

Symmetric preferences, asymmetric outcomes: Tipping dynamics in an open-city segregation model

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Schelling's model of segregation demonstrates that even in the absence of social or governmental interventions, individuals with *mild* in-group preferences can self-organize into strongly segregated neighborhoods. Many variants of this celebrated model have been proposed by assuming that agents tend to increase their satisfaction. Complementary to this traditional, utility-based approach, we model residential moves using *satisfaction-independent* reaction rates in a spatially extended chemical reaction network. The resulting model exhibits an emergent phenomenon: Despite symmetric in-group preferences, the system undergoes a tipping transition at a critical preference level, beyond which one agent type dominates. We characterize this asymmetric phase transition in details using mean-field analysis, numerical simulations, and finite-size-scaling methods. We find that while the transition shares key features with the Ising-universality class, such as $\mathbb{Z}_2$ symmetry breaking and similar exponent ratios, the full set of critical exponents does not match any known universality class.

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I. INTRODUCTION

Segregation is a widespread urban phenomenon in which a population is divided into subgroups along socioeconomic lines, such as ethnicity, income, education, or social economic status [1]. Since the pioneering work of Thomas Schelling [2], numerous studies have confirmed the robustness of segregation patterns arising from individual, satisfaction-driven movements [3–5]. While early works rely on agent-based simulations, offering a mechanistic view but limited theoretical insight, recent research has revealed intriguing analogies between the Schelling model and various physical phenomena, such as surface tension [6] and clustering in Ising- or Blume-Emery-Griffiths-type systems [7–10]. Such behavior is consistent with the broader class of nonequilibrium systems exhibiting Ising-type criticality, despite the absence of detailed balance [11]. Moreover, Zakine *et al.* [12] suggest

that the large-scale organization seen in social segregation may follow universal scaling laws similar to those observed in active-matter systems. Seara *et al.* [13] propose a hydrodynamic model with utility-driven diffusion whose parameters are calibrated by the U.S. census data.

In essence, these approaches rely on modeling, in one way or another, agents' preferences for neighborhoods of a certain demographic composition. Such preferences are often described by a so-called tolerance threshold—the maximal fraction of neighbors of another type that agents can tolerate (or transversely, an acceptance threshold defined as the minimal fraction of like neighbors below which agents consider moving out). The key conclusion of Schelling-type models is that *mild* in-group preference for same-type neighbors is sufficient to lead to segregation. Further results confirm this paradoxical behavior, establishing that a higher degree of tolerance can actually lead to an overall increase in segregation [14–16]. Even if agents have a preference for perfect integration, a best-response dynamics (where agents move if and only if they strictly improve their utility) may still lead to segregation [3]. In this regard, a central open question is whether segregation can arise in the *absence* of such explicit thresholds or utility-optimization rules.

This situation is often met in practice. Indeed, individual motivations and preferences are typically latent and difficult to infer, whereas aggregate quantities such as residential mobility rates are often accessible through administrative records

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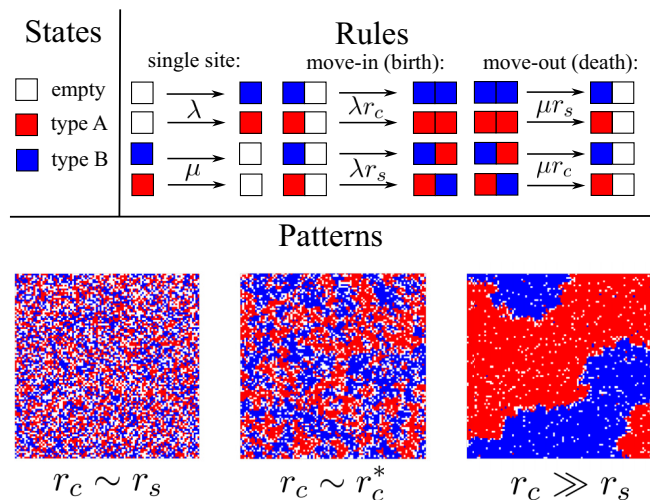


FIG. 1. Overview of the reactions of our segregation model. Each lattice site can be in three possible states, corresponding to either an empty house or a resident of type A or B. The reaction rules allow for random creation and annihilation of residents (moving into the region with rate λ or moving out with rate μ), alongside pairwise reactions grouped into move-in (birth) and move-out (death) reactions. For the pairwise moving-in reactions, we distinguish those that promote like neighbors (in-group, or “copy” with rate λr_c) from those that promote opposing neighbors (out-group, or “split” with rate λr_s). Similarly, we divide the pairwise moving-out reactions into two types: in-group reactions with rate μr_s and out-group reactions with rate μr_c . Some exemplary grid configurations are shown on a two-dimensional (2D) lattice with 100×100 sites. Increasing the in-group reaction rate r_c gradually leads to a segregating transition, where regions of both red and blue types are equally large. Further increasing r_c will lead to larger regions of one type dominating over smaller islands of the opposing type.

and statistical data. Motivated by the observation that such frequencies can be interpreted as reaction rates in a spatially extended chemical reaction network (CRN), we model the hopping dynamics using two pairwise reaction processes: in-group preference and out-group tolerance. The former encodes the tendency of agents to move into neighborhoods populated by others of the same type, while the latter captures the acceptance of agents of the opposite type into those neighborhoods. Each of these processes is accompanied by a *reverse* move, where one of the two interacting agents leaves the neighborhood (see Fig. 1). Within this picture, segregation emerges as a consequence of some random reactions occurring more (or less) frequently than others, generating a systematic tendency for same-type agents to cluster together. This mechanism is closely related to domain coarsening and phase-ordering processes studied in statistical physics [17].

The proposed approach does not assume that agents act to either maximize their utility or satisfy their own threshold, offering a perspective on the origin of segregation. The representation of agent mobility as reactions also comes with a number of advantages: (1) It utilizes the rich analytical and numerical treatment of phase separation [18] in CRNs—reminiscent of patterns observed in segregation; (2) it allows for modeling *open* neighborhoods, i.e., those without conservation of the number of agents of a certain type (although

the number of houses is conserved); and (3) the *ergodicity* of the CRN resolves the common problem of getting stuck in a suboptimal frozen state. Such a stochastic formulation is often adopted in a broad range of statistical-physics studies that model collective behavior through transition rates rather than explicit optimization [19].

In addition to the emergence of segregation, the open character of the model further permits a phase transition between a symmetric state, with equal representation of both agent types, and an asymmetric state in which one type dominates. This asymmetric tipping point occurs spontaneously at a critical level of in-group preference, even if the preference for like agents is symmetric between both types. By means of numerical simulations and finite-size-scaling analysis, we establish the continuous, i.e., second-order, nature of this phase transition.

We note that other tipping scenarios (including both second-order transition and crossover) have been considered previously, such as “type mobility” induced by a fraction of *type-switching* but *nonmoving* agents in 2D (regardless of their satisfaction [20]), or models where the entire population is able to switch their types on an one-dimensional ring [21]) or in a nonspatial setting [22]. While we also use probabilistic update rules, our reactions are mediated by empty sites, since we only include transitions between blue (red) and empty sites. As discussed in Ref. [23], empty sites facilitate the “effective” diffusion of red and blue agents, resulting in different segregation patterns. We therefore expect significant difference between our model and the works of Refs. [20–22] in terms of macroscopic behavior. Furthermore, the explicit role of vacancies resonates with sociological vacancy-chain theories of residential mobility, where empty housing units mediate population redistribution [24].

The rest of the article is organized as follows. In Sec. II, we present the model definition in terms of microscopic processes responsible for the in-group preference and out-group tolerance. Next, we derive a mean-field theory and its second-order correction via moment closure in Sec. III. This allows us to show a crossover in energy and a bifurcation in magnetization. The latter is responsible for the tipping point. In Sec. IV, we provide simulation results on 2D Moore neighborhoods, accompanied by finite-size-scaling analysis of the observables (energy, net magnetization), from which we extract the critical exponents for comparison with known universality classes. Finally, in Sec. V we discuss our results in the broader context of the emerging field of social physics [19,25] and statistical physics of cities [26]. All other technical details are provided in the Appendixes.

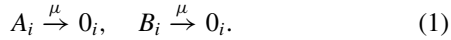
II. SEGREGATING CHEMICAL REACTION NETWORK

To make the connection to Schelling’s original work on segregation, we formulate a chemical reaction network on a two-dimensional square lattice with $N = L \times L$ total lattice sites. Each site $i \in ([1, 2, \dots, L] \otimes [1, 2, \dots, L])$ can be in any of three distinct states denoted by A_i , B_i , and 0_i , signifying that either an agent of type A or B, or a vacant site is present at that location on the lattice.

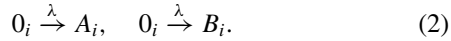
Next, we specify all possible reactions in our network as illustrated in Fig. 1. Any change in the macroscopic state

of the lattice is determined by move-in (*birth*) or move-out (*death*) reactions of its agents. These birth-death reactions can be split into (spontaneous) neighbor-independent and (interacting) neighbor-dependent processes. More concretely, in the first category we have the following:

(1) *Death reactions.* Agents moving out of the neighborhood with rate μ :

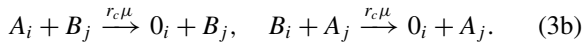
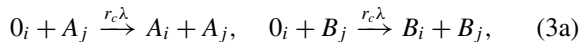


(2) *Birth reactions.* Agents moving into the neighborhood with rate λ :

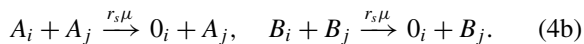
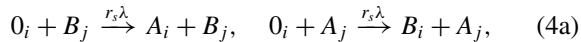


In addition to such a birth-death process at each lattice site i , we also consider interactions between i and its nearest neighbors within the Moore neighborhood. The conclusions of this work do not differ if we use other neighborhoods, although the exact quantitative results will. The neighbor-dependent processes are of two types:

(1) *In-group reactions.* These encode the tendency of agents to attract similar agents. This happens in two ways. First, by adding move-out pressure on opposite agents with rescaled death rate $r_c\mu$. Second, by adding move-in pressure on similar agents with rescaled birth rate $r_s\lambda$. Neighboring sites $i, j \in (L, L)$ thus interact as



(2) *Out-group reactions.* Encode the tendency of agents to attract opposite agents. This again happens in two ways. First, by adding move-out pressure on similar agents with rescaled death rate $r_s\mu$. Second, by adding move-in pressure on opposite agents with rescaled birth rate $r_s\lambda$. Neighboring sites $i, j \in (L, L)$ thus interact as



Besides the neighborhood definition, the parameters of our model are μ, λ, r_c , and r_s . Without loss of generality, we can always rescale time to set one of the parameters to one, leaving three free parameters in total.

The CRN defined by Eqs. (1)–(4) is ergodic; the out-group reactions (4) are the inverse of the in-group reactions (3), albeit with reaction rates replaced as $\mu \leftrightarrow \lambda$ and $r_s \leftrightarrow r_c$. The ergodicity of the system makes sure that Markov chain of the CRN converges to a unique steady-state distribution. In the next section, we will use a mean-field approach to investigate how this steady-state distribution depends on the model parameters.

III. MEAN-FIELD EQUATIONS AND THE COLLECTIVE PHASES

To gain insight into this model, we first analyze the mean-field equations, whose solutions correspond to different collective phases of the stochastic model. The mean-field equations are readily derived from the law of mass action, i.e., from the rate equations of the CRN [Eqs. (1)–(4)], where we

treat each node as an independent and identically distributed variable. Alternatively, they can also be derived from the master equation of the CRN by assuming statistical independence of pair probabilities for nearest neighbors in the lattice (see Appendix C 1). The resulting equations are expressed in terms of the probabilities $\rho^X(t)$ for any site of the lattice to be in state $X \in \{A, B, 0\}$ at time t :

$$\frac{d\rho^A}{dt} = -T\rho^A(t) + \rho^0(t) - 8T\rho^A(t)(r_s\rho^A(t) + r_c\rho^B(t)) + 8\rho^0(t)(r_c\rho^A(t) + r_s\rho^B(t)), \quad (5a)$$

$$\frac{d\rho^B}{dt} = -T\rho^B(t) + \rho^0(t) - 8T\rho^B(t)(r_s\rho^B(t) + r_c\rho^A(t)) + 8\rho^0(t)(r_c\rho^B(t) + r_s\rho^A(t)), \quad (5b)$$

and, owing to the vacancy constraint on each site,

$$\rho^0(t) = 1 - \rho^A(t) - \rho^B(t). \quad (5c)$$

Here, we have rescaled time $\tilde{t} = \lambda t$ and defined $T = \mu/\lambda$ for notational convenience. The probabilities $\rho^X(t)$ are equivalent to mean density of type X on the lattice.

The mean-field equations (5) have four steady-state solutions. One is always unphysical, giving negative probabilities. Whenever

$$r_c < r_c^* \equiv \frac{1}{4} + 3r_s + \frac{T}{8}, \quad (6)$$

there is a single physical solution to the mean-field equations (the others give negative or imaginary probability densities). In this state, the probabilities ρ^A and ρ^B are exactly equal, and there is no net overdensity on the lattice. For this reason, we call the phase associated to this state the *symmetric phase*, with the associated densities:

$$\rho^0 = \frac{T}{2+T}, \quad \rho^A = \frac{1}{2+T} = \rho^B. \quad (7)$$

When condition (6) is not met, the symmetric solution becomes linearly unstable, while two other stable branches of solutions become physical (i.e., real-valued and bounded $\in [0, 1]$). The system will settle into one of these branches, depending on initial conditions. On both branches, the $\mathbb{Z}_2$ symmetry exchanging $A \leftrightarrow B$ is explicitly broken, as one type of agent becomes more abundant. Since here $\rho^A \neq \rho^B$, we call this the *asymmetric phase*. The densities in this phase are given by

$$\rho^A = \frac{8(r_c - r_s) - T \pm \sqrt{D}}{16(r_c + r_s(T - 1))}, \quad (8a)$$

$$\rho^B = \frac{8(r_c - r_s) - T \mp \sqrt{D}}{16(r_c + r_s(T - 1))}, \quad (8b)$$

$$\rho^0 = \frac{T(r_s + 1/8)}{r_c + r_s(T - 1)}, \quad (8c)$$

where,

$$D = (-2 + 8r_c - 24r_s - T)(2 + 8r_c + 8r_s - T). \quad (9)$$

Note that the two branches are the same solution with the densities for ρ^A and ρ^B switched. In analogy with spin models, the symmetric and asymmetric phases of our model correspond to

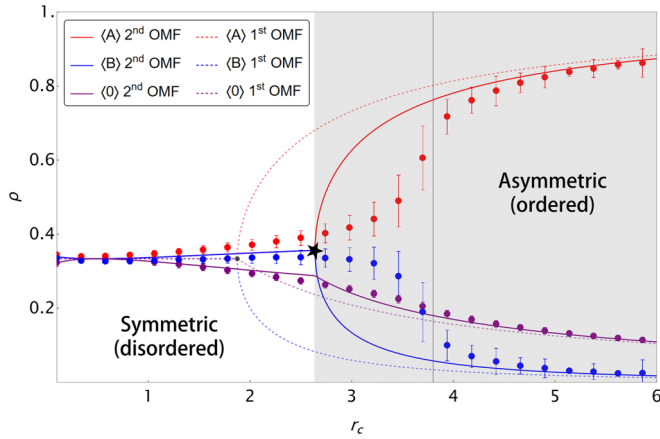


FIG. 2. The densities in the stable steady states for $T = \mu/\lambda = 1$ and $r_s = 0.5$, as a function of r_c , obtained from the second-order (full lines) and first-order (dashed lines) mean-field approximations. We show the density of vacant sites (purple), of agents of type A (red), and of agents of type B (blue). The scatter data correspond to the averages of 10^4 independent instances of the stochastic model, with grid size $N = 50 \times 50$. The mean-field analysis shows a pitchfork bifurcation at a critical value r_c (marked by a black star), where the solution changes from symmetric to asymmetric one. The mean-field solutions agree with the stochastic model away from r_c^* (marked by a vertical line $r_c \approx 3.8$). Note that there is also an asymmetric solution for which the densities of A and B are flipped with respect to the densities shown here.

disordered and ordered phases of a magnetic system. Therefore, we can use $M(t) = \rho^A(t) - \rho^B(t)$ as an order parameter to quantify the overdensities of one type of agent. In the asymmetric phase, there is a net overdensity $M(t \rightarrow \infty) \neq 0$ at stationarity:

$$|M| = |\rho^A - \rho^B| = \frac{\sqrt{D}}{8(r_c + r_s(T - 1))}, \quad (10)$$

which implies that one of the agent types will start to dominate in the neighborhood. Which agent type will dominate depends on initial conditions in the deterministic mean-field model, or on fluctuations in the stochastic model.

The densities ρ^A , ρ^B , and ρ^0 as predicted by the (first-order) mean-field approximation are plotted as dashed lines in Fig. 2. Here, we also compare the mean-field results with expectation values obtained from numerical simulations of the stochastic model for a grid of size $N = 50 \times 50$. We confirm two distinct phases, the disordered (symmetric) and the ordered (asymmetric) ones, which are separated by a pitchfork bifurcation at the critical point. As expected, away from the critical point, mean-field predictions are accurate, but long-range correlations spoil their validity when $r_c \sim r_c^*$. In order to improve upon the first-order mean-field theory, we have performed a second-order moment closure [27,28] in Appendix C.2. As shown by the solid lines in Fig. 2, the second-order moment closure significantly improves the estimation of r_c^* , although it still deviates from the numerical value. The second-order moment closure qualitatively agrees with the stochastic model.

Besides the tipping point at $r_c = r_c^*$, we identify an additional regime when interpreting the model in socioeconomic

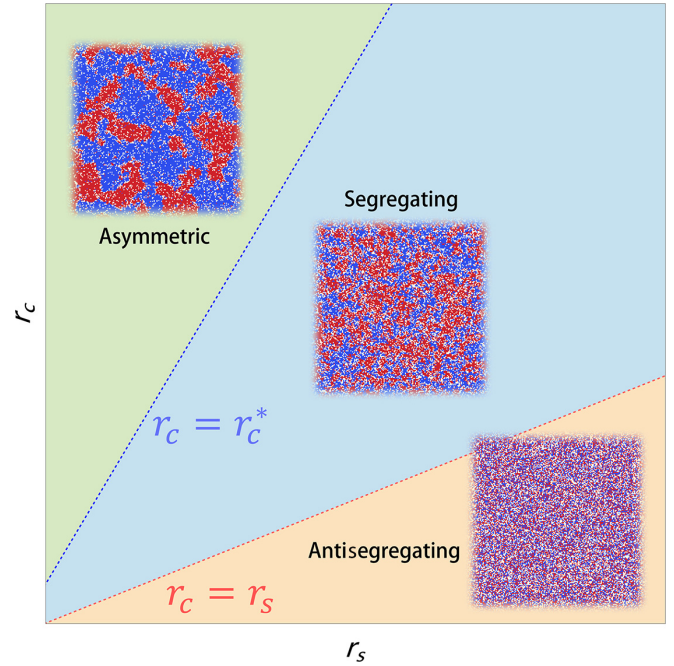


FIG. 3. Schematic phase diagram of our model in the (r_s, r_c) plane. The symmetric and asymmetric phases are separated by a tipping transition at r_c^* , while a crossover transition from antisegregating to segregating neighborhoods takes place when $r_c = r_s$. The inlay shows typical configurations for each phase, obtained from simulations of the model using Gillespie's algorithm.

terms. Specifically, whenever $r_s > r_c$, lattice sites in state A are more likely to be neighbored by lattice sites in state B and vice versa. This corresponds to a type of antiferromagnetic configuration, which can be interpreted as agents' preference for having diverse neighborhoods. In what follows, we refer to this phase as *antisegregating*. Moreover, we observe that a global measure of the net abundance of like neighbors, defined in a similar fashion to the (Ising) energy, passes through a zero at $r_c = r_s$, when the system transits from segregating to antisegregating patterns. However, as seen in the next section, such a transition is not critical, but rather a simple crossover from negative to positive energy.

In the remainder of the paper, we will analyze the dynamics and critical behavior of the asymmetric tipping point using exact stochastic simulations (Gillespie's algorithm [29]). Details on the simulation algorithm are given in Appendix A and a comparison between the mean-field solution and a stochastic simulation run is detailed in Appendix B. We show the mean-field phase diagram of the model in Fig. 3, along with typical configurations from each of the three phases. The transitions displayed in the figure are as follows:

(1) A crossover from an *antisegregating* neighborhood to a *segregating* one. Here, we define *antisegregating* as a neighborhood in which agents of opposite types are more likely to be neighbors than not. A *segregating* neighborhood represents the opposite situation. This transition is characterized by a crossover from negative to positive energy, where the energy is defined as the relative abundance of like neighbors.

(2) The asymmetric *tipping* transition from a neighborhood with an (approximately) equal number of agents of each

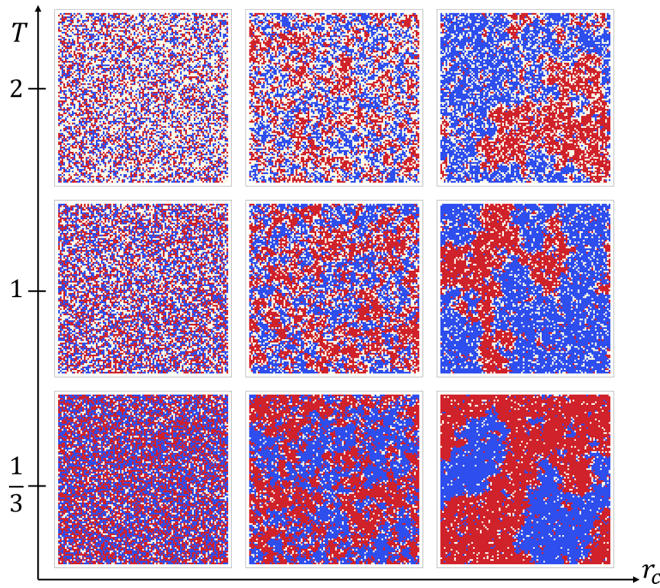


FIG. 4. Snapshots of steady-state solutions of the stochastic model obtained by varying $T = \mu/\lambda$ and r_c independently while fixing r_s (here $r_s = 0.5$). Note that this plot is merely meant for schematic purposes; the axes are not to scale.

type to one in which one of the two types gains the upper hand. This transition is akin to the transition from a paramagnetic to a ferromagnetic phase in the Ising model, characterized by a shift from zero magnetization to nonzero net magnetization and a peak in the variance of the energy. From a sociological point of view, this transition is “maximally” segregating as the whole neighborhood starts to be dominated by one type of agent.

Of these two transitions, only the second one is critical, in the sense that the correlation length ξ diverges at the critical point in the thermodynamic limit.

We conclude this section with a short discussion on varying the effective move-out (death) rate $T = \mu/\lambda$. With our choice of parameters, we have intentionally made sure that T controls to overall density of agents, and does not affect the qualitative behavior of the model. The dependence of the stochastic model on T is captured quite accurately by the mean-field solution, confirming our understanding that T is an overall tuning parameter of the model. As shown in Fig. 4, increasing T lowers the density of agents of both types, while preserving the three phases described above.

IV. PHASE TRANSITIONS, FINITE-SIZE SCALING, AND CRITICALITY

To further characterize the model and investigate its universality class, we will now find the critical exponents of the asymmetric tipping transition. To do so, we use the well-established finite-size-scaling method. In what follows, we briefly introduce this method before discussing the results for our chemical reaction network.

A. Finite-size scaling

At a continuous phase transition, the correlation length ξ of an infinite system diverges. Therefore, at fixed μ , λ , and r_s ,

we expect our system near the critical point r_c^* to have

$$\xi \sim |r_c - r_c^*|^{-\nu}, \quad (11)$$

where ν is the critical exponent for the correlation length. An observable O scaling with a critical exponent ζ near the transition thus depends on the correlation length as

$$O \sim |r_c - r_c^*|^{-\zeta} \sim \xi^{\zeta/\nu}. \quad (12)$$

In our simulations with lattices of finite size $N = L \times L$, typically, $L \ll \xi$, so the system size effectively bounds the maximum correlation length. In this regime, we conclude

$$O \sim L^{\zeta/\nu}. \quad (13)$$

This logic leads to the finite-size-scaling ansatz, which holds near the phase transition:

$$O \sim L^{\zeta/\nu} f(L^{1/\nu}(r_c - r_c^*)). \quad (14)$$

Here, $f(x)$ is a universal scaling function determining how finite-size effects influence the value of observables. By our previous discussion, we know that $f(x) \rightarrow \text{constant}$ as $x \rightarrow 0$, and that $f(x) \rightarrow x^{-\zeta}$ as $|x| \rightarrow \infty$. The finite-size ansatz in Eq. (14) allows us to determine the critical exponents of observables. In particular, the correct values of the exponents uniquely collapse the data of $L^{-\zeta/\nu} O$ plotted against $L^{1/\nu}(r_c - r_c^*)$ onto each other for different system sizes. To do this effectively, we need a way to determine the value of r_c^* . We can do this independently by using the Binder cumulant.

The results shown in this section are obtained by performing numerical simulations of our stochastic model using a Gillespie-type algorithm. We start by initializing the lattice randomly and evolving for 180 Monte Carlo (MC) sweeps to find an approximate steady state for a given set of parameters. We compute expectation values of observables by taking measurements every subsequent 18 MC steps (to avoid a strong autocorrelation) and averaging over a total of n_{sim} such measurements.

B. Binder cumulant

To make the analogy with spin models explicit (as originally suggested for Schelling’s model in Ref. [9]), we link the overdensity M defined in Eq. (10) with the microscopic (stochastic) variables

$$M = \frac{1}{N} \sum_i \sigma_i, \quad (15)$$

where M represents the net “magnetization” on the lattice and $\sigma_i = 1, 0, -1$, if site i is occupied by agent A , empty, or occupied by agent B , respectively. Another interesting observable is the average energy on the lattice,

$$E = \frac{1}{4N} \sum_{(i,j)} \sigma_i \sigma_j, \quad (16)$$

which is a measure of how segregated a neighborhood is locally, as it measures the net discrepancy between neighbors of like and opposing types. While its definition suggests that E is related to the energy in pairwise interacting spin models, it is not used here to specify the dynamical evolution of the system, but only as a measure of segregation. Furthermore,

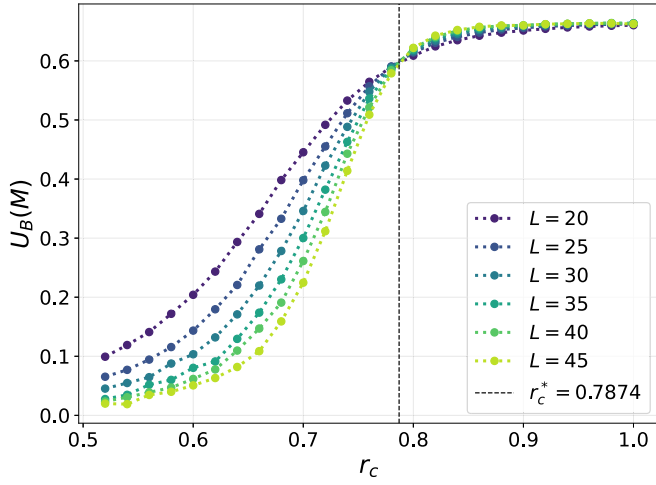


FIG. 5. The Binder cumulant for the magnetization characterizes the continuous phase transition to the asymmetric (ferromagnetic) phase where agents of one type become the dominant group in the region. The critical value, shown here for $r_s = 0$, is determined at the crossing point for the Binder cumulants as $r_c^* = 0.7874(2)$.

when measuring this energy, we use the von Neumann (left, right, up, down) neighborhood, conventional in Ising-like spin systems.

To determine the exact value of r_c^* , we use the Binder cumulant of the magnetization,

$$U_B(M) = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}, \quad (17)$$

which goes to a scale-independent universal value at the critical value r_c^* . In Fig. 5, we plot $U_B(M)$ for different system sizes $N = L \times L$ and determine r_c^* by identifying the crossing points of all the curves. The value for the Binder cumulant at the crossing is found to be $U_B(M)|_{r_c^*} = 0.60$, which is close to that of the Ising model (there $U_B(M)|_{r_c^*} = 0.61$ with periodic boundary conditions [30]).

Earlier, we characterized the transition from antisegregating to segregating as a smooth crossover occurring at $r_c = r_s$, corresponding to the absence of a net preference in neighbor type. At this point, the average energy defined in Eq. (16) changes sign, as shown in Fig. 10. Since this crossover is not associated with a diverging correlation length, the energy fluctuations remain approximately Gaussian in shape around this point. Phenomenologically,

$$P(E) \sim \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(E - \langle E \rangle)^2}{2\sigma^2}\right], \quad (18)$$

which implies a vanishing Binder cumulant $U_B(E) = 0$ for a perfectly Gaussian distribution. This behavior is observed numerically in Fig. 6, where the Binder cumulant exhibits a pronounced dip near $r_c = r_s$, independently of system size. Away from this point, the distribution becomes sharply peaked such that $U_B(E)$ approaches $2/3$ for increasing system sizes.

We emphasize that the near-Gaussian form of $P(E)$ at the crossover does not preclude critical scaling of energy fluctuations at the true critical point $r_c = r_c^*$. In particular, while the shape of the distribution remains close to Gaussian, its variance $\text{Var}(E)$ can still exhibit system-size scaling, giving

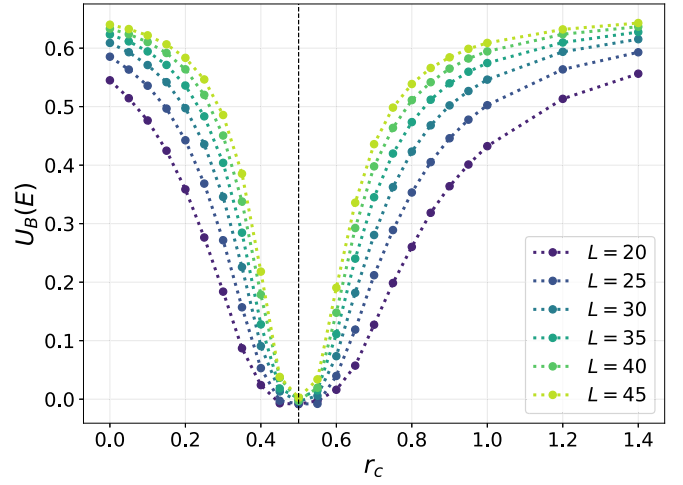


FIG. 6. The Binder cumulant for the energy for $r_s = 0.5$ and $T = \mu/\lambda = 1$ dips to zero at $r_c = r_s$. Dotted line indicates the value of r_s .

rise to critical behavior in the heat capacity. In summary, the Binder cumulant of the energy probes the Gaussian character of fluctuations at the crossover, whereas the Binder cumulant of the magnetization displays the standard crossing behavior characteristic of a continuous phase transition.

C. Critical exponents

Now that we have determined r_c^* belonging to the infinite system size limit, we can use the finite scaling ansatz to determine the value of the critical exponents for various observables. We focus here on the scaling of the absolute magnetization per site $|M| \sim L^{\beta/\nu}$, the susceptibility, computed as

$$\chi = L^2(\langle M^2 \rangle - \langle M \rangle^2), \quad (19)$$

and its exponent γ . Additionally, we characterize two more exponents at the critical point (α and ϵ) by determining the critical scaling of the variance in energy (or, the heat capacity):

$$C_V = L^2(\langle E^2 \rangle - \langle E \rangle^2) \quad (20)$$

and the variance in the density of vacant sites $\text{Var}(\rho_0)$, respectively. After determining the scaling behavior at the critical point, we can determine ν using Eq. (14) for the magnetization $|M|$. In practice, this boils down to rescaling the r_c axis as $(r_c - r_c^*)L^{1/\nu}$ and optimizing for ν that minimizes the variance in $|M|L^{-\beta/\nu}$ for various system sizes, effectively “collapsing” all magnetization curves on top of each other. This is shown for the observables of interest in Fig. 7. We perform this analysis for three values of r_s (shown in Table I) while varying r_c and N . The number of independent simulation runs n_{sim} for each (r_c, r_s, N) choice is shown in the bottom row of the table. The error margins in the reported critical exponents are obtained by the bootstrapping method, where 100 datasets were created by sampling with replacement from the obtained data and the finite-size-scaling analysis was repeated for each of these datasets. Note that this analysis omits other types of systematic errors, which may arise from the finite system size window chosen here ($L = 20\text{--}45$) or possible insufficient equilibration near the critical point resulting in the

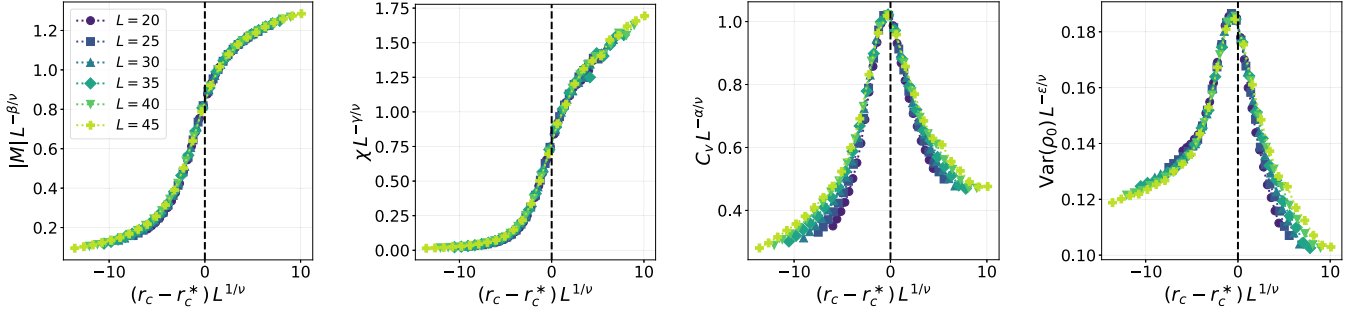


FIG. 7. Rescaling observables by their exponents at r_c^* shows a collapse of the curves onto a single line, shown here at $r_s = 0$ for the observables $|M|$, χ , C_V , and $\text{Var}(\rho_0)$, respectively. The value of ν was estimated by determining the best overlap for $|M|L^{-\beta/\nu}$, fixing it for the other three graphs.

underestimation of the autocorrelation. The ratios of critical exponents such as β/ν and γ/ν are close to the Ising values, suggesting that the transition is governed by a $\mathbb{Z}_2$ symmetry-breaking mechanism akin to Ising-like criticality. This feature is consistent with a broad class of nonequilibrium systems in which Ising universality emerges despite the absence of detailed balance [11,31]. From this perspective, the nonequilibrium nature of the CRN dynamics does not, by itself, preclude Ising-like criticality. However, the measured critical exponents in Table I differ from those of the standard 2D Ising and Potts universality classes. In particular, the heat capacity shows a power-law divergence, rather than the logarithmic behavior characteristic of the 2D Ising model. In addition, the independently measured exponents do not satisfy standard hyperscaling relations when interpreted within an equilibrium framework. These deviations indicate that the critical behavior of the present model cannot be straightforwardly identified with that of a standard equilibrium universality class such as the Ising or Potts models.

To assess whether these conclusions are affected by the chosen range of L , in Appendix D we performed additional robustness analysis for moderate system sizes $L = 50, 55, 60$. In brief, we repeated the full finite-size-scaling procedure for a family of fitting windows, recomputing the Binder estimate of r_c^* and the critical exponents independently for each window. This included both an upper-cutoff scan, in which the maximum system size was varied as $L_{\text{max}} = 45, 50, 55, 60$, and a lower-cutoff scan, in which the smallest system sizes were successively removed. Across these windows, the estimates of

β/ν and γ/ν remain stable within the statistical uncertainty of the bootstrap analysis. The heat-capacity exponent α/ν shows the largest window dependence, as expected for an energylike observable with stronger finite-size corrections, but it remains incompatible with a purely logarithmic Ising-like divergence over the accessible range of system sizes. The collapse estimate of ν also varies only moderately under changes of the fitting window. We therefore find no evidence that the deviations from 2D Ising-universality class reported in Table I can be removed by extending the analysis from $L \leq 45$ to $L \leq 60$.

One possible interpretation is that these deviations arise from strong corrections to scaling, which may still be significant even over the extended system-size range. Another possibility is that the critical behavior is influenced by an additional coupled field, namely, the local vacancy density, which plays an explicit dynamical role in the CRN formulation. Although the vacancy density is not globally conserved in the present model, it is constrained at the single-site level and is dynamically coupled to the order parameter. Such coupling may lead to effective renormalization of scaling behavior, particularly if density fluctuations evolve on timescales comparable to those of the order parameter near criticality [32,33].

In equilibrium statistical mechanics, analogous situations are known to give rise to renormalized critical behavior when thermodynamic constraints or hidden variables couple to the energylike scaling field [34]. Whether a similar mechanism is operative in the present nonequilibrium setting remains an open question. Establishing such a connection would require demonstrating that density fluctuations act as an effective constraint or slow mode at criticality, for instance, through timescale separation, ensemble comparisons, or controlled modifications of the dynamics that fix or suppress density fluctuations.

V. DISCUSSION

In this work, we have proposed and analyzed a nonequilibrium CRN model of segregation with macroscopic pattern formation resulting from interactions between agents. Unlike traditional approaches, the present model does not rely on utility maximization, explicit thresholds, or deterministic relocation rules. Instead, segregation emerges from purely stochastic, nearest-neighbor reaction processes in the Moore neighborhood of a square lattice. We have shown that,

TABLE I. Critical exponents for several observables, assuming $O \sim L^{\zeta/\nu}$ for the exponent ζ . Uncertainties are reported as the standard deviation over 100 exponents, obtained by resampling the simulated outputs with replacements (i.e., bootstrapping).

Observable	ζ	$r_s = 0$	$r_s = 0.1$	$r_s = 0.5$
r_c^*		0.7874(2)	1.3870(4)	3.8039(16)
$ M $	β	-0.1407(8)	-0.1420(8)	-0.1295(14)
χ	γ	1.8766(15)	1.9013(16)	1.8693(55)
C_V	α	0.2937(35)	0.2967(35)	0.2845(65)
$\text{Var}(\rho_0)$	ϵ	-1.975(4)	-2.003(4)	-1.944(10)
ξ	ν	1.0816(9)	1.0928(5)	1.0660(31)
n_{sim}		256 000	200 064	128 000

beyond a critical level of in-group preference, the system undergoes a continuous tipping transition. This transition spontaneously breaks the $\mathbb{Z}_2$ symmetry between the two agent types and shows diverging correlation lengths and susceptibilities. Finite-size-scaling analysis yields a consistent set of critical exponents describing this transition.

The observed phenomenology is qualitatively reminiscent of Ising-like critical behavior: The order parameter is non-conserved, interactions are short ranged, and exponent ratios such as β/ν and γ/ν are close to their two-dimensional Ising counterparts. Quantitatively, however, the measured thermal exponents ν and α deviate from their equilibrium Ising values, and the heat capacity exhibits a clear power-law divergence rather than the logarithmic behavior expected in the 2D Ising model. Taken together, these observations indicate that while the present model shares important features with Ising universality at the level of symmetry and exponent ratios, the full set of measured exponents cannot be straightforwardly assigned to a known equilibrium universality class. We have tested the robustness of this conclusion by extending the finite-size-scaling analysis to $L = 50, 55, 60$ and by repeating the exponent extraction over multiple sliding fitting windows (see Appendix D). This analysis shows that the main exponent ratios remain stable under changes of the fitting window, while the strongest residual finite-size dependence appears in the heat-capacity exponent. Thus, although corrections to scaling cannot be ruled out, the extended analysis does not indicate a drift toward the standard two-dimensional Ising values within the accessible system-size range. Possible explanations therefore include strong corrections to scaling persisting beyond the simulated sizes, or the presence of an additional coupled field—such as the vacancy density—that modifies the effective scaling behavior. Disentangling these scenarios requires further investigation, in particular, a systematic analysis of finite-size effects and the transient density fluctuations near criticality.

Several other directions for future research naturally arise from the present work. While our analysis demonstrates a systematic improvement of the analytical description through the inclusion of higher-order moment closures (see Appendix C 1), a general closed-form characterization of the steady-state distribution remains an open problem. One possible avenue is to explore approaches based on nonlinear graph Laplacians, whose minimizers have been shown to capture segregationlike structures in related settings [35].

Another important direction concerns the role of network structure in shaping the nature and location of the tipping transition [36,37]. Extending the present analysis to heterogeneous networks using degree-based mean-field theory and higher-order moment closures could provide analytical insight into how connectivity, degree fluctuations, and spatial structure influence segregation dynamics and critical behavior. Beyond stationary properties, it would also be of interest to study dynamical aspects of the transition, such as the statistics of tipping times or the time required for a minority group to become dominant. These questions can be naturally formulated as first-passage time problems and, in suitable limits, may be addressed through mappings to Schrödinger-type equations, as done for variants of the voter model in Ref. [38].

It is worth emphasizing that the CRN framework adopted here admits a continuous-time Markov chain (CTMC) description in terms of transition probabilities between global configurations. In contrast to most CTMC-based models of social dynamics [39], these transition rates do not rely on predetermined thresholds or explicit utility functions [40–42]. This feature opens the possibility of incorporating state-dependent utilities or decision rules directly into the reaction rates, thereby providing a natural bridge between the present stochastic formulation and utility-based or hydrodynamic approaches to social dynamics [12,13,43].

More broadly, the CRN formulation offers a promising route toward data-driven modeling of segregation dynamics. Because the model is expressed in terms of reaction rates rather than latent individual utilities, it is possibly more directly comparable to empirical data. In an urban context, these rates can be interpreted as coarse-grained propensities for residential mobility, local demographic turnover, or changes in neighborhood composition. National statistics bureaus, population registers, census agencies, and administrative records often provide data on residential mobility across urban regions and, in some cases, at the national level, together with demographic information on households and neighborhoods. Such data could be used to estimate effective transition rates between local population states, for example, by applying Bayesian parameter-inference methods for continuous-time Markov models. This would make it possible to test whether observed local demographic changes are adequately described by uncorrelated birth-death processes, or whether models including nearest-neighbor interactions of the type proposed here provide a better statistical fit to empirical residential dynamics.

Another natural direction concerns the spatial substrate on which the dynamics takes place. In the present work, the model is defined on a regular two-dimensional lattice, which is chosen for its convenience in studying local interactions, symmetry breaking, and systematic finite-size scaling. Real urban systems, however, are not regular Euclidean grids: Their geometry is shaped by street networks, land-use constraints, transport infrastructure, and historical development, and often displays heterogeneous or fractal-like spatial structure [44–48]. Extending the CRN model to irregular urban networks, weighted graphs, or fractal-like geometries would therefore be an important step toward more realistic applications. In such extensions, lattice sites could represent neighborhoods, census tracts, or spatial cells, while edges would encode adjacency, mobility flows, commuting connections, or other forms of spatial interaction. This would allow one to investigate how the heterogeneous connectivity and multiscale geometry characteristic of real cities modify the critical behavior and segregation patterns identified here.

Finally, it would be interesting to explore the interplay between the residential dynamics considered here and additional social processes such as type switching or opinion dynamics, including voterlike models [49,50] or state-swapping dynamics [51]. Previous studies, including Schelling-voter hybrids [52–54], have shown that such couplings can lead to rich and nontrivial pattern formation, suggesting a wide landscape of collective behaviors yet to be explored within the CRN framework.

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DATA AVAILABILITY

The software used to produce the data that support the findings of this article are openly available [55].

APPENDIX A: DESCRIPTION OF NUMERICAL ALGORITHM

In this Appendix, we describe the stochastic simulation algorithm used to generate the numerical results presented in this work, in particular, those underlying the finite-size-scaling analysis. The dynamics are simulated using a standard Gillespie stochastic simulation algorithm [29] applied to a chemical reaction network defined on a two-dimensional square lattice of size $N = L \times L$.

Each lattice site i can be in one of three states, $\sigma_i \in \{-1, 0, 1\}$, corresponding to an agent of type B , a vacant site, or an agent of type A , respectively. Interactions occur within the Moore neighborhood, such that each site has $n = 8$ neighbors. At any instant, a site may undergo (1) neighborhood-independent birth-death reactions or (2) interaction-dependent in-group and out-group reactions involving its neighbors. The precise reaction channels and their associated rates are specified in Eqs. (1)–(4).

For a given lattice configuration $\Sigma = \{\sigma_i\}$, all admissible reactions define a set of independent reaction channels indexed by α , each characterized by a propensity (rate) $p_\alpha(\Sigma)$. The total propensity is given by

$$P(\Sigma) = \sum_{\alpha} p_{\alpha}(\Sigma), \quad (\text{A1})$$

which determines both the waiting time to the next reaction event and the relative probability with which each reaction channel is selected. The resulting continuous-time Markov process is simulated using the Gillespie algorithm summarized in Algorithm 1.

Initial conditions are generated by fixing an initial vacancy density ρ_0 . Each lattice site is independently assigned to be vacant with probability ρ_0 , and otherwise occupied by an A or a B agent with equal probability. We verified that the choice of ρ_0 does not affect steady-state observables. Without loss of generality, the death rate is fixed to $\mu = 1$, thereby setting the unit of time; all other rates are expressed relative to this value.

To ensure convergence to the steady state, each simulation is first evolved for $n_{\text{therm}} = 180N$ reaction events. Observables are subsequently recorded every $18N$ reaction events. We verified that this sampling interval is sufficient to eliminate temporal correlations between successive measurements. Expectation values are obtained by averaging over many statistically independent steady-state configurations. For the finite-size-scaling analysis, simulations were parallelized over 128 CPU cores, each providing an independent realization

ALGORITHM 1. Gillespie algorithm for lattice reaction dynamics.

-
- 1: Initialize lattice configuration Σ and set $t = 0$
 - 2: **while** simulation not terminated **do**
 - 3: Enumerate all admissible reaction channels $\{\alpha\}$ and compute propensities $p_{\alpha}(\Sigma)$
 - 4: Compute total propensity $P = \sum_{\alpha} p_{\alpha}$
 - 5: Draw $r_1, r_2 \sim \mathcal{U}(0, 1)$
 - 6: Advance time: $t \leftarrow t + \frac{1}{P} \ln(1/r_1)$
 - 7: Select reaction α such that

$$\sum_{\beta < \alpha} p_{\beta} < r_2 P \leq \sum_{\beta \leq \alpha} p_{\beta}$$
 - 8: Update configuration $\Sigma \leftarrow \Sigma + \Delta \Sigma_{\alpha}$
 - 9: Record observables if required
-

evolving according to Algorithm 1. All code created to run the simulation and analyze the results are published in an accompanying GitHub repository [55].

APPENDIX B: COMPARING MEAN-FIELD TO STOCHASTIC SOLUTION

To investigate the validity of the mean-field results, we perform numerical simulations as outlined in Appendix A. We focus on two observables, to distinguish the different phases of the model, namely, the net overdensity, explicitly defined on the lattice as

$$\hat{M} = \frac{1}{N} \sum_i \sigma_i, \quad (\text{B1})$$

where the sum runs over all lattice sites and $\sigma_i = 1, -1, 0$ if lattice site i is in state A_i, B_i , or 0_i , respectively. This observable should accurately distinguish between the symmetric and asymmetric phases and therefore serves as a useful order parameter. Next, we need an observable that accurately tracks the segregation on the lattice. Note that one can have segregation with no net magnetization. For this, we define something akin to the total internal energy of the lattice,

$$\hat{E} = \frac{1}{8N} \sum_{i,j} \sigma_i C^{ij} \sigma_j, \quad (\text{B2})$$

where C^{ij} is the adjacency matrix. This should be an accurate proxy for segregation as it directly measures how many distinct agents are “living” next to each other.

We perform simulations of the full stochastic model. To investigate the transition between the two phases, we initially set $r_c = \frac{r_s}{2}$, with $r_s = 0.2$. Then, after we have simulated $\mathcal{N}_{\text{sim}} = 5 \times 10^6$ distinct reactions on our lattice, we change the value of r_c to $r_c = 2.5 > r_c^*$. We do this for an $N = 50 \times 50$ size lattice, with homogeneous initial conditions. In Fig. 8, we show the results of the average density of the different species in the lattice, and the average magnetization as defined in Eq. (B1). It is clear that, before the quench, the system is in the symmetric phase (no net magnetization). After the quench, the species A starts to dominate and the system becomes ordered, agreeing with the (second-order) mean-field results.

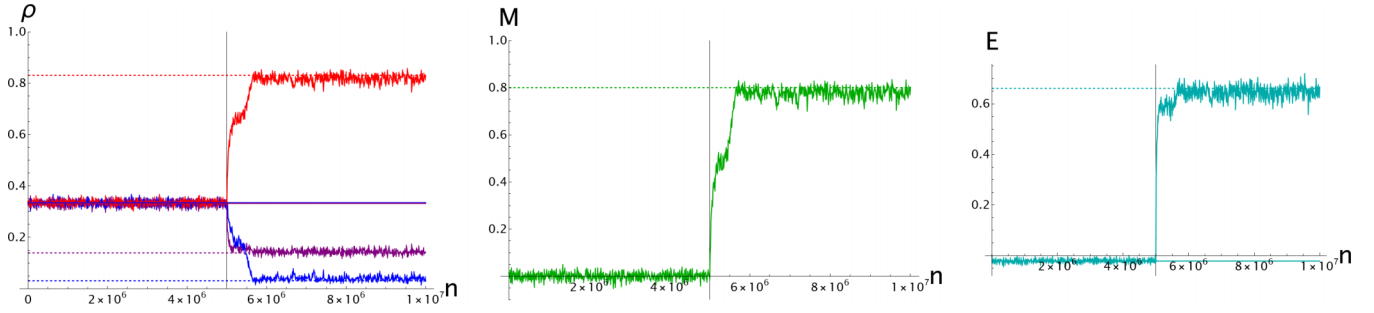


FIG. 8. The average densities (left), net magnetization (middle), and the Ising energy (right) for the quench experiment described in the text. The mean-field expectation (second order) in the symmetric (full) and asymmetric (dashed) phases is also shown. The gridline correspond to the step at which we change the value of r_c .

In the right panel of Fig. 8, we show the energy as defined in Eq. (B2) for this same experiment. We see that initially the energy is slightly negative when $r_c < r_s$. This signals a different type of phase, which we refer to as antisequesting. In this phase, agents are more likely to live next to distinct neighbors, signaling a situation in which a neighborhood is very homogeneously mixed. This behavior is only visible in the second-order mean-field approach (see Appendix C). Note that the transition from negative to positive energy at $r_c = r_s$ is not critical. Nonetheless, we argue that this crossover deserves some attention because of the socioeconomic context of this model.

APPENDIX C: FOCK SPACE FORMULATION OF THE MODEL

In this Appendix, we will work in the Fock space formulation of the CRN, as described in, for instance, Refs. [56–58]. This allows us to derive the first- and second-order mean-field equations mentioned in the main text explicitly. We start with defining the master equation for the CRN. The state of the system is a vector $|\rho(t)\rangle$, which is expanded as

$$|\rho(t)\rangle = \sum_{\Sigma} P(\Sigma, t) \prod_i (\hat{a}_i^\dagger)^{n_i^A} (\hat{b}_i^\dagger)^{n_i^B} (\hat{v}_i^\dagger)^{n_i^0} |0\rangle, \quad (C1)$$

where $P(\Sigma, t)$ gives the probability of observing the microscopic configuration $\Sigma = \{\sigma_i\}$ of lattice sites, each of which may be in any of the three states $A, B, 0$. We denote $\hat{a}_i, \hat{a}_i^\dagger, \hat{b}_i, \hat{b}_i^\dagger$, and $\hat{v}_i, \hat{v}_i^\dagger$ as the annihilation (creation) operators for A_i, B_i , and 0_i particles, respectively. They satisfy commutation relations:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} = [\hat{b}_i, \hat{b}_j^\dagger] = [\hat{v}_i, \hat{v}_j^\dagger]. \quad (C2)$$

The master equation is given as

$$\partial_t |\rho(t)\rangle = \hat{H} |\rho(t)\rangle, \quad (C3)$$

where the infinitesimal generator $\hat{H}$ is split into three parts:

$$\hat{H} = \hat{H}_0(T) + \hat{H}_c(T) + \hat{H}_s(T). \quad (C4)$$

Each term represents a set of reactions; the first corresponds to the birth/death process at a given level of T (we again set

$\lambda = 1$ by rescaling time):

$$\hat{H}_0 = \sum_i \{T(\hat{v}_i^\dagger - \hat{a}_i^\dagger)\hat{a}_i + T(\hat{v}_i^\dagger - \hat{b}_i^\dagger)\hat{b}_i + (\hat{a}_i^\dagger + \hat{b}_i^\dagger - 2\hat{v}_i^\dagger)\hat{v}_i\}. \quad (C5)$$

The generator $\hat{H}_c$ represents the cloning reactions:

$$\begin{aligned} \hat{H}_c = r_c \sum_{\langle i, j \rangle} & \{T(\hat{v}_i^\dagger \hat{a}_j^\dagger - \hat{b}_i^\dagger \hat{a}_j^\dagger)\hat{b}_i \hat{a}_j + T(\hat{v}_i^\dagger \hat{b}_j^\dagger - \hat{a}_i^\dagger \hat{b}_j^\dagger)\hat{a}_i \hat{b}_j \\ & + (\hat{a}_i^\dagger \hat{a}_j^\dagger - \hat{v}_i^\dagger \hat{a}_j^\dagger)\hat{v}_i \hat{a}_j + (\hat{b}_i^\dagger \hat{b}_j^\dagger - \hat{v}_i^\dagger \hat{b}_j^\dagger)\hat{v}_i \hat{b}_j\}. \end{aligned} \quad (C6)$$

Finally, the generator $\hat{H}_s$ represents the splitting reactions:

$$\begin{aligned} \hat{H}_s = r_s \sum_{\langle i, j \rangle} & \{T(\hat{v}_i^\dagger \hat{b}_j^\dagger - \hat{b}_i^\dagger \hat{b}_j^\dagger)\hat{b}_i \hat{b}_j + T(\hat{v}_i^\dagger \hat{a}_j^\dagger - \hat{a}_i^\dagger \hat{a}_j^\dagger)\hat{a}_i \hat{a}_j \\ & + (\hat{b}_i^\dagger \hat{a}_j^\dagger - \hat{v}_i^\dagger \hat{a}_j^\dagger)\hat{v}_i \hat{a}_j + (\hat{a}_i^\dagger \hat{b}_j^\dagger - \hat{v}_i^\dagger \hat{b}_j^\dagger)\hat{v}_i \hat{b}_j\}. \end{aligned} \quad (C7)$$

Here, $\langle i, j \rangle$ denotes nearest neighbors (within the Moore neighborhoods).

1. Mean-field equations

From the master equation (C3), one may derive equations for the expectation value for node i to be in the state A , denoted as $\rho_i^A(t) = \langle 1 | \hat{a}_i^\dagger \hat{a}_i | \rho(t) \rangle$. The equation is easily derived using that $\partial_t \rho_i^A = \langle 1 | \hat{a}_i \hat{H} | \rho(t) \rangle$, together with the commutation relations (C2):

$$\begin{aligned} \frac{d}{dt} \rho_i^A &= \rho_i^0 - T \rho_i^A + r_c \sum_{j \in \partial i} (\langle \hat{v}_i \hat{a}_j \rangle - T \langle \hat{a}_i \hat{b}_j \rangle) \\ &+ r_s \sum_{j \in \partial i} (\langle \hat{v}_i \hat{b}_j \rangle - T \langle \hat{a}_i \hat{a}_j \rangle). \end{aligned} \quad (C8a)$$

Here, ∂i is the local neighborhood of node i and $\langle \hat{v}_i \hat{a}_j \rangle$ denotes the probability that nodes i and j form a (0A) pair (and likewise for the other two-point correlators). Using similar computations for ρ_i^B and ρ_i^0 gives

$$\begin{aligned} \frac{d}{dt} \rho_i^B &= \rho_i^0 - T \rho_i^B + r_c \sum_{j \in \partial i} (\langle \hat{v}_i \hat{b}_j \rangle - T \langle \hat{b}_i \hat{a}_j \rangle) \\ &+ r_s \sum_{j \in \partial i} (\langle \hat{v}_i \hat{a}_j \rangle - T \langle \hat{b}_i \hat{b}_j \rangle), \end{aligned} \quad (C8b)$$

$$\frac{d}{dt}\rho_i^0 = T(\rho_i^A + \rho_i^B) - 2\rho_i^0 + 8r_s(\rho^0\rho^B - T\rho^A\rho^A), \quad (C9a)$$

$$+ r_c \sum_{j \in \partial i} (T\langle \hat{b}_i \hat{a}_j \rangle + T\langle \hat{a}_i \hat{b}_j \rangle - \langle \hat{v}_i \hat{a}_j \rangle - \langle \hat{v}_i \hat{b}_j \rangle) + 8r_s(\rho^0\rho^A - T\rho^B\rho^B), \quad (C9b)$$

$$+ r_s \sum_{j \in \partial i} (T\langle \hat{b}_i \hat{b}_j \rangle + T\langle \hat{a}_i \hat{a}_j \rangle - \langle \hat{v}_i \hat{a}_j \rangle - \langle \hat{v}_i \hat{b}_j \rangle). \quad (C8c)$$

$$\frac{d}{dt}\rho^0 = T(\rho^A + \rho^B) - 2\rho^0 + 8r_c(2T\rho^A\rho^B - \rho^0(\rho^A + \rho^B)) + 8r_s(T(\rho^A\rho^A + \rho^B\rho^B) - \rho^0(\rho^A + \rho^B)). \quad (C9c)$$

These equations are exact, but do not form a closed system. In order to arrive at a single set of closed ordinary differential equations, we make the mean-field assumption that each site is statistically independent and identically distributed. If this were true, we may write any pair expectation values as the product of the single nodes expectation values, i.e., $\langle \hat{v}_i \hat{a}_j \rangle = \rho_i^0 \rho_j^A$. Furthermore, we can invoke translational invariance by supposing that each lattice site is identical, such that the labels i, j become irrelevant. In that case, we arrive at the mean-field equations for the probability of any site to be in the state A, B , or 0 :

$$\frac{d}{dt}\rho^A = \rho^0 - T\rho^A + 8r_c(\rho^0\rho^A - T\rho^A\rho^B)$$

These formulas are equivalent to Eq. (5).

2. Second-order mean-field equations

In the previous subsection, we closed the equations at the first order, but a better mean-field approximation can be obtained when closing the equations at the second order [27,28]. To this end, we first formulate the equations for the second moments (or the two-point correlation functions), which represent the joint probabilities of forming a pair of specific agents. These equations are expressed in terms of pairs and triplets on the lattice:

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}_i \hat{a}_j \rangle &= (1 + r_c)(\langle \hat{a}_i \hat{v}_j \rangle + \langle \hat{v}_i \hat{a}_j \rangle) - 2T(1 + r_s)\langle \hat{a}_i \hat{a}_j \rangle + r_c \left[\sum_{k \in \partial i} \{ \langle \hat{v}_i \hat{a}_k \hat{a}_j \rangle - T\langle \hat{a}_i \hat{b}_k \hat{a}_j \rangle \} + \sum_{k \in \partial j} \{ \langle \hat{a}_i \hat{a}_k \hat{v}_j \rangle - T\langle \hat{a}_i \hat{b}_k \hat{a}_j \rangle \} \right] \\ &+ r_s \left[\sum_{k \in \partial i} \{ \langle \hat{v}_i \hat{b}_k \hat{a}_j \rangle - T\langle \hat{a}_i \hat{a}_k \hat{a}_j \rangle \} + \sum_{k \in \partial j} \{ \langle \hat{a}_i \hat{b}_k \hat{v}_j \rangle - T\langle \hat{a}_i \hat{a}_k \hat{a}_j \rangle \} \right], \end{aligned} \quad (C10a)$$

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}_i \hat{v}_j \rangle &= \langle \hat{v}_i \hat{v}_j \rangle + T(1 + r_c)\langle \hat{a}_i \hat{b}_j \rangle + T(1 + r_s)\langle \hat{a}_i \hat{a}_j \rangle - (T + 2 + r_c + r_s)\langle \hat{a}_i \hat{v}_j \rangle \\ &+ \sum_{k \in \partial i} \{ r_c \langle \hat{v}_i \hat{a}_k \hat{v}_j \rangle + r_s \langle \hat{v}_i \hat{b}_k \hat{v}_j \rangle - Tr_c \langle \hat{a}_i \hat{b}_k \hat{v}_j \rangle - Tr_s \langle \hat{a}_i \hat{a}_k \hat{v}_j \rangle \} \\ &+ \sum_{k \in \partial j} \{ T[r_c \langle \hat{a}_i \hat{b}_k \hat{a}_j \rangle + r_c \langle \hat{a}_i \hat{a}_k \hat{b}_j \rangle + r_s \langle \hat{a}_i \hat{a}_k \hat{a}_j \rangle + r_s \langle \hat{a}_i \hat{b}_k \hat{b}_j \rangle] - (r_s + r_c)[\langle \hat{a}_i \hat{a}_k \hat{v}_j \rangle + \langle \hat{a}_i \hat{b}_k \hat{v}_j \rangle] \}, \end{aligned} \quad (C10b)$$

$$\begin{aligned} \frac{d}{dt}\langle \hat{v}_i \hat{v}_j \rangle &= T[\langle \hat{a}_i \hat{v}_j \rangle + \langle \hat{b}_i \hat{v}_j \rangle + \langle \hat{v}_i \hat{a}_j \rangle + \langle \hat{v}_i \hat{b}_j \rangle] - 4\langle \hat{v}_i \hat{v}_j \rangle \\ &+ \sum_{k \in \partial i} \{ T[r_c \langle \hat{a}_i \hat{b}_k \hat{v}_j \rangle + r_c \langle \hat{b}_i \hat{a}_k \hat{v}_j \rangle + r_s \langle \hat{a}_i \hat{a}_k \hat{v}_j \rangle + r_s \langle \hat{b}_i \hat{b}_k \hat{v}_j \rangle] - (r_c + r_s)[\langle \hat{v}_i \hat{a}_k \hat{v}_j \rangle + \langle \hat{v}_i \hat{b}_k \hat{v}_j \rangle] \} \\ &+ \sum_{k \in \partial j} \{ T[r_c \langle \hat{v}_i \hat{b}_k \hat{a}_j \rangle + r_c \langle \hat{v}_i \hat{a}_k \hat{b}_j \rangle + r_s \langle \hat{v}_i \hat{a}_k \hat{a}_j \rangle + r_s \langle \hat{v}_i \hat{b}_k \hat{b}_j \rangle] - (r_c + r_s)[\langle \hat{v}_i \hat{a}_k \hat{v}_j \rangle + \langle \hat{v}_i \hat{b}_k \hat{v}_j \rangle] \}, \end{aligned} \quad (C10c)$$

$$\begin{aligned} \frac{d}{dt}\langle \hat{a}_i \hat{b}_j \rangle &= (1 + r_s)(\langle \hat{a}_i \hat{v}_j \rangle + \langle \hat{v}_i \hat{b}_j \rangle) - 2T(1 + r_c)\langle \hat{a}_i \hat{b}_j \rangle + r_c \left[\sum_{k \in \partial i} \{ \langle \hat{v}_i \hat{a}_k \hat{b}_j \rangle - T\langle \hat{a}_i \hat{b}_k \hat{b}_j \rangle \} + \sum_{k \in \partial j} \{ \langle \hat{a}_i \hat{b}_k \hat{v}_j \rangle - T\langle \hat{a}_i \hat{a}_k \hat{b}_j \rangle \} \right] \\ &+ r_s \left[\sum_{k \in \partial i} \{ \langle \hat{v}_i \hat{b}_k \hat{b}_j \rangle - T\langle \hat{a}_i \hat{a}_k \hat{b}_j \rangle \} + \sum_{k \in \partial j} \{ \langle \hat{a}_i \hat{a}_k \hat{v}_j \rangle - T\langle \hat{a}_i \hat{b}_k \hat{b}_j \rangle \} \right]. \end{aligned} \quad (C10d)$$

The equations for $\langle \hat{b}_i \hat{b}_j \rangle, \langle \hat{b}_i \hat{v}_j \rangle$ may be obtained from these expressions by replacing $\hat{a} \leftrightarrow \hat{b}$. Likewise, expressions for $\langle \hat{v}_i \hat{a}_j \rangle, \langle \hat{v}_i \hat{b}_j \rangle$, and $\langle \hat{b}_i \hat{a}_j \rangle$ are obtained by swapping lattice labels $i \leftrightarrow j$. In addition to these equations, the finite

volume constraint imposes that the sum of all possible two-point correlation functions is 1; hence, we may always choose to express one two-point correlation function in terms of the others. Together with Eq. (C8), these equations provide the

exact description of the one- and two-point correlation functions of the system. However, to form a closed set of equations, one would have to approximate the three-point correlation functions in terms of the one- and two-point correlators.

Before closing the set of equations, we note that owing to our choice of Moore neighborhoods, there are two statistically distinct geometries possible for the pair correlation functions: The pair could form a straight edge (if they are left, right, up, or down neighbors), or the pair could form a diagonal edge (if they are neighbors connecting to the four corners of the Moore neighborhood). These choices are distinct, as straight edges are part of four closed triangles, while diagonal edges are part of only two closed triangles. So for each possible pair of nodes, we separate the correlation functions into $\langle \rangle_s$ if they form up, down, left, or right neighbors and $\langle \rangle_d$ if they form diagonal neighbors.

When approximating the three-point functions to close the set of equations, we distinguish between two further scenarios. First, the triplet i, j, k may form an open triangle, such that one node is connected to the two others, but its neighbors are not connected to each other. The second case is when the triplet i, j, k forms a closed triangle and all nodes are connected to each other. The rules for closing the moment equations in these two cases are detailed as examples (1) and (2) in Ref. [27]. Specifically, for an open triangle with j being central node, we approximate the third moment as

$$\langle \hat{x}_i \hat{y}_j \hat{z}_k \rangle \sim \frac{\langle \hat{x}_i \hat{y}_j \rangle \langle \hat{y}_j \hat{z}_k \rangle}{\langle \hat{y}_j \rangle}. \quad (\text{C11})$$

When instead nodes i, j, k form a closed triangle (such that all nodes are connected to each other), we approximate the correlation function as

$$\langle \hat{x}_i \hat{y}_j \hat{z}_k \rangle \sim \frac{\langle \hat{x}_i \hat{y}_j \rangle \langle \hat{y}_j \hat{z}_k \rangle \langle \hat{z}_k \hat{x}_i \rangle}{\langle \hat{x}_i \rangle \langle \hat{y}_j \rangle \langle \hat{z}_k \rangle}, \quad (\text{C12})$$

which is a common closure scheme in epidemic models [59].

Combining the two distinct geometries for the two-point functions (straight or diagonal) with the two closure schemes for three-point functions (open or close triangles) leads to the following replacement rules for the sum over neighbors k of any node pair i and j (see also Fig. 9). Whenever the nodes i and j are straight neighbors, the node $k \in \partial i$ can form a closed triangle with i and j in four ways and an open triangle in three ways. Keeping track of the straight and diagonal connections leads to

$$\begin{aligned} \sum_{k \in \partial i} \langle \hat{x}_i \hat{y}_j \hat{z}_k \rangle &= 2 \frac{\langle \hat{x} \hat{y} \rangle_s \langle \hat{x} \hat{z} \rangle_d}{\langle \hat{x} \rangle} + \frac{\langle \hat{x} \hat{y} \rangle_s \langle \hat{x} \hat{z} \rangle_s}{\langle \hat{x} \rangle} \\ &+ 2 \frac{\langle \hat{x} \hat{y} \rangle_s (\langle \hat{y} \hat{z} \rangle_s \langle \hat{x} \hat{z} \rangle_d + \langle \hat{y} \hat{z} \rangle_d \langle \hat{x} \hat{z} \rangle_s)}{\langle \hat{x} \rangle \langle \hat{y} \rangle \langle \hat{z} \rangle}. \end{aligned} \quad (\text{C13})$$

However, if the node pair i and j are diagonal neighbors, the node $k \in \partial i$ forms a closed triangle with i and j in only two ways, and creates an open triangle in five ways. The replacement rule is then

$$\begin{aligned} \sum_{k \in \partial i} \langle \hat{x}_i \hat{y}_j \hat{z}_k \rangle &= 2 \frac{\langle \hat{x} \hat{y} \rangle_d \langle \hat{x} \hat{z} \rangle_s}{\langle \hat{x} \rangle} + 3 \frac{\langle \hat{x} \hat{y} \rangle_d \langle \hat{x} \hat{z} \rangle_d}{\langle \hat{x} \rangle} \\ &+ 2 \frac{\langle \hat{x} \hat{y} \rangle_d \langle \hat{y} \hat{z} \rangle_s \langle \hat{x} \hat{z} \rangle_s}{\langle \hat{x} \rangle \langle \hat{y} \rangle \langle \hat{z} \rangle}. \end{aligned} \quad (\text{C14})$$

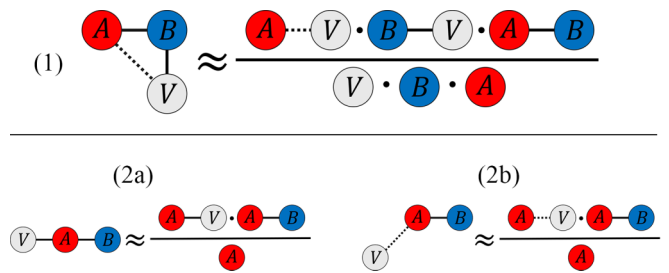


FIG. 9. A sketch illustrating how we perform moment closure of three-point correlation functions. (1) If the sites in the three-point function form a closed triangle, it consists of two straight edges (full black line) and one diagonal edge (dashed black line). The expectation value is approximated as in Eq. (C12). (2) If on the other hand the three-point function is open, we close as in Eq. (C11). An open triangle can consist of two straight edges [panel (2a)], one diagonal and one straight edge [panel (2b)] or two diagonal edges (not shown).

Substituting these replacements rules in Eq. (C10) and separating out the straight and diagonal edges is straightforward albeit tedious. It results in a set of 15 equations; three for the one point functions $\langle \hat{a} \rangle, \langle \hat{b} \rangle, \langle \hat{v} \rangle$, six for all possible straight edges $\langle \hat{a} \hat{a} \rangle_s, \langle \hat{a} \hat{b} \rangle_s, \langle \hat{a} \hat{v} \rangle_s, \langle \hat{b} \hat{b} \rangle_s, \langle \hat{b} \hat{v} \rangle_s, \langle \hat{v} \hat{v} \rangle_s$, and six for the diagonal edges. Among these equations, three are solved by imposing the volume exclusion constraints $\langle \hat{v} \rangle = 1 - \langle \hat{a} \rangle - \langle \hat{b} \rangle$, $\langle \hat{v} \hat{v} \rangle_s = 1 - \langle \hat{a} \hat{a} \rangle_s - \langle \hat{b} \hat{b} \rangle_s - 2 \langle \hat{a} \hat{b} \rangle_s - 2 \langle \hat{a} \hat{v} \rangle_s - 2 \langle \hat{b} \hat{v} \rangle_s$, and likewise for the diagonal edges. The resulting set of equations are solved by numerical integration, using the first-order mean-field solution as initial conditions. This allows us to obtain better estimates of the absolute magnetization and the energy, as shown in Fig. 10. The code for numerical integration of these equations is included in the GitHub repository [55].

APPENDIX D: FINITE-SIZE ROBUSTNESS ANALYSIS

To assess the sensitivity of the finite-size-scaling analysis to the range of lattice sizes included in the fits, we performed additional robustness checks for the cases $r_s = 0.1$ and $r_s = 0$. In the main analysis, the critical exponents were extracted from simulations with linear system sizes $L = 20, 25, 30, 35, 40, 45$. We extended these datasets by including additional simulations at $L = 50, 55, 60$, and repeated the full finite-size-scaling procedure.

For each fitting window, the critical point r_c^* was reestimated from the Binder cumulant crossing using the same variance-minimization procedure as in the main text. The observables were then interpolated to the corresponding window-specific value of r_c^* , and the exponent ratios were obtained from log-log fits at criticality. The correlation-length exponent ν was determined independently by optimizing the collapse of the magnetization curves. Note that values reported here are obtained from the analysis on the simulated data, without bootstrapping for estimating uncertainties. This may result in deviations from the values reported in Table I.

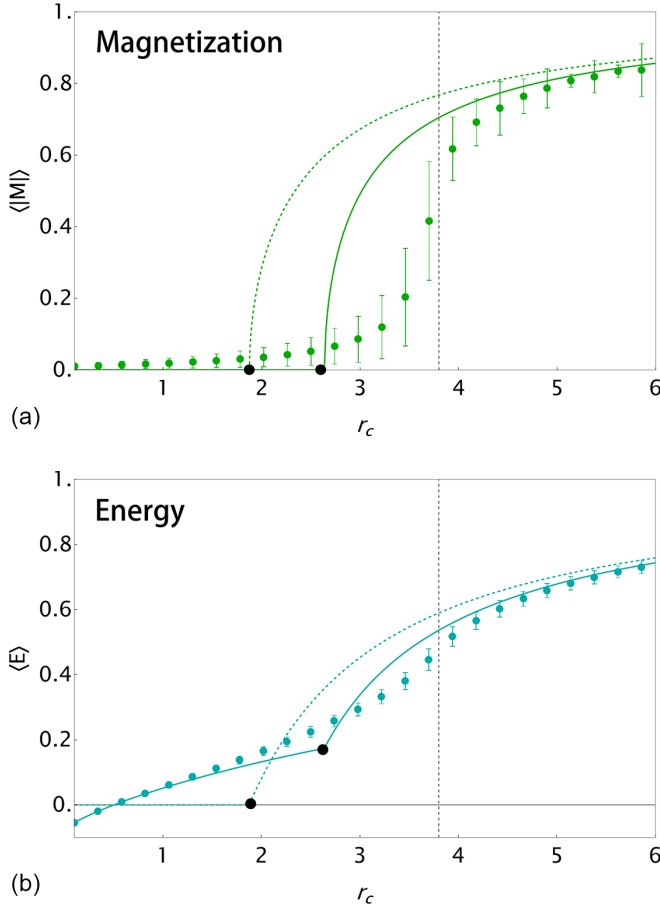


FIG. 10. Comparison in expectation value for the absolute magnetization per site [panel (a)] and the energy per site [panel (b)] between the moment closure of the mean-field equations at first (dashed line) and second (full line) order. Plots shown here are for $r_s = 0.5$ and $T = 1$. The scatter data correspond to results from the stochastic model, averaged over 10^4 independent grids with grid size $N = 50 \times 50$. We indicate the critical value of r_c^* with the dashed gridline.

1. Robustness at $r_s = 0.1$

For $r_s = 0.1$, the comparison between the original window and the full extended window is summarized as follows. Using the original range $L = 20$ –45, we find

$$\begin{aligned} r_c^* &= 1.386\,95, & \beta/\nu &= -0.130\,07, & \gamma/\nu &= 1.739\,55, \\ \alpha/\nu &= 0.264\,63, & \epsilon/\nu &= -1.832\,56, & \nu &= 1.031\,55. \end{aligned} \quad (\text{D1})$$

Including the additional larger system sizes up to $L = 60$ gives

$$\begin{aligned} r_c^* &= 1.389\,88, & \beta/\nu &= -0.128\,24, & \gamma/\nu &= 1.744\,53, \\ \alpha/\nu &= 0.262\,87, & \epsilon/\nu &= -1.828\,92, & \nu &= 1.060\,547. \end{aligned} \quad (\text{D2})$$

The corresponding relative changes are small: approximately 1.4% for β/ν , 0.3% for γ/ν , 0.7% for α/ν , and 2.8% for ν . Thus, extending the upper end of the system-size range does not significantly alter the exponent estimates. In particular, the enlarged dataset does not show a systematic drift of the exponents toward the two-dimensional Ising values.

A more detailed overview is given in Table II. We report both an upper-cutoff scan, in which the smallest system size is kept fixed while progressively increasing $L_{\max}$, and a lower-cutoff scan, in which the largest available system size is kept fixed while progressively increasing $L_{\min}$.

The upper-cutoff scan shows that the exponent ratios are stable when larger system sizes are added. Across these fits, β/ν remains close to 0.13, γ/ν remains close to 1.74, and α/ν remains positive and close to 0.26–0.28. This indicates that the previously reported exponents are not an artifact of having stopped the analysis at $L = 45$.

The lower-cutoff scan shows a stronger drift once the smallest sizes are removed, especially for $L_{\min} \geq 35$. However, these restricted fits are based on a much narrower range of system sizes. For example, the window $L = 45$ –60 contains only four data points and has a small logarithmic lever arm for determining power-law slopes. In addition, the Binder estimate of r_c^* shifts noticeably in these restricted windows, from $r_c^* \simeq 1.392$ for $L_{\min} = 30$ to $r_c^* \simeq 1.407$ for $L_{\min} = 45$. Since the observables are evaluated at the window-specific critical point, this drift in r_c^* feeds directly into the fitted exponent ratios.

2. Robustness at $r_s = 0$

We performed the same analysis for $r_s = 0$, which is the case shown explicitly in the Binder-cumulant crossing and scaling-collapse figures in the main text. Using the original range $L = 20$ –45, we find

$$\begin{aligned} r_c^* &= 0.787\,39, & \beta/\nu &= -0.130\,13, & \gamma/\nu &= 1.734\,74, \\ \alpha/\nu &= 0.271\,61, & \epsilon/\nu &= -1.826\,18, & \nu &= 1.096\,89. \end{aligned} \quad (\text{D3})$$

Including the additional larger system sizes up to $L = 60$ gives

$$\begin{aligned} r_c^* &= 0.789\,48, & \beta/\nu &= -0.125\,49, & \gamma/\nu &= 1.749\,13, \\ \alpha/\nu &= 0.270\,14, & \epsilon/\nu &= -1.830\,38, & \nu &= 1.102\,83. \end{aligned} \quad (\text{D4})$$

The corresponding changes are again modest: r_c^* shifts by approximately 0.27%, while the exponent ratios change by approximately 3.6% for β/ν , 0.8% for γ/ν , 0.5% for α/ν , and 0.5% for ν . The vacancy-fluctuation exponent ratio is similarly stable. Thus, the inclusion of $L = 50, 55, 60$ does not qualitatively change the finite-size scaling estimates for $r_s = 0$. A detailed overview of the window dependence for $r_s = 0$ is given in Table III.

The behavior at $r_s = 0$ is consistent with the conclusions drawn from $r_s = 0.1$. In the upper-cutoff scan, adding larger system sizes leads only to a weak drift of the exponent estimates. The estimate of β/ν decreases slightly when the full range up to $L = 60$ is included, while γ/ν increases slightly; however, both remain close to the values obtained from the original fitting window. The heat-capacity exponent ratio remains positive and close to $\alpha/\nu \simeq 0.27$, again showing no indication of crossing over to a logarithmic divergence over the accessible size range.

The lower-cutoff scan for $r_s = 0$ displays a smoother drift than the corresponding scan at $r_s = 0.1$. As the smallest system sizes are removed, β/ν decreases from 0.123 at

TABLE II. Finite-size robustness of the exponent estimates for $r_s = 0.1$. The upper-cutoff scan keeps the lower end of the fitting range fixed and progressively includes larger system sizes. The lower-cutoff scan keeps the largest available size fixed and progressively removes smaller system sizes. Each row uses a window-specific estimate of r_c^* .

Window type	L window	r_c^*	β/ν	γ/ν	α/ν	ν
Original	20–45	1.386 946	−0.130 075	1.739 547	0.264 627	1.031 553
Full	20–60	1.389 884	−0.128 239	1.744 529	0.262 874	1.060 547
Upper cutoff	20–35	1.386 026	−0.129 605	1.742 797	0.275 581	1.048 964
Upper cutoff	20–40	1.384 625	−0.132 275	1.735 856	0.270 319	1.022 459
Upper cutoff	20–50	1.387 805	−0.129 058	1.742 665	0.263 814	1.042 221
Upper cutoff	20–55	1.388 086	−0.131 328	1.739 259	0.262 499	1.044 124
Lower cutoff	25–60	1.391 114	−0.128 173	1.745 158	0.258 669	1.069 071
Lower cutoff	30–60	1.392 337	−0.127 948	1.745 961	0.259 500	1.081 009
Lower cutoff	35–60	1.404 472	−0.096 191	1.788 940	0.265 058	1.191 127
Lower cutoff	40–60	1.406 022	−0.095 630	1.789 244	0.266 174	1.227 072
Lower cutoff	45–60	1.407 224	−0.093 081	1.795 524	0.273 589	1.243 050

$L_{\min} = 25$ to 0.111 at $L_{\min} = 45$, while γ/ν increases from 1.755 to 1.766. The estimate of ν also increases mildly, reaching $\nu \simeq 1.14$ for the narrowest window. As before, these restricted windows have a reduced logarithmic lever arm and should therefore be interpreted primarily as a diagnostic of residual fitting-window dependence.

Taken together, the $r_s = 0$ and $r_s = 0.1$ robustness checks support the same conclusion. The most relevant comparison

is the extension of the original window $L = 20$ –45 to the full window $L = 20$ –60, for which the exponent estimates change only weakly. Within the enlarged system-size range, we find no evidence that the measured critical behavior drifts toward the standard two-dimensional Ising-universality class. In particular, the heat-capacity scaling remains compatible with a positive power-law exponent rather than the logarithmic behavior expected for the two-dimensional Ising model.

TABLE III. Finite-size robustness of the exponent estimates for $r_s = 0$. The upper-cutoff scan keeps the lower end of the fitting range fixed and progressively includes larger system sizes. The lower-cutoff scan keeps the largest available size fixed and progressively removes smaller system sizes. Each row uses a window-specific estimate of r_c^* .

Window type	L window	r_c^*	β/ν	γ/ν	α/ν	ν
Original	20–45	0.787 393	−0.130 129	1.734 736	0.271 607	1.096 894
Full	20–60	0.789 478	−0.125 486	1.749 127	0.270 144	1.102 832
Upper cutoff	20–35	0.783 618	−0.136 797	1.721 781	0.278 827	1.075 518
Upper cutoff	20–40	0.785 211	−0.134 085	1.725 577	0.275 139	1.086 573
Upper cutoff	20–50	0.788 744	−0.126 559	1.744 422	0.273 748	1.103 205
Upper cutoff	20–55	0.788 747	−0.126 691	1.746 169	0.272 193	1.092 804
Lower cutoff	25–60	0.790 744	−0.123 081	1.754 983	0.267 911	1.102 954
Lower cutoff	30–60	0.791 287	−0.121 284	1.757 778	0.264 650	1.105 099
Lower cutoff	35–60	0.792 204	−0.117 930	1.763 339	0.266 301	1.108 719
Lower cutoff	40–60	0.793 413	−0.112 685	1.768 435	0.267 661	1.113 491
Lower cutoff	45–60	0.792 703	−0.110 769	1.766 294	0.263 455	1.142 352

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