

Super-Klein tunneling in two-dimensional Lorentzian-type barriers in graphene

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We introduce a two-dimensional model of spin-1/2 Dirac fermions in graphene subjected to a highly tunable electric field, which exhibits super-Klein tunneling. The electric field can be continuously interpolated between two limiting configurations: a uniform electrostatic Lorentzian barrier with translational invariance and a chain of well-separated electrostatic scatterers. We demonstrate that super-Klein tunneling arises naturally as a direct consequence of the intrinsic connection of the model to free-particle dynamics, a relation that is established through methods of supersymmetric quantum mechanics, which provide an elegant and analytically tractable framework. Besides the mentioned super-Klein tunneling, scale invariance of the model and invisibility of the potential for particles of specific energy are revealed, and possible routes toward experimental realization are discussed.

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I. INTRODUCTION

Super-Klein (or omnidirectional) tunneling is a phenomenon where a relativistic particle of specific energy passes through an electrostatic barrier without reflection, regardless of its incidence angle. This effect results from complex interference between reflected and transmitted waves at the barrier. It contrasts with standard Klein tunneling that occurs at any energy but for normal incidence only and its granted by conservation of pseudo-spin [1,2]. The phenomenon has been explored in many physical scenarios [3–20]. Let us mention line-centered square lattices [3], dice lattices [4], spin-1 systems with spin-orbit interaction [7], and transverse-polarized electromagnetic waves described by a spin-1 equation [6]. It was also theoretically analyzed in time-periodic dice lattices [13], excited electromagnetic waves in bi-isotropic media [12], and superlattices [15], and modeled in acoustic media [14]. Recently, super-Klein tunneling was observed in phononic crystals [16] and Lieb lattices [18].

Although its existence is often attributed to spin-1 systems with a flat band in the spectrum, super-Klein tunneling has also been shown to occur in spin-0 and spin-1/2 systems. In

Ref. [9], it was observed for spin-0 particles described by the Klein–Gordon equation and further analyzed for spin-1/2 Dirac fermions in graphene [8,10] as well as in the context of solitons [11].

In all these scenarios, the effect arises in the presence of short-range interactions. Most studies considered an electric field in the form of a translationally invariant rectangular barrier, while in Ref. [10] it was demonstrated for an exponentially decaying electric barrier lacking translational invariance.

In this article, we demonstrate that super-Klein tunneling occurs in spin-1/2 systems with a potential barrier exhibiting x^{-2} asymptotics at large distances. We derive a one-parametric model of super-Klein tunneling, where the electric field can vary from the Lorentzian-type barrier to highly structured comb of scatterers.

II. THE MODEL

Similar to Ref. [10], we employ a combination of time-dependent supersymmetric (SUSY) transformations [21] and a Wick rotation in the derivation of our model. The SUSY transformation produces a specific deformation of the potential in the evolution equation while preserving its solvability. It maps the solutions of the original system to those of the deformed one through the so-called intertwining operator. A subsequent Wick rotation then converts the $1 + 1$ evolution equation into a stationary equation of a planar system. The construction is described in more detail below.

Let us take the free Dirac particle in $1 + 1$ dimensions with mass m as the initial system described by the following equation:

$$H_0\psi = (i\partial_t - i\sigma_2\partial_z - m\sigma_3)\psi = 0, \quad (1)$$

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where we set $v_F \hbar = 1$, with v_F being Fermi velocity of Dirac fermions in graphene. The equation can be solved in a straightforward manner by fixing the following two solutions:

$$\begin{aligned}\phi_1(z, t) &= \left(iz e^{-imt}, \frac{i}{2m}(-e^{-imt} + 2\alpha e^{imt}) \right)^T, \\ \phi_2(z, t) &= \sigma_1 \phi_1(z, -t), \quad H_0 \phi_{1,2} = 0,\end{aligned}\quad (2)$$

where $\sigma_{1,2,3}$ are Pauli matrices, α is a real constant, and T denotes transposition. Now, we perform a SUSY transformation as described in the Appendix. The two spinors (2) can be used to compose the matrix U ,

$$U = (\phi_1, \phi_2), \quad H_0 U = 0. \quad (3)$$

Using U , we define the operators H_1 and L through

$$\begin{aligned}H_1 &= H_0 - i[\sigma_2, (\partial_z U)U^{-1}] \\ &= i\partial_t - i\sigma_2 \partial_z - m \left(1 - \frac{(1 - 2\alpha \cos(2mt))}{F(-iz, t)} \right) \sigma_3\end{aligned}\quad (4)$$

and

$$\begin{aligned}L &= U \partial_z U^{-1} \\ &= \partial_z + \frac{m}{F(-iz, t)} \left(m z \sigma_0 + \left(\frac{1}{2} - \alpha \cos(2mt) \right) \sigma_1 \right. \\ &\quad \left. - \alpha \sin(2mt) \sigma_2 \right),\end{aligned}\quad (5)$$

where

$$F(x, y) = m^2 x^2 + \frac{1}{4} + \alpha(\alpha - \cos(2my)). \quad (6)$$

These operators satisfy the following intertwining relation:

$$LH_0 = H_1L \quad (7)$$

(see Ref. [21] for more details). The operator L , commonly referred to as the supersymmetric or Darboux transformation, intertwines the operators H_0 and H_1 . The latter corresponds to the deformed evolution equation $H_1 \psi_1 = 0$, whose solutions can be generated by L ,

$$H_1 \psi_1 = 0, \quad \psi_1 = L \psi_0, \quad H_0 \psi_0 = 0, \quad (8)$$

where ψ_0 is an arbitrary zero mode of H_0 .

We now transform the $(1+1)$ -dimensional system with a time-dependent potential into a stationary planar system [22] by the Wick rotation $z \rightarrow ix, t \rightarrow y$. Additionally, we multiply the equation $H_0 \psi = 0$ from the left by σ_3 and subsequently perform a unitary transformation with the constant matrix $u = \sigma_3 e^{i\pi/4 \sigma_1}$. This yields

$$h_0 = u \sigma_3 H_0 u^{-1} = -i\sigma_1 \partial_x - i\sigma_2 \partial_y - m \sigma_0, \quad (9)$$

$$\begin{aligned}h_1 &= u \sigma_3 H_1 u^{-1} = -i\sigma_1 \partial_x - i\sigma_2 \partial_y - m \sigma_0 \\ &\quad + \frac{m(1 - 2\alpha \cos(2my))}{F(x, y)} \sigma_0,\end{aligned}\quad (10)$$

and

$$\begin{aligned}\mathcal{L} &= u L u^{-1} = -i \partial_x - \frac{m}{F(x, y)} \left(-imx \sigma_0 \right. \\ &\quad \left. + \frac{1}{2}(1 - 2\alpha \cos(2my)) \sigma_1 - \alpha \sin(2my) \sigma_3 \right),\end{aligned}\quad (11)$$

where σ_0 is 2×2 identity matrix.

As a result, the intertwining relation (7) between the three operators is modified and takes the form

$$\sigma_2 \mathcal{L} \sigma_2 h_0 = h_1 \mathcal{L}. \quad (12)$$

Due to the asymmetry of the intertwining operators on the left and right sides of the equality, the relation provides the mapping between the zero modes of h_0 and those of h_1 only. The operator h_1 is Hermitian and represents the Hamiltonian of the planar system. Its potential term is regular provided that $F(x, y)$ has no zeros, which can be easily ensured by choosing $\alpha \neq 1/2$.

Extracting the constant part of the potential term in h_1 , we obtain

$$\tilde{h}_1 = -i\sigma_1 \partial_x - i\sigma_2 \partial_y + V(x, y), \quad (13)$$

where (for $\alpha \in \mathbb{R}$)

$$V(x, y) = \frac{m(1 - 2\alpha \cos(2my))}{m^2 x^2 + \frac{1}{4} + \alpha(\alpha - \cos(2my))}. \quad (14)$$

The stationary equation associated with $\tilde{h}_1$ then reads

$$(-i\sigma_1 \partial_x - i\sigma_2 \partial_y + V(x, y)) \tilde{\psi}(x, y) = m \tilde{\psi}(x, y), \quad (15)$$

where $\tilde{\psi}(x, y)$ can be obtained from the zero modes of h_0 [see Eq. (8)]. The potential $V(x, y)$ is periodic in y and decays as x^{-2} for large $|x|$,

$$V\left(x, y + \frac{\pi}{m}\right) = V(x, y), \quad V(x, y) \sim \frac{1}{m x^2}. \quad (16)$$

The system possesses scale invariance, as under the rescaling

$$x = \tilde{x}/q, \quad y = \tilde{y}/q, \quad m = q \tilde{m}, \quad (17)$$

Eq. (15) remains the same. Therefore, we can set $m = 1$ without loss of generality, since other values of m can be achieved by rescaling the coordinates. The potential is illustrated in Fig. 1 for different values of the parameter α . In the limit $\alpha \rightarrow 0$, we obtain a one-dimensional Lorentzian barrier

$$V_L(x) = \lim_{\alpha \rightarrow 0} V(x, y) = 4m \frac{\left(\frac{1}{2m}\right)^2}{x^2 + \left(\frac{1}{2m}\right)^2} \quad (18)$$

centered at $x = 0$, with a peak height of $4m$ and half-maximum points located at $x_{hm} = \pm 1/(2m)$. In the subspace $p_y = 0$, the Hamiltonian $\tilde{h}_1 \tilde{\psi} = E \tilde{\psi}$, with $\alpha = 0$, is unitarily equivalent to the one-dimensional free-particle energy operator [24]. Therefore, its explicit eigenstates of arbitrary energy E can be found as

$$\tilde{\psi}_E = A_+ f^+(x) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i A_- f^-(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (19)$$

where $f^\pm(x) = \exp[\pm i \int^x (V_L(x') - E) dx'] = \frac{(i \mp 2mx)^2}{1 + 4m^2 x^2} e^{\mp i E x}$.

A. Scattering states

Since h_1 was derived from the free-particle Hamiltonian, there is no back-scattering of Dirac fermions with energy m on the potential barrier $V(x, y)$. Indeed, the scattering solutions of Eq. (15) can be obtained from the plane waves:

$$\psi_\phi = e^{i\mathbf{k}\cdot\mathbf{x}} (1, e^{i\phi})^T, \quad |\mathbf{k}| = m, \quad (20)$$

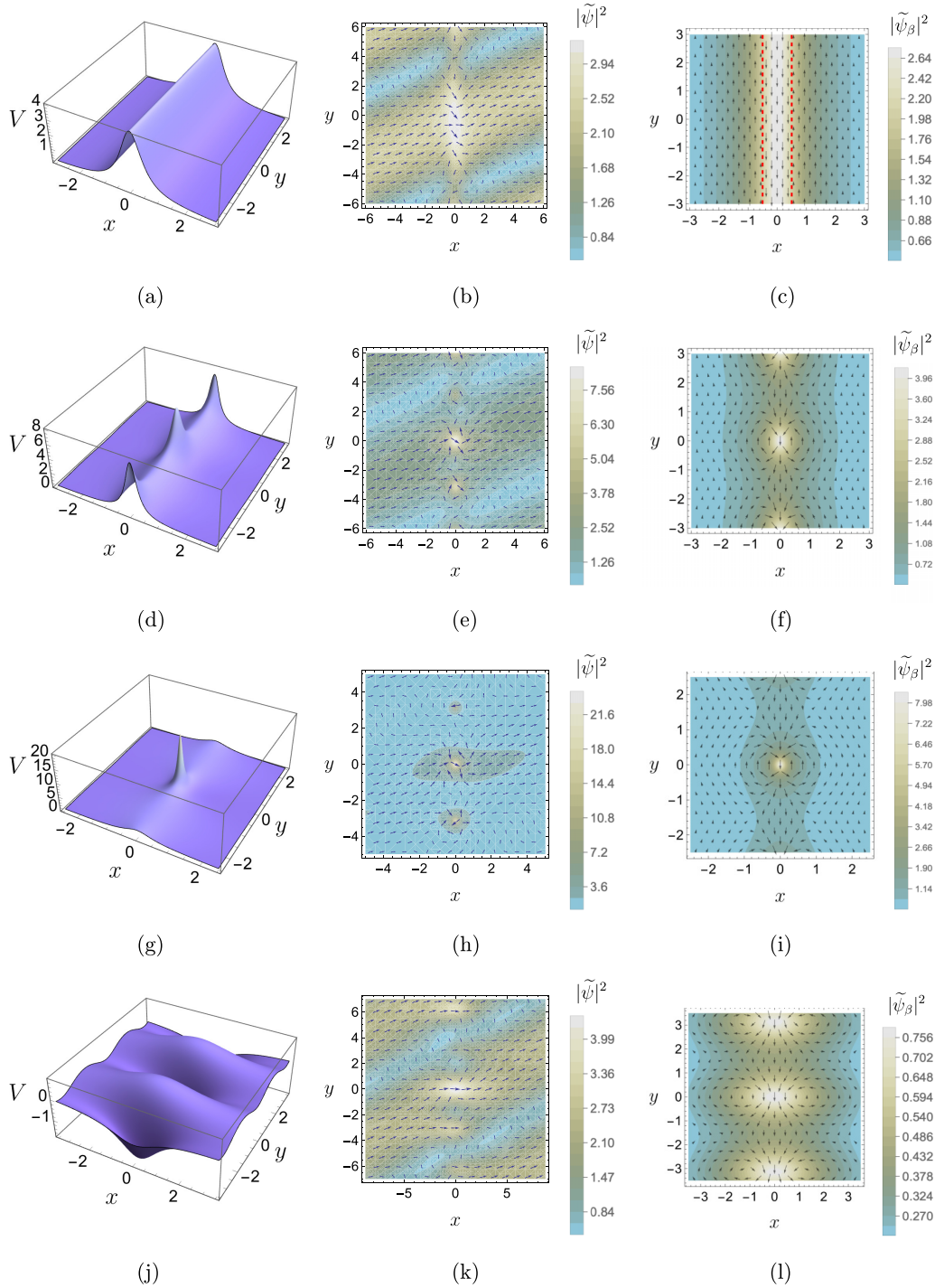


FIG. 1. (Left column) Potential $V(x, y)$ in $v_F \hbar$ units. (Middle column) Probability density of $\tilde{\psi} = \mathcal{L}(\psi_0 + \psi_{\pi/4})$ together with its current $\vec{j} = (\tilde{\psi})^\dagger \vec{\sigma} \tilde{\psi}$. The red lines in panel (c) are borders of the areas where the probability current changes direction. (Right column) Probability density of localized state $\tilde{\psi}_\beta = \mathcal{L}\psi_\beta$ together with its current $\vec{j} = (\tilde{\psi}_\beta)^\dagger \vec{\sigma} \tilde{\psi}_\beta$ (right column). We fixed (a)–(c) $\alpha = 0$, (d)–(f) $\alpha = 0.25$, (g)–(i) $\alpha = 0.4$, and (j)–(l) $\alpha = 1.64$. In all panels, $m = 1$ and $\beta = 1$. The coordinates x and y are in nanometers.

where $h_0\psi_\phi = 0$ and $\tilde{h}_1(\mathcal{L}\psi_\phi) = m(\mathcal{L}\psi_\phi)$. The scattering states of $\tilde{h}_1$ with generic incidence angle ϕ can be generated from the free-particle plane waves ψ_ϕ upon acting with the operator $\mathcal{L}$. They have the following asymptotic form for $|x| \rightarrow \infty$:

$$\mathcal{L}\psi_\phi = (ik + O(x^{-1})\sigma_0 + O(x^{-2})\sigma_1 + O(x^{-2})\sigma_3)\psi_\phi. \quad (21)$$

The wave function is not reflected as there is no component with the opposite momentum. This implies that there is super-Klein tunneling in the system described by $\tilde{h}_1$ for the particles of energy $E = m$.

Additionally, the wave function does not acquire any phase shift when going through the potential. Therefore, the potential $V(x, y)$ is completely invisible at this energy [25].

However, the invisibility is only asymptotic: The scattering state approaches a plane wave with corrections that are vanishing rather slowly as x^{-1} [see Eq. (21)]. This is illustrated in Fig. 1, where the probability density of a linear combination of asymptotic plane waves $\tilde{\psi} = \mathcal{L}(\psi_0 + \frac{1}{2}\psi_{\pi/4})$ is presented. The scattering states are affected by the presence of the barrier. For $\alpha \sim 1/2$, when the barrier is rather formed by a chain of sharply peaked scatterers, the probability density of the scattering states has sharp peaks at these scattering centers (see Fig. 1).

B. Localized states

Besides the scattering states, there also exist localized states with normalizable localization along x axis, $\int_{-\infty}^{\infty} |\tilde{\psi}_\beta|^2 dx < \infty$. We can derive such a localized state when applying the intertwining operator $\mathcal{L}$ on the following specific nonphysical eigenstate of h_0 corresponding to zero energy:

$$\psi_\beta = \frac{1}{\sqrt{2}} \begin{pmatrix} -xe^{-imy} - \frac{1}{2m}(2\beta e^{imy} - e^{-imy}) \\ ixe^{-imy} - \frac{i}{2m}(2\beta e^{imy} - e^{-imy}) \end{pmatrix}. \quad (22)$$

The state $\tilde{\psi}_\beta = \mathcal{L}\psi_\beta$ is an eigenstate of $\tilde{h}_1$ with energy m , from which further bound states follow by exploiting the symmetries of $\tilde{h}_1$, e.g., $\sigma_1 \tilde{\psi}_\beta(x, -y)$. In Fig. 1, we plot the nonnormalized modulus

$$|\tilde{\psi}_\beta|^2 = \frac{4(\alpha - \beta)^2}{1 + 4m^2x^2 + 4\alpha^2 - 4\alpha \cos(2my)}. \quad (23)$$

Furthermore, due to the periodicity of the solutions, we can always interpret the system as an infinite cylinder defined by the restricted space $\Omega = \{(x, y) | x \in \mathbb{R}, y \in (0, \frac{\pi}{m})\}$, with the identification of the boundaries $y = 0$ and $y = \frac{\pi}{m}$. In this case, the integration of the probability density can be performed exactly, and it yields

$$\int_{\Omega} |\psi_\beta|^2 d\Omega = 4(\alpha - \beta)^2 \frac{\pi K\left(-\frac{8\alpha}{(1-2\alpha)^2}\right)}{(1-2\alpha)m^2}, \quad (24)$$

where $K(k)$ is the complete elliptic integral of the first kind. Note that when $\alpha \rightarrow 0$, this value reduces to $\frac{2\pi\beta^2}{m}$, obtaining in this way the bound state for one-dimensional Lorentzian barrier (18).

It is worth noticing that the direction of the probability current $\vec{j} = (\tilde{\psi}_\beta)^\dagger \vec{\sigma} \tilde{\psi}_\beta$ changes the sign in the area of strong potential. In the limit case of a Lorentzian barrier [see Eq. (18)], the current in the x -direction vanishes everywhere, while the y component has a change of sign exactly at the half-maximum point $x_{hm} = \pm 1/(2m)$, where the potential $V(x_{hm}, y) = 2m$. This behavior is generic for the current j_y of the state $\tilde{\psi}_\beta$: for any fixed value of y , the position x_0 such that condition $j_y = 0$ holds (where there is a change of sign in the current):

$$x_{0,\pm} = \pm \frac{1}{2m} \sqrt{4\alpha^2 - 4\alpha \cos(2my) + 1} \quad (25)$$

is equivalent to finding the half-maximum (or minimum) point of the potential $V(x_{0,\pm}, y)/V(0, y) = 1/2$.

TABLE I. Maximum values of the electric potential $U(x, y)$ in graphene evaluated at $x = 0$ and $\cos(2my) = 1$ for different values of the parameter α .

α	$U_{\max}$ (V)	Regime description
0.0	0.40	Positive maximum potential (barrier)
0.1	0.50	Intensity increase
0.2	0.67	Peak narrowing
0.3	1.00	Near resonance
0.4	2.00	Highly localized peak
0.6	-2.00	Inversion to deep potential well
0.7	-1.00	Potential well
0.8	-0.67	Depth decrease
0.9	-0.50	Stabilization
1.0	-0.40	Negative maximum well

III. EXPERIMENTAL PROPOSAL AT REALISTIC SCALES

To analyze the system at a realistic scale, we assume that x , y , and m^{-1} have dimensions of length and multiply the equation $h\mathcal{L}\psi = m(\mathcal{L}\psi)$ by $\hbar v_f$. This identifies the physical energy as $\mathcal{E} = \hbar v_f m$, while the applied voltage is given by $U = \frac{\hbar v_f}{e} V(x, y)$, where e is the elementary charge. The maximal voltage is then obtained as $U_{\max} = \frac{\hbar v_f}{e} V_{\max} = 4 \frac{\mathcal{E}}{e} \frac{1}{1-2\alpha}$, where the ratio $\frac{\mathcal{E}}{e}$ sets the voltage scale, while α acts as a modulation parameter. As long as $\alpha \neq \frac{1}{2}$, the maximal voltage can be tuned according to the system scale determined by the parameter m . For example, if $\mathcal{E} = 0.1$ (eV), then $m = 0.152$ (nm⁻¹), resulting in a natural scale of $\pi/m = 20.6$ (nm), which exceeds the interatomic scale and thus validates the Dirac approximation. Furthermore, the effective channel length $1/m = 6.6$ (nm) defines a confining region accessible to a scanning tunneling microscope (STM). In Table I, we show how the maximum voltage behaves for different values of α between 0 and 1.

The barrier represented by $V(x, y)$ can be constructed using the line-charged model of an STM tip placed in the proximity of a graphene sheet [26]. Suppose that graphene is encapsulated between hexagonal boron-nitride (hBN) layers. Misalignment of the layers preserves the electronic properties of Dirac fermions in graphene [27,28]. The grounded plate is attached to one side of the hBN-sandwiched graphene, whereas the horizontal STM tip, parallel with the graphene sheet, lies on the other side. The graphene sheet is in the plane (x, y, z_0) with $z_0 > 0$. The STM tip is represented by an infinite wire (e.g., carbon nanotube) with constant charge density τ that passes through the points $(0, y, z_1)$, where $z_1 > z_0$ (see Fig. 2 for illustration). The plane $(x, y, 0)$ is grounded; i.e., the electrostatic potential generated by the wire must vanish there. The electric potential U on the graphene sheet can then be obtained using the method of mirror charges, yielding

$$V_{el} \equiv \frac{e}{v_f \hbar} U(x) = \frac{e}{v_f \hbar} \frac{\tau}{4\pi \epsilon_0 \epsilon_r} \ln \left(\frac{x^2 + (z_0 + z_1)^2}{x^2 + (z_0 - z_1)^2} \right), \quad (26)$$

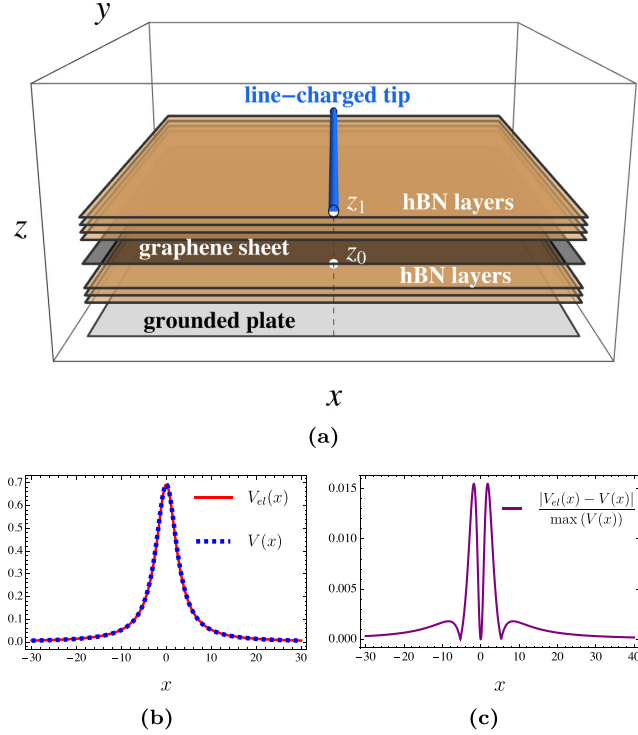


FIG. 2. (a) Graphene encapsulated by layers of hexagonal boron nitride. Grounded plane is attached below, and the line-charged tip lies on the upper surface. (b) Potential $V_{el}(x, y)$ in Eq. (26) with charge density (27) (red) and V from Eq. (14) for $\alpha = 0$ (blue dotted), measured in $v_F \hbar$ units. (c) The difference $\frac{|V_{el}(x) - V(x)|}{\max(V(x))}$. We fixed $z_0 = 1$ nm, $z_1 = 3$ nm, and $m = 0.173$ nm $^{-1}$. Then, we get from Eq. (27) the linear charge density $\tau = 0.77e$ nm $^{-1}$. The number of the hBN layers is for illustrative purposes only.

where [29] $4\pi\epsilon_0\epsilon_r v_F \hbar \sim 1.55e^2$. To reproduce the barrier profile (14), we fix the charge density of the line as

$$\tau = \frac{4\pi\epsilon_0\epsilon_r v_F \hbar}{e} \frac{4m}{\ln\left(\frac{(z_0+z_1)^2}{(z_0-z_1)^2}\right)}. \quad (27)$$

With this choice, the electric potential of the line-charged tip (26) is accurately approximated by the barrier (14) for $\alpha = 0$, as illustrated in Figs. 2(b) and 2(c).

IV. SUMMARY

We derived a one-parameter family of models describing pseudo-spin- $\frac{1}{2}$ Dirac fermions in the presence of an electric potential (14) that has an asymptotic $1/x^2$ dependence. Depending on the value of the parameter, the barrier either is translationally invariant along the y axis ($\alpha = 0$) or forms a y -periodic chain of scatterers. The construction is based on a time-dependent supersymmetric transformation that makes the stationary equation analytically solvable for a specific value of the energy. It allowed us to find the scattering states and show that the system possesses super-Klein tunneling for the particles with $E = m$. It is worth mentioning that the stationary equation (15) is relevant in both Dirac valleys; i.e.,

super-Klein tunneling is witnessed by Dirac fermions from the vicinity of both Dirac points.

Contrary to the electrostatic barrier found in Ref. [10], the transmitted plane waves acquire no phase shift [see Eq. (21)]. Consequently, the barrier is completely undetectable at $E = m$, and it is effectively invisible to these scattering states. Comparing the experimental feasibility, the electric field behaves like $1/x^2$ that is easier to prepare in the experiments than the exponentially vanishing one [30–32]. We demonstrated that the electric field in the form of the Lorentzian barrier approximates very well the field of a prolonged, line-charged STM tip placed in the proximity of the graphene sheet.

It is worth noting that Dirac fermions in electric fields with $1/r^2$ asymptotic behavior have been studied, e.g., in Refs. [30,31], where confinement of zero-energy charge carriers by the electric field was observed. In Ref. [33], the supersymmetric framework underlying the existence of zero modes in generalized Lorentzian quantum dots was discussed.

Unlike in the case of standard Klein tunneling, no symmetry-based argument is known that would justify the existence of super-Klein tunneling at a specific energy level. Nevertheless, our results provide a concrete unifying link between the two phenomena. In the present model (and also in Ref. [10]), super-Klein tunneling emerges because the dynamics at the special energy can be mapped to the free-particle one via a supersymmetric transformation. Standard Klein tunneling can be explained in a similar manner: The subsystem corresponding to normal incidence can be unitarily transformed into that of a free particle [24], providing a straightforward explanation for the absence of backscattering.

Be it either in the subspace of fixed energy in case of super-Klein tunneling or in the subspace of normal incidence in case of standard Klein tunneling, the system retains a memory of the free-system properties. Recent results relate super-Klein tunneling to the concept of integrability [11]. A symmetry generator I of the free particle, be it chiral, rotational, or translational symmetry, satisfies $[h_0, I] = 0$. Following Ref. [11], the “Darboux dressed” operator is then defined as $\mathcal{I} = \mathcal{L}I\mathcal{Y}$, where $\mathcal{Y} = \sigma_2 \mathcal{L}^\dagger \sigma_2$. This operator does not commute with h_1 ; however, it generates a new solution of Eq. (15) from a known one. To show this, we first consider the adjoint equation of Eq. (12), $h_0 \mathcal{Y} = \mathcal{L}^\dagger h_1$. The operator $\mathcal{Y}$ maps solutions of $h_1 \tilde{\psi} = 0$ into solutions of $h_0 \tilde{\psi} = 0$ via $\tilde{\psi} = \mathcal{Y}\psi$. Then,

$$[h_1, \mathcal{I}]\tilde{\psi} = h_1 \mathcal{L}I\mathcal{Y}\tilde{\psi} = \sigma_2 \mathcal{L} \sigma_2 [I h_0 (\mathcal{Y}\tilde{\psi})] = 0. \quad (28)$$

Therefore, $\mathcal{I}$ corresponds to a quasisymmetry; it commutes with the Hamiltonian in a given subspace only. This subspace is the most symmetric one in the sense that it inherits all the symmetries of the free particle.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: SUSY TRANSFORMATION FOR TIME-DEPENDENT DIRAC EQUATIONS

For completeness, we briefly summarize the supersymmetric technique for time-dependent Dirac equations (for further details, see Ref. [21]). We start from the intertwining relation $LH_0 = H_1L$, where the operators H_j , $j = 0, 1$, represent two quantum systems. The operator L is known as the *intertwining operator*. We fix the form of H_j as time-dependent Dirac operators:

$$H_j = i\partial_t - i\sigma_2\partial_z - V_j(z, t), \quad j = 0, 1,$$

where $V_j(z, t)$ are 2×2 matrix potentials usually referred to as *supersymmetric partners potentials*, which may also include a mass term. The intertwining relation ensures that any solution ψ_0 of $H_0\psi_0 = 0$ is mapped to a solution ψ_1 of $H_1\psi_1 = 0$ via $\psi_1 = L\psi_0$.

Assuming that the potential $V_0(z, t)$ and the solutions of $H_0\psi_0 = 0$ are known, the problem reduces to constructing

L and H_1 . We take L as a first-order differential operator of the form $L = \partial_z + B$, where $B = B(z, t)$ is a matrix to be determined. Substituting the explicit forms of H_0 , H_1 , and L into the intertwining relation yields

$$i[B, \sigma_2] = V_1 - V_0, \quad (\text{A1})$$

$$(V_1 - V_0)B = i\partial_t B - i\sigma_2\partial_z B + \partial_z V_0 + [B, V_0]. \quad (\text{A2})$$

Eliminating $V_1 - V_0$ from Eqs. (A1) and (A2), we obtain a single equation for B :

$$i\partial_t B - i\sigma_2\partial_z B + \partial_z V_0 + [B, V_0] - i[B, \sigma_2]B = 0.$$

This equation can be linearized using the ansatz $B = -(\partial_z U)U^{-1}$, which leads, after integration, to the following equation for U :

$$i(\partial_t U) - i\sigma_2(\partial_z U) - V_0 U = 0.$$

This equation formally has the structure of a Dirac equation for the initial problem $H_0 U = 0$; however, in this case U is a 2×2 matrix rather than a spinor. The matrix U can be constructed from two spinor solutions ϕ_1 and ϕ_2 of $H_0\phi = 0$ by taking them as columns, $U = (\phi_1, \phi_2)$.

Once $U(z, t)$ is determined, the intertwining operator L and the potential V_1 can be expressed as

$$L = \partial_z - (\partial_z U)U^{-1} = U\partial_z U^{-1}, \\ V_1 = V_0 + i[\sigma_2, (\partial_z U)U^{-1}].$$

Finally, the SUSY-transformed Dirac operator becomes

$$H_1 = H_0 - i[\sigma_2, (\partial_z U)U^{-1}].$$

Substituting U as given in Eqs. (2) and (3), one directly recovers the operator H_1 in Eq. (4).

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