reorganizations reported here constitute a spectral response reconfiguration of the Liouvillian residues without the appearance of slow eigenmodes. This constitutes a finite- $\aleph$ phenomena that can produce abrupt observable switching in the response without a Liouvillian-gap closure or a nonanalytic change of $\hat{\rho}_0$. Consequently, experiments may observe sharp, switchlike changes in the measured chirality response even when the Liouvillian gap remains finite.

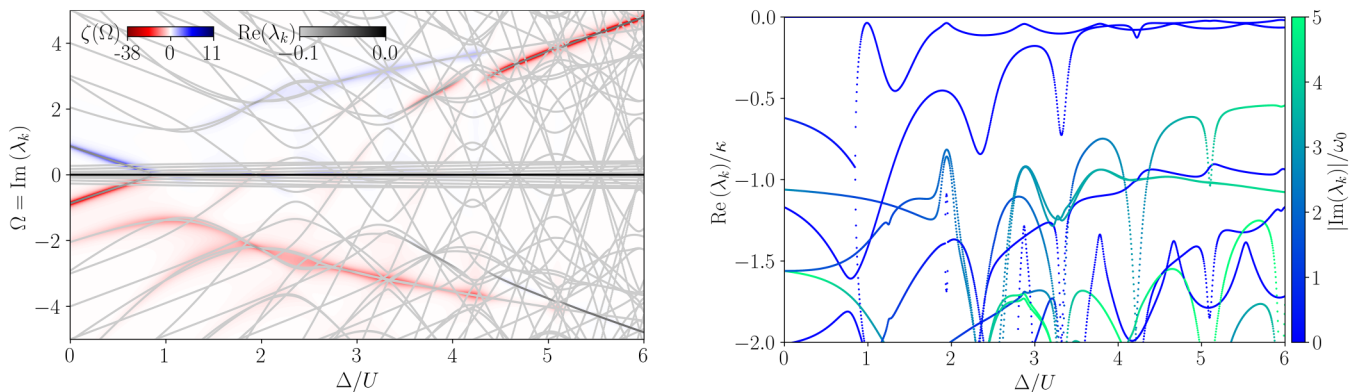


FIG. 9. Chirality spectrum and Liouvillian modes in the parametric (KPO) limit. Same plotting conventions as in Fig. 8, but for $F/U = 0$ and $G/U = 1$.

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