

Bipartite entanglement measures for anyonic systems

Wenhao Ye,¹ Li You^{1,2,3,4,*} and Cheng-Qian Xu^{1,5,†}¹State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China²Frontier Science Center for Quantum Information, Beijing, China³Beijing Academy of Quantum Information Sciences, Beijing 100193, China⁴Hefei National Laboratory, Hefei, Anhui 230088, China⁵School of Physics and Optoelectronic Engineering, Hainan University, Haikou 570228, China

(Received 25 August 2025; accepted 13 March 2026; published 10 April 2026)

As promising candidates for realizing fault-tolerant quantum computing, systems composed of anyons feature nontensor product state space due to their distinctive fusion rules, resulting in entanglement properties that differ fundamentally from those of conventional quantum systems. However, a quantitative characterization of entanglement in such systems remains elusive. In this Letter, we address this issue in bipartite settings using the framework of quantum resource theory. We propose three measures that quantify total entanglement, conventional entanglement, and anyonic charge entanglement (ACE) of a bipartite anyonic state, respectively. Our central result is summarized as a theorem: Total entanglement is composed of ACE and conventional entanglement. Geometrically, ACE measures the distance between the state and its charge-decorrelated version, while conventional entanglement is the minimal distance between the latter and the set of separable states, characterizing Bell nonlocality in the conventional sense. Our result confirms the long-standing intuition that ACE stems from the reduced dimensionality of the separable state space due to superselection and fusion constraints. Furthermore, we prove that ACE is equivalent to a previously proposed probe termed the entropy of ACE, thereby providing rigorous theoretical support for the latter's monotonicity as an entanglement measure and extending the known correspondence between the geometric and operational characterizations of bipartite correlations to anyonic systems.

DOI: [10.1103/4nd3-k6rw](https://doi.org/10.1103/4nd3-k6rw)

Introduction. Entanglement is a fundamental resource [1,2] that underlies a wide range of applications in quantum technologies, including quantum communication [3,4], computation [5–7], and metrology [8–10]. Furthermore, entanglement theory has found broad applications in other areas of physics—most notably in condensed matter physics [11]—where it has provided new insights into several phenomena in many-body systems [12–14]. Characterizing and quantifying entanglement is crucial from both practical and fundamental perspectives.

While significant efforts have been devoted to quantifying entanglement in systems whose state space admits a tensor product structure [15–18], characterizing entanglement in systems that lack such a property [19,20] remains largely unexplored. A prominent example is anyons [21,22], quasiparticle excitations that emerge in two-dimensional systems with topologically ordered phases [11]. Anyons not only serve as the building blocks of topological quantum com-

putation [23], where information is encoded and processed in the fusion space under anyonic superselection constraints, but also play a crucial role in capturing certain long-range entanglement, known as topological entanglement entropy (TEE) [24,25]. The state space of anyonic systems possesses a peculiar structure, shaped by two defining features: anyonic superselection rules [26] that prohibit superpositions between different topological charges and fusion algebra [27,28] that governs the combinatorial rules of anyons. A specific form of entanglement—anyonic charge entanglement (ACE) [29]—emerges from this structure and manifests as anyonic charge lines connecting the subsystems, motivating a refined characterization that separates ACE from the conventional part associated with the usual tensor-product structure.

Several works have investigated the entanglement properties of anyonic systems. For example, the bipartite entanglement entropy for a pure state in $SU(2)_k$ Chern-Simons theory was derived via skein theory [30], and later generalized to anyon models on surfaces of arbitrary genus [31]. From an information-theoretic perspective, asymptotic entanglement entropy (AEE) of bipartite pure states was proposed as an operational measure related to entanglement distillation and dilution [32]. For mixed states, Ref. [33] introduced the entropy of ACE as a probe of this unique entanglement, and further used it to derive TEE under specific assumptions. Despite these advances, a general theory of anyonic entanglement remains lacking—one that can systematically

*Contact author: lyou@mail.tsinghua.edu.cn†Contact author: chengqianxu@mail.tsinghua.edu.cn

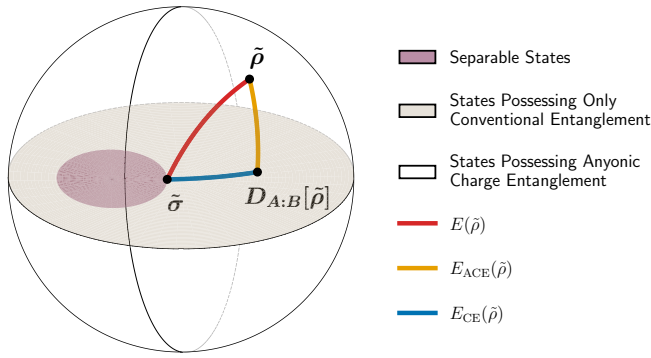


FIG. 1. Illustration of entanglement structure in anyonic state space. The set of separable states (SEP) and the set of states possessing only conventional entanglement (CENT) share the same dimension, jointly forming a low-dimensional convex subset. The subset $\text{SEP} \cup \text{CENT}$ is subspace with tensor product structure in anyonic state space. The entanglement $E(\tilde{\rho})$ of a bipartite state $\tilde{\rho}$, defined as the minimal distance from $\tilde{\rho}$ to SEP based on anyonic relative entropy, admits a decomposition into two components: the anyonic charge entanglement $E_{\text{ACE}}(\tilde{\rho})$ and the conventional entanglement $E_{\text{CE}}(\tilde{\rho})$. The measure $E_{\text{ACE}}(\tilde{\rho})$ is the minimum distance from $\tilde{\rho}$ to $\text{SEP} \cup \text{CENT}$ (see Theorem 2), achieved at $D_{A:B}[\tilde{\rho}]$, while the measure $E_{\text{CE}}(\tilde{\rho})$ is the minimum distance from $D_{A:B}[\tilde{\rho}]$ to SEP.

quantify both the total entanglement in generic anyonic states and the unique contribution beyond conventional quantum entanglement.

In this Letter, we propose the anyonic relative entropy of entanglement $E(\tilde{\rho})$, a generalization of the relative entropy of entanglement [17,34], as an entanglement measure for bipartite anyonic state $\tilde{\rho}$. We show that, within the framework of resource theory [35], $E(\tilde{\rho})$ satisfies the standard axioms of entanglement measures [36], and further admits a decomposition: $E(\tilde{\rho}) = E_{\text{ACE}}(\tilde{\rho}) + E_{\text{CE}}(\tilde{\rho})$, where both E_{ACE} and E_{CE} are also established as valid entanglement measures. E_{ACE} captures entanglement arising from the nontensor product structure of the state space, while E_{CE} quantifies entanglement within a subspace that admits tensor product structure. The latter coincides with the relative entropy of entanglement for a conventional quantum state under certain constraints and characterizes the Bell nonlocality of anyonic states [37]. This decomposition reveals the distinct entanglement structure in anyonic state space, as illustrated in Fig. 1. The measure $E_{\text{ACE}}(\tilde{\rho})$ has a closed form by definition, which facilitates ACE computation and also possesses a geometric interpretation that aids in theoretical analysis. Moreover, we prove its equivalence to the entropy of ACE, thereby rigorously establishing the monotonicity required for the entropy of ACE to serve as a proper entanglement measure. This result underscores the connection between geometric and operational descriptions of this unique anyonic entanglement.

Anyon models. We first give a brief review of anyon models [27] and their diagrammatic representation [29]. An anyon model consists of a finite set of topological charges obeying fusion rules: $a \times b = \sum_c N_{ab}^c c$, where non-negative integers N_{ab}^c (the fusion coefficients) indicate the number of ways that charges a and b can fuse into charge c . A unique vacuum charge, denoted by 1, exists such that both fusion and braiding

with 1 are trivial. An anyon carrying charge a is Abelian if the fusion outcome of a with any charge b is unique; otherwise, it is non-Abelian. The N_{ab}^c distinct fusion channels correspond to an orthonormal basis of the fusion (splitting) Hilbert space $V_c^{ab}(V_{ab}^c)$. Their diagrammatic representations are shown below [38]:

$$|a, b; c, \mu\rangle = \left(\frac{d_c}{d_a d_b}\right)^{1/4} \begin{array}{c} a \quad b \\ \searrow \quad \swarrow \\ \mu \\ \uparrow \\ c \end{array},$$

$$\langle a, b; c, \mu| = \left(\frac{d_c}{d_a d_b}\right)^{1/4} \begin{array}{c} c \\ \uparrow \\ \mu \\ \searrow \quad \swarrow \\ a \quad b \end{array},$$

where d_a is the quantum dimension of charge a and $\mu = 1, \dots, N_{ab}^c$. The normalization factors are included so that diagrams are in the isotopy invariant convention. Diagrams can be stacked together by connecting lines with same charge to represent states and operators involving multiple anyons. For non-Abelian anyons, the Hilbert space does not admit a tensor product structure due to the emergence of additional degrees of freedom when anyons fuse together.

An anyonic state can be represented by an anyonic density operator $\tilde{\rho}$ acting on the fusion Hilbert space $\mathcal{H}_{\{a_i\}} = \bigoplus_c V_{a_1 \dots a_n}^c$, normalized by $\text{Tr}[\tilde{\rho}] = 1$, where $\text{Tr}[\tilde{\rho}] = \sum_c d_c \text{Tr}[\tilde{\rho}_c]$ with $\tilde{\rho}_c$ the projection onto the total-charge sector c (here Tr is the ordinary matrix trace on $V_{a_1 \dots a_n}^c$). Anyonic superselection rules enforce the direct-sum block structure $\tilde{\rho} = \bigoplus_c \tilde{\rho}_c$. For two Fibonacci anyons τ (the Fibonacci model has charges $\{1, \tau\}$ with fusion $\tau \times \tau = 1 + \tau$ and quantum dimension $d_\tau = (1 + \sqrt{5})/2$ [39,40]), we have $\mathcal{H}_{\tau\tau} = V_{\tau\tau}^1 \oplus V_{\tau\tau}^\tau$.

We focus on the bipartite setting in this Letter. Unlike conventional quantum systems, the state space of a bipartite anyonic system does not possess a tensor product structure, as a consequence of both the nontrivial fusion algebra and the anyonic superselection rules. An anyonic density operator is said to admit a tensor product structure iff it can be decomposed as

$$\tilde{\rho} = \sum_{i,j} p_{ij} \tilde{\sigma}_A^i \otimes \tilde{\sigma}_B^j, \quad (1)$$

where $\{\tilde{\sigma}_A^i\}_i$ and $\{\tilde{\sigma}_B^j\}_j$ form complete bases for the anyonic density operator spaces of subsystems A and B , respectively. It is important to distinguish between the Hilbert space and the state space: The former refers to a vector space, while the latter refers to an operator space. Notably, the state space of a bipartite Abelian anyon system also lacks a tensor product structure.

Decomposition of entanglement. We present the central result of this work: a formula that decomposes the total entanglement of an anyonic state into two distinct components—conventional entanglement and ACE.

Theorem 1 (Decomposition of entanglement). The entanglement of a bipartite anyonic state $\tilde{\rho}$ can be decomposed as

$$E(\tilde{\rho}) = E_{\text{ACE}}(\tilde{\rho}) + E_{\text{CE}}(\tilde{\rho}), \quad (2)$$

TABLE I. A list of acronyms and their full names.

Acronym	Full name
AEE	Asymptotic entanglement entropy
ACE	Anyonic charge entanglement
CE	Conventional entanglement
SEP	The set of separable states
CENT	The set of states possessing only CE
AENT	The set of states possessing ACE

where $E(\tilde{\rho}) = \min_{\tilde{\sigma} \in \text{SEP}} \tilde{S}(\tilde{\rho}||\tilde{\sigma})$ represents the measure of total entanglement of $\tilde{\rho}$, $E_{\text{ACE}}(\tilde{\rho}) = \tilde{S}(\tilde{\rho}||D_{A:B}[\tilde{\rho}])$ represents the measure of ACE, and $E_{\text{CE}}(\tilde{\rho}) = \min_{\tilde{\sigma} \in \text{SEP}} \tilde{S}(D_{A:B}[\tilde{\rho}]||\tilde{\sigma})$ represents the measure of conventional entanglement. The notations in these expressions will be introduced later.

Next we prove that $E(\tilde{\rho})$, $E_{\text{ACE}}(\tilde{\rho})$, and $E_{\text{CE}}(\tilde{\rho})$ are valid measures of the corresponding quantities within the framework of resource theory. Our analysis begins with the resource theory of ACE for two reasons. First, ACE captures the correlation that is intrinsic to anyonic systems and has no analog in conventional quantum systems. Second, once the resource theory of ACE is established, the corresponding frameworks for total and conventional entanglement follow in a straightforward manner.

For the convenience of the readers, we list the key acronyms in Table I.

Resource theory of ACE. A resource theory consists of three key ingredients: free states, free operations, and resource measure [35].

(1) *Free states.* We define a bipartite state $\tilde{\rho}$ to be a free state if there is no anyonic charge line connecting the two subsystems, which is equivalent to the condition $\tilde{\rho} = D_{A:B}[\tilde{\rho}]$. Here, $D_{A:B}$ is a superoperator that acts on the system by applying the ω_1 loop [33]:

$$D_{A:B} \left[\begin{array}{c} a \quad b \\ \uparrow \quad \uparrow \\ e \\ \downarrow \quad \downarrow \\ a' \quad b' \end{array} \right] = \begin{array}{c} a \quad b \\ \uparrow \quad \uparrow \\ \omega_1 \\ \downarrow \quad \downarrow \\ a' \quad b' \end{array} = \delta_{e1} \delta_{a'a} \delta_{b'b} \begin{array}{c} a \quad b \\ \uparrow \quad \uparrow \\ \downarrow \quad \downarrow \\ a' \quad b' \end{array}, \quad (3)$$

where we omit fusion trees inside subsystems A and B . The ω_1 loop is a linear superposition of loops carrying topological charge a with a weight of d_a/D^2 and projects the enclosed charge lines onto the vacuum [41]. The action of $D_{A:B}$ can be regarded as a projection onto the operator subspace where there is no anyonic charge correlation between two subsystems. A state is said to be a resource state if it is not a free state.

(2) *Free operations.* Similar to the entanglement theory in conventional quantum system [36], we define free operations to be local operations and classical communications (LOCC). Here, local operations include (a) adding anyons with trivial total charge to any subsystem; (b) tracing out a part of anyons; (c) braiding anyons inside subsystems; and (d) performing projective measurements that respect anyonic superselection rules on subsystems.

(3) *Measure of ACE.* Quantum relative entropy [2] is a well-established measure of distinguishability between quantum states and possesses several desirable properties for quantifying resources. We adopt its anyonic counterpart to quantify ACE for an anyonic state $\tilde{\rho}$:

$$E_{\text{ACE}}(\tilde{\rho}) = \tilde{S}(\tilde{\rho}||D_{A:B}[\tilde{\rho}]), \quad (4)$$

where anyonic relative entropy is defined as $\tilde{S}(\tilde{\sigma}||\tilde{\tau}) = \tilde{\text{Tr}}(\tilde{\sigma} \ln \tilde{\sigma}) - \tilde{\text{Tr}}(\tilde{\sigma} \ln \tilde{\tau})$.

A valid resource measure needs to satisfy (a) vanishing only for free states, which is obvious for E_{ACE} and (b) monotonicity under any free operation Φ : $E_{\text{ACE}}(\Phi(\tilde{\rho})) \leq E_{\text{ACE}}(\tilde{\rho})$. To prove monotonicity, we need two propositions.

Proposition 1. Superoperator $D_{A:B}$ and free operation Φ are commutative:

$$D_{A:B}[\Phi(\tilde{\rho})] = \Phi(D_{A:B}[\tilde{\rho}]). \quad (5)$$

Proposition 2. Anyonic relative entropy is nonincreasing under any completely positive and trace-preserving (CPTP) map $\tilde{\mathcal{E}}$ that respects anyonic superselection rules:

$$\tilde{S}(\tilde{\mathcal{E}}(\tilde{\rho})||\tilde{\mathcal{E}}(\tilde{\sigma})) \leq \tilde{S}(\tilde{\rho}||\tilde{\sigma}). \quad (6)$$

Here, trace refers to the quantum trace $\tilde{\text{Tr}}$. Proposition 2 extends Uhlmann's theorem [42] to the context of anyonic relative entropy. The proofs of two propositions are presented in Ref. [41]. Combining these two propositions, we immediately get the monotonicity of $E_{\text{ACE}}(\tilde{\rho})$ under Φ since all free operations belong to the set of CPTP maps.

In some contexts, an entanglement measure is required to satisfy the condition of nonincrease on average under LOCC [43,44]. It accounts for scenarios in which one may have access to information about the measurement outcomes. We prove that E_{ACE} also satisfies this condition:

$$\sum_i p_i E_{\text{ACE}} \left(\frac{\tilde{\rho}_i}{p_i} \right) \leq E_{\text{ACE}}(\tilde{\rho}), \quad (7)$$

where $\tilde{\rho}_i = \tilde{K}_i \tilde{\rho} \tilde{K}_i^\dagger$, $\tilde{K}_i$ are the Kraus operators corresponding to a given LOCC protocol, and $p_i = \tilde{\text{Tr}}(\tilde{\rho}_i)$ is the probability of obtaining outcome i . Since E_{ACE} is a convex measure [41], this condition is stronger than monotonicity under LOCC.

Resource theories of total and conventional entanglement. The resource theories of total and conventional entanglement can now be formulated naturally. In both cases, free states are defined to be separable states (SEP), which take the form $\tilde{\rho} = \sum_i p_i \tilde{\rho}_i^A \otimes \tilde{\rho}_i^B$. Free operations are taken to be LOCC, as in the resource theory of ACE. We define the measure of total (conventional) entanglement for an anyonic state $\tilde{\rho}$ as $E(\tilde{\rho}) = \min_{\tilde{\sigma} \in \text{SEP}} \tilde{S}(\tilde{\rho}||\tilde{\sigma})$ [$E_{\text{CE}}(\tilde{\rho}) = \min_{\tilde{\sigma} \in \text{SEP}} \tilde{S}(D_{A:B}[\tilde{\rho}]||\tilde{\sigma})$], which is a generalization of relative entropy of entanglement [17]. The interpretation of $E_{\text{CE}}(\tilde{\rho})$ as conventional entanglement is justified by the fact that $D_{A:B}[\tilde{\rho}]$ is the projection of $\tilde{\rho}$ onto the subspace without ACE, which admits a tensor product structure in anyonic state space. Indeed, $E_{\text{CE}}(\tilde{\rho})$ does not depend on quantum dimension d_a , and one can construct a map from $D_{A:B}[\tilde{\rho}]$ to the density matrix ρ of a conventional quantum system under certain constraints, such that $E_{\text{CE}}(\tilde{\rho}) = \min_{\sigma \in \text{SEP}} S(\rho||\sigma)$ [41], where $S(\cdot||\cdot)$ denotes quantum relative entropy.

The monotonicity of both $E_{CE}(\tilde{\rho})$ and $E(\tilde{\rho})$ under free operations follows directly from Proposition 2. Moreover, both measures satisfy the stronger condition of nonincrease on average under LOCC. For $E_{CE}(\tilde{\rho})$, it follows from the monotonicity of relative entropy of entanglement under LOCC. The same conclusion holds for $E(\tilde{\rho})$ since it is the sum of $E_{ACE}(\tilde{\rho})$ and $E_{CE}(\tilde{\rho})$, both of which individually satisfy the condition.

Entanglement structure in anyonic state space. We have now established the validity of the three entanglement measures introduced in Theorem 1. The proof of Theorem 1 is given in Ref. [41]. Theorem 1 implies that the state space of an anyonic system can be partitioned into three distinct subsets: (1) SEP; (2) subset of states possessing only conventional entanglement (CENT) with conditions: $E_{CE}(\tilde{\rho}_{AB}) \neq 0$ and $E_{ACE}(\tilde{\rho}_{AB}) = 0$; and (3) subset of states possessing ACE (AENT), characterized by $E_{ACE}(\tilde{\rho}_{AB}) \neq 0$. Our framework unveils a profound geometric structure underlying anyonic entanglement. As proven in Theorem 2, the total entanglement $E(\tilde{\rho})$ can be interpreted as the distance to SEP, which decomposes exactly into the sum of two distances: E_{ACE} , the distance to the charge-decorated state $D_{A:B}[\tilde{\rho}]$, and E_{CE} , the distance from $D_{A:B}[\tilde{\rho}]$ to SEP. This relationship is a manifestation of the Pythagorean theorem for the quantum relative entropy [37,45]. While such a powerful geometric analogy offers an intuitive understanding, the term “distance” is used loosely to refer to the relative entropy divergence, which, though not a true metric, still provides a solid foundation for our resource theory. Figure 1 provides an intuitive illustration of this partition. Unlike conventional quantum systems, where SEP attains full dimensionality within the state space, SEP here forms a lower-dimensional convex subset in anyonic state space (here dimension refers to the number of independent parameters required to characterize a set). CENT has the same dimensionality as SEP, while the remaining space is occupied by AENT. Notably, SEP and CENT together form the subspace without ACE, which is exactly the set of free states defined in the resource theory of ACE.

The dimensionality gap between $\text{SEP} \cup \text{CENT}$ and the full anyonic state space arises from two factors. First, the anyonic superselection rules forbid coherent superpositions between different charges within each subsystem. Second, the non-trivial fusion rules of non-Abelian anyons generate additional degrees of freedom in the full state space, corresponding to the fusion space associated with the total charges of subsystems A and B . A more detailed explanation is provided in Ref. [41].

We use isotropic states as a typical example to further illustrate how $E_{ACE}(\tilde{\rho})$ reveals the unique entanglement structure of anyonic systems. In conventional quantum systems, an isotropic state is defined as a symmetric bipartite state that is a combination of the maximally mixed state and the maximally entangled state:

$$\rho_\alpha = (1 - \alpha)\rho_M + \alpha|\Psi^+\rangle\langle\Psi^+|, \quad (8)$$

for the real parameter $\alpha \in [\frac{-1}{d^2-1}, 1]$, where d is the dimensionality of the Hilbert space of each subsystem, $\rho_M = \frac{1}{d^2}I$ denotes the maximally mixed state, and $|\Psi^+\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |ii\rangle$ represents the maximally entangled state. It has been proven that ρ_α is separable for $\alpha \in [\frac{-1}{d^2-1}, \frac{1}{d+1}]$ [46]. We adapt

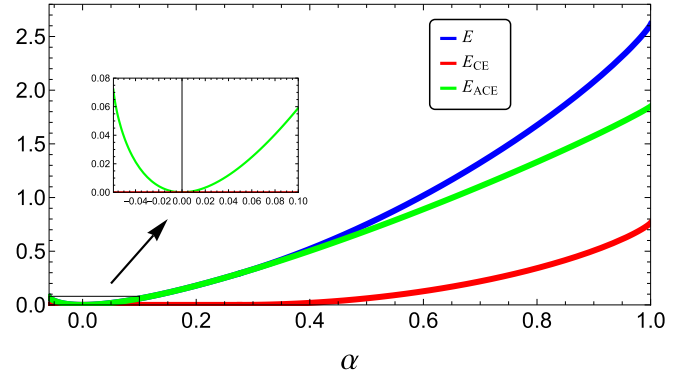


FIG. 2. Three measures E_{ACE} , E_{CE} , and E as functions of α for the anyonic isotropic state $\tilde{\rho}_\alpha$ consisting of six Fibonacci anyons. E_{ACE} is nonzero except at $\alpha = 0$, while E_{CE} remains zero at a range of α .

this definition to anyonic systems and compute the ACE of isotropic states based on Fibonacci anyon model.

Figure 2 shows the values of three entanglement measures for the isotropic state consisting of six Fibonacci anyons, as functions of the parameter α (see Ref. [41] for the explicit expressions). Note that except for $\alpha = 0$, $E_{ACE}(\tilde{\rho})$ remains nonzero for all α , indicating that the corresponding isotropic state $\tilde{\rho}_\alpha$ is entangled. The difference between anyonic and conventional scenarios can be interpreted from a geometrical perspective. In the conventional setting, there exists a separable ball around the maximally mixed state [47], so that any state sufficiently close to the maximally mixed state is separable. In contrast, such a separable ball does not exist in anyonic state space because SEP is no longer full dimensional; it resides in the subspace without ACE. Consequently, the combination of the maximally mixed state with any state containing ACE necessarily leaves this subspace and results in an entangled state.

Properties of $E_{ACE}(\tilde{\rho})$. $E(\tilde{\rho})$ and $E_{CE}(\tilde{\rho})$ are generally difficult to compute, as their evaluation requires minimizing the anyonic relative entropy over SEP. In contrast, calculating $E_{ACE}(\tilde{\rho})$ is a straightforward task. Nevertheless, $E_{ACE}(\tilde{\rho})$ also admits a similar geometric interpretation, as formalized in the following theorem.

Theorem 2. For any bipartite anyonic state $\tilde{\rho}$,

$$E_{ACE}(\tilde{\rho}) = \min_{\tilde{\sigma} \in \text{SEP} \cup \text{CENT}} \tilde{S}(\tilde{\rho}||\tilde{\sigma}), \quad (9)$$

where $\text{SEP} \cup \text{CENT}$ is the set of free states in the resource theory of ACE.

Theorem 2 shows that E_{ACE} equals the minimal distance between the state $\tilde{\rho}$ and any free state $\tilde{\sigma}$, with the minimum achieved at $D_{A:B}[\tilde{\rho}]$. Given Theorem 2 and the joint convexity of anyonic relative entropy [41], it is straightforward to derive the convexity of $E_{ACE}(\tilde{\rho})$:

$$E_{ACE}(\lambda\tilde{\rho}_1 + (1 - \lambda)\tilde{\rho}_2) \leq \lambda E_{ACE}(\tilde{\rho}_1) + (1 - \lambda)E_{ACE}(\tilde{\rho}_2),$$

which implies that mixing anyonic states never increases the amount of ACE. The same argument applies to $E(\tilde{\rho})$ and $E_{CE}(\tilde{\rho})$ as well.

Last but not least, we demonstrate the equivalence between E_{ACE} and the entropy of ACE, which is defined

as $\tilde{S}_{\text{ACE}}(\tilde{\rho}) = \tilde{S}(D_{A:B}[\tilde{\rho}]) - \tilde{S}(\tilde{\rho})$ [33], thereby providing a resource-theoretic interpretation to $\tilde{S}_{\text{ACE}}(\tilde{\rho})$.

Theorem 3. For any bipartite anyonic state $\tilde{\rho}$,

$$E_{\text{ACE}}(\tilde{\rho}) = \tilde{S}_{\text{ACE}}(\tilde{\rho}). \quad (10)$$

Equation (10) demonstrates the equivalence between two distinct definitions of anyonic charge correlations. The left-hand side of this equation, based on anyonic relative entropy, quantifies the correlations by using the minimal distance from anyonic state $\tilde{\rho}$ to the set $\text{SEP} \cup \text{CENT}$. The right-hand side represents the entropy difference between two anyonic states, which, according to Ref. [48], is equivalent to the maximum entropy difference between anyonic state $\tilde{\rho}$ and a set of states indistinguishable from it under local tomography, thus defining the nonlocal correlations in anyonic states that cannot be determined through LOCC. These two definitions, one geometric and the other operational, are demonstrated to be equivalent here. The same correspondence holds for correlations in conventional quantum systems [49], and Theorem 3 extends this equivalence to anyonic systems.

It was proved that the entropy of ACE, $\tilde{S}_{\text{ACE}}(\tilde{\rho})$, is nonlocal information undetectable if the parties sharing the state are restricted to LOCC operations [48]. Alternatively speaking, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ always yield identical expected value of any local observable if $D_{A:B}[\tilde{\rho}_1] = D_{A:B}[\tilde{\rho}_2]$. In this sense, $E_{\text{ACE}}(\tilde{\rho})$ quantifies the amount of information that is protected from local parties under LOCC constraints. It is natural to come up with data hiding protocols [50–52] based on this distinguishability of anyonic states. Take Fibonacci anyon model as an example; one bit of information can be encoded into two orthogonal states: $|\tau, \tau; 1\rangle$ and $|\tau, \tau; \tau\rangle$, such that only global operations could reveal this information.

On the other hand, the presence of such undetectable entanglement implies the existence of pure anyonic entangled states that are nevertheless local, which is impossible for conventional pure states [53]. Specifically, as shown in a recent work [37], all pure anyonic states with $E_{\text{ACE}}(\tilde{\rho}) \neq 0$ but $E_{\text{CE}}(\tilde{\rho}) = 0$ are local, in the sense that they do not violate any Bell inequalities. In other words, $E_{\text{CE}}(\tilde{\rho})$ characterizes the Bell nonlocality of anyonic state $\tilde{\rho}$. However, Bell nonlocality can be activated when multiple copies of such states

are considered—a phenomenon known as the superactivation of nonlocality [54]. This arises from the conversion of ACE into conventional entanglement as the number of copies increases. In the asymptotic limit, the average conventional entanglement approaches the total entanglement, while the average ACE vanishes. Furthermore, the average conventional entanglement and total entanglement both converge to AEE, which serves as a unified measure of entanglement cost and distillable entanglement for pure anyonic states [32].

Conclusion. While the division of entanglement into conventional and anyonic parts is intuitively natural, our work presents the first formal theorem that establishes rigorously this additive decomposition. This transforms a qualitative expectation into a quantitatively verifiable principle, providing a solid theoretical foundation for future research on anyonic entanglement.

Our work advances the field by introducing the first complete resource-theoretic framework for anyonic entanglement. This framework not only validates the “entropy of ACE” through rigorous analysis but also converts it into a practical tool. Key advances include providing a comprehensive operational context and offering a novel geometric interpretation, which together transform ACE from an abstract concept into a quantifiable and visualizable resource for future theoretical and experimental explorations.

In summary, our work establishes a comprehensive framework for understanding and quantifying entanglement in anyonic systems. This framework lays the groundwork for future explorations of the unique properties of anyons and their potential for realizing fault-tolerant quantum computing. We anticipate that this work will stimulate further research in the field of anyonic entanglement, leading to a deeper understanding of the fundamental nature of entanglement and its role in topological quantum computation.

Acknowledgments. We thank Dr. Duanlu Zhou for insightful discussions. This work was supported by NSFC (Grants No. 12361131576 and No. 92265205) and by the Quantum Science and Technology—National Science and Technology Major Project (2021ZD0302104). C.-Q.X. acknowledges support from NSFC (Grant No. 12405018) and the Postdoctoral Fellowship Program of CPSF (Grant No. GZC20231366).

Data availability. No data were created or analyzed in this study.

-
- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
 - [3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, *Phys. Rev. Lett.* **70**, 1895 (1993).
 - [4] A. K. Ekert, Quantum cryptography based on Bell’s theorem, *Phys. Rev. Lett.* **67**, 661 (1991).
 - [5] D. Deutsch and R. Jozsa, Rapid solution of problems by quantum computation, *Proc. R. Soc. London A* **439**, 553 (1992).
 - [6] L. K. Grover, Quantum mechanics helps in searching for a needle in a haystack, *Phys. Rev. Lett.* **79**, 325 (1997).
 - [7] P. W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, *SIAM Rev.* **41**, 303 (1999).
 - [8] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, *Phys. Rev. Lett.* **96**, 010401 (2006).
 - [9] L. Pezzè, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Quantum metrology with nonclassical states of atomic ensembles, *Rev. Mod. Phys.* **90**, 035005 (2018).
 - [10] Y.-Q. Zou, L.-N. Wu, Q. Liu, X.-Y. Luo, S.-F. Guo, J.-H. Cao, M. K. Tey, and L. You, Beating the classical precision limit

- with spin-1 Dicke states of more than 10,000 atoms, *Proc. Natl. Acad. Sci. USA* **115**, 6381 (2018).
- [11] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, *Quantum Information Meets Quantum Matter* (Springer, New York, 2019).
 - [12] T. J. Osborne and M. A. Nielsen, Entanglement in a simple quantum phase transition, *Phys. Rev. A* **66**, 032110 (2002).
 - [13] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in quantum critical phenomena, *Phys. Rev. Lett.* **90**, 227902 (2003).
 - [14] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, *Rev. Mod. Phys.* **80**, 517 (2008).
 - [15] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of noisy entanglement and faithful teleportation via noisy channels, *Phys. Rev. Lett.* **76**, 722 (1996).
 - [16] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying entanglement, *Phys. Rev. Lett.* **78**, 2275 (1997).
 - [17] V. Vedral and M. B. Plenio, Entanglement measures and purification procedures, *Phys. Rev. A* **57**, 1619 (1998).
 - [18] P. M. Hayden, M. Horodecki, and B. M. Terhal, The asymptotic entanglement cost of preparing a quantum state, *J. Phys. A: Math. Gen.* **34**, 6891 (2001).
 - [19] S. Hollands and K. Sanders, *Entanglement Measures and Their Properties in Quantum Field Theory* (Springer, Cham, 2018), Vol. 34.
 - [20] E. Witten, APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory, *Rev. Mod. Phys.* **90**, 045003 (2018).
 - [21] F. Wilczek, Quantum mechanics of fractional-spin particles, *Phys. Rev. Lett.* **49**, 957 (1982).
 - [22] F. Wilczek, Magnetic flux, angular momentum, and statistics, *Phys. Rev. Lett.* **48**, 1144 (1982).
 - [23] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian anyons and topological quantum computation, *Rev. Mod. Phys.* **80**, 1083 (2008).
 - [24] A. Kitaev and J. Preskill, Topological entanglement entropy, *Phys. Rev. Lett.* **96**, 110404 (2006).
 - [25] M. Levin and X.-G. Wen, Detecting topological order in a ground state wave function, *Phys. Rev. Lett.* **96**, 110405 (2006).
 - [26] A. Kitaev, D. Mayers, and J. Preskill, Superselection rules and quantum protocols, *Phys. Rev. A* **69**, 052326 (2004).
 - [27] A. Kitaev, Anyons in an exactly solved model and beyond, *Ann. Phys.* **321**, 2 (2006).
 - [28] B. Coecke, *New Structures for Physics* (Springer Science & Business Media, Berlin, Heidelberg, 2011), Vol. 813.
 - [29] P. Bonderson, K. Shtengel, and J. Slingerland, Interferometry of non-Abelian anyons, *Ann. Phys.* **323**, 2709 (2008).
 - [30] K. Hikami, Skein theory and topological quantum registers: Braiding matrices and topological entanglement entropy of non-Abelian quantum Hall states, *Ann. Phys.* **323**, 1729 (2008).
 - [31] R. N. C. Pfeifer, Measures of entanglement in non-Abelian anyonic systems, *Phys. Rev. B* **89**, 035105 (2014).
 - [32] K. Kato, F. Furrer, and M. Murao, Information-theoretical formulation of anyonic entanglement, *Phys. Rev. A* **90**, 062325 (2014).
 - [33] P. Bonderson, C. Knapp, and K. Patel, Anyonic entanglement and topological entanglement entropy, *Ann. Phys.* **385**, 399 (2017).
 - [34] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Unified view of quantum and classical correlations, *Phys. Rev. Lett.* **104**, 080501 (2010).
 - [35] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
 - [36] M. B. Plenio and S. S. Virmani, An introduction to entanglement theory, *Quantum Information and Coherence* (Springer, Cham, 2014), pp. 173–209.
 - [37] C.-Q. Xu, W. Ye, and L. You, Superactivation of Bell nonlocality in pure anyonic states, *Phys. Rev. A* **112**, 062232 (2025).
 - [38] P. Bonderson, M. Freedman, and C. Nayak, Measurement-only topological quantum computation via anyonic interferometry, *Ann. Phys.* **324**, 787 (2009).
 - [39] J. K. Pachos, *Introduction to Topological Quantum Computation* (Cambridge University Press, Cambridge, 2012).
 - [40] S. Trebst, M. Troyer, Z. Wang, and A. W. Ludwig, A short introduction to Fibonacci anyon models, *Prog. Theor. Phys. Suppl.* **176**, 384 (2008).
 - [41] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/4nd3-k6rw> for detailed proofs and calculations, which also includes Refs. [2,17,27,33,42,55,56].
 - [42] A. Uhlmann, Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory, *Commun. Math. Phys.* **54**, 21 (1977).
 - [43] M. Horodecki, Entanglement measures, *Quantum Inf. Comput.* **1**, 3 (2001).
 - [44] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, *Phys. Rev. Lett.* **95**, 090503 (2005).
 - [45] D. Petz, *Quantum Information Theory and Quantum Statistics, Theoretical and Mathematical Physics* (Springer, Berlin, 2007).
 - [46] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, *Phys. Rev. A* **59**, 4206 (1999).
 - [47] L. Gurvits and H. Barnum, Largest separable balls around the maximally mixed bipartite quantum state, *Phys. Rev. A* **66**, 062311 (2002).
 - [48] C.-Q. Xu and D. L. Zhou, Topological correlation in anyonic states constrained by anyonic superselection rules, *Phys. Rev. A* **108**, 052221 (2023).
 - [49] D. Zhou, Irreducible multiparty correlations can be created by local operations, *Phys. Rev. A* **80**, 022113 (2009).
 - [50] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, Hiding bits in Bell states, *Phys. Rev. Lett.* **86**, 5807 (2001).
 - [51] T. Eggeling and R. F. Werner, Hiding classical data in multipartite quantum states, *Phys. Rev. Lett.* **89**, 097905 (2002).
 - [52] F. Verstraete and J. I. Cirac, Quantum nonlocality in the presence of superselection rules and data hiding protocols, *Phys. Rev. Lett.* **91**, 010404 (2003).
 - [53] N. Gisin and A. Peres, Maximal violation of Bell's inequality for arbitrarily large spin, *Phys. Lett. A* **162**, 15 (1992).
 - [54] C. Palazuelos, Superactivation of quantum nonlocality, *Phys. Rev. Lett.* **109**, 190401 (2012).
 - [55] E. M. Rains, Bound on distillable entanglement, *Phys. Rev. A* **60**, 179 (1999).
 - [56] G. Lindblad, Expectations and entropy inequalities for finite quantum systems, *Commun. Math. Phys.* **39**, 111 (1974).