

Extracting randomness from magic quantum states

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Magic quantum states (nonstabilizer states) play a pivotal role in fault-tolerant quantum computation. Simultaneously, random resources have emerged as a key element in various randomized techniques within contemporary quantum science. In this study, we establish a direct connection between these two notions. More specifically, our research demonstrates that when a subsystem of a quantum state is measured, the resultant unmeasured part of the system can exhibit a high degree of randomness that can be enhanced by the inherent correlations of the underlying magic quantum state. Our findings suggest an approach to quantifying correlations within magic quantum states beyond the conventional paradigm of entanglement, and introduce an efficient approach for leveraging such correlations to generate random quantum resources.

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Introduction. The field of quantum computing has experienced remarkable advancements in recent decades [1–4], yet the essential source of power driving the quantum speedup continues to be elusive. This complex issue is inherently multifaceted, with its resolution dependent on the theoretic framework for characterizing the underlying resources. For instance, entanglement was known to be insufficient: Clifford circuits produce entanglement among qubits but nevertheless can be efficiently simulated classically [5]. Alternatively, the same set of operations constitutes the basic building block for the stabilizer formalism of quantum computing, which, when augmented with the so-called *magic quantum states* [6–9], makes a blueprint for universal fault tolerant quantum computation. This paradigm thus recognizes magic states as a crucial quantum resource. The exploration and exploitation of the physical properties of quantum magic states have recently emerged as an active area of research [10–13].

Randomness stands as another pivotal resource in quantum information science, playing a crucial role in numerous protocols for quantum information processing. Various applications, including quantum device benchmarking [1,14–17], tomography [18,19] and quantum channel approximation [20–22], rely on random quantum resources. In a recent breakthrough [23,24], it was demonstrated that random resources can be extracted from complex many-body quantum states. Specifically, when a subsystem of a strongly correlated many-body state undergoes projective measurements, the unmeasured subsystem yields a highly random quantum state ensemble, which nevertheless may not be directly prepared efficiently. This result not only offers an efficient protocol for generating random quantum states as a practical resource but

also opens up new avenues for exploring fundamental physical problems, such as nonequilibrium quantum dynamics [25–27], and, as will be delved into in this work, characterization of quantum states.

In this context, we unveil a profound connection between randomness and magic (the degree of nonstabilizerness of a quantum state, as will be defined in the following). Specifically, we demonstrate that typical quantum states exhibiting higher magic values yield post-measurement projected ensembles with greater randomness. The degree of magic and randomness are precisely quantified using proper metrics, namely, stabilizer Rényi entropy and quantum state t -designs.

The significance of this result is twofold: On a conceptual level, it unravels an intrinsic randomness associated with magic quantum states. This randomness is a manifestation of many-body correlations that are not captured by conventional measures such as entanglement. Consequently, these findings shed new light on the role of magic as a quantum resource. Simultaneously, it proposes an efficient scheme for leveraging such correlations for random quantum state generation with only Clifford circuit operations. The later comprises operations that are naively fault tolerant in quantum computing architectures protected by stabilizer error-correcting codes, and therefore operationally “cost free.” In the following, we will lay down the essential components of the setup and present the main theorem, as well as supporting evidence from numerical simulations.

Projected ensemble. For any ensemble of pure quantum states, $\mathcal{E} = \{(p_i, |\psi_i\rangle)\}$, the level of its randomness can be quantified rigorously using the notion of quantum state design. Define the t th moment of $\mathcal{E}$ as

$$\rho_{\mathcal{E}}^{(t)} = \sum_i p_i |\psi_i\rangle\langle\psi_i|^{\otimes t}, \quad (1)$$

the average of its t th tensor power, and denote especially the t th moment of the Haar random ensemble by $\rho_{\text{Haar}}^{(t)}$. We say that $\mathcal{E}$ is a *quantum state t -design* if $\rho_{\mathcal{E}}^{(t)} = \rho_{\text{Haar}}^{(t)}$, that is, $\mathcal{E}$ is

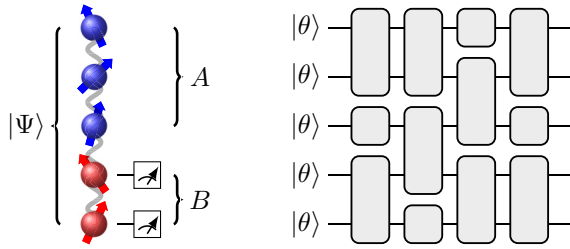


FIG. 1. Left: A subsystem B of a complex many-body state $|\Psi\rangle$ is subject to projective measurement. The unmeasured subsystem A is projected to an ensemble of random states. Right: The qubits are prepared in a product magic state, parametrized by θ , which is then transformed by a random Clifford circuit, generating a state with the same magic value.

indistinguishable from Haar measure up to its t th moment. In practice, an exact t -design can never be reached. It is therefore useful to have a measure of the approximate closeness of the ensemble's distribution to the Haar measure. This idea is captured by an ε -approximate quantum state t -design, an ensemble $\mathcal{E}$ for which

$$d^{(t)}(\mathcal{E}) \equiv \|\rho_{\mathcal{E}}^{(t)} - \rho_{\text{Haar}}^{(t)}\| \leq \varepsilon. \quad (2)$$

Here, $\|\cdot\|$ can be any proper norm over density matrices. Throughout this work, we will employ two commonly used norms, namely, trace norm $\|\rho\|_1 \equiv \text{tr}\sqrt{\rho^\dagger \rho}$ and the Hilbert-Schmidt norm $\|\rho\|_2 \equiv \sqrt{\text{tr}\rho^\dagger \rho}$, and denote the corresponding distance as $d_{\text{trace}}^{(t)}(\mathcal{E})$ and $d_{\text{HS}}^{(t)}(\mathcal{E})$, respectively.

A basic ingredient of our work is the use of a subsystem measurement to generate an ensemble of random states (Fig. 1, left). Consider a bipartite pure state $|\Psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ over systems A and B . We take the A and B subsystems to be composed of N_A and N_B qubits, respectively. Denote $N \equiv N_A + N_B$, $d_A \equiv 2^{N_A}$, $d_B \equiv 2^{N_B}$, and $d \equiv 2^N$, and let $\{|i\rangle_B\}_{i=1}^{d_B}$ be a basis of product states for $\mathcal{H}_B$. The outcomes of a projective measurement on the B subsystem with respect to this basis, followed by a discarding of the B subsystem, form an ensemble of states $\mathcal{E}_{|\Psi\rangle} \equiv \{(p_i, |\psi_i\rangle)\}$ on the A subsystem, where the probability p_i is

$$p_i = \langle \Psi | (I_A \otimes |i\rangle\langle i|) | \Psi \rangle \quad (3)$$

and the corresponding projected state of A is

$$|\psi_i\rangle = (I_A \otimes |i\rangle\langle i|) |\Psi\rangle / \sqrt{p_i}. \quad (4)$$

We will refer to the ensemble $\mathcal{E}_{|\Psi\rangle}$ as the *projected ensemble* [28] generated by $|\Psi\rangle$, whose statistics can be quantified by the t -design framework discussed above.

Conceivably, the distribution of the projected ensemble reflects the intrinsic randomness of the underlying generating state; a more complex many-body state, e.g., with stronger entanglement, may produce a projected ensemble with a higher degree of randomness. Indeed, this intuition was captured formally in a recent result in [23,24]. That is, if the premeasurement state $|\Psi\rangle$ itself is sampled from an ε' -approximate t' -design (greater t' indicates higher complexity), the projected ensemble $\mathcal{E}_{|\Psi\rangle}$ will form an ε -approximate t -design with probability $1 - \delta$ for any positive ε , δ , and integer $t \geq 1$, given that N_A and N_B are sufficiently large. This provides an

efficient scheme for generating random state ensembles from a single (complex) many-body state with projective measurements without having to sample from random quantum circuit realizations, as in the conventional approach.

Magic states. The notion of magic states stems from the stabilizer formalism of fault-tolerant quantum computing. The latter was built upon a finite and nonuniversal set of unitary operations, the Clifford gates. Due to their nonuniversality, quantum circuits consisting of only Clifford operations can only produce a proper subset of all quantum states (from a fixed initial state). States that cannot be produced by Clifford circuits applied on initial computational basis states are called nonstabilizer states. Remarkably, it is known that by supplementing stabilizer operations with certain nonstabilizer states, one may achieve universal quantum computation. The degree of nonstabilizerness of a state is typically referred to as its “magic,” and may be quantified by various measures [10–12,29–33]. Several methods have been developed for numerically computing resource-theoretic measures of magic [30,34–36], and the relationships between these measures and entanglement properties of quantum states have been explored [37–39]. These concepts have found applications across a variety of areas, including classical simulation of quantum circuits with few non-stabilizer components [29,31,32] and the study of measurement-induced phase transitions [40,41]. Of particular interest to our work is a recently developed family of measures of magic known as stabilizer entropies [13,42–44], which have the advantage of being defined in an efficiently computable way rather than through a variational definition. In particular, we will employ the stabilizer linear entropy from this family.

More precisely, denote by $\mathcal{P}_n$ the set of all tensor products $P_1 \otimes \cdots \otimes P_n$, where P_i is a single qubit Pauli operator for $1 \leq i \leq n$. Note that $\mathcal{P}_n$ is merely the set of all Pauli strings on n systems and not the whole Pauli group, which includes global phase factors of $\pm 1, \pm i$. For any given state $|\psi\rangle$ over n qubits and Pauli string $P \in \mathcal{P}_n$, define $\Xi_P(|\psi\rangle) \equiv \frac{1}{2^n} \langle \psi | P | \psi \rangle^2$, the (normalized) square of the expectation value of P over state $|\psi\rangle$. From this, we may define the vector $\Xi(|\psi\rangle) \equiv (\Xi_P(|\psi\rangle))_{P \in \mathcal{P}_n}$, which has 4^n real components. The *stabilizer linear entropy* is defined to be

$$M_{\text{lin}}(|\psi\rangle) = 1 - 2^n \|\Xi(|\psi\rangle)\|_2^2, \quad (5)$$

where $\|\cdot\|_2$ is simply the Euclidean norm on $\mathbb{R}^{4^n}$. This measure satisfies the properties desired for good measures of nonstabilizerness from the point of view of resource theory [13] and offers the computational advantages for the purpose of the present work.

Connecting magic and randomness. We are now ready to establish the link between magic and randomness. Suppose that we generate an ensemble $\mathcal{E}_{|\Psi\rangle}$ from a state $|\Psi\rangle$ via the subsystem measurement introduced above. One might ask how the randomness of the resulting projected ensemble, as measured by distance $d^{(t)}(\mathcal{E}_{|\Psi\rangle})$ in (2), depends on the magic of state $|\Psi\rangle$, as quantified by the stabilizer linear entropy $M_{\text{lin}}(|\Psi\rangle)$ in (5). It is worth stressing that “magic” is an operational notion defined with respect to the stabilizer formalism: states that are transformable by Clifford operations have the same magic value. This notion underlies other resource the-

ories, such as entanglement: states transformable by local operation and classical communications are defined to have the same entanglement. Therefore, we treat magic as a property of a *class* of states; the magic-randomness relationship is then established by averaging states with the same magic value, i.e.,

$$\langle d^{(t)}(\mathcal{E}_{C|\Psi}) \rangle_{C_N} \equiv \int_{C \in C_N} dC d^{(t)}(\mathcal{E}_{C|\Psi}). \quad (6)$$

Note that in the above equation the average is equivalently performed over the Clifford group C_N , rather than a general unitary group. The task is then to determine the relationship between $\langle d^{(t)}(\mathcal{E}_{C|\Psi}) \rangle_{C_N}$ and $M_{\text{lin}}(|\Psi\rangle)$. This leads to the key result of this study.

Theorem 1. Up to an error that decays with respect to d_A , the average randomness extracted from $|\Psi\rangle$ is described by

$$[\langle d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi}) \rangle]_{C_N}^2 = \alpha - \beta M_{\text{lin}}(|\Psi\rangle) + \varepsilon, \quad (7)$$

where α and β are numerical factors that depend on both subsystem dimensions.

Furthermore, we estimate the bound on the error ε to be

$$\frac{|\varepsilon|}{[\langle d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi}) \rangle]_{C_N}^2} = \Theta\left(\frac{1}{d_A}\right), \quad (8)$$

which implies that the error ε decays exponentially faster in N_A than the exact quantity $[\langle d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi}) \rangle]_{C_N}^2$. This error term comes from estimating the average of a quotient by separately averaging the numerator and denominator. Proof of the theorem and derivation of the error estimate are relegated to the Supplemental Material (SM) [45] (see also Refs. [52,53] therein). We add that we consider the average of the square of $d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi})$ rather than the Hilbert-Schmidt distance itself because doing so allows us to avoid the difficulty of averaging the square root of the operator trace, and hence deduce a rigorous result. The average of squares $[\langle d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi}) \rangle]_{C_N}^2$ is expected to approximate the square of average $\langle d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi}) \rangle_{C_N}^2$ for large systems due to concentration of measure, with an error suppressed exponentially fast with the system size. Rigorously proving this conjecture is challenging due to the complexity of evaluating higher moments of the Clifford group. Instead, we provide supporting numerical evidence in the SM.

While previous work has analyzed the magic content of states in the context of quantum typicality [46,47], this theorem implies a direct connection between the randomness one may extract from quantum states and their magic. It gives a new meaning to the stabilizer linear entropy of an arbitrary state in terms of the extractable randomness from that state. We note that the “stabilizer purification” framework of [41] may provide a useful approach for analyzing the mechanism by which this relationship between randomness and magic arises. Finally, we add that since the coefficients tend to zero as $d_A, d_B \rightarrow \infty$, Theorem 1 confirms that as one increases the size of the subsystem measurement and of the surviving systems, the values of $d_{\text{HS}}^{(2)}(\mathcal{E}_{C|\Psi})$ tend to zero, i.e., the randomness becomes near perfect.

Simulations. Due to the rapid growth in the size of the Clifford group C_N with respect to N , it is difficult to numerically compute our quantifier of randomness

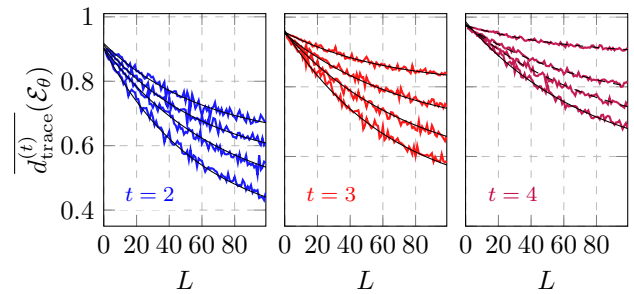


FIG. 2. We numerically estimate $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ (colored solid lines) with $N_A = 2$ and $N_B = 4$ and provide an exponential fit (black solid line). For each circuit depth, the numerical average is taken over 100 realizations of a random Clifford circuit of the given depth initialized to $|\Psi(\theta)\rangle$. We plot $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ as a function of circuit depth assuming initial states $|\Psi(\theta)\rangle$ of magic value $M_{\text{lin}}(|\Psi(\theta)\rangle) = 0, 0.25, 0.5, 0.75$, with lower curves corresponding to higher magic values.

$\langle d^{(t)}(\mathcal{E}_{C|\Psi}) \rangle_{C_N}$. To circumvent this issue, we estimate $\langle d^{(t)}(\mathcal{E}_{C|\Psi}) \rangle_{C_N}$ by averaging across random Clifford circuits of restricted length and extrapolating the results. Furthermore, another advantage of considering simulations of random Clifford circuits together with subsystem measurements is that they represent a physical quantum information processing task in which nonstabilizerness can affect the formation of approximate state designs.

We consider an N -qubit product state $|\Psi(\theta)\rangle = 2^{-N/2}(|0\rangle + e^{i\theta}|1\rangle)^{\otimes N}$, which has a magic value $M_{\text{lin}}(|\Psi(\theta)\rangle) = 1 - (1 + \cos^4 \theta + \sin^4 \theta)^N / d$ [37]. The state $|\Psi(\theta)\rangle$ is evolved according to a random Clifford circuit $C_L = \prod_{i=1}^L U_i$ of depth L . Each layer U_i of this circuit (Fig. 1, right) acts on two registers randomly sampled from the N total registers and is itself randomly selected from the two-qubit gate set $\{\text{CNOT}, H \otimes I, S \otimes I, I \otimes H, I \otimes S\}$, where H is the Hadamard gate and $S = \text{diag}(1, e^{i\pi/4})$ is the phase shift gate. The projected ensemble $\mathcal{E}_{C_L|\Psi(\theta)}$ over the A subsystem is then obtained by simulating measurements on the state $C_L|\Psi(\theta)\rangle$ over the B subsystem. Define $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ to be the average of $d_{\text{trace}}^{(t)}(\mathcal{E}_{C_L|\Psi(\theta)})/2$ over all possible Clifford circuits C_L of depth L composed of two qubit unitaries in the previously described manner. Here, we insert a normalization factor of $1/2$ to ensure that $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ lies in $[0, 1]$ for illustrative purposes. We emphasize that the bar is used to indicate an average over all Clifford circuits of a specified length, rather than the entire Clifford group as in (6). A numerical estimate for the trace distance $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ is obtained by averaging over many realizations of a random Clifford circuit. By varying the depth of the simulated circuit, we obtain data for $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ as a function of circuit depth L .

Numerically, we see that the average distance $\overline{d_{\text{trace}}^{(2)}(\mathcal{E}_\theta)}$ of the post-measurement ensemble decays exponentially with respect to the circuit depth L , with values of θ corresponding to a higher magic value producing a faster decay (Fig. 2). The variation in the decay curves shows that the magic of many-body quantum states influences the degree of randomness that may be extracted from them via local measurements. Furthermore, it was observed that the asymptotic value of the

decay in average trace distance falls exponentially with respect to N_B for fixed N_A and magic, while it decreases linearly with respect to $M_{\text{lin}}(|\Psi(\theta)\rangle)$ for fixed N_A and N_B . The latter is consistent with Theorem 1.¹

The exponential decay of the $\overline{d_{\text{trace}}^{(t)}(\mathcal{E}_\theta)}$ with respect to circuit depth may be seen as a general property of random circuits: For many random circuit configurations, the behavior of $d^{(t)}(\mathcal{E})$ has been studied [48,49], where in this case $\mathcal{E}$ is the ensemble of all possible states that may be evolved from a fixed initial state of the random circuit. Generally, any positive value ε of $d^{(t)}(\mathcal{E})$ may be achieved provided the circuit depth L scales at least linearly with respect to t , number of registers N , and $\log(1/\varepsilon)$. The linear growth of this threshold with respect to $\log(1/\varepsilon)$ suggests that $d^{(t)}(\mathcal{E})$ decays exponentially with respect to circuit depth in these random circuit paradigms. Since the measurements in our scheme introduce an extra degree of randomness, our value of $\overline{d^{(t)}(\mathcal{E}_\theta)}$ should decay at least exponentially fast, as was observed numerically. Furthermore, the exponential decay of the asymptotic values with respect to N_B follows from Theorem 2 of Ref. [23], where the bounds on the value of N_B necessary to produce a post-measurement ensemble $\mathcal{E}$ with a value of $d_{\text{trace}}^{(t)}(\mathcal{E}) \leq \varepsilon$ require that N_B scale exponentially faster than $\log(1/\varepsilon)$.

We conclude our numerical study with a demonstration of Theorem 1. As before, we define $\langle [d_{\text{HS}}^{(t)}]^2(\mathcal{E}_\theta) \rangle$ to be the average of $[d_{\text{HS}}^{(t)}(\mathcal{E}_{C_L|\Psi(\theta)})]^2$ over all possible Clifford circuits C_L of depth L composed according to the aforementioned two-qubit gate prescription. By numerically estimating $\langle [d_{\text{HS}}^{(t)}]^2(\mathcal{E}_\theta) \rangle$ with many iterations of a finite-depth Clifford circuit, we obtain data for the exponential decay of $\langle [d_{\text{HS}}^{(t)}]^2(\mathcal{E}_\theta) \rangle$ as a function of circuit depth. Fitting this data to an exponential decay function, we are able to extrapolate to infinite depth so that we obtain an approximation for $\langle [d_{\text{HS}}^{(t)}(\mathcal{E}_{C|\Psi(\theta)})]^2 \rangle_{C_N}$, which we abbreviate to $\langle [d_{\text{HS}}^{(t)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$. By repeating this process for various angles θ , we obtain $\langle [d_{\text{HS}}^{(t)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$ as a function of $M_{\text{lin}}(|\Psi(\theta)\rangle)$, as seen in Fig. 3 for $t = 2$. Note the linear nature of the relationship between $\langle [d_{\text{HS}}^{(2)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$ and magic. Furthermore, the numerically simulated values for $\langle [d_{\text{HS}}^{(2)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$ fall below those of the plotted analytical approximation (7). We have numerically verified that these data fall within the error threshold given by (8). Further analysis shows that we can expect the deviation between the values of $\langle [d_{\text{HS}}^{(2)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$ and its analytical approximation to be uniformly negative (see the SM [45]). Lastly, our numerical simulations (Fig. 3, right) suggest that the linear relationship between $\langle [d_{\text{HS}}^{(t)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$

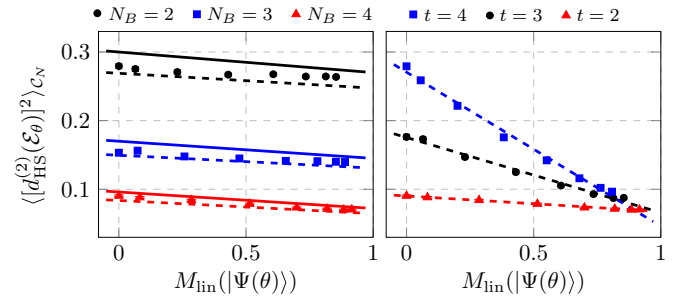


FIG. 3. Numerical estimates for the average extracted second degree randomness $\langle [d_{\text{HS}}^{(2)}(\mathcal{E}_\theta)]^2 \rangle_{C_N}$ of the initial state $|\Psi(\theta)\rangle$ as a function of its magic value $M_{\text{lin}}(|\Psi(\theta)\rangle)$ (data points) for various system sizes and moments t . The numerical estimates were obtained by averaging over 100 realizations per circuit depth of random Clifford circuits with depth ranging from 1 to 200, and extrapolating to larger depth via an exponential fit. Left: Solid lines are the analytic approximation (7). The deviation from the analytic approximation (see the SM [45]) by an amount given by our error estimate (8) is shown by dashed lines. Error bars indicating the standard error of the asymptotic fitting parameter were plotted and are obscured by the data points. All results were obtained for $N_A = 5$. Right: Data were obtained for $N_{A(B)} = 2(4)$ at $t = 4$, $N_{A(B)} = 4(3)$ at $t = 3$, and $N_{A(B)} = 5(4)$ at $t = 2$. Dashed lines represent the best linear fits.

and $M_{\text{lin}}(|\Psi(\theta)\rangle)$ holds for higher values of t . Moreover, the enhancement of randomness by magic is more prominent for larger values of t , further justifying the efficiency of the protocol for random state generation.

Discussion. One of the most notable aspects of this study is the transformative influence of the magic inherent in the initial state on the resulting randomness in the projected ensemble. It is crucial to stress that the Clifford group constitutes a quantum unitary 3-design [50]. This implies that the produced ensemble (of the total system before projective measurement) by random Clifford circuits forms a quantum state 3-design as well, irrespective of the initial input state. Consequently, one might naturally anticipate that the projected ensemble's randomness level should be independent of the initial state, contrary to our observation.

The observed augmentation of the randomness level of the projected ensemble due to the amount of magic reflects internal correlations among subsystems within each *single* quantum state of the ensemble. These internal correlations are not manifest in the quantum state without subsystem projective measurements; they also go *beyond* what conventional metrics can capture. For example, the entanglement of a bipartite pure quantum state is completely determined by the reduced density matrix of a subsystem [51], representing a first-moment average of the projected ensemble [$t = 1$ in (1)]. The connection between magic and randomness thus suggests a novel approach for characterizing and harnessing correlations in magic states.

Simultaneously, leveraging the benefits of high-order quantum state designs necessitates multiple copies of each states within the ensemble. This poses a significant constraint on the precision of random state generation protocols, challenging both the conventional methods, such as sampling random uni-

¹At great depth, the random circuit average $\overline{d_{\text{trace}}^{(2)}(\mathcal{E}_\theta)}$ approximates the Clifford group average $\frac{1}{2} \langle d_{\text{trace}}^{(2)}(\mathcal{E}_{C|\Psi(\theta)}) \rangle_{C \in C_N}$, whose value we can expect to approximate $\sqrt{[\frac{1}{2} \langle d_{\text{trace}}^{(2)}(\mathcal{E}_{C|\Psi(\theta)}) \rangle^2]_{C \in C_N}}$ due to measure concentration. Furthermore, we may then use the inequality $d_{\text{trace}}(\cdot) \leq \sqrt{d} d_{\text{HS}}(\cdot)$ to estimate $\langle d_{\text{trace}}^{(2)}(\mathcal{E}_{C|\Psi(\theta)}) \rangle_{C \in C_N}$ by a constant multiple of the square root of the right-hand side of (7). Due to the small slope in the linear function (7), we can then deduce that the asymptotic values of $\overline{d_{\text{trace}}^{(2)}(\mathcal{E}_\theta)}$ should decrease approximately linearly with respect to $M_{\text{lin}}(|\Psi(\theta)\rangle)$ (See Fig. 3). Finally, we note the scaling factor of the exponential decay appears to be roughly constant with respect to $M_{\text{lin}}(|\Psi(\theta)\rangle)$ when holding N_A and N_B fixed.

tary circuits, and the recently proposed projected ensemble approach. The strategy proposed in this work, involving the use of Clifford circuits, marks a crucial first step toward developing a fault tolerant protocol for random state generation.

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Data availability. The data that support the findings of this article are not publicly available because of legal restrictions preventing unrestricted public distribution. The data are available from the authors upon reasonable request.

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