

Tuning between a fractional topological insulator and competing phases at $\nu_T = 2/3$

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We study a spinful, time-reversal symmetric lowest Landau level model for a flatband quantum spin Hall system at total filling fraction $\nu_T = 2/3$. Such models are relevant, e.g., for spin-valley locked moiré transition metal dichalcogenides. The opposite Chern number of the two spins hinders the formation of a quantum Hall ferromagnet, instead favoring other phases. We study the phase diagram in dependence on different short-range Haldane pseudopotentials v_m and uncover several phases: a fractional topological insulator, a phase-separated state, a spin-polarized fractional quantum Hall state, and the partially particle-hole transformed Halperin (111) state. The effect of the pseudopotentials v_m depends on the parity of m , the relative angular momentum.

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I. INTRODUCTION

Moiré materials are distinguished by their remarkable tunability. Due to their two-dimensional nature, electrostatic doping can be used to tune the density. Moreover, the constituent layers can be chosen from a broad variety of stackable two-dimensional van der Waals materials, offering access to a wide range of underlying band structures [1]. Crucially, these band structures can be further engineered through control over twist angle, displacement field, substrate choice (which may break inversion symmetry, as in the case of hexagonal boron nitride, or induce spin-orbit coupling, as in transition metal dichalcogenides), and perturbations such as strain. Strain, in particular, has a pronounced influence on the electronic structure of moiré systems [2–5]. While *in situ* control over strain remains limited, spatial variations naturally present in fabricated devices can be imaged with high resolution [6–8], allowing for the identification of regions with desirable strain profiles [9]. These capabilities illustrate the significant versatility available for tailoring the single-particle band structure in moiré heterostructures. However, the many-body physics of these systems depends equally on the interaction term in the Hamiltonian, whose controllability is of growing experimental and theoretical interest.

Engineering electron-electron interactions is a key step toward realizing correlated and topological phases. One of the most direct means of tuning interactions is through screening. In twisted bilayer graphene, for example, the distance to metallic gates has been employed as a handle to modulate screening [10,11]. Proximity to a tunable metallic layer presents another avenue [12], as does the use of substrates

with high and controllable dielectric constants. Strontium titanate, with its tunable dielectric constant reaching values on the order of 10 000, exemplifies this approach and offers *in situ* tunability [13]. Additionally, multilayers of exfoliable high-dielectric materials such as HfO₂ can be used to engineer the spatial profile of the interaction potential [14]. Interactions are also influenced by electron-phonon coupling, which may mediate effective attraction and influences the phase diagram of twisted bilayer graphene [15,16]. This coupling is sensitive to substrate alignment and thus experimentally tunable [17].

In future experiments, interaction engineering promises to elevate moiré materials into a regime akin to tunable quantum simulators, as realized in cold atom systems [18]. Among the most intriguing theoretical targets is the fractional topological insulator (FTI)—a time-reversal (TR) symmetric state exhibiting both topological order and fractionalization [19–24]. This state realizes the long-sought fractional quantum spin Hall effect. Moiré systems offer a promising platform for realizing this elusive phase, but interaction control is essential for its stabilization [14].

Twisted transition metal dichalcogenide heterostructures, and specifically twisted MoTe₂, host pairs of TR-related bands near charge neutrality for small twist angles [25]. These bands are flat, of opposite Chern number, and contain electrons of opposite spin flavor due to the spin-orbit coupling induced spin-valley locking. This band structure at small twist angles $\theta \sim 2^\circ$ has been proposed to be very close to a Landau level description [26–28], allowing for a simplified modeling of the system using TR symmetric Landau level “bilayers.” The layers correspond to the spin index that by TR symmetry directly matches the sign of the Chern number associated with each spin species through $\text{sgn}(\sigma_z) = \text{sgn}(C)$ [23,26]. (Note that unlike physical bilayers the spin species are not spatially separated.) Such quantum Hall systems with additional degrees of freedom are known to exhibit exotic phases [29–32]. Unlike the model of interest above, however, where the two spin species correspond to opposite Chern numbers C , the multicomponent systems [33,34] mostly studied so far consist of bands of equal Chern number. The magnetic

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field setup hinders TR symmetric phases and, as a consequence, Landau levels of electrons of both spin species with $(C_\uparrow, C_\downarrow) = (1, -1)$ were not investigated beyond a few initial studies [35–37]. However, with the newly gained access to TR-related Chern bands in transition metal dichalcogenide band structures, there have been many developments toward understanding the opposite Chern setup [14,38–53]. Within this configuration, fractional Chern insulators (FCIs) [54–56] and fractional topological insulators might arise, where the former have already been realized experimentally in twisted MoTe_2 [57–62]. As of yet, fractional topological insulators have been experimentally elusive.

An exact diagonalization study of a TR-invariant lowest Landau level (LLL) revealed that the existence of the FTI depends strongly on interaction amplitudes, specifically the suppression of the on-site part of the Coulomb interactions. Further, they showed that the long-range interactions for $\nu_\uparrow = \nu_\downarrow = 1/3$ filling are beneficial to the FTI [14]. This indicates a clear departure from the limit of two decoupled FCIs, whose ground state is the tensor product of TR-conjugated zero-energy eigenstates of the Laughlin parent Hamiltonian, i.e., the Laughlin state $|\frac{1}{3}\rangle \otimes |\frac{1}{3}\rangle$ [63]. In Ref. [52], it was found that there are no zero-energy eigenstates of the coupled $\nu_T = 2/3$ TR symmetric quantum Hall Hamiltonian and the only tractable parent Hamiltonian is the above limit of two decoupled Laughlin Hamiltonians, despite complete absence of interspin correlations. This Hamiltonian only exhibits short-range intraspin correlations, but has been shown to be stable for small short-range interspin interactions, before transitioning into a phase-separated state upon increased amplitude [35]. Phase separation (PS) is also observed in Refs. [38,64], where the latter reference additionally studied the competition between the FTI and a superconducting phase.

These findings motivate the investigation of phase diagrams in terms of pseudopotential interactions to find ways to stabilize FTIs and illuminate the competing phases. To achieve this, we draw on the description of the LLL in the symmetric gauge and investigate the interaction Hamiltonian in terms of its coefficients, the Haldane pseudopotentials v_m , where m is the relative angular momentum between the two interacting particles [65]. We use the pseudopotential formalism to define toy Hamiltonians at $\nu_\uparrow = \nu_\downarrow = 1/3$ filling featuring few active pseudopotentials in order to investigate their individual action on the TR symmetric LLL. They manifest in the nature of the ground state for given pseudopotential configuration, which may change upon variation of the amplitudes. This information can be gathered in two-dimensional phase diagrams in terms of the on-site and a single long-range pseudopotential (v_0, v_m) . We observe the FTI in the phase diagrams for all m . Furthermore, an even-odd effect in m is apparent. Phase diagrams with equal m -parity are similar to each other, but different for opposite parities, suggesting a qualitative equivalence $v_{2n} \sim -v_{2n+1}$. We show the schematic phase diagram in Fig. 1.

II. METHODS

We investigate a spinful $\nu_T = 2/3$ filled lowest Landau level where the two spin species carry opposite Chern number. The kinetic energy is fully quenched by virtue of being in the

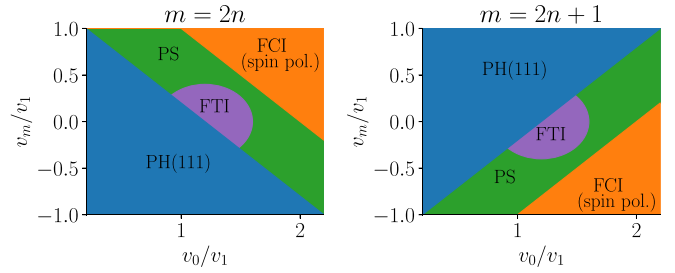


FIG. 1. Schematic phase diagram at $\nu_T = 2/3$ as a function of Haldane pseudopotentials v_0/v_1 and v_m/v_1 for even (left) and odd (right) angular momentum m . The phases listed in the diagram are a partially particle-hole transformed Halperin 111 state PH(111), an FTI, PS, and a spin-polarized FCI.

LLL. The model interaction Hamiltonian is built by defining the Haldane pseudopotentials

$$v_m = \langle m | \hat{V} | m \rangle, \quad (1)$$

where $|m\rangle$ are two-body eigenstates of the LLL Hamiltonian in the relative angular momentum $m > 0$ basis. The Hamiltonian can be written using projectors $\mathcal{P}_m^{\sigma\sigma'}$ into states of relative angular momentum m and spins σ and σ' as

$$H_{\text{int}} = \sum_{m,\sigma} v_m \mathcal{P}_m^{\sigma\sigma} + \frac{1}{2} \sum_{m,\sigma,\bar{\sigma}} v_m \mathcal{P}_m^{\sigma\bar{\sigma}}, \quad (2)$$

where $\bar{\sigma}$ indicates the opposite spin species to σ . Due to fermionic antisymmetry, intraspin interactions only consist of odd m pseudopotentials, whereas interspin interactions contain the full set $v_{|m|}$.

Using exact diagonalization, we investigate the model on the torus with aspect ratio 1 and an angle $\pi/2$ between the two axes. The system size is determined by $N_\Phi = 15$ flux quanta threading the torus, with $N_e = 10$ electrons. We restrict to the LLL and neglect Landau level mixing. However, we note that band mixing in twisted MoTe_2 cannot always be neglected and is often relevant in realistic systems [14]. The torus manifests the ground-state degeneracy (GSD) of topologically ordered states in the $E(\mathbf{k})$ spectra, where $\mathbf{k} = (k_x, k_y)$ is the many-body momentum of the system [56]. For the system size of $N_\Phi = 15$ flux quanta, the momentum components take values in $\{0, \dots, 14\}$, due to periodic boundary conditions in both directions. For a fully spin-polarized system, additional symmetries reduce the many-body momentum phase space as discussed in the Supplemental Material [66,67].

The parent Hamiltonian for the FTI in the limit of decoupled FCIs has a single intraspin pseudopotential $v_1 > 0$. The corresponding spectrum exhibits an exact ninefold degenerate ground state. This degeneracy can be captured by the FTI quality parameter s_9/Δ_9 , which measures the maximal energy difference between neighboring ground states, i.e., the spread $s_9 = \max(E_{i+1} - E_i)$, $i \in \{0, \dots, 8\}$, and compares it to the spectral gap that separates the ground-state manifold from the excited states, $\Delta_9 = E_9 - E_8$, where E_i are eigenstate energies ordered by magnitude. In the above limit, the degeneracy of the ground states is exact such that $s_9 = 0$ and, thus, the FTI quality parameter is zero, implying an FTI of optimal quality [Fig. 2(a)]. Long-range interactions induce departure from the

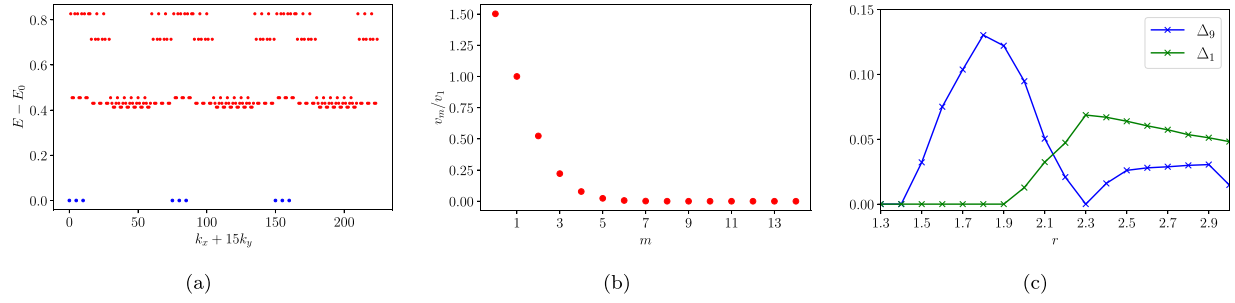


FIG. 2. (a) Spectrum of the reference state of two decoupled Laughlin states on the torus for the system size $N_\Phi = 15$ at $\nu_\uparrow = \nu_\downarrow = 1/3$, where N_Φ is the number of flux quanta that thread the torus. The nine degenerate ground states are marked in blue color and aid in identifying the ground-state momentum sectors of the FTI. (b) Numerical values for the normalized pseudopotentials v_m/v_1 as a function of relative angular momentum for a hard-core interaction $V_{\text{HC}} = \Theta(R - |z|)$, where $\Theta(z)$ is the Heaviside step function, z are the relative holomorphic coordinates of the two particles, and R is the range of the interaction. The radius is measured by the dimensionless quantity $r = R^2/4l_B^2$, where l_B is the magnetic length. We show the pseudopotentials for $r = 1.9$. (c) The gap size $\Delta_{\text{GSD}} = E_{\text{GSD}} - E_{\text{GSD}-1}$ for the FTI at $\text{GSD} = 9$ (blue) and the PH(111) state at $\text{GSD} = 1$ (green) as a function of the distance parameter r for the fermionic hard-core interaction defined in panel (b). The phase is identified according to the GSD featuring the largest gap.

ideal FTI limit and result in nonzero s_9 up to total loss of topological order when the gap closes at $s_9/\Delta_9 \sim 1$.

For some of the phases we find, there are symmetry-enforced degeneracies in the spectrum. For such phases, the size of the quality parameter defined above is not informative as it vanishes by symmetry. Also for phases with a single ground state the quality parameter has to be adapted as the ground-state manifold does not feature a spread. Instead, we therefore use the size of the gap as the identifying property and assign the dimension of the ground-state manifold according to the maximal gap $\max_{\text{GSD}} \Delta_{\text{GSD}} = \max_{\text{GSD}} (E_{\text{GSD}+1} - E_{\text{GSD}})$. Figure 2(b) displays the Haldane pseudopotentials for a hard-core interaction $V_{\text{HC}} = \Theta(R - |z|)$, where $\Theta(z)$ is the Heaviside step function, z are the relative holomorphic coordinates of the two particles, and R is the range of the interaction. Using the method of largest gap, $\text{GSD} = 9$ is identified for a large range of interaction ranges, indicating the FTI [Fig. 2(c)]. The FTI is accompanied by a phase that exhibits a single ground state upon increase of R , which acts to suppress on v_0 .

In some cases, especially for phase diagrams containing multiple phases, the quality parameter has to be extended further for a consistent measure of stability inside the phases. After the identification of the ground-state degeneracy for given pseudopotential configuration according to $\max_{\text{GSD}} \Delta_{\text{GSD}}$, the quality parameter is then given by the relative gap size within given phase, i.e., $\Delta_{\text{GSD}(\{v_m\})} / \max_{\{v_m\}} \Delta_{\text{GSD}(\{v_m\})}$. The phase is therefore the most stable where its quality parameter takes the value 1 and the instability is measured by how quickly the quality parameter falls off toward a transition. As this method also captures accidental ground-state degeneracies, the spectra are inspected by eye and only phase-specific GSDs are taken into account.

III. RESULTS

Our main findings are presented in Figs. 3 and 4, which show the ground-state spin polarization and ground-state degeneracy, respectively. The plots are computed for a system with the three active pseudopotentials $v_0, v_1 = 1$, and v_m with

$m \in \{2, 3, 4, 5\}$. The diagrams for the remaining v_m with $m \in \{6, \dots, 14\}$ are featured in the Supplemental Material [66].

In Fig. 3, we observe that large repulsive v_{2n} pushes the system to spontaneously spin polarize to avoid energy penalties. For attractive v_{2n} , the spin-unpolarized case is favored. It is worth noting that the v_{2n} pseudopotentials are purely interspin interactions due to fermion statistics. The v_{2n+1} pseudopotentials, however, contribute as both intraspins and interspin interactions and hence the system prefers spin polarization for attractive v_{2n+1} and the unpolarized state for repulsive v_{2n+1} . This reflects the fact that for given v_{2n+1} the intraspins interactions dominate over the interspin interactions due to the form factors of electrons of opposite Chern number being smaller. For this reason, there is an energy penalty for spin polarization for large repulsive v_{2n+1} . Interestingly, even though the FTI requires a vanishing spin polarization, the FTI quality parameter is smallest close to the phase boundary between the spin-polarized and spin-unpolarized regions.

TABLE I. Summary of the emergent phases on the TR symmetric LLL on the torus. The abbreviation PS stands for “phase-separated state.” The values for the λ_i columns represent the eigenvalues for the C_4 , y mirror (M_y), and center of mass (C.M.) symmetries, respectively. The set of the three complex cube roots of 1 are denoted by $\{\sqrt[3]{1}\}$. How the symmetry eigenvalues dictate the ground-state degeneracy is discussed in the Supplemental Material [66]. There, also representative spectra for the listed phases are provided. The last column lists the composition of the GSD in an abbreviated manner, where the threefold redundancy from the center of mass symmetry in the case of full spin polarization and the decomposition into the distinct states for $\text{GSD} = 56$ are highlighted.

Phase	Spin pol.	λ_{C_4}	λ_{M_y}	$\lambda_{\text{C.M.}}$	GSD
PH(111)	0	1	1	1	1
FCI	Max	1	1	$\{\sqrt[3]{1}\}$	$3 = 1 \times 3$
PS I	0	$(\pm 1, \pm i)$	1	1	4
PS II	0	$(\pm 1, \pm i)$	± 1	1	8
FTI	0	1	1	1	9
PS III	0	$(\pm 1, \pm i)$	$1, (\pm 1)$	1	$56 = 2 \times 4 + 6 \times 8$

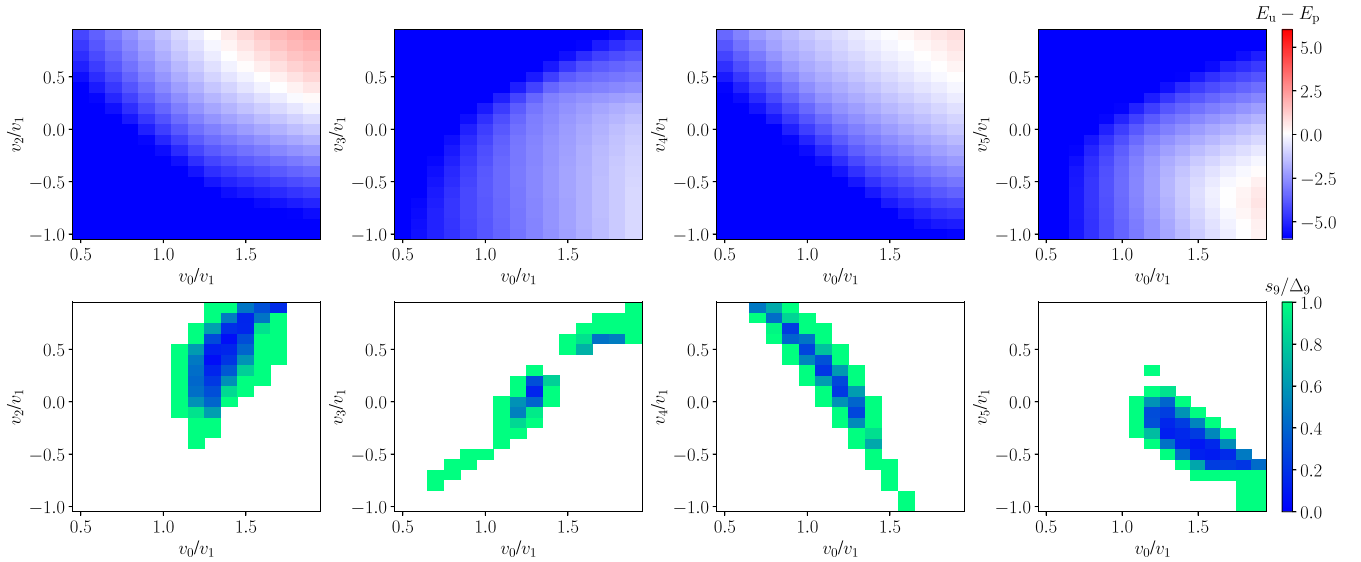


FIG. 3. Top: Phase diagrams for the ground-state spin polarization as a function of v_m and v_0 (for fixed v_1) measured by the difference between the ground-state energies of the spin-unpolarized and spin-polarized sectors, $E_u - E_p$. We compute all spin polarization sectors, but the spin is always either fully polarized or fully unpolarized. The system parameters are $N_\Phi = 15$ and $N_e = 10$. Inside regions of full spin polarization, an FCI with threefold degeneracy arises. Bottom: The FTI quality parameter $0 \leq s_9/\Delta_9 < 1$ is shown to illustrate its alignment with equi-energy-difference lines. Competition with full spin polarization appears to be beneficial to the FTI. Consideration of the full set of pseudopotentials v_m in the Supplemental Material [66] reveals an apparent even-odd effect with respect to the favored polarization at repulsive or attractive interactions, which is owed to the fact that v_{2n} are purely interspin interactions and hence the energy penalty for repulsive interactions can be avoided by polarization. For v_{2n+1} , which include intraspin interactions, polarization is favored for attractive interactions.

Figure 4 shows the effect of individual pseudopotentials on the TR symmetric LLL in more detail by considering the dimension of the ground-state manifold. The phase diagram on the torus, apart from gapless regions, can be divided into regions with different GSDs and four families of phases arise. One of these families contains the spontaneously spin-polarized phases discussed above. For even m , there is a stable spontaneously magnetized $v_\sigma = 2/3$, $v_{\bar{\sigma}} = 0$ FCI as these even m pseudopotentials do not affect the single spin species Laughlin state due to fermionic antisymmetry.

Regions in the phase diagrams of relatively low v_0 exhibit a phase with a single ground state at $\mathbf{k} = (0, 0)$ and a

comparatively large spectral gap. We compute

$$O = \sum_k |\langle \Psi_{\text{ED}} | n_{\uparrow, \mathbf{k}} n_{\downarrow, \mathbf{k}} | \Psi_{\text{ED}} \rangle|^2, \quad (3)$$

where $|\Psi_{\text{ED}}\rangle$ is the exact diagonalization ground state. O probes the double occupancy of orbitals and for small v_0 we find a large value of O , indicating a paired state. Figure 5 shows that for $v_{m>1} = 0$ the state is adiabatically connected to the phase at large negative v_0 , which was shown to be an interspin paired s -wave superconductor of electrons by means of flux insertion [64]. Borrowing concepts from quantum Hall ferromagnetism, one can find an alternative description of

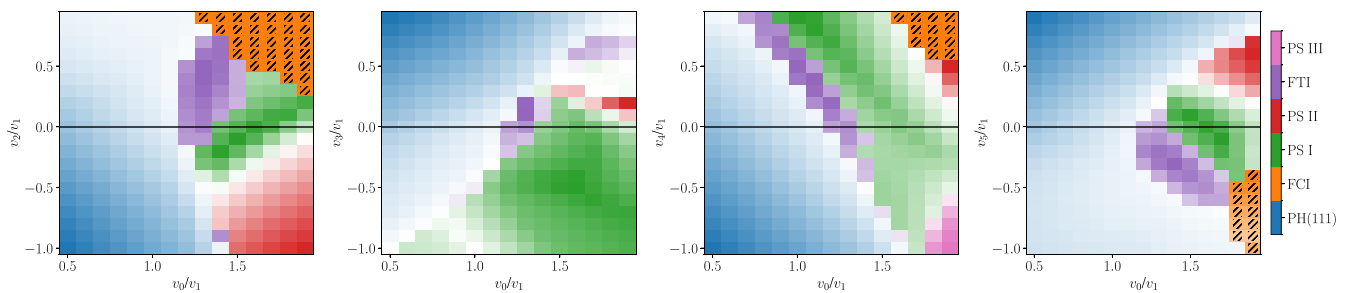


FIG. 4. Phase diagrams for the ground-state degeneracy in terms of individually tuned v_m and v_0 for fixed v_1 . The system parameters are again $N_\Phi = 15$ and $N_e = 10$. The colors determine the phases according to their respective GSD and the transparency reflects the size of the quality parameter $\Delta_{\text{GSD}} / \max_{(v_0, v_m)} \Delta_{\text{GSD}}$, being fully opaque when it is 1. Hatched regions indicate full spin polarization and white regions reflect spectra for which a GSD cannot be identified conclusively or are gapless. Featured in the phase diagrams are the partially particle-hole transformed Halperin (111) state (GSD = 1), the FCI in the polarized region (GSD = 3), the FTI (GSD = 9), and phase-separated states, i.e., PS I-III, of different ground-state momentum sectors (GSD mod 4 = 0). Details of the phases in terms of composition and symmetry-induced degeneracies are given in Table I. To discriminate against false detection of ground-state degeneracies during transitions, the condition of featuring the correct momentum sectors was added. Upon inspection of the full set of pseudopotentials v_m in the Supplemental Material [66], there is an evident even-odd effect in m , which dictates the direction and extent of the FTI phase in the phase diagram.

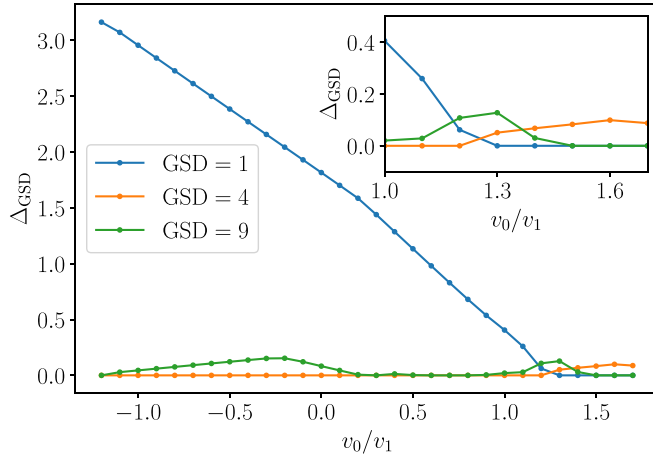


FIG. 5. The gap size of given ground-state dimension $\Delta_{\text{GSD}} = E_{\text{GSD}+1} - E_{\text{GSD}}$ for a system where $v_{m \geq 2} = 0$ along v_0/v_1 into large attractive amplitudes (right to left). The single ground-state phase at moderate repulsive interactions $0 < v_0/v_1 \lesssim 1.2$ is clearly connected to the state at large negative v_0 . The gaps for GSD = 4 and 9 are added to illustrate the phase transitions for large repulsive on-site interactions. In the inset, the region of v_0 relevant to the FTI is shown more closely to illustrate the similar gap size of the FTI and phase-separated phase and where the transitions occur.

this state. Particle-hole transforming the spin-down species in the relevant limit $v_0 \rightarrow -\infty$ yields spins with equal Chern number $C_\uparrow = C_\downarrow = 1$, filling fractions $\nu_\uparrow = 1/3$, $\nu_\downarrow = 2/3$, and interaction $v_0 \rightarrow +\infty$ [68]. This has the Halperin (111) state [69] as the ground state. Note that the Halperin (111) state has $\nu_{\text{tot}} = \nu_\uparrow + \nu_\downarrow = 1$, but indefinite $(\nu_\uparrow - \nu_\downarrow)$. In the following, we will always be considering the projection of the Halperin (111) state onto the relevant $(\nu_\uparrow, \nu_\downarrow) = (1/3, 2/3)$ sector. Particle-hole transforming one spin species to go back to the opposite Chern scenario we are considering, one obtains the partially particle-hole transformed Halperin (111) state. The state with GSD = 1 will therefore in the following be labeled PH(111). The 111 state has a counterflow supercurrent; this means that after the partial particle-hole transformation, the PH(111) state will have a supercurrent.

The phases with $\text{GSD} \bmod 4 = 0$ have ground states that individually break the fourfold rotational symmetry. The multiplicity of 4 is readily explained by having nonzero wave vectors for charge-ordered phases and the orbit of this wave vector under the action of the symmetry group $C_4 \times M_y$, where M_y is the mirror symmetry group associated with the y-axis. For spin-polarized phase-separated states, the GSD due to symmetry considerations is tripled. A more detailed discussion is given in the Supplemental Material [66]. The symmetry-related ground states lead to an exact degeneracy in the spectrum. The most prominent representative in the phase diagrams is the state of GSD = 4 and shows clear charge order upon investigation on the sphere (see the Supplemental Material [66]) and the wave vector of the charge order scales with system size. This is in agreement with the phase-separated states found in Refs. [35,38]. The eightfold degenerate state is likely gapless and connected to the phase-separated state of Ref. [35]. The fourfold degenerate state is potentially gapless

too, although further investigation is necessary. In the phase diagrams, these phase-separated states are abbreviated as PS I–III, distinguished by their GSD. The PS III phase is composed of eight individual phase-separated states that are close in energy compared to the gap size and is therefore identified by their combined GSD of $2 \times 4 + 6 \times 8 = 56$.

Finally, the GSD = 9 phase is the FTI. For unscreened Coulomb-type interactions, the on-site interaction $v_0/v_1 \approx 2$ and the stabilization of the FTI therefore requires strong v_0 suppression [14]. It is adiabatically connected to the limit of decoupled FCIs and survives isotropic interspin interactions at sufficiently low v_0 with respect to its Coulomb value, as seen in the Supplemental Material [66]. Interspin coupling and nonzero $m \geq 2$ pseudopotentials deteriorate the quality parameter and the FTI is unstable against stabilization of the PH(111) state or phase-separated phases. The ideal value of v_0/v_1 for the FTI drifts as a function of $v_{m \geq 2}$. The trend with $v_{m \geq 2}$ follows an even-odd pattern in m , according to the boundaries of the PH(111) phase. The identified phases can be summarized in a table according to the respective GSD and spin polarization (Table I).

In the case of tuning v_3 (second panel from the left in Fig. 4), we observe two different FTI regions. For a single spinless Landau level at filling $\nu = 1/3$, there is a transition between the Laughlin state and a nematic fractional quantum Hall state at $v_3/v_1 \sim 0.6$ [70]. The FTI at small v_3 is adiabatically connected to decoupled (isotropic) Laughlin states, while the FTI at large v_3 is adiabatically connected to decoupled nematic fractional quantum Hall states.

Finally, we comment on the nature of the phase transitions between the FTI and the different spin-unpolarized phases: the PS and the PH(111) phases. The FTI-PH(111) transition is continuous and can be shown to be in the three-dimensional XY universality class [35]. On the other hand, the FTI-PS transition is first order.

IV. CONCLUSION

In this work, we study how the individual Haldane pseudopotentials affect the FTI in TR symmetric lowest Landau levels. The different regions in the (v_m, v_0) phase diagrams have been identified and classified in order to understand the competing phases to the FTI. We found the partially particle-hole transformed Halperin (111) state that is adiabatically connected to a superconductor, a family of polarized and unpolarized phase-separated states, a spin-polarized FCI, and the sought-after FTI. It is evident that there is an even-odd pattern in the relative angular momentum m , owing to the effect of v_m as an interspin or intraspin interaction. Large nonzero $v_{m \geq 3}$ generally destroys the FTI; however, for small interaction amplitudes, the FTI can be stabilized for appropriate v_0 . For a pure Coulomb interaction, $v_0/v_1 \approx 2$, such that it has to be diminished drastically in order to stabilize the FTI at $v_0/v_1 \sim 1.25$. This condition can be relaxed if small long-range interactions are included. To achieve this suppression and access the relevant regions of the phase diagram in terms of the FTI, an additional short-range interaction has to be introduced. One way to accomplish such an additional background potential is by dielectric engineering [14].

There are two differences between our LLL model and a realistic model of twisted MoTe_2 . On the one hand, in the realistic model there will be a finite dispersion of the Chern bands. In the limit of large dispersion, a phase that did not appear in the phase diagram in Fig. 1 will appear: the Fermi liquid phase. On the other hand, there will be a nonuniform quantum geometry. It is often assumed that the most favorable condition for realizing a fractional Chern insulator is when the band dispersion vanishes; however, this is not the case in the presence of nonuniform quantum geometry. The reason is that the nonuniform quantum geometry will lead to a Hartree renormalization of the band structure. So, in fact, the optimal condition for the fractional Chern insulator is when the bandwidth of the Hartree-renormalized band structure is as small as possible [14].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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