

Parastatistics and a secret communication challenge

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One of the most unconventional features of topological phases of matter is the emergence of quasiparticles with exotic statistics, such as non-Abelian anyons in two-dimensional systems. Recently, a different type of exotic particle statistics that is consistently defined in any dimension, called R parastatistics [Z. Wang *et al.*, *Nature (London)* **637**, 314 (2025)], is also shown to be possible in a special family of topological phases. However, the physical significance of emergent parastatistics still remains elusive. Here, we demonstrate a distinctive physical consequence of parastatistics by proposing a challenge game that can only be won in a noise-robust way using physical systems hosting paraparticles, as passing the challenge requires the two participating players to secretly communicate in an indirect way by exploiting the nontrivial exchange statistics of the quasiparticles. The winning strategy using emergent paraparticles is robust against noise, as well as the most relevant class of eavesdropping via local measurements. This provides both an operational definition and an experimental identity test for paraparticles, alongside a potential application in secret communication.

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Introduction. Topological phases [1–7] are long-range-entangled phases of matter, exhibiting fundamentally new physical properties compared to conventional symmetry-breaking phases, and have attracted tremendous interest in the condensed matter community. A key unconventional feature of such phases is the emergence of exotic quasiparticle statistics beyond the celebrated Bose-Einstein and Fermi-Dirac statistics, which reflects the pattern of long-range entanglement [8–10] in the topologically ordered ground states. For example, non-Abelian anyonic statistics is possible in two dimensions (2D) [1,2,11,12], where spatially braiding anyons leads to a unitary evolution in the topologically degenerate states with n identical anyons, realizing a representation of the braid group B_n , and has a potential application in topological quantum computation [1,12].

In higher spatial dimensions, it has long been believed that all pointlike particles are either fermions or bosons [13–15]. There is actually a straightforward generalization of non-Abelian statistics to arbitrary spatial dimension, known as parastatistics [16], in which identical particles transform in higher-dimensional representations of the symmetric group S_n (instead of the braid group B_n) under exchange of their spatial positions [17]. Although parastatistics has been considered since the dawn of quantum mechanics [18] and extensively studied by the high-energy physics community in the second

half of the last century [17,19–22], it is now widely believed to be physically equivalent to ordinary particle statistics [13,20,22]. In other words, the theory of parastatistics, while mathematically consistent, was long thought to predict no new physical phenomena beyond what is already known from the theory of fermions and bosons.

Recently, a paper [23] proposed a different theory of parastatistics, now called R parastatistics, which exhibit exchange behaviors dramatically different from our conventional physical picture of fermions and bosons. These R paraparticles have been shown to emerge in a family of exactly solvable quantum spin models [23] realizing an exotic class of topological phases. Despite this significant progress, many important questions are open, such as what are the most distinctive physical (experimental) consequences of R paraparticles and how to characterize their universal topological properties independently of the microscopic realization.

In this Letter, we demonstrate a distinctive physical consequence of R parastatistics from a quantum information viewpoint. Roughly speaking, we show that emergent R parastatistics enable long-range communication of information by only exchanging the spatial positions of paraparticles, without either perturbing the environment or leaving any trace of information behind. We formulate this idea by proposing a secret communication challenge game, designed in a way that can only be won by topological phases hosting emergent paraparticles [24], as winning the challenge requires the two participating players to send a message to each other in a noise-robust way by exploiting the nontrivial exchange statistics of the quasiparticles. We illustrate the winning strategy using exactly solvable quantum spin models with emergent R paraparticles (in both two and three dimensions) and show that this strategy is robust against local noise and eavesdropping. We give a simple description of the universal topological properties of R paraparticles that involves only basic quantum

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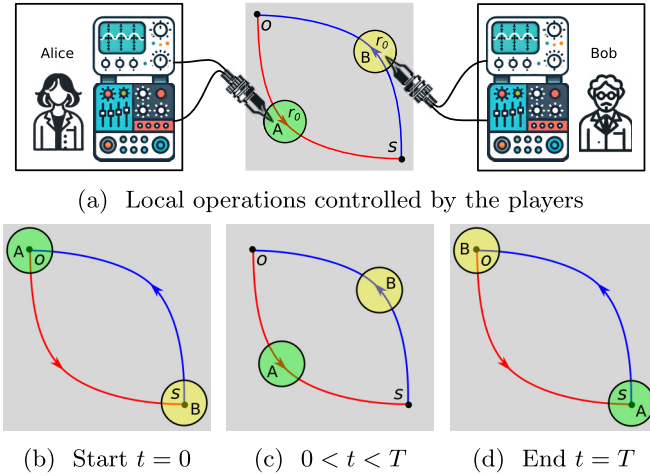


FIG. 1. Illustration of the game process. Here, we draw the 2D version for simplicity; in 3D, the circles become spheres. The two circles have radius r_0 defined by the players. During the game, the Referees move the two circles along their respective paths to complete an exchange of positions. The two far-separated points o, s are chosen by the players, while the two paths are chosen by the Referees.

mechanics and show that the winning strategy only relies on these universal properties. We then systematically investigate which three-dimensional (3D) topological phases can win the game, using the established fact that pointlike topological quasiparticles in 3D are universally described by symmetric fusion categories (SFCs) [13–15]. This leads to a categorical description of R paraparticles and the winning strategy, and singles out a special class of 3D topological phases that can win the game.

Rules of the game. The participants of the game consist of two parties: two players, whom we call Alice (A) and Bob (B), and a group of Referees (R). To win the game, the two players are required to send a message to each other using a very restricted class of local operations on a common quantum many-body system, and the Referees' role is to initiate the challenge and monitor the whole process of the game to prevent cheating.

Before the game starts, the players are allowed to discuss a winning strategy, and then they are required to

(1) submit a locally interacting Hamiltonian $\hat{H}$ on a 2D or 3D lattice that has a unique, gapped, and frustration-free ground state $|G\rangle$;

(2) choose the radius r_0 of the circles (or spheres in 3D) in Fig. 1, and two far-separated [25] points o, s in the lattice (which are allowed to be on the boundary);

(3) experimentally prepare the ground state $|G\rangle$ for a sufficiently large system size $L \gg r_0$ [L is determined by the Referees after they receive (1) and (2)].

A few technical remarks are in order here. Gapped means that there is a uniform lower bound (independent of the system size L) on the energy difference between the first excited state and ground state of $\hat{H}$. Frustration-free means that $\hat{H}$ can be written in the form $\hat{H} = \sum_i \hat{h}_i$ such that $\hat{h}_i \geq 0$ and $\hat{h}_i |G\rangle = 0$, which is a technical requirement imposed to simplify dis-

cussion, and we relax this requirement in the Supplemental Material [26]. A few other technical regularity conditions are also stated in the Supplemental Material [26].

When the Referees receive all above (1)–(3), they experimentally check that $|G\rangle$ is prepared correctly, by verifying that $\hat{h}_i |G\rangle = 0$ for all i . Then, Alice and Bob are led to separate rooms and the Referees randomly pick two numbers $a, b \in \{1, 2, 3, 4\}$, and give them to Alice and Bob, respectively, who do not know each other's number. The goal for the players is to gain information about each other's number through a restricted set of local operations on the physical system they have prepared, as we describe below.

Once the game starts, any form of direct communication between the players is prohibited. Throughout the game, Alice and Bob are confined in separate rooms, where the only access to the outside world is through local unitary operations and measurements on the system they prepared, restricted to their respective circle areas, as shown in Fig. 1(a). When the game starts at $t = 0$, the state of the physical system is initialized to be $|\Psi(0)\rangle = |G\rangle$, and Alice's circle starts at the special point o , while Bob's circle starts at the other special point s , as shown in Fig. 1(b). Then, the Referees select the two paths [27] shown in Fig. 1 and slowly move the two circles along their respective paths simultaneously, ensuring that the two circles remain far apart at all times. Throughout the process, the Referees frequently check the local ground-state condition $\hat{h}_i |\Psi(t)\rangle = 0$ everywhere beyond the two circle areas to make sure that the players are not cheating by leaving any trace of information behind. If at any moment the Referees detect an excitation beyond the circle areas, the challenge fails. The game ends at $t = T$ in the configuration shown in Fig. 1(d), when the two circles complete an exchange of positions. After this, the Referees move both circles out of the system and perform one last check of the local ground-state condition $\hat{h}_i |\Psi(T)\rangle = 0$ everywhere. If this final check is passed, the Referees ask Alice about b , and ask Bob about a , and the players win if they both answer correctly.

What can the players do to win the game? They can create some excitations inside their respective circle areas and make measurements on them, but throughout the experiment they are obliged to move the excitations to follow the circle movement, and clean up whatever excitations inside the circles before the game ends at $t = T$, to make sure that the Referees cannot detect an excitation anywhere beyond the circle areas at any time. At first glance, the challenge appears insurmountable under such restrictive conditions, as the players need to communicate some information without either leaving a trace behind or sending any physical particles to each other as information carriers. Nevertheless, we will see that it is possible to succeed if the ground state $|G\rangle$ hosts some emergent quasiparticles with nontrivial exchange statistics. The players can create such quasiparticles at $t = 0$ and encode their numbers in the internal state of the quasiparticles. Then, they move the quasiparticles following the circle movement, and right before the game ends at $t = T$, the quasiparticles have exchanged their positions, inducing a unitary rotation on their internal states, which the players can measure. We will see that some information can be sent this way if the exchange statistics is nontrivial. In the following, we illustrate

this winning strategy in detail, using emergent paraparticles in the some exactly solvable models.

Solvable quantum spin models with emergent R paraparticles. The models can be defined in both 2D and 3D, as we present explicitly in the Supplemental Material [26]. Both models have unique, gapped, and frustration-free ground state $|G\rangle$ and therefore satisfy the game requirement. The microscopic details of these models are not important, as it turns out that the winning strategy depends only on certain universal topological properties of the emergent paraparticles that are common to both models, which we summarize below:

(1) *The state space.* In general, an excited state of $\hat{H}$ with n identical paraparticles is denoted as $|G; i_1^{a_1} i_2^{a_2} \dots i_n^{a_n}\rangle$. Here, $i_1, i_2, \dots, i_n$ are the positions of the paraparticles, and we assume that they are mutually different [28]. The numbers $a_1, a_2, \dots, a_n \in \{1, 2, \dots, m\}$ label the *internal states* of the paraparticles, where $m \geq 2$ is an integer called the quantum dimension of the paraparticle. When the positions $i_1, \dots, i_n$ are fixed, there are m^n linearly independent internal states, spanning an m^n -dimensional subspace, which we denote as $\mathcal{H}_{i_1 i_2 \dots i_n}$.

(2) *Topological degeneracy.* The internal states of the paraparticles cannot be accessed using any local measurements when all particles are far away from the boundary and from each other. Formally, this means that any local operator $\hat{O}$ is diagonal in the subspace $\mathcal{H}_{i_1 i_2 \dots i_n}$,

$$\langle G; i_1^{b_1} \dots i_n^{b_n} | \hat{O} | G; i_1^{a_1} \dots i_n^{a_n} \rangle = C_{i_1 \dots i_n}^O \prod_{j=1}^n \delta_{a_j b_j}, \quad (1)$$

where $C_{i_1 \dots i_n}^O$ is a constant. Equation (1) implies that states in $\mathcal{H}_{i_1 i_2 \dots i_n}$ are topologically degenerate in energy [29].

(3) *Particle movements.* There exists a local unitary operator $\hat{U}_{i_k j_k}$ that moves the particle at position i_k to a nearby position j_k , for any $k = 1, 2, \dots, n$:

$$\hat{U}_{i_k j_k} |G; i_1^{a_1} \dots i_k^{a_k} \dots i_n^{a_n}\rangle = |G; i_1^{a_1} \dots j_k^{a_k} \dots i_n^{a_n}\rangle. \quad (2)$$

(4) *Exchange statistics.* Here, we focus on the case $n = 2$, which is enough for our purpose; the generalization to arbitrary number of paraparticles is straightforward [26]. For $i \neq j$, $\mathcal{H}_{ij}$ must be equal to $\mathcal{H}_{ji}$, since both describe the subspace of excited states of two identical paraparticles at positions i and j . Therefore, the two different bases $\{|G; i^a j^b\rangle | 1 \leq a, b \leq m\}$ and $\{|G; j^b i^a\rangle | 1 \leq a, b \leq m\}$ must be related by a unitary transformation:

$$|G; i^a j^b\rangle = \sum_{a', b'} R_{ab}^{b'a'} |G; j^{b'} i^{a'}\rangle, \quad (3)$$

where the four-index tensor $R_{ab}^{b'a'}$ (called the R -matrix) must satisfy the Yang-Baxter equation [Eq. (S8) in the Supplemental Material [26]] so that Eq. (3) can be consistently generalized to any number of particles [23]. Each R -matrix defines a type of parastatistics [23]. There are many different types of parastatistics that can emerge in condensed matter systems, and for the two solvable models we present in the Supplemental Material [26], the relevant paraparticles have quantum dimension $m = 4$, and the nonzero elements of the

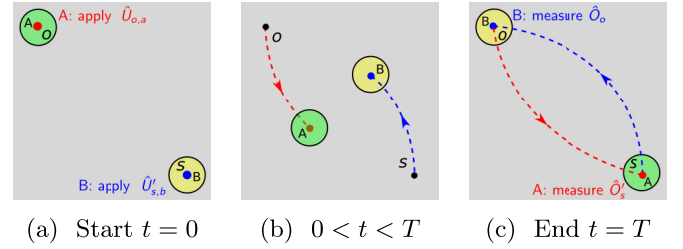


FIG. 2. Illustration of the winning strategy using emergent paraparticles. o and s are chosen to be the two special points where a paraparticle can be locally created and measured, as described in properties (5) and (6). The dashed lines represent the paths traversed by the circles, along which the players have applied unitary operators $\hat{U}_{ij}$ in Eq. (2).

R -matrices are given by

$$R_{ab}^{b'a'} = \pm 1, \quad \text{if } (b', a') = \begin{pmatrix} 43 & 12 & 24 & 31 \\ 21 & 34 & 42 & 13 \\ 14 & 41 & 33 & 22 \\ 32 & 23 & 11 & 44 \end{pmatrix}_{ab}, \quad (4)$$

where (4, 3) is abbreviated as 43 and similarly for others, and -1 (+1) is for the 2D (3D) model.

(5) *Creation and annihilation of paraparticles.* Paraparticles can be locally created at the two special points o, s shown in Fig. 2. More precisely, there exist unitary operators $\hat{U}_{o,a}, \hat{U}_{s,a}'$ localized around o, s , respectively, satisfying

$$\begin{aligned} \hat{U}_{o,a} |G; i_1^{a_1} \dots i_n^{a_n}\rangle &= |G; o^a i_1^{a_1} \dots i_n^{a_n}\rangle, \\ \hat{U}_{s,a}' |G; i_1^{a_1} \dots i_n^{a_n}\rangle &= |G; i_1^{a_1} \dots i_n^{a_n} s^a\rangle. \end{aligned} \quad (5)$$

Since unitary processes are reversible, one can also annihilate paraparticles at o, s using $\hat{U}_{o,a}^\dagger, \hat{U}_{s,a}'^\dagger$, respectively.

(6) *Measurement of the internal state.* When a paraparticle is close to either o or s , one can measure its internal state locally: There exist observables $\hat{O}_o, \hat{O}_s'$ localized around o, s , respectively, satisfying

$$\begin{aligned} \hat{O}_o |G; i_1^{a_1} \dots i_n^{a_n}\rangle &= a_1 |G; i_1^{a_1} \dots i_n^{a_n}\rangle, \quad \text{if } i_1 = o, \\ \hat{O}_s' |G; i_1^{a_1} \dots i_n^{a_n}\rangle &= a_n |G; i_1^{a_1} \dots i_n^{a_n}\rangle, \quad \text{if } i_n = s. \end{aligned} \quad (6)$$

Note that if we instead have $i_k = o$ for some $k > 1$ (or $i_k = s$ for some $k < n$), then we need to use a basis transformation in Eq. (3) [or more generally, Eq. (S14) in the Supplemental Material [26]] to swap i_k all the way to the front (back) before applying Eq. (6) to compute $\langle \hat{O}_o \rangle$ (or $\langle \hat{O}_s' \rangle$). We will see an example soon in Eq. (7).

All these properties can be derived from the exact solution of these two models, as we show in the Supplemental Material [26]. We emphasize that in order to win the challenge, it is crucial to have two special points o and s in the lattice where a single paraparticle can be created and measured using local operations, and these are the two points the players choose at which they begin and end their journey. In the 2D model, o and s are chosen to be at the intersection between two different types of gapped boundaries, while in the 3D model, they are chosen to be the positions of some special pointlike topological defects in the bulk.

Winning strategy with emergent paraparticles. We now detail the strategy that ensures success in the challenge. The pregame preparation is done using either of the two solvable models mentioned above: The players submit the model Hamiltonian $\hat{H}$, choose o, s to be the two special points in Eqs. (5) and (6), prepare its ground state $|G\rangle$, and let the Referees verify. After the game starts and the players obtain their numbers from the Referees, they use the following strategy to win, as illustrated in Fig. 2:

(a) At $t = 0$, Alice applies $\hat{U}_{o,a}$ to create a paraparticle with internal state a at o , and similarly Bob applies $\hat{U}'_{s,b}$ at s . The state of the whole system is $|\Psi(0)\rangle = |G; o^a s^b\rangle$.

(b) Throughout the game process, Alice and Bob move their paraparticles along the paths using $\hat{U}_{ij}$ in Eq. (2), closely following the circle movement.

(c) When the exchange is complete, the state evolves to

$$|\Psi(T)\rangle = |G; s^a o^b\rangle = \sum_{a',b'} R_{ab}^{b'a'} |G; o^{b'} s^{a'}\rangle, \quad (7)$$

where we use Eq. (3). Then, Alice and Bob measure $\hat{O}'_s$ and $\hat{O}'_o$ in Eq. (6) and obtain a' and b' , respectively, after which the state collapses into $|G; o^{b'} s^{a'}\rangle$. Finally, Alice and Bob annihilate their paraparticles using $\hat{U}_{s,a'}^{\dagger}$ and $\hat{U}_{o,b'}^{\dagger}$, respectively, so that the Referees cannot detect any excitations anywhere after they leave the game.

According to the R -matrix in Eq. (4), the measurement results a', b' are definite, and importantly, knowing b and b' completely determines a , and similarly, knowing a and a' completely determines b [30]. For example, if Alice has $(a, a') = (2, 3)$, then she searches in the second row of the matrix for the column that has $a' = 3$. This turns out to be the fourth column, so she determines that $b = 4$ and $b' = 1$. This allows the players to win the game with a 100% success rate.

This winning strategy highlights a clear physical distinction between paraparticles and ordinary fermions or bosons with an internal symmetry, such as color or flavor. The statistics of the latter can also be described by Eq. (3), but with $R_{ab}^{b'a'} = \pm \delta_{a'a} \delta_{b'b}$. Therefore, if ordinary fermions or bosons are used in this strategy instead of paraparticles, Alice will simply obtain $a' = a$ and Bob will obtain $b' = b$, which contains no information about each other's number, and therefore cannot win the game with a probability better than pure guessing. A more careful argument that ordinary fermions and bosons cannot win the game in a noise-robust manner using any possible strategy is given in the Supplemental Material [26], and a formal proof is given in Ref. [31].

It is important to emphasize again that this strategy relies only on the topological properties (1)–(6) of emergent paraparticles, which are universal characteristics of the underlying topological phase independent of the detailed microscopic realization. In particular, any quantum system hosting quasiparticles satisfying properties (1)–(6) with a nontrivial R -matrix (i.e., $R_{ab}^{b'a'}$ is not of the trivial product form $R_{ab}^{b'a'} = p_{a'a} q_{b'b}$) can pass the challenge. Therefore, we can use properties (1)–(6) as a definition of R paraparticles [31,32].

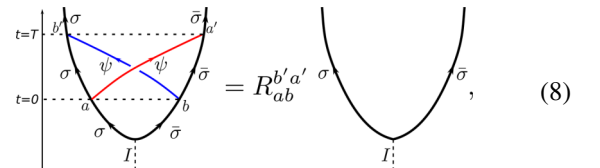
Finally, while winning the challenge in 3D unambiguously demonstrates nontrivial R parastatistics [27], in 2D it is possible that some special types of non-Abelian anyons can also win. In the Supplemental Material [26], we design a twisted

version of the game that can prevent a class of non-Abelian anyons from winning, and more general antianyon twists are given in Ref. [31].

Robustness against noise and eavesdropping. Importantly, the above winning strategy using emergent paraparticles is robust against noise, which is a consequence of Eq. (1). More precisely, the susceptibility to local noise decays exponentially as $e^{-l/\xi}$, where l is the minimal distance between the paraparticles and the boundary. This means that even when noise is present, the above strategy still has a high chance of success that is independent of the system size L . Equation (1) also implies the robustness against the most physically relevant class of eavesdropping via local measurements: When both players are in the bulk, an eavesdropper cannot obtain any information about the players' numbers a and b using any local measurements on the system, even including measurements at o, s or inside the circle areas.

Symmetric fusion category description. So far, we have presented solvable models in 2D and 3D with emergent paraparticles that can pass the challenge. We now give an alternative description of the winning strategy using the framework of SFC, which gives a universal description of pointlike quasiparticles in 3D topological phases [13–15,33–35]. We will see that certain 3D topological phases described by a special class of SFCs can win the game; an example is the 3D solvable model given in the Supplemental Material [26].

The SFC description of the winning strategy is based on the following space-time diagram:



$$= R_{ab}^{b'a'} \sigma \bar{\sigma}, \quad (8)$$

where σ represents a special type of topological quasiparticle, with $\bar{\sigma}$ being its antiparticle. In this case, the players need to prepare a state with topological quasiparticles σ and $\bar{\sigma}$ at positions o and s , respectively, which can still be the unique and gapped ground state of a locally interacting (but not translationally invariant) Hamiltonian that qualifies the requirement of the game, as shown in the Supplemental Material [26]. Importantly, we assume that the quasiparticles have fusion rules of the form [36]

$$\sigma \times \psi = m \sigma, \quad \psi \times \bar{\sigma} = m \bar{\sigma}, \quad (9)$$

where m is the quantum dimension of ψ . Equation (9) implies that ψ can be locally created and measured in the vicinity of σ or $\bar{\sigma}$, and the tensor $R_{ab}^{b'a'}$ in Eq. (8) defines the statistics of the paraparticle ψ in the sense of Eq. (3). Indeed, Eq. (8) exactly describes the winning strategy similar to that depicted in Fig. 2, in which at $t = 0$ Alice creates a paraparticle with internal state a in the vicinity of σ , and meanwhile Bob creates a paraparticle with internal state b in the vicinity of $\bar{\sigma}$; at $t = T$ when the exchange process is over, Alice measures the internal state of her paraparticle in the vicinity of $\bar{\sigma}$ and obtains a' , and then fuses her paraparticle into $\bar{\sigma}$, and meanwhile Bob performs similar operations in the vicinity of σ and obtains b' . As long as R is nontrivial, some information can be transferred between Alice and Bob, and by using multiple identical layers

of the same system, the players can transfer more information to each other, thereby arbitrarily enhancing the chance of winning. We therefore conclude that a gapped ground state $|G\rangle$ described by an SFC $\mathcal{C}$ can win the challenge if $\mathcal{C}$ has fusion rules of the form (9) such that the R -matrix defined by Eq. (8) is nontrivial [37]. The 3D solvable model given in the Supplemental Material [26] is described by an SFC of this type. A more detailed categorical analysis is presented in a separate paper [31].

Discussion. We have presented a secret communication challenge that demonstrates a distinctive physical feature of paraparticles. In a nutshell, emergent parastatistics in topological phases enables long-range communication between the two players solely through an exchange of their positions, without either perturbing the environment or leaving any trace of information behind. Such a winning strategy using parastatistics is robust against any local noise and eavesdropping, which cannot be achieved using ordinary fermions or bosons. We can therefore use this challenge game as a criterion and an operational definition for nontrivial parastatistics.

We emphasize that our secret communication challenge games fundamentally differ from prior nonlocal games in topological phases [38,39] in goal and design. These prior nonlocal games are played on the 2D toric code; their main goal is to demonstrate noise-robust quantum advantage, while our goal is to demonstrate the nontrivial physical consequence of R parastatistics, an intrinsically quantum many-body phenomenon that only occurs in certain exotic topological phases.

Finally, we discuss some potential future directions. One interesting direction is to experimentally implement this chal-

lenge game and its winning strategy using systems that host emergent paraparticles. The first critical step would be engineering such a system and preparing its ground state. The latter can potentially be achieved by generalizing the recent topological quantum state preparation protocol [40,41] involving measurements and feedforward to quantum double ground states based on an exotic class of solvable groups [42]. Another interesting direction is to generalize the challenge game and its winning strategy to quantum many-body systems at finite temperature (and more generally to mixed states). In 2D, there is no topological order at finite temperature [43–46]. In 3D, however, some nontrivial features of topological order persist at finite temperature [47,48]. Successfully achieving this challenge at finite temperatures is crucial for experimentally observing emergent parastatistics. Additionally, this could highlight a nontrivial feature of finite-temperature topological order and its potential application in quantum information.

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Data availability. The data that support the findings of this article are openly available [51].

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- [24] Throughout this paper, whenever we claim that the game can only be won by emergent paraparticles, we refer to the 3D version of the game, since the 2D version can also be won by a special class of non-Abelian anyons, and additional challenges need to be introduced to single out paraparticles in 2D, as we discuss in Sec. S4 of the Supplemental Material [26].
- [25] For example, we can require that the distance between o and s is at least half of the system size.
- [26] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/35db-174g> for a list of technical requirements on the Hamiltonian $\hat{H}$ (Sec. S1), details about the 2D and the 3D exactly solvable models with emergent paraparticles (Secs. S2 and S3), a preliminary example of antianyon twist (Sec. S4), a clarification on the robustness of the winning strategy against local noise and eavesdropping (Sec. S5), and a sketch of the argument that only nontrivial paraparticles can pass the 3D version of the challenge (Sec. S6).
- [27] We emphasize that, to establish the claim that only paraparticles can win the challenge in 3D, it is very important that we ask the Referees to choose the two paths in Fig. 1. If we instead allow the players to choose their paths, then they can win the game using non-Abelian anyons even in 3D. For example, they can construct a 3D system by stacking identical layers of some 2D topological phase hosting non-Abelian anyons, and then choose the two paths to lie entirely in the same layer, which reduces the quantum dynamics to a 2D problem, allowing them to use a 2D winning strategy with non-Abelian anyons. Asking the Referees to choose the paths (after the players submit the physical system) forces the players to use excitations with unrestricted mobility in 3D, thereby preventing the above strategy with non-Abelian anyons (whose mobility is necessarily restricted to some 2D surface).
- [28] If not, then the exclusion statistics of the paraparticles have to be taken into account [23]. This is unnecessary for our discussion here, since we never move two paraparticles to the same position in this paper.
- [29] In a general topological phase, Eq. (1) only holds up to a small correction of order $O(e^{-l/\xi})$ [49], where ξ is the correlation length of $|G\rangle$ and l is the minimal distance between the paraparticles (and the boundary). Consequently, the topological degeneracy in energy generally also has a small splitting up to $O(e^{-l/\xi})$. In this solvable model, however, we can tune the model parameters such that $\xi = 0$, and this small correction exactly vanishes.
- [30] Indeed, this R -matrix is an example of a perfect tensor; i.e., if we group any two indices of the R -matrix as input and the other two as output, the resulting matrix we get is unitary (it is an invertible classical gate in our case). This allows the players to obtain complete information about each other's number by inverting this matrix.
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