

# Analytical expression for fracture profile in viscoelastic crack propagation

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We derive an analytical expression for the strain field during steady-state crack propagation in viscoelastic solids described by the standard linear solid (Zener) model. This expression reveals three regions in the fracture profile and in the strain field ahead of the crack tip, each distinguished by power-law exponents that evolve with distance from the crack tip. These features explain the experimentally observed crack-tip sharpening in rubbers and gels as the crack-propagation velocity increases, often associated with catastrophic failure triggered by a velocity jump. Furthermore, we establish de Gennes' viscoelastic trumpet on a continuum-mechanical foundation, previously based only on a scaling argument.

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**Introduction.** Viscoelastic rubberlike materials are used in diverse applications, including industry, medical instruments, and food products. Understanding their fracture characteristics [1–4] possesses both scientific and practical significance. Rubbers [5–11] and gels [12–14] can fail catastrophically due to abrupt increases in crack-propagation velocity; such velocity jumps are strongly correlated with a rise in crack-tip sharpness. This sharpening cannot be explained by the linear elastic fracture mechanics (LEFM) [15], which predicts a parabolic profile for the surface of static cracks. Experiments on rubbers [8,16,17] and gels [18–24] report that increased crack-propagation velocity leads to a sharper, nonparabolic profile near the crack tip.

The weakly nonlinear theory of dynamic fracture [19], an extension of LEFM that accommodates second-order nonlinearity but omits viscoelasticity, may explain this sharpening in some experiments [8,18,20]. Nevertheless, it fails to explain the tip sharpening in which cracks propagate rapidly in filler-reinforced synthetic rubber [8]. This highlights the need to include viscoelasticity, which has not yet been fully incorporated into continuum fracture mechanics.

A pioneering fracture theory that incorporates viscoelasticity is de Gennes' "viscoelastic trumpet" [25,26], which adopts a standard linear solid model characterized by the frequency ( $\omega$ ) dependence of the complex modulus

$$\mu(\omega) = \mu_0 + (\mu_\infty - \mu_0) \frac{i\omega\tau}{1 + i\omega\tau}. \quad (1)$$

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Here,  $\tau$  is the relaxation time, and  $\mu_0$  and  $\mu_\infty$  are the low- and high-frequency elastic moduli, respectively, with the ratio  $\lambda \equiv \mu_\infty/\mu_0 \sim 10^2 - 10^3$  [1,25,26]. Rather than adopting the traditional fracture-mechanics approach that directly addresses continuum mechanics problems, de Gennes applied scaling analysis and energy balance. Through this, he conjectured a trumpet-shaped fracture profile comprising three separate regions, each distinguished by characteristic power-law exponents.

However, in the absence of direct solutions in continuum mechanics, it is unclear how the viscoelastic trumpet is related to traditional fracture mechanics. Efforts to validate the theory, whether by finite element methods (FEM) or experiments [27,28], have been limited, leaving a decisive confirmation unresolved.

In this Letter, we investigate how viscoelasticity leads to crack-tip sharpening as crack-propagation velocity increases. We use the traditional fracture-mechanics approach, which seeks to explain all observed phenomena directly from the first principles of continuum mechanics. Following de Gennes [25,26], we simplify the problem by introducing certain assumptions. We derive an analytical expression for the strain field and fracture profile during steady-state crack propagation, identifying three distinct regions within the fracture profile with different power-law exponents. Notably, these exponents are consistent with those of the viscoelastic trumpet. Our analytical results validate the viscoelastic trumpet and elucidate its position in continuum fracture mechanics. They also illuminate the origin of crack-tip sharpening, which prior studies [8,18–20] attributed to the nonlinear stress-strain relationship; our analysis reveals that, even in the linear regime, viscoelasticity can amplify sharpening at elevated crack-propagation velocities.

**Setup.** As shown in Fig. 1(a), we consider a two-dimensional viscoelastic sheet of height  $2L$  and infinite width. A semi-infinite linear crack is located at its center, aligned with the  $x$  axis in the undeformed state. The sheet is subjected

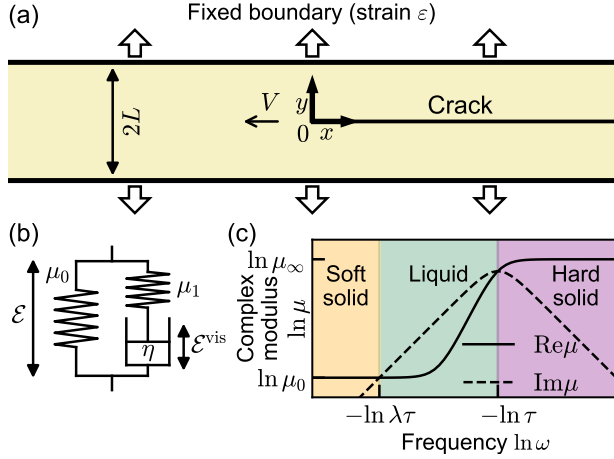


FIG. 1. Setup of viscoelastic crack propagation. (a) Two-dimensional viscoelastic sheet with a semi-infinite linear crack at its center, subjected to a strain  $\varepsilon$  due to fixed boundary conditions. The crack propagates at a constant velocity  $V$  in the  $-x$  direction. (b) Zener model: two springs with shear moduli  $\mu_0$  and  $\mu_1 = \mu_\infty - \mu_0$ , interconnected with a dashpot with viscosity  $\eta$ . The strains of the left spring and dashpot are  $\varepsilon_{ij}$  and  $\varepsilon_{ij}^{\text{vis}}$ , respectively. (c) Complex modulus  $\mu(\omega)$  of the Zener model [Eq. (1)] characterizing three types of dynamic response. The relaxation times are  $\tau \equiv \eta/\mu_1$  and  $\lambda\tau$ , where  $\lambda \equiv \mu_\infty/\mu_0$ .

to a strain  $\varepsilon$  induced by a fixed boundary condition, which causes the crack to propagate in the  $-x$  direction at a constant velocity  $V$ . By focusing on steady-state crack propagation, all fields at position  $(x, y)$  and time  $t$  depend only on the comoving coordinate  $(x + Vt, y)$ , allowing us to analyze the fields at time  $t = 0$  with the crack tip located at  $x = 0$ . The fixed boundary and stress-free boundary conditions are [29,30]

$$\begin{aligned} u_x(x, \pm L) &= 0 & (\text{for } -\infty < x < \infty), \\ u_y(x, \pm L) &= \pm \varepsilon L & (\text{for } -\infty < x < \infty), \\ \sigma_{xy}(x, \pm 0) &= \sigma_{yy}(x, \pm 0) = 0 & (\text{for } 0 < x < \infty), \end{aligned} \quad (2)$$

where  $u_i = u_i(x, y)$  is the displacement field and  $\sigma_{ij} = \sigma_{ij}(x, y)$  ( $i, j = x, y$ ) is the Cauchy stress tensor. For  $x > 0$ , the coordinates  $(x, \pm 0)$  correspond to the upper and lower crack surfaces, respectively [31].

Our main objective is to determine the fracture profile (i.e., the displacement field on the crack surface)  $\mathcal{U}(x) \equiv u_y(x, +0)$  for  $x > 0$ , assuming the symmetry  $u_y(x, +0) = -u_y(x, -0)$ . Following Okumura and de Gennes [32], we neglect  $u_x(x, y)$  as it has only a minor influence on  $\mathcal{U}(x)$ , and thus assume

$$u_x(x, y) = 0, \quad (3)$$

for  $-\infty < x < \infty$  and  $-L \leq y \leq L$ .

In continuum mechanics, the momentum balance equation in the absence of inertia [15] reads

$$0 = \sum_{j=x,y} \partial_j \sigma_{ji} \quad (\text{for } i = x, y), \quad (4)$$

which is a valid approximation when  $V \ll C_s$ , where  $V$  and  $C_s$  are the crack-propagation and shear wave velocities, respectively [1,25,26].

To describe viscoelasticity, we employ the Zener model [33], shown in Fig. 1(b). Its complex modulus  $\mu(\omega)$ ,

given by Eq. (1), captures the three types of dynamic response of viscoelastic solids: soft solid, liquid, and hard solid [25,26], as shown in Fig. 1(c). The viscoelastic stress-strain relationship under plane-stress conditions is

$$\sigma_{ij} \equiv 2\mu_0 \varepsilon_{ij} + \frac{2\mu_0 \nu}{1-\nu} \delta_{ij} \sum_{k=x,y} \varepsilon_{kk} + \sigma_{ij}^{\text{vis}}, \quad (5)$$

$$\begin{aligned} \sigma_{ij}^{\text{vis}} &\equiv 2\eta \partial_t \varepsilon_{ij}^{\text{vis}} + \frac{2\eta \nu}{1-\nu} \delta_{ij} \sum_{k=x,y} \partial_t \varepsilon_{kk}^{\text{vis}} \\ &= 2\mu_1 (\varepsilon_{ij} - \varepsilon_{ij}^{\text{vis}}) + \frac{2\mu_1 \nu}{1-\nu} \delta_{ij} \sum_{k=x,y} (\varepsilon_{kk} - \varepsilon_{kk}^{\text{vis}}), \end{aligned} \quad (6)$$

where  $\varepsilon_{ij} \equiv (\partial_i u_j + \partial_j u_i)/2$  is the Cauchy strain, and  $\sigma_{ij}^{\text{vis}}$  and  $\varepsilon_{ij}^{\text{vis}}$  are the Cauchy stress and strain in the dashpot, respectively. The details of Eqs. (5) and (6) are provided Sec. S1 in the Supplemental Material [34]. Since all fields are functions of  $(x + Vt, y)$  in steady-state crack propagation, it follows that  $\partial_t = V \partial_x$  [19], such as

$$\partial_t \varepsilon_{ij}^{\text{vis}} = V \partial_x \varepsilon_{ij}^{\text{vis}}. \quad (7)$$

*Analytical solution for steady-state crack propagation.* In the setup described by Eqs. (2)–(7), we derive the analytical solution  $u_y = u_y(x, y)$  for steady-state crack propagation with a propagation velocity  $V$ . This is achieved through a variable transformation [35] that reformulates the viscoelastic steady-state crack-propagation problem into an equivalent static-crack problem where the crack does not propagate. Accordingly, the corresponding static crack solution  $u_y^0 = u_y^0(x, y)$  is required. In a static-crack problem, Eqs. (5) and (6) are reduced to

$$\sigma_{ij} = 2\mu_0 \varepsilon_{ij} + \frac{2\mu_0 \nu}{1-\nu} \delta_{ij} \sum_{k=x,y} \varepsilon_{kk}, \quad (8)$$

because  $\sigma_{ij}^{\text{vis}} = 0$ . This static-crack problem has been solved [32] as follows:

*Lemma.* (Static crack) Given the boundary conditions in Eq. (2) with Eq. (3), the displacement field satisfying Eq. (8) is given by  $u_y(x, y) = u_y^0(x, y)$ , where

$$u_y^0(x, y) \equiv \frac{2\varepsilon L}{\pi} \text{Im} \left[ \ln \left( e^{-\pi z^*/(2L)} + \sqrt{e^{-\pi z^*/L} - 1} \right) \right]. \quad (9)$$

Here,  $z^* \equiv x\sqrt{2/(1-\nu)} - yi$  is the complex conjugate of  $z \equiv x\sqrt{2/(1-\nu)} + yi$ . In Eq. (9), the branch of the logarithmic function is selected such that  $\log z$  has a zero argument for real positive  $z$ , and the square root is defined such that  $z^{1/2} = i$  for  $z = -1$ .

By employing the static crack solution in Eq. (9), we obtain the solution for steady-state crack propagation as follows:

*Theorem.* (Steady-state crack propagation) Given the boundary conditions in Eq. (2) with Eq. (3), the displacement field satisfying Eqs. (4)–(7) is given by

$$\begin{aligned} u_y(x, y) &= u_y^0(x, y) - \left(1 - \frac{1}{\lambda}\right) \\ &\times \int_{-\infty}^x \frac{\partial u_y^0(\xi, y)}{\partial \xi} \exp\left(-\frac{x-\xi}{\lambda \tau V}\right) d\xi, \end{aligned} \quad (10)$$

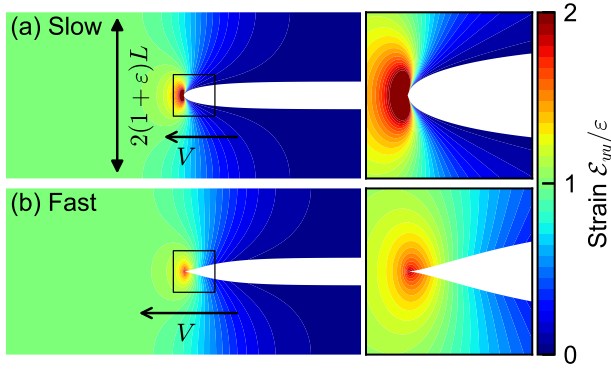


FIG. 2. Strain field  $\mathcal{E}_{yy}$  and fracture profile  $\mathcal{U}(x)$  for (a) slow ( $\mathcal{V} = 10^{-2}$ ) and (b) fast ( $\mathcal{V} = 1$ ) crack propagations at a constant velocity  $V$  for  $\lambda = 10^3$ , based on the analytical expression provided in the theorem.

where  $\lambda \equiv \mu_\infty/\mu_0$  and  $\tau \equiv \eta/\mu_1$ . Here,  $u_y^0(x, y)$  is given by Eq. (9).

A proof of the theorem is provided in Appendix A. As  $V \rightarrow 0$ , the integrand in Eq. (10) vanishes, resulting in  $u_y(x, y)$  converging to  $u_y^0(x, y)$  in Eq. (9).

*Analytical expression for fracture profile.* Before investigating the fracture profile  $\mathcal{U}(x) \equiv u_y(x, +0) = -u_y(x, -0)$  for  $x > 0$  in steady-state crack propagation in Eq. (10), it is instructive to first confirm that the fracture profile  $\mathcal{U}^0(x) \equiv u_y^0(x, +0)$  in the static crack solution in Eq. (9) exhibits a parabolic contour near the crack tip. In the limit  $y \rightarrow +0$ , Eq. (9) yields [32]

$$\mathcal{U}^0(x) = \frac{2\varepsilon L}{\pi} \arctan \left[ \sqrt{e^{\sqrt{2/(1-\nu)}\pi x/L} - 1} \right] \quad (11)$$

for  $x > 0$ . By introducing the dimensionless coordinate  $X \equiv \sqrt{2/(1-\nu)}\pi x/L$ , we deduce that  $\mathcal{U}^0(x) = (2\varepsilon L/\pi)\sqrt{X} + O(X^{3/2})$  for small  $X$ , corroborating the anticipated parabolic contour near the crack tip.

To determine  $\mathcal{U}(x)$ , we introduce the dimensionless velocity  $\mathcal{V} \equiv \sqrt{2/(1-\nu)}\pi\lambda\tau V/L$ . In the limit  $y \rightarrow +0$ , Eq. (10) yields

$$\mathcal{U}(x) = \frac{2\varepsilon L}{\pi} \left[ \arctan(\sqrt{e^X - 1}) - \left(1 - \frac{1}{\lambda}\right) H_{\mathcal{V}}(X) \right], \quad (12)$$

where

$$\begin{aligned} H_{\mathcal{V}}(X) &\equiv \frac{1}{2} \int_0^X \frac{e^{-(X-\Xi)/\mathcal{V}}}{\sqrt{e^\Xi - 1}} d\Xi \\ &= e^{-X/\mathcal{V}} \sqrt{e^X - 1} {}_2F_1\left(\frac{1}{2}, 1 - \frac{1}{\mathcal{V}}; \frac{3}{2}; 1 - e^X\right). \end{aligned} \quad (13)$$

Here,  ${}_2F_1(a, b; c; z)$  is the hypergeometric function [36]. This result implies that a change in the Poisson ratio  $\nu$  simply rescales  $X$  and  $\mathcal{V}$  for  $\mathcal{U}(x)$ .

Based on Eq. (10), Fig. 2 shows  $\mathcal{E}_{yy}(x, y)$  and  $\mathcal{U}(x)$  for slow and fast crack propagations using an arbitrary-precision library [37] to avoid catastrophic cancellation. The fracture profile exhibits a parabolic contour at lower  $\mathcal{V}$ , transitioning to a sharper contour as  $\mathcal{V}$  increases. These findings are consistent with the experimental observations reported

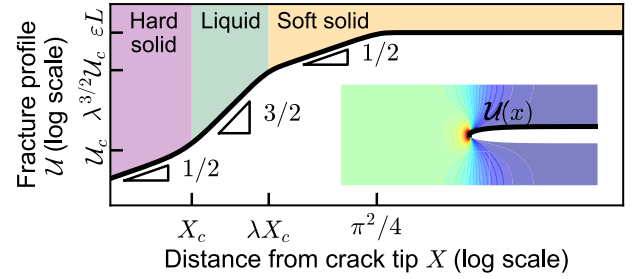


FIG. 3. Fracture profile  $\mathcal{U}(x)$  [Eq. (12)] on a log-log scale, revealing three regions characterized by power-law exponents 1/2, 3/2, and 1/2. The ticks on the horizontal axis indicate the crossover points,  $X_c$ ,  $\lambda X_c$ , and  $\pi^2/4$ , while the ticks on the vertical axis indicate the corresponding points of the respective asymptotic power laws,  $\mathcal{U}_c$ ,  $\lambda^{3/2}\mathcal{U}_c$ , and  $\varepsilon L$ . The condition  $\lambda X_c < \pi^2/4$  (equivalent to  $\lambda x_c < \pi L\sqrt{1-\nu}/4\sqrt{2}$ ) is necessary for the distinct manifestation of these regions. The inset shows the same profile on a linear scale, with colors corresponding to the strain field shown in Fig. 2.

for rubbers [8,16,17] and gels [18–24] explained in the introduction.

*Origin of crack-tip sharpening and viscoelastic trumpet.* To elucidate the origin of crack-tip sharpening as  $\mathcal{V}$  increases, we expand Eq. (12) in powers of  $\sqrt{X}$  as

$$\begin{aligned} \arctan(\sqrt{e^X - 1}) &= \sqrt{X} - \frac{X^{3/2}}{12} + O(X^{5/2}), \\ H_{\mathcal{V}}(X) &= \sqrt{X} - \frac{(\mathcal{V} + 8)X^{3/2}}{12\mathcal{V}} + O(X^{5/2}). \end{aligned}$$

These expansions yield the asymptotic behavior of  $\mathcal{U}(x)$ :

$$\mathcal{U}(x) \approx \frac{2\varepsilon L}{\pi} \begin{cases} \frac{1}{\lambda} \sqrt{X} & (\text{for } 0 < X < X_c), \\ \frac{2(\lambda-1)}{3\lambda\mathcal{V}} X^{3/2} & (\text{for } X_c < X < \lambda X_c), \\ \sqrt{X} & (\text{for } \lambda X_c < X < \frac{\pi^2}{4}), \end{cases} \quad (14)$$

where  $X_c \equiv 3\mathcal{V}/[2(\lambda-1)]$  and  $\lambda X_c$  delineate the crossover points among these regimes. Throughout this Letter, we assume  $\lambda X_c < \pi^2/4$ , corresponding to  $\mathcal{V} < \pi^2(1-\lambda^{-1})/6 \approx 1.6$ , to ensure that all three regions emerge before the sheet is fully relaxed [ $\mathcal{U}(x) \approx \varepsilon L$ ] at  $X \approx \pi^2/4$ . Figure 3 shows  $\mathcal{U}(x)$ , revealing distinct power-law exponents for each regime in Eq. (14): 1/2 near the crack tip, 3/2 beyond  $X_c$  up to  $\lambda X_c$ , and 1/2 again beyond  $\lambda X_c$  as  $H_{\mathcal{V}}(X)$  becomes negligible. Since  $X$  is proportional to  $x$ , these exponents hold in either coordinate.

The exponents (1/2, 3/2, and 1/2) and the crossover points separating each region of  $\mathcal{U}(x)$  [ $x_c \equiv \sqrt{1-\nu}X_cL/(\sqrt{2}\pi) = 3\lambda\tau V/[2(\lambda-1)] \sim \tau V$  and  $\lambda x_c \sim \lambda\tau V$  for  $\lambda \gg 1$ ] in Eq. (14) are consistent with those of the viscoelastic trumpet [25–27,38], which assumes  $\lambda \gg 1$ . Notably, Eq. (14) extends beyond the viscoelastic trumpet, by providing the prefactors of both the power law and the crossover points. Although both the viscoelastic trumpet and Eq. (14) rely on parameter constraints, Eq. (12), from which Eq. (14) is derived, holds more generally. Consequently, Eq. (12) reproduces the profile of the viscoelastic trumpet as a limiting case and elucidates its foundation in continuum fracture mechanics.

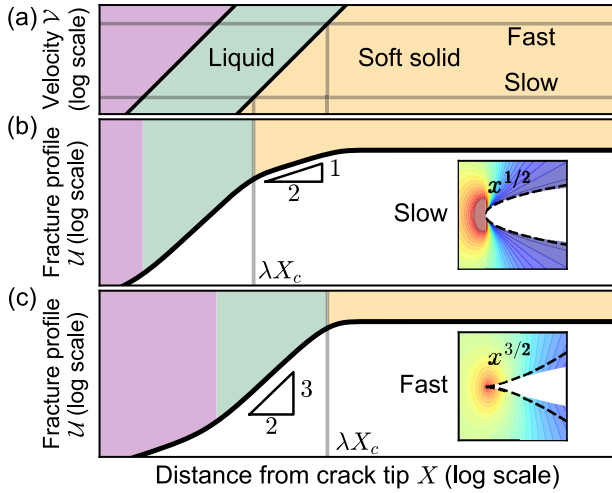


FIG. 4. Fracture profiles  $\mathcal{U}$  during slow ( $V = 10^{-2}$ ) and fast ( $V = 1$ ) crack propagation. (a) Viscoelastic states on the crack surface as a function of the distance  $x$  from the crack tip, varying with  $V$ , while gray horizontal lines correspond to the fast and slow crack propagation cases shown in panels (b) and (c). Gray vertical lines indicate  $\lambda X_c$  in panels (b) and (c). (b) At low  $V$ , a smaller  $\lambda X_c$  leads to the dominance of the  $1/2$  power-law region, resulting in a parabolic crack-tip profile. (c) At high  $V$ , a larger  $\lambda X_c$  extends the  $3/2$  power-law region, resulting in a sharper crack-tip profile.

Figure 4(a) shows how the viscoelastic state on the crack surface evolves with the distance  $x$  from the crack tip, which varies according to the crack-propagation velocity  $V$ . The black solid lines represent  $X_c$  ( $\propto V$ ) and  $\lambda X_c$  ( $\propto V$ ), indicating the crossover points between the soft-solid, liquid, and hard-solid regimes. The soft and hard solids exhibit power-law exponents of  $1/2$ , yielding parabolic fracture profiles, whereas the liquid exhibits a power-law exponent of  $3/2$ , resulting in a sharp profile. At lower  $V$  [Fig. 4(b)], the dominance of the  $1/2$  power-law exponent and the small magnitudes of  $\lambda X_c$  lead to nearly parabolic profiles. In contrast, at higher  $V$  [Fig. 4(c)], the increasing  $\lambda X_c$  expands the liquid region, where the prevailing  $3/2$  power-law exponent leads to progressive tip sharpening (note the logarithmic scale). Thus, our analytical expression elucidates the origin of the velocity-dependent crack-tip sharpening observed in rubbers [8,16,17] and gels [18–24], suggesting that crack-tip sharpening results from high velocity, rather than the cause of velocity jump.

To further demonstrate the relevance of our analytical solution, we compare it with fracture profiles experimentally observed in carbon-black-filled rubbers [8], which show systematic deviations  $\delta$  from a parabolic shape. Our solution predicts  $\delta$  as  $\lambda x_c \approx (3/2)\lambda\tau V$ , corresponding to the soft-solid–liquid crossover point (Figs. 3 and 4). We focus on crack propagation at  $V \approx 0.1C_s$  (where  $C_s$  is the shear wave velocity), where inertia is negligible but  $\delta$  is measurable. Using the reported relaxation time  $\lambda\tau$  [8], we find that the measured  $\delta$  is consistent in magnitude with our prediction  $(3/2)\lambda\tau V$ . Moreover, our analytical solution reproduces the experimentally observed increase of  $\delta$  with velocity. (See Sec. S3 in the Supplemental Material [34] for details.)

**Strain field ahead of the crack tip.** Using Eq. (10), we illustrate the strain field  $\mathcal{E}_{yy}(x, y) \equiv \partial_y u_y(x, y)$  ahead of the

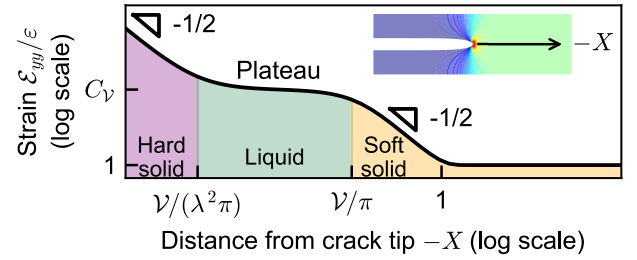


FIG. 5. Strain field  $\mathcal{E}_{yy}(x, 0)$  ahead of the crack tip ( $X < 0$ ) as a function of the normalized distance from the crack tip  $-X$  for  $\lambda \gg 1$ . The strain field exhibits two power-law regions,  $\mathcal{E}_{yy}(x, 0) \sim (-X)^{-1/2}$  for  $-X < V/(\lambda^2\pi)$  and  $V/\pi < -X < 1$ , and a plateau for  $V/(\lambda^2\pi) < -X < V/\pi$ . These regions correspond to the hard-solid, liquid, and soft-solid regimes in the complex modulus [Eq. (1)].

crack tip ( $x < 0$ ) along the  $x$  axis ( $y = 0$ ) in Fig. 5. Here, we assume  $\lambda \gg 1$ , as typical for rubberlike materials [1,25,26]. Figure 5 shows that  $\mathcal{E}_{yy}(x, 0)$  exhibits two distinct power-law regions,  $\sim (-x)^{-1/2}$ , separated by an intermediate plateau,  $\sim (-x)^0$ . These correspond to hard-solid, liquid, and soft-solid regimes in the viscoelastic response. They are manifested in the asymptotic form

$$\frac{\mathcal{E}_{yy}(x, 0)}{\varepsilon} \approx \begin{cases} \frac{1}{\lambda\sqrt{-X}} & (\text{for } 0 < -X < \frac{1}{\lambda^2 C_V^2}), \\ C_V & (\text{for } \frac{1}{\lambda^2 C_V^2} < -X < \frac{1}{C_V^2}), \\ \frac{1}{\sqrt{-X}} & (\text{for } \frac{1}{C_V^2} < -X < 1), \\ 1 & (\text{for } 1 < -X), \end{cases} \quad (15)$$

where

$$C_V \equiv \frac{\sqrt{\pi} \Gamma(\frac{1}{V} + 1)}{\Gamma(\frac{1}{V} + \frac{1}{2})} = \sqrt{\frac{\pi}{V}} + \frac{\sqrt{\pi V}}{8} + O(V^{3/2}), \quad (16)$$

and  $\Gamma(z)$  is the gamma function. The derivation of Eq. (15) is provided in Appendix B.

For  $V \ll 1$ , the crossover points in Eq. (15) are  $-X = 1/C_V^2 \approx V/\pi$  and  $-X = 1/(\lambda^2 C_V^2) \approx V/(\pi\lambda^2)$ . The former corresponds to the soft-solid–liquid crossover and gives  $-x \approx \lambda\tau V/\pi \sim \lambda\tau V$ , which coincides with the characteristic length scale in the fracture profile [Eq. (14)]. The latter corresponds to the liquid–hard-solid crossover and gives  $-x \approx \tau V/(\pi\lambda) \sim \tau V/\lambda$ , which is more localized by a factor of  $\lambda^{-1}$  relative to the scale of the fracture profile.

Equation (15) is consistent with strain-field measurements in carbon-black reinforced styrene butadiene rubber (SBR) [17], which exhibit a plateau near the crack tip followed by a power-law decay. Focusing on the low-velocity case ( $V \approx 0.03C_s \approx 0.6$  m/s), where inertia is negligible, we estimate the soft-solid–liquid crossover length as  $\lambda\tau V/\pi$  using a representative relaxation time  $\lambda\tau \approx 10^{-3}$  s. The resulting value ( $\approx 0.2$  mm) agrees with the experimentally observed plateau width, suggesting that the strain plateau arises from viscoelasticity rather than cohesive-zone effects. (See Sec. S4 in the Supplemental Material [34] for details.)

**Conclusion.** We have derived an analytical expression [Eq. (12)] for the fracture profile during steady-state crack propagation in viscoelastic solids, utilizing the standard linear solid model. The resulting fracture profile exhibits three distinct regions [Eq. (14) and Fig. 3], each characterized by

a power-law exponent, confirming de Gennes' viscoelastic trumpet [25,26]. This expression further provides the power-law prefactors and crossover locations beyond the original scaling argument. We also analyzed the strain field ahead of the crack tip [Eq. (15) and Fig. 5], revealing a plateau between two power-law regimes with  $(-x)^{-1/2}$ , a signature of viscoelasticity. The predictions capture features experimentally observed in fracture profiles [8] and strain fields [17] in filled rubbers. These findings bridge a long-standing gap in viscoelastic fracture mechanics by linking the crack-tip sharpness to propagation velocity [Fig. 4]. By emphasizing the critical role of viscoelasticity, our research could facilitate the development of tough polymer materials by controlling crack-tip processes.

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**Data availability.** No data were created or analyzed in this study.

**Appendix A: Proof of the theorem.** To prove Eq. (10), we introduce a transformation relating the intended displacement field  $u_i(x, y)$  to an auxiliary field  $u_i^0(x, y)$  (for  $i = x, y$ ), defined by

$$u_i^0(x, y) \equiv (1 + \lambda \tau V \partial_x) u_i^{\text{vis}}(x, y), \quad (\text{A1})$$

$$u_i^{\text{vis}}(x, y) \equiv \frac{1}{\tau V} \int_{-\infty}^x u_i(\xi, y) e^{-(x-\xi)/(\tau V)} d\xi. \quad (\text{A2})$$

Equation (A2) implies  $u_i(x, y) = (1 + \tau V \partial_x) u_i^{\text{vis}}(x, y)$ . First, we show that  $u_i^0(x, y)$  satisfies both the balance equation [Eq. (4)] and the boundary conditions in the static crack. Substituting Eq. (7) into Eq. (6), we obtain  $\mathcal{E}_{ij} = (1 + \tau V \partial_x) \mathcal{E}_{ij}^{\text{vis}}$ . Combining this relationship with Eq. (A2), we obtain  $\mathcal{E}_{ij}^{\text{vis}} = (\partial_j u_i^{\text{vis}} + \partial_i u_j^{\text{vis}})/2$ . Substituting Eqs. (6) and (7) into Eq. (5), we obtain  $\sigma_{ij} = 2\mu_0(1 + \lambda \tau V \partial_x) \mathcal{E}_{ij}^{\text{vis}}$ . Combining this equation with Eq. (A1), we obtain the stress-strain relationship in the static crack, Eq. (8) with  $\mathcal{E}_{ij}$  replaced by  $\mathcal{E}_{ij}^0 \equiv (\partial_j u_i^0 + \partial_i u_j^0)/2$ . Therefore,  $u_i^0(x, y)$  satisfies the balance equation in the static crack, as claimed above.

Applying Eq. (2) to Eqs. (A1) and (A2), we derive the boundary conditions for  $u_i^0$  as

$$\begin{aligned} u_x^0(x, \pm L) &= 0 \quad (\text{for } -\infty < x < \infty), \\ u_y^0(x, \pm L) &= \pm \varepsilon L \quad (\text{for } -\infty < x < \infty). \end{aligned} \quad (\text{A3})$$

These boundary conditions for  $u_i^0$  are thus the same as those for  $u_i$  in Eq. (2); additional stress conditions are the same as those in Eq. (2). Therefore,  $u_i^0(x, y)$  is simply the displacement field in the static crack, as described in the Lemma.

Next, we derive  $u_y(x, y)$ . Using Eqs. (3), (A1), and (A2), we obtain  $u_x^0(x, y) = 0$ . Substituting Eq. (8) into Eq. (4) with  $\mathcal{E}_{ij}$  replaced by  $\mathcal{E}_{ij}^0$ , we obtain

$$[(1 + \nu) \partial_x^2 + 2 \partial_y^2] u_y^0(x, y) = 0, \quad (\text{A4})$$

which has the form of the Laplace equation. The solution of Eq. (A4), subject to the boundary conditions [Eq. (A3)] and the

stress conditions in Eq. (2), is  $u_y^0(x, y) = u_y^0(x, y)$ , as provided in Eq. (9). To determine  $u_y(x, y)$ , we employ the variation of constants method. By using Eqs. (9), (A1), and (A2), we derive  $u_y(x, y)$  as given in Eq. (10). Q.E.D.

**Appendix B: Derivation of strain field ahead of the crack tip.** In this Appendix, we analyze the strain field  $\mathcal{E}_{yy}(x, y) \equiv \partial_y u_y(x, y)$  and derive its asymptotic form given in Eq. (15). Using the analytical expression for  $u_y(x, y)$  in Eq. (10), the strain field is expressed as

$$\begin{aligned} \mathcal{E}_{yy}(x, y) &= \mathcal{E}_{yy}^0(x, y) - \left(1 - \frac{1}{\lambda}\right) \int_{-\infty}^x \frac{\partial \mathcal{E}_{yy}^0(\xi, y)}{\partial \xi} \exp\left(-\frac{x-\xi}{\lambda \tau V}\right) d\xi, \end{aligned} \quad (\text{B1})$$

where  $\mathcal{E}_{yy}^0(x, y) \equiv \partial_y u_y^0(x, y)$  is the strain field for the static crack, and the second term represents the viscoelastic contribution arising from crack propagation. By integrating Eq. (B1) by parts, we obtain

$$\mathcal{E}_{yy}(x, y) = \frac{1}{\lambda} \mathcal{E}_{yy}^0(x, y) + \left(1 - \frac{1}{\lambda}\right) \mathcal{E}_{yy}^{\text{vis}}(x, y), \quad (\text{B2})$$

where

$$\mathcal{E}_{yy}^{\text{vis}}(x, y) = \frac{1}{\lambda \tau V} \int_{-\infty}^x \mathcal{E}_{yy}^0(\xi, y) \exp\left(-\frac{x-\xi}{\lambda \tau V}\right) d\xi \quad (\text{B3})$$

is the strain field in the dashpot. Along the line  $y = 0$ , Eq. (9) yields

$$\frac{\mathcal{E}_{yy}^0(x, 0)}{\varepsilon} = \begin{cases} \frac{1}{\sqrt{1-e^x}} & (\text{for } x < 0), \\ 0 & (\text{for } x > 0), \end{cases} \quad (\text{B4})$$

with the dimensionless coordinate  $X \equiv \sqrt{2/(1-\nu)} \pi x/L$ . Substituting Eq. (B4) into Eq. (B3), we obtain the strain field in the dashpot ahead of the crack tip ( $x < 0$ ) as

$$\begin{aligned} \frac{\mathcal{E}_{yy}^{\text{vis}}(x, 0)}{\varepsilon} &= \frac{1}{\mathcal{V}} \int_{-\infty}^X \frac{1}{\sqrt{1-e^\Xi}} \exp\left(-\frac{X-\Xi}{\mathcal{V}}\right) d\Xi \\ &= \sqrt{1-e^X} {}_2F_1\left(1, \frac{1}{2} + \frac{1}{\mathcal{V}}; 1 + \frac{1}{\mathcal{V}}; e^X\right), \end{aligned} \quad (\text{B5})$$

with the dimensionless crack-propagation velocity  $\mathcal{V} \equiv \sqrt{2/(1-\nu)} \pi \lambda \tau V/L$  and the hypergeometric function  ${}_2F_1(a, b; c; z)$ .

Figure 5 shows the strain field  $\mathcal{E}_{yy}(x, 0)$  ahead of the crack tip ( $x < 0$ ) using Eqs. (B2), (B4), and (B5) for  $\lambda \gg 1$ . The strain field exhibits two distinct power-law regimes separated by an intermediate plateau.

To clarify the asymptotic behavior near the crack tip, we expand  $\mathcal{E}_{yy}^0(x, 0)$  in Eq. (B4) for  $X < 0$  as

$$\frac{\mathcal{E}_{yy}^0(x, 0)}{\varepsilon} = \frac{1}{\sqrt{1-e^X}} = \frac{1}{\sqrt{-X}} + \frac{\sqrt{-X}}{4} + O((-X)^{3/2}), \quad (\text{B6})$$

indicating a singularity of the form  $(-X)^{-1/2}$  as  $X \rightarrow -0$ . Similarly, expanding Eq. (B5) for  $X < 0$  yields

$$\frac{\mathcal{E}_{yy}^{\text{vis}}(x, 0)}{\varepsilon} = C_{\mathcal{V}} - \frac{2\sqrt{-X}}{\mathcal{V}} + O((-X)^1), \quad (\text{B7})$$

where  $C_V$  is defined in Eq. (16). Equation (B7) indicates that  $\mathcal{E}_{yy}^{\text{vis}}(x, 0)$  remains finite as  $X \rightarrow -0$ . Substituting Eqs. (B6) and (B7) into Eq. (B2), we obtain the asymptotic form of the strain field as

$$\frac{\mathcal{E}_{yy}(x, 0)}{\varepsilon} = \frac{1}{\lambda\sqrt{-X}} + \left(1 - \frac{1}{\lambda}\right)C_V + O(\sqrt{-X}). \quad (\text{B8})$$

As  $X \rightarrow -0$ , the first term in Eq. (B8) dominates and diverges as  $(-X)^{-1/2}$ , corresponding to the hard-solid regime. As  $-X$  increases, the singular term in Eq. (B8) rapidly decays, par-

ticularly for large  $\lambda$ , and  $\mathcal{E}_{yy}(x, 0)$  begins to exhibit a plateau governed by the second term in Eq. (B8), corresponding to the viscoelastic liquid regime. As  $-X$  increases further, the second term (the viscoelastic contribution) in Eq. (B1) becomes negligible, and the first term becomes dominant. Thus, the static strain field  $\mathcal{E}_{yy}^0(x, 0) \approx 1/\sqrt{-X}$  emerges near  $-X \approx 1$  [see Eq. (B6)], corresponding to the soft-solid regime. These results show that the strain field ahead of the crack tip  $\mathcal{E}_{yy}(x, 0)$  exhibits three distinct regimes: a near-tip hard-solid regime, an intermediate liquid-like plateau, and a distant soft-solid regime, as captured by Eq. (15).

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