

Rise and fall of nonstabilizerness via random measurements

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We investigate the dynamics of nonstabilizerness—also known as “magic”—in monitored quantum circuits composed of random Clifford unitaries and local measurements. For measurements in the computational basis, we derive an analytical model for dynamics of the stabilizer nullity, showing that it decays in quantized steps and requires exponentially many measurements to vanish, which reveals the strong protection through Clifford scrambling. On the other hand, for measurements performed in rotated non-Clifford bases, measurements can both create and destroy nonstabilizerness. Here, the dynamics leads to a steady state with nontrivial nonstabilizerness, independent of the initial state. We find that Haar-random states equilibrate in constant time, whereas stabilizer states exhibit linear-in-size relaxation time. While the stabilizer nullity is insensitive to the rotation angle, stabilizer Rényi entropies expose a richer structure in their dynamics. Our results uncover sharp distinctions between coarse and fine-grained nonstabilizerness diagnostics and demonstrate how measurements can both suppress and sustain quantum computational resources.

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I. INTRODUCTION

In the quest for quantum advantage, significant effort has been devoted to identifying and formalizing the resources that enable quantum computational power, often through the framework of resource theories. While entanglement is a key signature of *quantumness*, it is not sufficient for quantum computational advantage. This is evidenced by the existence of states that can be efficiently simulated on classical computers despite being highly entangled, notably the class of stabilizer states. These are generated by Clifford operations [1–3], built from Hadamard, phase, and controlled-NOT (CNOT) gates, as well as by projective measurements in the computational basis. Stabilizer operations do not unlock universal quantum computation on their own, as they have to be augmented with non-Clifford operators—such as T gates [4,5]—for quantum advantage [6,7]. They constitute the main factor determining the runtime of quantum computers [8] and are exponentially hard to simulate on classical computers [9–11].

The amount of non-Clifford operations necessary to construct generic quantum states can be quantified through the resource theory of nonstabilizerness or “magic” [6,7,12–21]. Nonstabilizerness measures include, among others, the stabilizer nullity [22,23] and stabilizer Rényi entropies (SREs) [24]. The latter are particularly powerful: They can be analytically treated [25,26], numerically efficiently computed for different classes of states [27–33], and experimentally measured [34–37]. They offer a lens for understanding a wide variety of phenomena in many-body physics [38–41] and quantum chaos [11,42–45].

The interplay between quantum resources and quantum operations is now a rich area of exploration [16,24,46–48]. Clifford gates—free operations in the resource theory of magic—do not increase magic, yet they create a significant amount of entanglement. In contrast, projective measurements tend to suppress entanglement and destroy quantum information [49]. However, applying high-entanglement random unitaries can effectively spread the information throughout the Hilbert space, hiding it from local measurements [50,51]. This principle is the key feature used in quantum error correction to protect logical information within many redundant physical qubits [1,52]. Furthermore, systems combining unitary Clifford dynamics with random projective measurements have been shown to exhibit a measurement-induced phase transition in entanglement entropy [53–59] depending on the value of the measurement rate: Frequent measurements suppress entanglement, leading to area-law scaling and loss of information [60], while infrequent measurements allow unitary dynamics to dominate, resulting in volume-law growth and error suppression [50].

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Although much attention has been focused on the entanglement entropy [61–68], increasing interest has turned to nonstabilizerness in monitored quantum circuits, exploring its dynamics under measurements. Nonstabilizerness has been studied in different settings such as Clifford+T circuits [57,58], random quantum circuits [48,69–71], or monitored Gaussian fermions [72]. In Ref. [73], the authors investigated the decay of magic under sequences of local Cliffords and measurements, finding that the decay of magic can be characterized by two phases, depending on the rate of measurements. Further, Refs. [74,75] studied the injection of magical noise channels within a random Clifford basis followed by measurements of most of the qubits, in analogy to quantum error correction under coherent noise, which is shown to remove magic below a critical noise rate. Ref. [74] focuses mainly on the measurement-induced phase transition in magic, characterized by analytical, numerical, and experimental probes, through both SREs and basis-minimized measurement entropy. In contrast, Ref. [75] focuses on how such noisy channels influence nonstabilizerness, showing through the robustness of magic [16] that noise can generate magic depending on the initial state. However, past works concentrated on the regime where the number of measurements is extensive with system size, while the role of individual measurements in the dynamics has not been well understood.

Here, we study how random measurements create and destroy magic. We study circuits composed of global random Clifford unitaries combined with single-qubit measurements in the computational basis. We construct an analytical model to describe the dynamics of nullity, observing that it takes exponential time to decay, then extending it to the SRE. We then study a modified circuit where we rotate the single-qubit measurements into a non-Clifford basis, characterized by an angle parameter θ_M . In this setting, measurements can both and destroy magic and the system converges to a long-time steady state with nontrivial magic. The time to convergence depends on the initial state: States with high nullity require a constant time to relax, whereas low-nullity states require a linear time in the qubit number. For the nullity, the steady state is solely determined by the system size and independent of θ_M . In contrast, the SRE steady state scales as θ_M^2 for $\theta_M \ll 1$. Our work shows that magic can persist and thrive even under measurements.

The manuscript is organized as follows: In Sec. II, we review the magic resource measures of stabilizer nullity and SRE. In Sec. III, we explain our circuit and measurement protocol. In Sec. IV, we provide analytical models describing how magic evolves via nonmagic measurements, while in Sec. V we extend our model to measurements that can induce magic. In Sec. VB, we show numerical simulations of our protocol for both nullity and SRE for different states and comparing the numerics with analytical results. Finally, in Sec. VI we discuss our results and draw our conclusions.

II. NONSTABILIZERNESS

Let us consider N -qubit Pauli operators $P = \bigotimes_{i=1}^N \sigma_i^\alpha$ which are tensor products of single-qubit Pauli operators σ^x , σ^y and σ^z and the single-qubit identity $\sigma^0 = I_1$. We

consider $\mathcal{P}_N$ as the set of all unsigned N -qubit Pauli operators $P \in \mathcal{P}_N$, which is the Pauli group modulo global phases $\{\pm 1, \pm i\}$, with cardinality $|\mathcal{P}_N| = 4^N$. We denote $\mathcal{G}_S(|\psi\rangle)$ as the stabilizer group of a given state $|\psi\rangle$, i.e., the set of commuting Pauli operators with

$$P|\psi\rangle = \pm|\psi\rangle, \forall P \in \mathcal{G}_S. \quad (1)$$

Stabilizer states are pure quantum states where the stabilizer group has maximal size with $\mathcal{G}_S = 2^N$. Alternatively, any stabilizer state can be generated by applying on the computational basis state $|0\rangle^{\otimes N}$ the unitary gates of the Clifford group $U_C \in \mathcal{C}_N$, generated by the Hadamard gate, the CNOT gate and the S phase gate. Notably, such Clifford unitaries U_C map Pauli strings P into Pauli strings as $P' = U_C P U_C^\dagger$. Using the so-called tableau formalism [1,3], one can store the generators of the stabilizer group in a $N \times 2N$ tableau, allowing for efficient Clifford unitary operations at cost $\mathcal{O}(N^2)$.

Of course, not all states are stabilizer states, and indeed nonstabilizerness provides a way to quantify how far a given quantum state is from the set of stabilizer states. Various measures have been introduced to capture different aspects of this resource [9,19,34,76]. In the remainder of this section, we focus on two such quantities that are central to this work: the stabilizer nullity and the α stabilizer Rényi entropies.

The definition of the stabilizer nullity ν of a state [22,23] is based on the number of independent Pauli operators that stabilize it, formally

$$\nu(|\psi\rangle) = N - \log_2 |\mathcal{G}_S(|\psi\rangle)|, \quad (2)$$

where $|\mathcal{G}_S(|\psi\rangle)|$ is the cardinality of the stabilizer group. According to this definition, for a general pure state $|\psi\rangle$ its nullity is such that $\nu(|\psi\rangle) \leq N$, while $\nu(|\psi\rangle) = 0$ for pure stabilizer states. $\nu(|\psi\rangle)$ represents a genuine magic monotone, as it is nonincreasing under stabilizer operations, such as Clifford unitaries or measurements of Pauli operators. However, it exhibits sensitivity to small perturbations: Even infinitesimal nonstabilizer components in the state can reduce the number of exact Pauli stabilizers, resulting in a discontinuous jump in nullity. Despite this, its operational meaning and simplicity make it a useful and interpretable diagnostic of nonstabilizerness.

The second measure of magic we refer to are the α SREs [24]. They quantify the α th moments of the distribution of Pauli expectation values

$$\mathcal{M}_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \ln \left(\sum_{P \in \mathcal{P}_N} 2^{-N} |\langle \psi | P | \psi \rangle|^{2\alpha} \right). \quad (3)$$

α SREs can be interpreted as the α Rényi entropy of the classical probability distribution $\Xi_P = 2^{-N} |\langle \psi | P | \psi \rangle|^2$ since $\Xi_P \geq 0$ and $\sum_{P \in \mathcal{P}_N} \Xi_P = 1$ for any $|\psi\rangle$. While it is not a monotone under Clifford operations for $\alpha < 2$ and not a strong monotone for any α [27], it is a monotone for pure states for $\alpha \geq 2$ [77]. Note that the nullity can be used to bound the SREs as $\mathcal{M}_\alpha(|\psi\rangle) \leq \nu(|\psi\rangle) \ln 2$.

III. PROTOCOL

We study the evolution of magic in monitored circuits measured in random Clifford basis, which is sketched in Fig. 1(a).

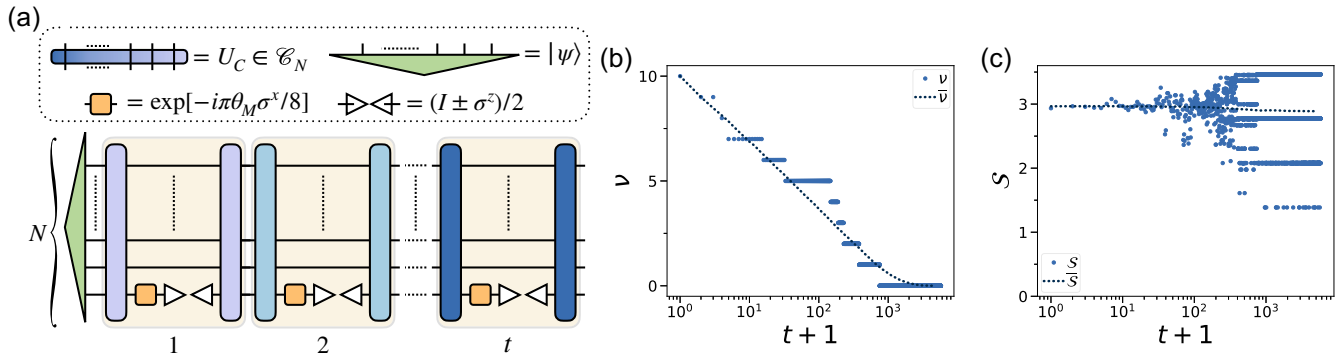


FIG. 1. (a) Circuit with random measurements in (rotated) Clifford basis. Starting from initial state $|\psi\rangle$, we apply t rounds of randomly chosen global Clifford unitaries $U_C \in \mathcal{C}_N$, single-qubit rotation $R_x(\theta_M)$ with angle $\theta_M \in [0, 1]$, projection onto single-qubit computational basis state $|b\rangle$ sampled according to its measurement probability p_s , and inverse Clifford $U_C^\dagger$. For $\theta_M = 0$, circuit is completely Clifford, while it induces magic for $\theta_M > 0$. (b) Representative trajectory of stabilizer nullity ν for different measurement steps t and $\theta_M = 0$ for $N = 10$, together with the average $\bar{\nu}$ over $N_r \approx 1000$ trajectories in dotted line. We start from initial Haar random state, and each measurement either leaves ν invariant or decreases it, yielding a characteristic stepwise decay toward stabilizer states, where the probability for a step to occur decreases exponentially with t . (c) Entanglement as measured by von-Neumann entropy $\mathcal{S}$ for half-bipartition against t for the same representative trajectory. The average entanglement $\bar{\mathcal{S}}$ barely changes with t ; however, individual trajectories of $\mathcal{S}$ fluctuate strongly with increasing t . Approaching low magic, and at the end of the dynamics when the state can be described as a random stabilizer, $\mathcal{S}$ can only acquire particular values, between which the trajectory jumps rapidly. For the representative trajectory, for $t \geq 10^3$ we see that $\nu = 0$, and average $\bar{\mathcal{S}}$ corresponds to the entanglement entropy of a random stabilizer state.

We consider a monitored circuit where we apply a random N -qubit Clifford unitary $U_C \in \mathcal{C}_N$, followed by a single-qubit rotation in the x basis acting on the first qubit

$$R_x(\theta_M) = \exp(-i\theta_M\pi/8\sigma_1^x). \quad (4)$$

Note that, due to the Clifford unitary randomizing the basis, we would get the equivalent dynamics if we choose any other qubit. We then measure the same qubit in the computational basis σ_1^z , record the measurement outcome $\in \{\pm 1\}$. Note that here measurement and rotation are orthogonal, so nonstabilizer-ness can remain after measurement, in contrast to the case where rotation and measurement in the same basis as studied in Ref. [57].

Then, we apply the inverse Clifford unitary $U_C^\dagger$ on the full state. We repeat this process t times, where we sample a random Clifford unitary U_C at every step. Operationally, this process can be described by measurement operators [4]

$$\Pi_\pm(\theta_M) = \frac{1}{2} \left[\cos \frac{\theta_M\pi}{8} (I_N \pm P^z) - i \sin \frac{\theta_M\pi}{8} (P^x \mp iP^y) \right], \quad (5)$$

where $P^\gamma \equiv U_C^\dagger \sigma_1^\gamma U_C$ is a Pauli operator rotated into a random Clifford basis with $\gamma \in \{x, y, z\}$, and we have the positive operator-valued measure $\{\Pi_b^\dagger \Pi_b\}_{b=\pm}$. For each step, the probability of measuring the outcome ± 1 in the state $|\psi\rangle$ is given by $p_\pm = \langle \psi | \Pi_\pm^\dagger \Pi_\pm | \psi \rangle$. After each measurement, we normalize the postmeasurement state as $|\psi_\pm\rangle = \frac{1}{\sqrt{p_\pm}} \Pi_\pm |\psi\rangle$.

For $\theta_M = 0$, the process is a Clifford operation and does not induce any magic. In this case, Eq. (5) simplifies to a projector of rank 2^{N-1}

$$\Pi_\pm(\theta_M = 0) = \frac{1}{2} (I_N \pm P), \quad (6)$$

with random Pauli $P \in \mathcal{P}_N / \{I\}$, and the superscript z has been discarded for simplicity. In contrast, for $\theta_M > 0$, the process can induce magic due to the single-qubit rotation $R_x(\theta_M)$ before the measurement, where the induced magic is maximal for $\theta_M = 1$.

To give a glimpse of the effect of such measurement protocol, we show a representative trajectory and the average over N_r trajectories for $\theta_M = 0$ in Figs. 1(b) and 1(c), respectively, the stabilizer nullity and the von Neumann entropy, defined as $\mathcal{S}(\rho_A) = -\text{tr}(\rho_A \log \rho_A)$ with bipartite state $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|)$ over bipartition A . In the two panels, we show the evolution of the two quantities starting from the same initial Haar-random state according to the same random unitaries. In Fig. 1(b), the nullity ν starts from its maximal value and decays at random times in a discrete stair case, yielding an average which takes an exponential number of measurements to vanish. In contrast, in Fig. 1(c) the average von Neumann entanglement entropy $\bar{\mathcal{S}}$ barely changes on average in time t . The trajectory shows initially small fluctuations that become more pronounced and discrete for large time t , rapidly jumping between different values. At exponential times, the state becomes a random stabilizer state ($\nu = 0$), for which $\mathcal{S}$ can only assume a linear number of different values [78–81]. Thus, magic uncovers a complementary picture on the dynamics in randomly monitored circuits, which we proceed to study in more detail in the following sections.

We test our protocols on four different initial states: random states $|\psi\rangle_{\text{Haar}}$ drawn from the Haar measure [82], tensor products of T states $|T\rangle^{\otimes N}$, where $|T\rangle \propto |0\rangle + e^{i\pi/4}|1\rangle$, states $|\psi\rangle_{\text{GUE}} = \exp(-iH_{\text{GUE}}t)|0\rangle^{\otimes N}$ evolved with a random Hamiltonian H_{GUE} drawn from the Gaussian unitary ensemble (GUE) with normalized spectrum $\in [-2, 2]$ [47,83] where we choose small $t = 0.1$, and finally the computational basis state $|0\rangle^{\otimes N}$.

IV. NONMAGIC MEASUREMENTS

A. Analytical model

We now present an analytical model of how measurements affect magic by describing the dynamics as a Markov chain, where we first consider $\theta_M = 0$, while $\theta_M > 0$ is deferred to Sec. V. We start with an initial state $|\psi_v\rangle$ with nullity v , then apply a random Clifford unitary, measure in the z basis, then apply the inverse of the Clifford unitary, which corresponds to Eq. (6). One can see from Eq. (6) that there are $|\mathcal{P}_N/\{I_N\}| = 4^N - 1$ possible choices of basis, from which we draw randomly due to the random Clifford transformation. At each round of measurement, v either decreases or remains unchanged. Any state $|\psi_v\rangle$ with nullity v can be written as [73]

$$|\psi_v\rangle = U_C |0\rangle^{\otimes N-v} |M\rangle, \quad (7)$$

where we have Clifford unitary U_C and some v -qubit magic state $|M\rangle$. Now, let us consider the state after measuring in Pauli basis P

$$\begin{aligned} \Pi_{\pm} |\psi_v\rangle &\propto (I_N \pm P) U_C |0\rangle^{\otimes N-v} |M\rangle \\ &= \frac{1}{2} U_C (I_N \pm P') |0\rangle^{\otimes N-v} |M\rangle, \end{aligned} \quad (8)$$

where we have the transformed Pauli operator $P' = U_C^\dagger P U_C$. The measurement $\Pi_{\pm}$ can affect stabilizer nullity v in three distinct ways depending on the chosen Pauli P' [73]:

(1) First, let us consider a Pauli basis $P' = P_{N-v} \otimes I_v$, where $P_{N-v} \in \mathcal{P}_{N-v}/\{I_{N-v}\}$ acts on the first $N-v$ qubits, while I_v is the identity acting on the last v qubits. Then, we can neglect any prefactors and the initial Clifford U_C as they do not affect the measurement. Further,

$$(I_N \pm P_{N-v} \otimes I_v) |0\rangle^{\otimes N-v} |M\rangle \quad (9)$$

acts trivially on the magic part $|M\rangle$ of the state, and only induces a Clifford transformation which leaves v invariant, i.e., $v \rightarrow v$. The case (1) has $(4^{N-v} - 1)$ possible Paulis P' .

(2) As a second case, let us consider $P' = \{I, \sigma^z\}^{\otimes N-v} \otimes P_v$, where we have a tensor product of identity and σ^z Pauli operators on the first $N-v$ qubits, and $P_v \in \mathcal{P}_v/\{I_v\}$. Let us assume that we have σ^z on the first $N-v$ qubits, while all other cases follow similarly. Applying the measurement gives us then

$$\begin{aligned} (I_N \pm \sigma^z)^{\otimes N-v} \otimes P_v |0\rangle^{\otimes N-v} |M\rangle \\ = (I_N \pm I_{N-v} \otimes P_v) |0\rangle^{\otimes N-v} |M\rangle. \end{aligned} \quad (10)$$

This is equivalent to measuring a qubit on a magic state $|M\rangle$, which reduces the stabilizer nullity $v \rightarrow v - 1$. Note that here we neglect the probability of the nullity decreasing by more than 1, as such events are extremely unlikely. Case (2) has $2^{N-v}(4^v - 1)$ possible Pauli operators.

(3) Finally, we have $P' = \{I, \sigma^z, \sigma^x, \sigma^y\}^{\otimes N-v} \otimes P_v$, where for the first $N-v$ qubits we have a tensor product of identity, σ^z , σ^x , and σ^y Pauli operators, where at least one Pauli operator is either σ^x or σ^y , and $P_v \in \mathcal{P}_v/\{I_v\}$. For simplicity, let us assume we have σ^x on the first $N-v$ qubits, while all other cases follow similarly. Then,

$$\begin{aligned} (I_N \pm \sigma^x)^{\otimes N-v} \otimes P_v |0\rangle^{\otimes N-v} |M\rangle \\ = (|0\rangle^{\otimes N-v} |M\rangle \pm I_{N-v} \otimes P_v |1\rangle^{\otimes N-v} |M\rangle). \end{aligned} \quad (11)$$

Now, we add the control-Pauli cP_v (by using $cP_v^\dagger cP_v = I$), where the control acts on one of the first $N-v$ qubits. Note that cP_v is a Clifford unitary. This gives us

$$\begin{aligned} (I_N \pm \sigma^x)^{\otimes N-v} \otimes P_v |0\rangle^{\otimes N-v} |M\rangle \\ = cP_v^\dagger cP_v (|0\rangle^{\otimes N-v} |M\rangle \pm I_{N-v} \otimes P_v |1\rangle^{\otimes N-v} |M\rangle) \\ = cP_v^\dagger (|0\rangle^{\otimes N-v} |M\rangle \pm |1\rangle^{\otimes N-v} |M\rangle) \\ = cP_v^\dagger (|0\rangle^{\otimes N-v} \pm |1\rangle^{\otimes N-v}) |M\rangle. \end{aligned} \quad (12)$$

The final state is a GHZ state for the first $N-v$ qubits with cP_v applied, which is a stabilizer state, with only $|M\rangle$ containing magic. Thus, the final state has unchanged nullity, i.e., $v \rightarrow v$. Case (3) has $(4^{N-v} - 2^{N-v})(4^v - 1)$ possible Paulis. One can easily verify that (1) to (3) cover all $P \in \mathcal{P}_N/\{I\}$ with in total $4^N - 1$ Pauli operators.

Now, after one time step, the nullity can only decrease for case (2) with probability

$$\text{Pr}_z(v \rightarrow v - 1) = \frac{2^N}{4^N - 1} (2^v - 2^{-v}), \quad (13)$$

where for $v \gg 1$ we have $\text{Pr}_z(v \rightarrow v - 1) \approx 2^{v-N}$, i.e., exponentially small with decreasing v . The nullity remains unchanged with probability $\text{Pr}_z(v \rightarrow v) = 1 - \text{Pr}_z(v \rightarrow v - 1)$.

Our model now describes, in agreement with what was derived in Ref. [48], a Markov chain for the random variable v , with the update rule of the probability distribution $\rho(v)$ for each time step t given by

$$\rho(v) \rightarrow \sum_{v'} \rho(v') \text{Pr}_z(v' \rightarrow v). \quad (14)$$

Motivated by this, we now derive an approximate model for the dynamics of v . To simplify the analytical calculations, let us define the nonstabilized dimension y as

$$y(v) = 2^v, \quad (15)$$

and rewrite Eq. (13) as

$$\text{Pr}_z(y \rightarrow y/2) = A_N \left(y - \frac{1}{y} \right), \quad (16)$$

where

$$A_N = \frac{2^N}{4^N - 1}. \quad (17)$$

Then, each step of the Markov chain changes the value of $\bar{y}$ as

$$\bar{y} \rightarrow \bar{y} - \frac{A_N}{2} (\bar{y}^2 - 1), \quad (18)$$

where the $\bar{y}$ indicates the statistical average over all realizations y . Let us assume that the probability distribution of y is sharply concentrated about the mean value $\bar{y}$, such that $\bar{y}^2 \approx \bar{y}^2$. In Appendix A, we find that the fluctuations of the nullity concentrate around its mean. In particular, we find that it has fluctuations constant in N , while v is usually linear in N . This implies that $\bar{y}^2 \approx c \bar{y}^2$ with some c , which numerically we find to be within $1 \leq c \leq 1.4$ during the evolution. Analytically, the choice $c = 1$ leads to a model that describes the actual dynamics with high accuracy and gives the correct steady state.

Now, we apply the approximation $\bar{y}^2 \approx \bar{y}^2$ to construct an approximate continuous-time model

$$\frac{d\bar{y}}{dt} \approx -\frac{A_N}{2}(\bar{y}^2 - 1). \quad (19)$$

In this model, the number of measurement rounds t is approximated by a continuous variable, which can be interpreted as the evolution time of the system. Solving this differential equation yields

$$\bar{y}(t) \approx 1 - \frac{2b}{e^{A_N t} + b}, \quad (20)$$

where

$$b = \frac{1 - \bar{y}(0)}{1 + \bar{y}(0)}. \quad (21)$$

For large N and $\bar{y}(0) \gg 1$, we have $A_N \approx 2^{-N}$ and $b \approx -1$, which gives

$$\bar{y}(t) \approx 1 + \frac{2}{e^{t/2^N} - 1} \approx \coth\left(\frac{t}{2^N}\right). \quad (22)$$

Note that $\bar{v}(t) \leq \log_2 \bar{y}(t)$ from Jensen's inequality. However, since we assumed that the distribution of y is sharply concentrated in deriving Eq. (19), we can approximate $\bar{v}(t) \approx \log_2 \bar{y}(t)$. Evaluating Eq. (22) for the regimes $t \ll 2^N$ and $t \gg 2^N$ yields the leading-order behavior

$$\bar{y}(t) \approx \begin{cases} a \frac{2^N}{t}, & t \ll 2^N, \\ 1 + 2e^{-t/2^N}, & t \gg 2^N. \end{cases} \quad (23)$$

Equivalently, we have

$$\bar{v}(t) \approx \begin{cases} N - \log_2 t, & t \ll 2^N, \\ \frac{2}{\ln 2} e^{-t/2^N}, & t \gg 2^N. \end{cases} \quad (24)$$

This implies that the stabilizer nullity (on average) decreases very slowly with the number of measurement rounds, up to an exponential timescale $t \sim 2^N$. At late times $t \gg 2^N$, the stabilizer nullity decreases exponentially to zero, where the decay factor 2^{-N} is exponentially small. We stress that these analytical results are meant to provide qualitative insights into the dynamics of the stabilizer nullity, and we do not expect this to exactly reproduce the discrete-time dynamics generated by the Markov chain due to the approximations used.

B. Numerical results

We now perform numerical simulations of our circuit and compare the results to our Markov chain model. We first focus on the stabilizer nullity v . The probability that the nullity decreases with measurement is exponentially small in v , as derived in Eq. (13). In Fig. 1(b), we plot one representative trajectory. We find that the first measurement always induces $v \rightarrow v - 1$, and then v decreases exactly N times in steps of 1, creating plateaus whose length increases exponentially with time, on average. The average $\bar{v}$ takes an exponential time to decay, which is plotted as a dotted line.

We apply the measurement protocol to three different initial state types with initial maximal nullity $\bar{v}(0) \equiv \bar{v}_0 = N$, as

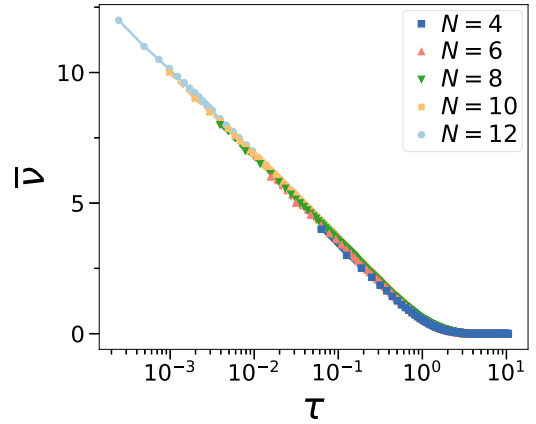


FIG. 2. Average nullity $\bar{v}$ against τ , where the rescaled time for each curve is $\tau = A_N(t + 1)$, A_N as defined in Eq. (17). The points represent the numerical simulation of the full quantum model via Eq. (5), while the lines are a simulation of our Markov chain model as given by Eq. (14).

listed in Sec. III. Note that here we do not study the initial computational basis state $|0\rangle^{\otimes N}$, as our protocol leaves stabilizer states trivially unchanged. The agreement between the exact numerical simulation and a simulation of our Markov chain model of Eq. (14) is shown in Fig. 2. The average $\bar{v}$ over $N_t \approx 1000$ trajectories behaves in the same way regardless of the initial state type. The results depicted are obtained starting from initial Haar-random states, but the other two states yield completely equivalent results. In particular, different points correspond to the numerical simulation of Haar-random states for different sizes N , and lines with matching colors correspond to the dynamics simulated by the Markov chain rule of Eq. (14) with the corresponding transition probabilities given by Eq. (13). Each curve is plotted as a function of time rescaled by its size-dependent timescale A_N as defined in Eq. (17), so that $t \rightarrow A_N t$ [we call $\tau = A_N(t + 1)$ to show what happens at $t = 0$ in a semilog scale]. As we can see, the model correctly predicts the average decay toward random stabilizer states with $\bar{v} = 0$, and all curves collapse on top of each other. We also find that the analytical solution to the differential equation in Eq. (20) agrees well with the exact numerical simulations, which we show in detail in Appendix B.

We now describe the dynamics of the SRE $\mathcal{M}_2$ in our circuit. Similar to v , we observe discrete changes in SRE with t , decreasing N times until it vanishes. However, for the SRE, the difference between consecutive plateaus is not constant, but can change; we show representative trajectories in Appendix C. Further, unlike stabilizer nullity v , the dynamics depends on the initial state. We now discuss in detail what happens for Haar-random states. We can characterize the distribution of differences in SRE between two consecutive plateaus using the following ansatz: After $N - v$ reductions in nullity, the remaining state can be written as v -qubit Haar-random state $|\psi\rangle_{\text{Haar}}^v$ [73] combined with $(N - v)$ -qubit stabilizer state in a random Clifford basis U_C

$$|\psi_v\rangle = U_C |\psi\rangle_{\text{Haar}}^v |0\rangle^{\otimes N-v}, \quad (25)$$

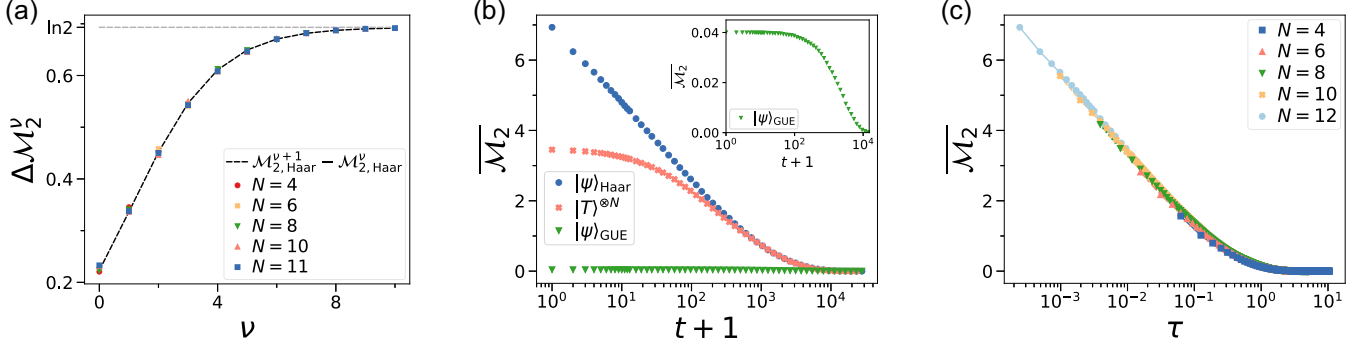


FIG. 3. (a) Change in SRE $\Delta \mathcal{M}_2^\nu$ at the ν th measurement that yields a change in magic, averaged over $N_r \approx 1000$ trajectories, for different N and for $\theta_M = 0$. The dashed line is $\overline{\mathcal{M}}_{\text{Haar}}^{\nu+1} - \overline{\mathcal{M}}_{\text{Haar}}^\nu$, where $\mathcal{M}_{\text{Haar}}^N$ is the SRE of Haar random state of N qubits as defined in Eq. (26). (b) Average SRE $\overline{\mathcal{M}}_2$ over $N_r \approx 1000$ trajectories vs measurement step t for three different initial states, $|\psi\rangle_{\text{Haar}}$, $|T\rangle^{\otimes N}$, and $|\psi\rangle_{\text{GUE}}$ for $\theta_M = 0$ for $N = 12$ qubits. Inset: Average magic $\overline{\mathcal{M}}_2$ for $|\psi\rangle_{\text{GUE}}$ with different axis ranges. (c) Dots show numerical $\overline{\mathcal{M}}_2$ against rescaled $\tau = A_N(t+1)$ for initial Haar-random states averaged over $N_r \approx 1000$ trajectories where $A_N \approx 2^{-N}$. Solid lines show $\overline{\mathcal{M}}_2$ obtained using nullity from analytical prediction of the Markov chain in Eq. (20) inserted into SRE of Haar-random states of Eq. (26).

where its average magic is given by [24]

$$\mathcal{M}_{2,\text{Haar}}(\nu) = \ln \left(\frac{3 + 2^\nu}{4} \right). \quad (26)$$

We find that the change in SRE from the step with nullity $\nu + 1$ to nullity ν , which we define as $\Delta \mathcal{M}_2^\nu$, is on average

$$\Delta \mathcal{M}_{2,\text{Haar}}^\nu = \mathcal{M}_{2,\text{Haar}}^{\nu+1} - \mathcal{M}_{2,\text{Haar}}^\nu. \quad (27)$$

In particular, looking at Fig. 3(a), we can see that at the beginning of the evolution ($\nu \gg 1$), the average difference between two consecutive plateaus is almost constant as $\Delta \mathcal{M}_2^\nu \rightarrow \ln 2$, which is predicted by Eq. (26), while it becomes smaller for $\nu \approx 0$ due to finite-size effects.

As previously stated, the SRE is initial-state dependent. In general, also for $|T\rangle^{\otimes N}$ and $|\psi\rangle_{\text{GUE}}$ states, $\theta_M = 0$ measurements can either reduce SRE or keep it constant, with rare exceptions better explained in Appendix D. In Fig. 3(b), we then show how the average SRE $\overline{\mathcal{M}}_2$ evolves with this measurement protocol for different initial states with initial maximal nullity $\nu_0 = N$. Despite all three still taking an exponential number of measurements to vanish, they behave differently for most part of the dynamics. In particular, the SRE for $|T\rangle^{\otimes N}$ states, after a certain number of measurements, overlaps with the SRE of $|\psi\rangle_{\text{Haar}}$ states. For the $|\psi\rangle_{\text{GUE}}$ states, $\overline{\mathcal{M}}_2$ is much lower than the one of the other two states as shown in the inset in Fig. 3(b) with a different range on the y axis, overlapping with the other two at the very end of the dynamics.

We can now show that the model derived for the decay of nullity can also accurately capture that of the SRE for Haar-random states. Indeed, for such states, Eq. (26) holds and maps the nullity $\nu \rightarrow \mathcal{M}_2$ at all times. In Fig. 3(c), we show for Haar-random states the average SRE $\overline{\mathcal{M}}_2$ over $N_r \sim 1000$ trajectories as different points of different colors representing increasing N . The solid lines of the same color represent the mapping through Eq. (26) of the ν as predicted by the Markov chain rule in Eq. (20). Time is rescaled as $t \rightarrow A_N t$ [we again call $\tau = A_N(t+1)$] consistently with the scaling observed for stabilizer nullity as shown in Fig. 2, the different curves collapse onto a single universal one. The SRE of both $|T\rangle^{\otimes N}$

and $|\psi\rangle_{\text{GUE}}$ states does not respect such functional relation, as further explained in Appendix E. Despite no analytical predictions on them can be made using this model, we can still use this relation as an effective model to phenomenologically describe SREs for different ensembles (in Appendix B).

V. MEASUREMENT IN MAGIC BASIS

A. Analytical model

Next, we develop a model for our circuit with $\theta_M > 0$; i.e., the random measurements can induce magic via the rotation $R_x(\theta_M)$ before the measurement. First, we compute the probability that the rotation $R_x(\theta_M)$ changes the nullity ν . Here, we can assume the rotation

$$\begin{aligned} R_P(\theta_M) &= \exp(-i\theta_M \pi / 8 P) \\ &= \cos(\theta_M \pi / 8) I - i \sin(\theta_M \pi / 8) P \end{aligned}$$

acts in a random Pauli basis $P = U_C \sigma^x U_C^\dagger$ due to a random Clifford unitary U_C . We can write the initial state for nullity ν as $|\psi_\nu\rangle$ with Clifford unitary U_C applied to a tensor product of stabilizer state $|0\rangle^{\otimes N-\nu}$ and some magic state of ν qubits $|M\rangle$ [73]

$$|\psi_\nu\rangle = U_C |0\rangle^{\otimes N-\nu} |M\rangle. \quad (28)$$

Now, the rotation $R_P(\theta_M)$ does not increase ν when P acts trivially on $|0\rangle^{\otimes N-\nu}$. These are Paulis of the form $P \in \{I, \sigma^z\}^{\otimes N-\nu} \otimes P_\nu$, where $P_\nu \in \mathcal{P}_\nu$ and $P \neq I_N$. The corresponding probability is given by

$$\Pr_R(\nu \rightarrow \nu) = \frac{2^{N+\nu} - 1}{4^N - 1}. \quad (29)$$

The probability that the nullity increases is then given by

$$\Pr_R(\nu \rightarrow \nu + 1) = 1 - \Pr_R(\nu \rightarrow \nu) = \frac{4^N - 2^{N+\nu}}{4^N - 1}. \quad (30)$$

Here, we neglected irrelevant fringe cases: For example, ν could also potentially decrease due to the rotation on $|M\rangle$, which however is extremely unlikely. Note that the nullity only measures whether stabilizers are broken (even by the

slightest amounts), and thus is practically independent of rotation angle θ_M as long as it is non-Clifford, i.e., $\theta_M \neq 0$.

After the rotation, the measurement is applied. Assuming sufficient entanglement, the measurement is nearly independent of the rotation and only depends on the nullity (after the rotation). For measurement, the same probabilities as in the previous section (with $\theta_M = 0$) apply. Thus, we can simply multiply the probabilities of change in v for rotation and measurement, giving us

$$\Pr_f(v \rightarrow v') = \sum_{v''} \Pr_R(v \rightarrow v'') \Pr_Z(v'' \rightarrow v'). \quad (31)$$

This gives us the explicit transition rules

$$\Pr_f(v \rightarrow v - 1) = \frac{4^N}{(4^N - 1)^2} (2^v - 2^{-N})(2^v - 2^{-v}), \quad (32)$$

$$\Pr_f(v \rightarrow v + 1) = \frac{4^N - 2^{N+v}}{4^N - 1} \left(1 - \frac{2^N(2^{v+1} - 2^{-v-1})}{4^N - 1} \right), \quad (33)$$

and

$$\Pr_f(v \rightarrow v) = 1 - \Pr_f(v \rightarrow v - 1) - \Pr_f(v \rightarrow v + 1). \quad (34)$$

These probabilities now define the update rules for the Markov chain for distribution $\rho(v)$ via

$$\rho(v) \rightarrow \sum_{v'} \rho(v') \Pr_f(v' \rightarrow v). \quad (35)$$

Now, we establish an analytical model. For that, let us define the normalized nonstabilized dimension as

$$w(v) \equiv y(v)/2^N = 2^{v-N}. \quad (36)$$

In the regime $\{v, N\} \gg 1$, the transition probabilities follow

$$\Pr_f(w \rightarrow w) \approx 3w - 3w^2, \quad (37)$$

$$\Pr_f(w \rightarrow 2w) \approx 1 - 3w + 2w^2, \quad (38)$$

and

$$\Pr_f(w \rightarrow w/2) \approx w^2. \quad (39)$$

At each step of the Markov chain, the mean value $\bar{w}$ changes as

$$\bar{w} \rightarrow 2\bar{w} - 3\bar{w}^2 + \frac{3}{2}\bar{w}^3. \quad (40)$$

To derive this, we used the update rule of the Markov chain

$$\rho(w) \rightarrow \sum_{w'} \rho(w') \Pr_f(w' \rightarrow w) \quad (41)$$

for the probability distribution $\rho(w)$, to obtain the corresponding update rule for $\bar{w}$,

$$\begin{aligned} \bar{w} &\rightarrow \sum_{w, w'} w \rho(w') \Pr_f(w' \rightarrow w) \\ &= \sum_w [w \rho(2w) \Pr_f(2w \rightarrow w) + w \rho(w/2) \Pr_f(w/2 \rightarrow w) \\ &\quad + w \rho(w) \Pr_f(w \rightarrow w)] \\ &= \sum_w \rho(w) \left[\frac{w}{2} \Pr_f(w \rightarrow w/2) + 2w \Pr_f(w \rightarrow 2w) \right. \end{aligned}$$

$$\begin{aligned} &\left. + w \Pr_f(w \rightarrow w) \right] \\ &= \frac{1}{2} \overline{w \Pr_f(w \rightarrow w/2)} + 2 \overline{w \Pr_f(w \rightarrow 2w)} \\ &\quad + \overline{w \Pr_f(w \rightarrow w)} = 2\bar{w} - 3\bar{w}^2 + \frac{3}{2}\bar{w}^3. \end{aligned} \quad (42)$$

Equation (40) motivates the continuous-time model

$$\frac{d\bar{w}}{dt} \approx \bar{w} - 3\bar{w}^2 + \frac{3}{2}\bar{w}^3 \quad (43)$$

analogous to Eq. (19), assuming that the probability distribution $\rho(w)$ is concentrated close to $\bar{w}$, i.e., $\overline{w^2} \approx \bar{w}^2$ and $\overline{w^3} \approx \bar{w}^3$. This model has one stable fixed point at $\bar{w} \approx 1 - 1/\sqrt{3} \approx 0.423$. Assuming that $\bar{w} \approx 2^{\bar{v}-N}$, this predicts the steady-state value for the mean stabilizer nullity $\bar{v} \approx N - 1.24$; i.e., $\bar{v}$ is asymptotically maximum up to a small additive shift. Physically, this implies that the injection of magic even for an arbitrarily small θ_M overwhelms the decay of magic due to the measurement.

We can get further insights into the dynamics by solving the continuous-time model. By inspecting Eq. (43), we can see that $\bar{w}(t)$ converges to the stable fixed point $\bar{w}_\infty = 1 - 1/\sqrt{3} \approx 0.423$ from above (below) if the initial value $\bar{w}_0$ is above (below) $\bar{w}_\infty$. As we will now show, the convergence timescales in both cases can be drastically different. First, let us consider the case where $\bar{w}_0 > \bar{w}_\infty$, and write

$$\bar{w}(T_\downarrow) = w_\infty + \epsilon, \quad (44)$$

for some small $\epsilon > 0$. The convergence time $T_\downarrow$ is given by

$$T_\downarrow \approx \frac{1}{\sqrt{3} - 1} \ln \left(\frac{\bar{w}_0 - \bar{w}_\infty}{\epsilon} \right), \quad (45)$$

which does not explicitly depend on the size of the system N . For example, if we initialize in a Haar-random state such that $\bar{w}_0 \approx 1$, the convergence time

$$T_\downarrow \approx \frac{1}{\sqrt{3} - 1} \ln \left(\frac{1}{\epsilon} \right) \quad (46)$$

is independent of N . On the other hand, if we consider the case where $\bar{w}_0 < \bar{w}_\infty$ and write

$$\bar{w}(T_\uparrow) = w_\infty - \epsilon, \quad (47)$$

again for a small $\epsilon > 0$, the convergence time $T_\uparrow$ becomes

$$T_\uparrow \approx \ln \left(\frac{\bar{w}_\infty}{\bar{w}_0} \right) + \frac{1}{\sqrt{3} - 1} \left(\frac{\bar{w}_\infty - \bar{w}_0}{\epsilon} \right). \quad (48)$$

For example, if we initialize in a stabilizer state, $\bar{w}_0 = 2^{-N}$, and the convergence time

$$T_\uparrow \approx N \ln 2 + \frac{1}{\sqrt{3} - 1} \ln \left(\frac{1}{\epsilon} \right) \quad (49)$$

grows linearly with N . The difference in timescales between Eqs. (46) and (49) can be intuitively understood as follows: The steady-state value of the stabilizer nullity v is extensive in N (with a small constant correction), and each measurement round only changes v by at most 1. Thus, we need $\sim N$ number of measurements to inject extensive magic into the initial

stabilizer state, and only a constant number of measurements to remove a constant amount of magic for Haar-random states.

Now, we get a more accurate estimate of the steady-state value of $\bar{\nu}$ by solving for the steady state of the Markov chain approximately. To this end, let us denote the steady-state probability distribution for w by $\rho_\infty(w)$. The steady state satisfies the recurrence relation

$$\begin{aligned} \rho_\infty(w) = & \Pr_f(w \rightarrow w) \rho_\infty(w) + \Pr_f(2w \rightarrow w) \rho_\infty(2w) \\ & + \Pr_f(w/2 \rightarrow w) \rho_\infty(w/2). \end{aligned} \quad (50)$$

It is straightforward to see from the Markov chain that $\rho_\infty(1) = 0$. Evaluating the recurrence relation for specific values of w , we get

$$\rho_\infty(1/4) = \frac{2}{3} \rho_\infty(1/2), \quad (51)$$

$$\rho_\infty(1/8) = \frac{2}{21} \rho_\infty(1/4), \quad (52)$$

and

$$\rho_\infty(1/16) = \frac{2}{105} \rho_\infty(1/8). \quad (53)$$

Now, since $\rho_\infty(2/105)$ is very small, we can approximate $\rho_\infty(2/105) \approx 0$. Applying the normalization condition $\rho_\infty(1/2) + \rho_\infty(1/4) + \rho_\infty(1/8) \approx 1$, we obtain the steady-state probability distribution

$$\rho_\infty(w) \approx \begin{cases} 0.578, & w = 1/2, \\ 0.385, & w = 1/4, \\ 0.037, & w = 1/8, \\ 0, & \text{else,} \end{cases} \quad (54)$$

rounded to three decimal places. From $\rho_\infty(w)$, we can estimate

$$\bar{w} \approx 0.39 \quad (55)$$

and

$$\nu_{\text{analytic}}^{\text{ss}} \equiv \bar{\nu} \approx N - 1.46, \quad (56)$$

which matches the actual steady-state nullity accurately as we show in the next section.

B. Numerical results

We now compare the analytical predictions with the numerical simulations. Considering single trajectories, which we show and analyze in more detail in Appendix C, nullity is bounded by $\nu \leq N - 1$ as the measurements reduce $\bar{\nu} \rightarrow \bar{\nu} - 1$ whenever ν approaches the maximum value. In Fig. 4, we show how this is reflected in the average $\bar{\nu}$, as it evolves, after a brief transient, toward the steady-state value ν^{ss} which solely depends on N and not on the angle parameter θ_M . In this figure, for each fixed size, the points correspond to the value of the average over $N_r \approx 1000$ trajectories of the nullity $\bar{\nu}$ computed numerically for two different classes of states. The first class is that of states with maximal initial nullity, $\bar{\nu}_0 = N$, such as $|\psi\rangle_{\text{Haar}}$ states, $|T\rangle^{\otimes N}$ and $|\psi\rangle_{\text{GUE}}$, which all yield the same effective dynamics. The second class starting from $\bar{\nu}_0 = 0$ corresponds to $|0\rangle^{\otimes N}$.

In addition to the exact numerical simulations, we plot in Fig. 4 also numerical simulations of the Markov chain of Eq. (35), with the transition probabilities described in

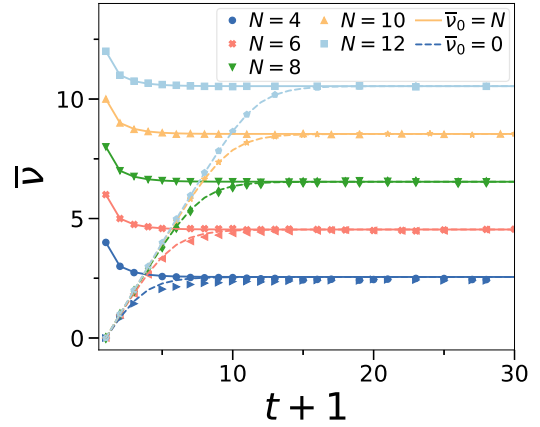


FIG. 4. Average stabilizer nullity $\bar{\nu}$ against measurement step t for $\theta_M = 0.001$ ($\bar{\nu}$ evolves the same for any $\theta_M > 0$). Points show average over $N_r \approx 1000$ trajectories with initial states given by Haar-random states (points starting from $\bar{\nu} = N$) and by stabilizer state $|0\rangle^{\otimes N}$ (starting from $\bar{\nu} = 0$). Solid lines are the numerical simulation of the Markov chain via Eq. (35) for initial $\bar{\nu}_0 = N$, while dashed lines are for initial $\bar{\nu}_0 = 0$.

Eqs. (29)–(34). Here, solid lines represent the result of the numerical simulations for initial high-nullity states, while dashed lines represent the same quantity for initial stabilizer states.

We numerically confirm that the convergence time to the steady state depends only on the initial nullity: The former states, i.e., initial states with nullity larger than the steady-state value, converge to it in a time t that is independent of N , as predicted in Eq. (46). In contrast, the latter with lower nullity converge in a time that is linear in N , our numerical findings matching the theoretical predictions of Eq. (49).

Furthermore, the analytical prediction in Sec. V gives $\nu_{\text{analytic}}^{\text{ss}} \simeq N - 1.46$. Numerically, we find that for all initial states, we converge to the same steady state with $\nu_{\text{num}}^{\text{ss}} \simeq N - 1.46 \pm 0.02$. Thus, we find that our theory and numerics are in excellent agreement.

We now analyze the evolution of the average $\mathcal{M}_2$ for the same set of four different initial states in the presence of measurements in the magical basis ($\theta_M \neq 0$). Due to the nonmonotonic behavior of SRE along individual trajectories, in the main text we primarily focus on the ensemble-averaged $\bar{\mathcal{M}}_2$ and in particular, we consider the density $\bar{m}_2 = \bar{\mathcal{M}}_2/N$. The behavior of individual trajectories for different θ_M and initial states is described in detail in Appendix C.

Unlike stabilizer nullity, which only depends on ν_0 of the initial state, $\bar{m}_2$ strongly depends on the properties of the initial states. This is due to the fact that SREs probe the structure of the Pauli spectrum, which differs for different types of states. However, the asymptotic steady state is independent of the chosen initial state, and only depends on the measurement angle parameter θ_M . In Figs. 5(a)–5(c), we show the evolution of the SRE density for three representative values of the parameter $\theta_M = 0.01, 0.2$, and 1 . For the smallest angle $\theta_M = 0.01$ in panel (a), the dynamics of both $|\psi\rangle_{\text{Haar}}$ and $|T\rangle^{\otimes N}$ states closely resembles the $\theta_M = 0$ one, while the other two state types have very little increasing SRE until they reach a low steady-state value. Increasing the angle parameter

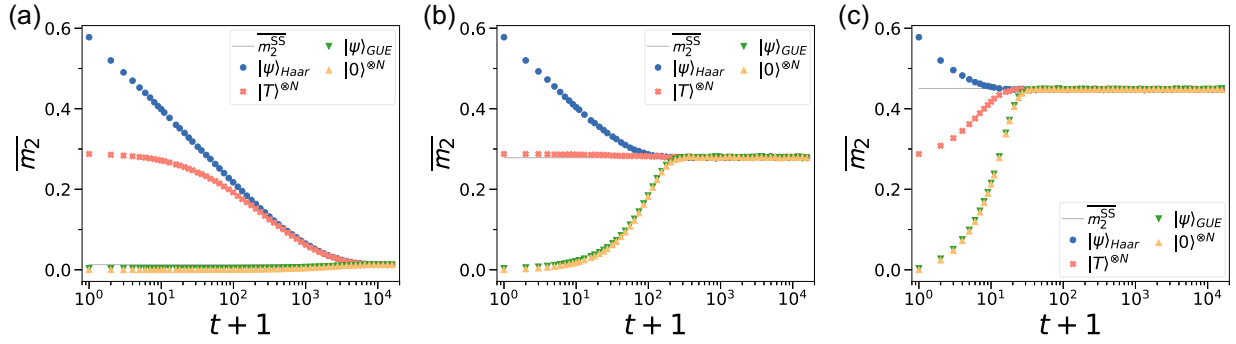


FIG. 5. (a) Average magic density $\bar{m}_2 = \bar{\mathcal{M}}_2/N$ over $N_t \approx 1000$ trajectories vs measurement steps for $\theta_M = 0.01$ for $N = 12$ for four different initial states. (b) Average magic density $\bar{m}_2 = \bar{\mathcal{M}}_2/N$ over $N_t \approx 1000$ trajectories vs measurement steps for $\theta_M = 0.2$ for $N = 12$ for four different initial states. (c) Average magic density $\bar{m}_2 = \bar{\mathcal{M}}_2/N$ over $N_t \approx 1000$ trajectories vs measurement steps for $\theta_M = 1$ for $N = 12$ for four different initial states.

to $\theta_M = 0.2$, in panel (b) we observe that the number of measurements required to get to the same steady state decreased, the Haar-random initial states still follow a similar decay profile, while the SRE for T states is nearly constant changing drastically the behavior with respect to the nonmagical basis case, and the other two state types behave almost the same increasing monotonically. Finally, at maximal $\theta_M = 1.0$, all four ensembles rapidly converge to the same-steady value within approximately $\sim N$ measurement steps. The number of measurements required to reach the steady state strongly depends on the angle parameter θ_M , going from exponentially big for $\theta_M \ll 1$ to linear for $\theta_M \rightarrow 1$ in the number of qubits N , which can be numerically extrapolated for Haar-random states, shown in Appendix F. Despite the difference in the timescales, in the three panels we keep the same range on both axes to better compare the dynamics. These different timescales barely depend on the initial state type, although we can still see how SRE of $|T\rangle^{\otimes N}$ states always tends to overlap with that of $|\psi\rangle_{Haar}$ before reaching the steady state, while the states with initial $\bar{m}_2(0) < \bar{m}_2^{SS}$ reach it in slightly more measurements than the ones with $\bar{m}_2(0) > \bar{m}_2^{SS}$.

The continuous growth of the steady-state value $\bar{m}_2^{SS}$ with the angle parameter θ_M can be better visualized in Fig. 6. We

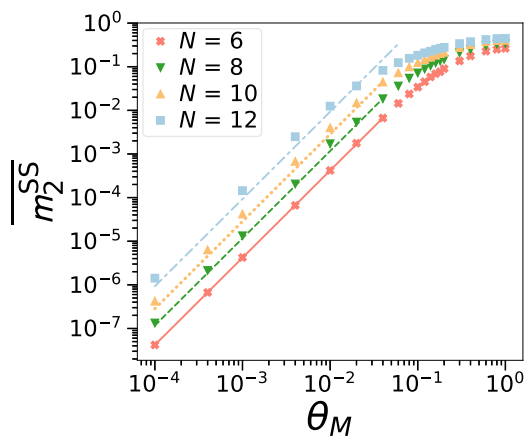


FIG. 6. Average magic density of the steady state $\bar{m}_2^{SS}$ for $t \rightarrow \infty$ as a function of θ_M for different N shown as dots. For $\theta_M \ll 1$, we find $\bar{m}_2^{SS} \propto \theta_M^2$ which is indicated as lines.

find that for “low-magic” basis measurements ($\theta_M \ll 1$), the steady-state value increases quadratically with the angle, as shown in the same figure by the dashed fit lines. We find that the steady state is in general not a Haar-random state of the form of Eq. (25), which is evidenced by the fact for any $\theta_M > 0$ the nullity is close to $N - 1$, while $\mathcal{M}_2$ can be relatively small, especially for $\theta_M \ll 1$ [84]. However, the steady state of SREs appears to be close to the corresponding SRE of Haar-random states of $N - 2$ qubits, as shown in Appendix G.

It is important to stress that in this protocol stabilizer nullity ν is independent of the choice $\theta_M > 0$, while SRE strongly depends on θ_M . This suggests that the mapping in Eq. (26) is not valid in this case, not even for Haar-random states. Therefore, despite the clarity of the numerical description, further investigations are required to analytically model the behavior of SRE in this protocol.

VI. DISCUSSION

In this work, we studied how sequences of measurements and random Clifford unitaries affect the nonstabilizerlessness—also known as magic—of quantum states. Starting from Haar-random initial states, we derive an analytical expression for the decay of stabilizer nullity under measurements in the computational basis. While measurements usually destroy magic, we find that random Clifford transformations protect it. In fact, we find that erasing all magic requires an exponential number of measurements in the system size, demonstrating the inherent robustness of magic. We note that this matches the time needed for purification [51]. We find that our analytical model for stabilizer nullity ν as well as for SREs $\mathcal{M}_\alpha$, accurately captures the quantitative behavior of the dynamics and matches our numerical simulations.

We then extended our analysis to single-qubit measurements in rotated bases, where measurements can also induce magic. Notably, rotated measurements can both generate and suppress magic, which converges to a steady state for sufficiently long evolution [47]. We construct an analytical model for the stabilizer nullity ν in this generalized case, accurately capturing the stabilizer nullity of the ensemble steady state. Starting from an initial ν , we observe that the average stabilizer nullity converges to about $N - 1$, independent of

measurement rotation angle θ_M and the initial state. Meanwhile, the steady-state value of the SRE $\mathcal{M}_2^{\text{ss}}$ depends on θ_M , with a quadratic scaling $\mathcal{M}_2^{\text{ss}} \propto \theta_M^2$ in the small-angle limit. Our findings for measurements in a rotated basis also provide a lens through which to view the landscape of phases in monitored circuits. Of particular interest is the recently discussed phase of matter characterized by high, volume-law entanglement but low, area-law magic [48]. Our model provides a tunable protocol to realize and stabilize such a state: By choosing $\theta_M \propto 1/\sqrt{N}$ our circuit naturally flows to a steady state with volume-law entanglement but constant nonzero magic. This provides a concrete and controllable setting to study the properties of this phase. To support our analytical results, we performed extensive numerical simulations for various initial states, such as Haar-random states, unentangled product states (e.g., T states), and short-time GUE-evolved states. Across all these initial conditions, we observed consistent behavior that confirms the validity of our conclusions and the generality of the observed phenomena.

Our findings reveal that measurements, while typically associated with the destruction of quantum resources, can also act as a source of magic under suitable conditions. This highlights avenues for manipulating nonstabilizerness in hybrid quantum protocols and raises intriguing questions about the interplay between measurement, randomness, and quantum computational resources.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: FLUCTUATIONS OF THE NULLITY DISTRIBUTION

In this Appendix, we study the fluctuations of distributions of the stabilized dimension $y = 2^\nu$ and the nonstabilized dimension $w = 2^{v-N}$ for our dynamics. In particular, we want to focus on the approximations made to get to the continuous-time models as defined in Eq. (19) for the variable y and in Eq. (43) for w .

We first perform numerical studies for the nullity ν , y , and w . For simulation, we use the update rule as defined in Eq. (14) as well as the full quantum simulation. We compute the average over the distribution $\rho(\nu)$ at a fixed time step t as

$$\bar{\nu} \equiv \mathbb{E}[\nu] = \sum_{\nu} \nu \rho(\nu) \quad (\text{A1})$$

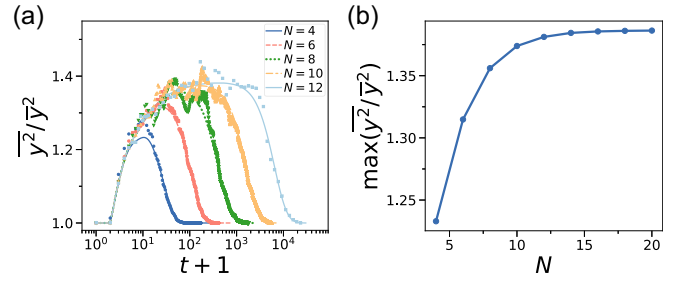


FIG. 7. (a) $\bar{y}^2/\bar{y}^2$ for different sizes of the system N for the measurements in the computational basis, computed using the Markov chain (lines) and full simulation of the quantum state (dots), with initial Haar-random state $|\psi\rangle_{\text{Haar}}$. (b) Maximum value of the ratio $\bar{y}^2/\bar{y}^2$ over time as a function of the system size N .

and the relative variance as

$$\sigma_v^2 \equiv \sigma^2[\nu] = \bar{\nu}^2 - \bar{\nu}^2. \quad (\text{A2})$$

We define $\bar{y}$, $\bar{y}^2$, $\bar{w}$, and $\bar{w}^2$ similarly. We find indeed that the variance σ_v^2 of its distribution is strongly centered around the mean value $\bar{\nu}$, with the variance independent of system size. As the mean is usually linear in N , we have $\sigma_v/\bar{\nu} \propto 1/N$; i.e., the nullity indeed concentrates around its mean.

Now, we consider the stabilized dimension $y = 2^\nu$, such that $dy = 2^\nu \ln(2)d\nu$, implying that in general the fluctuations of y are the same order as its mean value. In fact, we find numerically that

$$\frac{\bar{y}^2}{\bar{y}^2} = c(t) \quad (\text{A3})$$

is bounded. We can compute such quantity using the simulation of the quantum circuit, and compare them with the prediction of the Markov chain as defined in Eq. (14)

$$\bar{y} \equiv \mathbb{E}[y] = \sum_{\nu} 2^\nu \rho(\nu). \quad (\text{A4})$$

In Fig. 7(a), we show how the ratio defined in Eq. (A3) changes with each time step for fixed size of the system N . We start from a state with maximal initial nullity $\nu_0 = N$, then evolve such state with our dynamics for $\theta_M = 0$ for different system sizes N . The lines represent the ratios obtained using the evolution of the Markov chain, while the dots represent the same quantity obtained for the numerical data as discussed in Sec. IV B. One can see that the agreement between the Markov chain prediction and the numerical simulations increases with N . In particular, the maximum value of the ratio saturates with N to a constant value

$$\max_{t,N} c(t) = \max_{t,N} (\bar{y}^2/\bar{y}^2) \approx 1.4, \quad (\text{A5})$$

as shown in Fig. 7(b).

We can repeat such procedure for the nonstabilized dimension $w(\nu) = 2^{v-N}$ as defined in Eq. (36) for $\theta_M > 0$. In particular, in Fig. 8(a) we show how the ratio $\bar{w}^2/\bar{w}^2$ for initial states $|\psi\rangle_{\text{Haar}}$ changes in time when evolved according to the measurements in the rotated bases with $\theta_M = 1$, while in Fig. 8(b), we show the ratio $\bar{w}^3/\bar{w}^3$ for the same kind of initial states and the same rotation angle θ_M . The evolutions

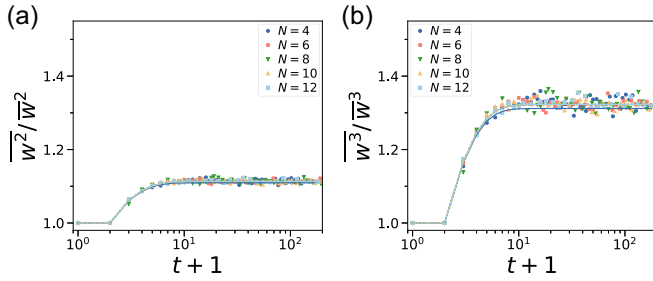


FIG. 8. (a) $\overline{w^2}/\overline{w^2}$ for different sizes of the system N for the measurements in the rotated basis with $\theta_M = 1$, computed using the Markov chain evolution (lines) and full simulation of the quantum state (dots), with initial Haar-random state $|\psi\rangle_{\text{Haar}}$. (b) $\overline{w^3}/\overline{w^3}$ for different sizes of the system N for the measurements in the rotated basis with $\theta_M = 1$, computed using the Markov chain evolution (lines) and full simulation of the quantum state (dots), with initial Haar-random state $|\psi\rangle_{\text{Haar}}$.

of both ratios are regular and almost size independent at each time step, converging to the same size and rotation angle independent steady-state value. We obtain the same qualitative result starting from states with $\nu_0 = 0$.

APPENDIX B: COMPARISON BETWEEN NUMERICAL AND ANALYTIC RESULTS

We show a more complete comparison between the analytical predictions in Sec. IV and the numerical data. We already showed that the probability of decay in Eq. (13) correctly predicts the decay of stabilizer nullity as shown in Fig. 2. Now, we want to explore the correctness of the predictions of the differential equation in Eq. (19) derived from the decay probabilities.

In Fig. 9, we directly compare the approximation in Eq. (22) for large N and $t \gg 2^N$ with the numerical simulation, and this approximate solution is already in good agreement with the numerical data.

To further investigate, we can now try and compare also the behavior of the approximate solution in Eq. (20) to the

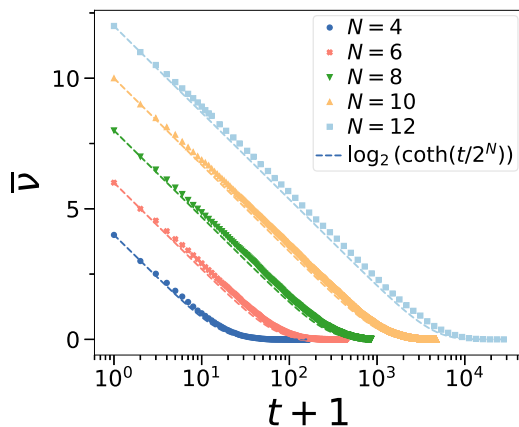


FIG. 9. Comparison of the average stabilizer nullity $\bar{\nu}$ for different sizes with the approximate differential equation solution in Eq. (22) for different sizes N as a function of measurement steps $t + 1$.

numerical data. In this case, using A_N and b as defined in Eq. (17) and in Eq. (21) the general behavior is well represented, but the agreement can be improved, and in order to do that we can use both parameters as fitting parameters. In Fig. 10(a), we show the numerical data $\bar{\nu}$ as dots, and in dashed lines of the same color the result of the fit using the function and parameters

$$\bar{\nu}_f = \log_2(\bar{\nu}_f(t)x) = \log_2\left(1 - \frac{2b_f}{e^{A_f t} + b_f}\right),$$

$$b_f = \frac{1 - y_{0,f}}{1 + y_{0,f}}. \quad (\text{B1})$$

In Fig. 10(b), we show, together with the analytical predictions for A_N and b , the fitting parameters as a function of the system size N . One can see that $y_{0,f}$ as obtained through the fitting procedure is consistent with the analytical prediction $y(0) = 2^N$ (therefore also b). In contrast, the factor A_f obtained from the fit slightly differs from the analytical prediction in Eq. (17). We find empirically that by multiplying the analytics with a factor of ≈ 1.65 , we get good agreement between fit and analytics. We believe that this difference arises from the continuous-time approximation of the discrete time steps.

We also use the mapping defined in Eq. (26) to describe how for $\theta_M = 0$ the SRE decays for initial states different from Haar. This relation is indeed valid for Haar-random states, but it can be extended if the SRE for a general state with maximal initial nullity depends on

$$\mathcal{M}_2(t) = \ln\left(\frac{2^{\nu_f} + 3}{4}\right) = \ln\left(\frac{2^{\nu_{\text{Haar}}(t)} 2^{\Delta\nu(t)} + 3}{4}\right), \quad (\text{B2})$$

where ν_f is the nullity obtained by fitting the SRE of a general initial state, and $\nu_{\text{Haar}}(t)$ is the analytical prediction of the nullity and is such that

$$\nu_f(t) = \nu_{\text{Haar}}(t) + (\nu_f(t) - \nu_{\text{Haar}}(t)) = \nu_{\text{Haar}}(t) + \Delta\nu(t). \quad (\text{B3})$$

Defining these quantities, we can finally compare the numerically computed SRE and the one obtained with the fit, and such results are shown in Fig. 10(c).

The same procedure can also be extended to the protocol for $\theta_M > 0$ in some cases. Considering that for the protocol in magic basis the $\mathcal{M}_2^{\text{ss}} > 0$, we can also introduce a constant factor in the effective nullity entering SRE as

$$\mathcal{M}_2(\nu) = \ln\left(\frac{2^{\nu_f} 2^C + 3}{4}\right). \quad (\text{B4})$$

Using now these three free fitting parameters, we are able to describe the behavior of the SRE for Haar-random states for every θ_M , as we show in Fig. 11. In general, this fitting model works in cases in which the nullity and the SRE both decrease with the measurement procedure. In the case where the steady-state SRE is larger than the initial one, this model seems to no longer describe the dynamics accurately.

APPENDIX C: SINGLE TRAJECTORIES IN MAGICAL MEASUREMENTS

We now study individual trajectories for our circuit where we consider measurements in a rotated basis, i.e., $\theta_M > 0$.

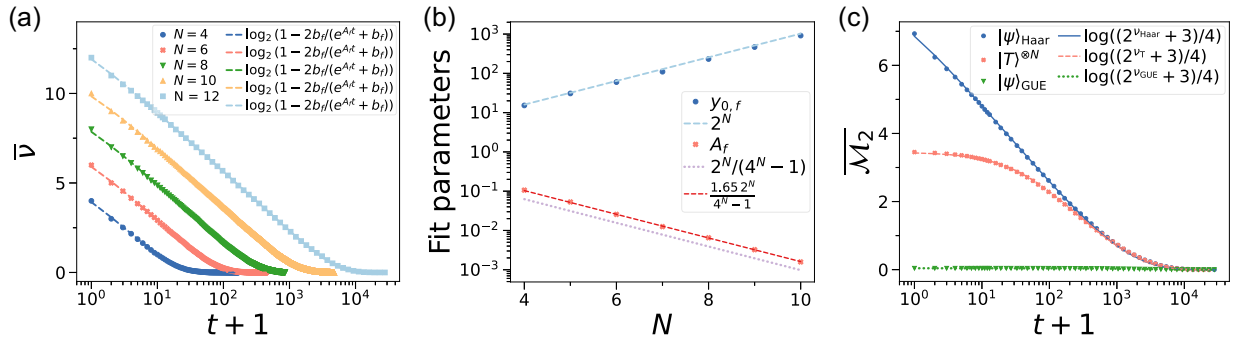


FIG. 10. (a) Comparison of the average stabilizer nullity $\bar{\nu}$ for different sizes with the approximate differential equation solution in Eq. (20) for different sizes N as a function of measurement steps $t + 1$, using $b_f = (1 - y_{0,f})/(1 + y_{0,f})$ and A_f as fit parameters. (b) Fit parameters $y_{0,f}$ and A_f vs the number of qubits in the system, compared with analytical predictions. In particular, we can see that the predicted value of $y_{0,f} \sim 2^N$, which is $2^{N(0)}$. The predicted value of A_N as $2^N/(4^N - 1)$ in Eq. (17) is displayed as a lilac dotted line. However, we find empirically that A_f can be better fitted with a factor ≈ 1.65 increased value, which is shown as the red dashed line. (c) Average SRE $\bar{M}_2$ vs time for $\theta_M = 0$ for $|\psi\rangle_{\text{Haar}}$, $|T\rangle^{\otimes N}$, and $|\psi\rangle_{\text{GUE}}$ for a system of $N = 12$ qubits. For all states, dots are the numerical calculations, while for the lines we choose fitting ansatz inspired from our analytics. In particular, they are the solution of the fitting procedure of Eq. (B2) with free parameters $y_{0,f}$ and A_f .

In Figs. 1(b) and 1(c), we show a single trajectory and the average value of the stabilizer nullity ν and von Neumann entanglement entropy $\mathcal{S}$ for initial Haar-random states for measurements in the computational basis. In this section, we show how these quantities change in single trajectories and on average with this measurement protocol in the magic basis and how the angle parameter θ_M affects the outcome of the procedure.

When measuring in the rotated bases, since the nullity evolution does not depend on the value of θ_M , we can see that nullity will oscillate regardless of the angle around the asymptotic value $\nu_{\text{analytic}}^{\text{ss}}$ creating plateaus for $\nu = N - 1, N - 2$ (and with low probability $N - 3$ and lower), which do not change with the angle. Two examples, one for initial Haar-random states and one for $|0\rangle^{\otimes N}$ states are shown in Figs. 12(a) and 12(b). The dynamics does not depend on the value of θ_M , and we show $\theta_M = 0.0001$ to highlight the nonperturbative behavior of the stabilizer nullity.

We already showed that at the end of the procedure described in Sec. IV, the resulting state is a random stabilizer

state; therefore, its entanglement entropy can only assume discrete values. In each panel of Fig. 13, it showed both a single representative trajectory and the average over $N_r \approx 1000$ different trajectories of entanglement entropy. In particular, the blue plots in the first row represent the said behavior for Haar-random states, while the yellow plots in the second row represent what happens for $|0\rangle^{\otimes N}$ states. In the plots (a) and (d), we show what happens for $\theta_M = 0.0001$: As one can see, starting from both Haar-random states and from the computational basis state, the typical plateau structure we see when the state approaches a random stabilizer. Moving to panels (b) and (e), for $\theta_M = 0.1$ the plateau structure begins to fade and the entanglement entropy at the end of the dynamics assumes more values continuously spread around its average value, computed over N_r different realizations. For the maximal value $\theta_M = 1$, the plateau structure is totally lost, and the entanglement entropy is concentrated very close to the average value. This can be visually seen by looking at the standard deviations of the mean $\bar{\sigma}_S = \sigma_S/\sqrt{N_r}$ in Fig. 14(a) for Haar-random states and in Fig. 14(b) for initial product states $|0\rangle^{\otimes N}$, for $N = 12$ qubits. We see that the steady-state value of the standard deviations is independent of the initial

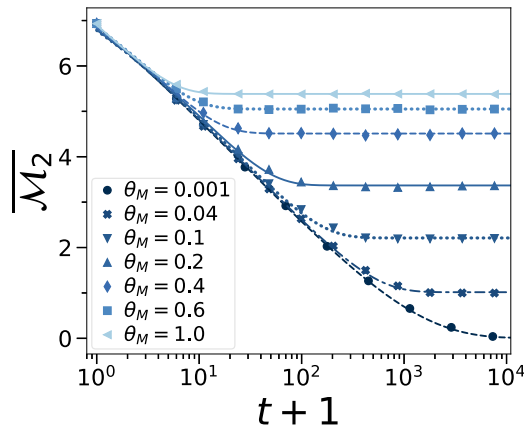


FIG. 11. Dots: numerical average SRE for initial $|\psi\rangle_{\text{Haar}}$ vs measurement steps for $N = 12$ qubits for different values of θ_M . Lines: results of the fits performed using Eq. (B4).

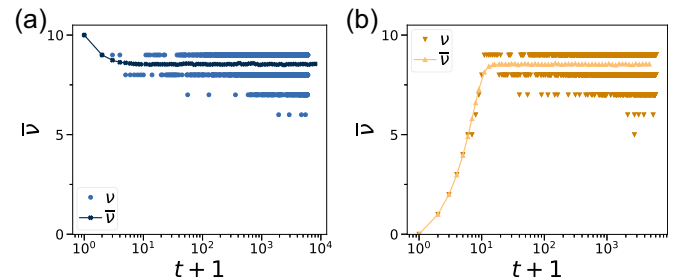


FIG. 12. (a) Average $\bar{\nu}$ and a single random trajectory ν as a function of measurements $t + 1$ for $N = 10$ and $\theta_M = 0.0001$ starting from Haar-random states. (b) Average $\bar{\nu}$ and a single random trajectory ν as a function of measurements $t + 1$ for $N = 10$ and $\theta_M = 0.0001$ starting from $|0\rangle^{\otimes N}$ states.

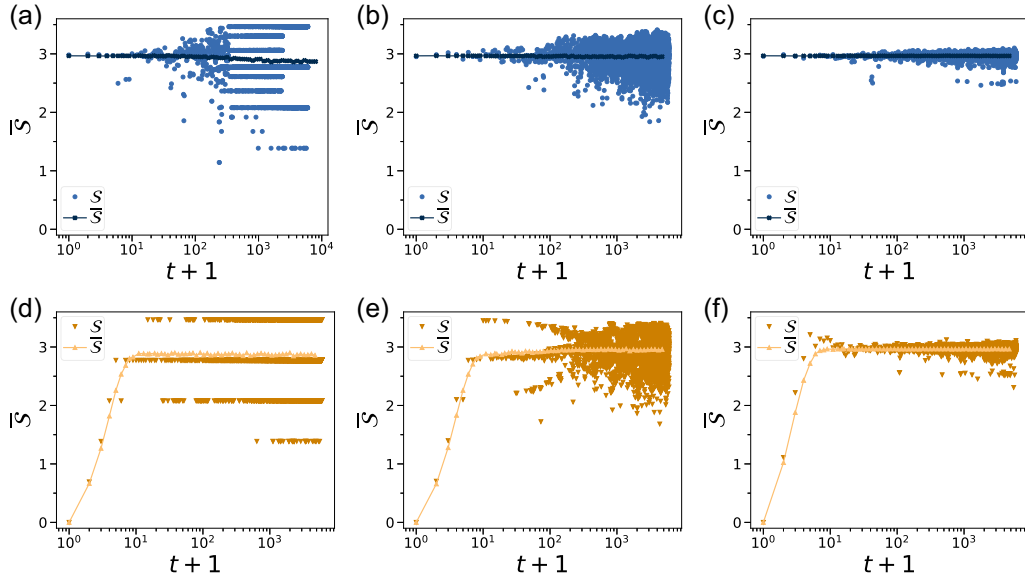


FIG. 13. Von Neumann entanglement entropy S for $N = 10$ qubits. The first row [panels (a)–(c)] contains Haar-random initial states and the second row [panels (d)–(f)] contains all $|0\rangle^{\otimes N}$. The first column [panels (a) and (d)] contains as dotted a random trajectory for $\theta_M = 0.0001$ and as a line the average over $N_r \sim 1000$ different evolutions: The plateau structure is still visible. The second column [panels (b) and (e)] contains the same information but for $\theta = 0.1$. In the last column [panels (c) and (f)], we show single trajectories and the average for the same three states for $\theta_M = 1$: The plateau structure is lost and the entanglement is concentrated around its average value.

state, which we show in Fig. 14(c). In all cases, the $\overline{\sigma_S}$ decreases with increasing angle parameter, and while fixing the angle parameter it decreases with increasing system size N . The average over time is

$$(\overline{\sigma_S})_T = 1/T \int_{t_{\min}}^{t_{\max}} \overline{\sigma_S} dt, \quad (\text{C1})$$

where the integration time $T = t_{\max} - t_{\min}$, $t_{\max}$ is the end of the evolution, and $t_{\min} = N$.

Next, we study trajectories of our circuit for the SRE. The behavior of the SRE is not very different from the nullity, except that the difference between two plateaus is not a fixed integer, as shown in Fig. 3(a). Note that the measurement protocol with $\theta_M > 0$ changes the single trajectories and therefore the average behavior. Also, it is worth noting that for every $\theta_M > 0$ the nullity changes on timescales that depend

linearly on N , while for the SRE we still need exponential timescales for $\theta_M < 1$. In Figs. 5(a)–5(c), we show how the SRE converges to the steady-state value for the different initial states; therefore, in Fig. 15 we show how this average dynamics compares to single evolutions. For Haar-random states in Figs. 15(a)–15(c) for increasing angle parameters, it is evident that the plateau structure of the dynamics is conserved up to the moment the SRE reaches its angle-dependent steady value. For $\theta_M = 0.1$ in Fig. 15(b), we can clearly see that the moment the average approaches its steady value coincides with the moment the plateau structure is loose. For $\theta_M = 1$ in Fig. 15(c), the SRE takes $\sim N$ measurements to reach $\mathcal{M}_2^{\text{ss}}$, which that means there is no time to create plateaus.

We repeat the same analysis for $|0\rangle^{\otimes N}$ in Figs. 15(d)–15(f), where the initial state has no entanglement or magic. In this case, as $\mathcal{M}_2^{\text{ss}}(\theta_M \ll 1) \ll 1$. For $\theta_M \ll 1$, we can see that measurement in the computational basis increases the SRE by

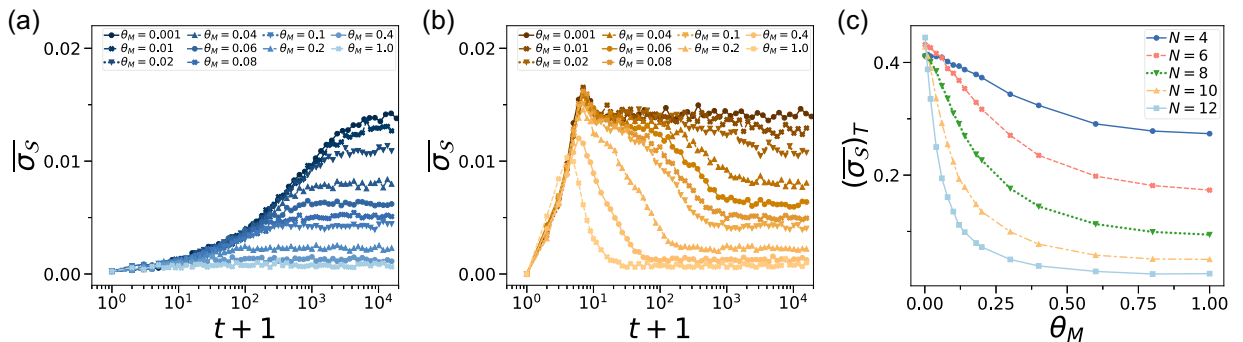


FIG. 14. Standard deviation σ_S of von Neumann entanglement entropy. (a) $\overline{\sigma_S} = \sigma_S / \sqrt{N_r}$ vs $t + 1$ for initial Haar-random states for $N = 12$ qubits, $N_r = 1000$, for different angle parameters fixed. (b) $\overline{\sigma_S} = \sigma_S / \sqrt{N_r}$ vs $t + 1$ for initial $|0\rangle^{\otimes N}$ states for $N = 12$ qubits, $N_r = 1000$. (c) Standard deviation $\overline{\sigma_S}$ averaged over time T as shown in Eq. (C1). We plot against angle parameter θ_M for different qubit numbers N for Haar-random initial states.

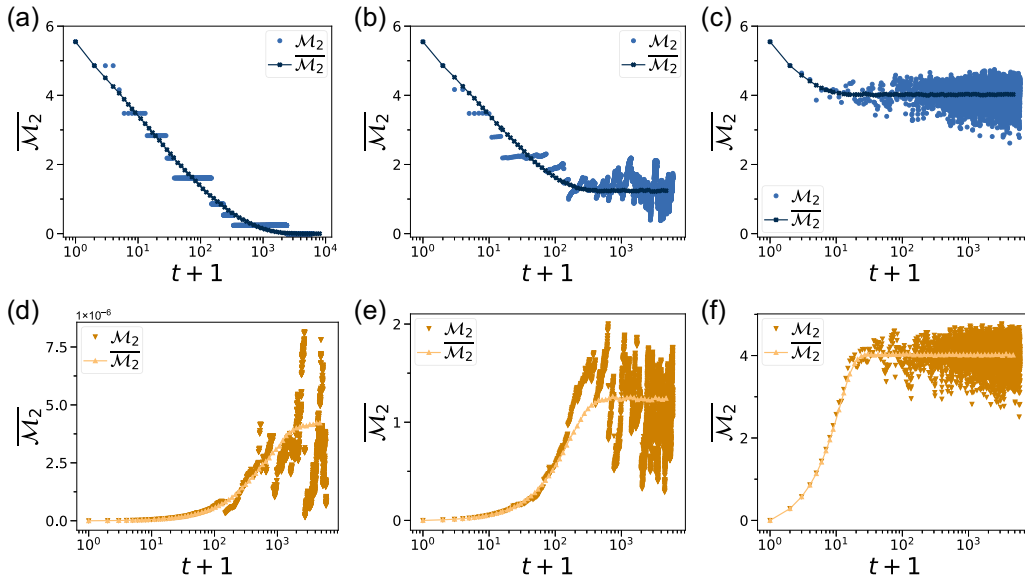


FIG. 15. SRE trajectories for $N = 10$ qubits. The first row [panels (a)–(c)] contains Haar-random initial states, and the second row [panels (d)–(f)] contains all $|0\rangle^{\otimes N}$. The first column [panels (a) and (d)] contains as dotted a random trajectory for $\theta_M = 0.0001$ and as a line the average over $N_t \sim 1000$ different evolutions, and the second column [panels (b) and (e)] contains the same kind of information for $\theta_M = 0.1$, while the third one [panels (c) and (f)] for $\theta_M = 1.0$.

a very small amount, which could be due to the structure of the entanglement entropy for such state as shown in Fig. 13(d): Such state is still very close to a stabilizer. Anyway, for $\theta_M < 1$ we can see that still the majority of measurement steps increase magic, which then rapidly drops and grows again. For $\theta_M = 1$, we see that the SRE reaches its steady value in $\sim N$ measurement and then oscillates around it.

APPENDIX D: INCREASE IN SRE AFTER MEASUREMENT

Under Clifford measurements, a (strong) magic monotone is expected not to increase on average. However, when looking at individual trajectories, magic can actually increase, as long as it is compensated by a more significant decrease by another trajectory. As SREs are not strong monotones, such increases can happen even when averaging over trajectories [27].

We observe significant increases in SREs in rare instances of trajectories specifically for $|\psi\rangle_{\text{GUE}}$ states. As an example, we plot the SRE $\mathcal{M}_2$ and nullity ν for three different random trajectories starting from three different states in Figs. 16(a) and 16(b), in which the same color curves correspond to the same initial state and same evolution. The results plotted correspond to systems of fixed size of $N = 10$ qubits, but such behavior can be seen for every size. As can be seen in Fig. 16(a), while all states at the beginning have more or less the same $\mathcal{M}_2 \approx 0.04$, throughout the evolution of $t \sim 10^3$ measurements the SRE increases after a measurement with very low probability ($\approx 1/10^3$), and their magnitude can reach up to $\sim 10^2$ times the premeasurement value of $\mathcal{M}_2$. For the same trajectories, in the correspondence of the change-inducing measurement ν decreases. Note that such outliers are averaged out over many trajectories, and we find a monotone decrease in SRE on average under Clifford measurements.

In contrast, we do not see any such spikes for Haar-random states, while for $|T\rangle^{\otimes N}$ states we find very small increases in SRE with very small probability.

APPENDIX E: DEVIATION FROM HAAR-RANDOM BEHAVIOR

For random Haar states under random Clifford measurements, we propose in the main text that the state given nullity ν can be written as

$$|\psi_\nu\rangle = U_C |0\rangle^{\otimes N-\nu} |M\rangle, \quad (\text{E1})$$

with an analytical relationship between the α SRE and the nullity ν with

$$\mathcal{M}_2(\nu) = \ln\left(\frac{3 + 2^\nu}{4}\right). \quad (\text{E2})$$

In Fig. 17(a), we plot the average $\mathcal{M}_2$ as a function of the average nullity for random Haar states. We can see that the relation in Eq. (E2) is therefore respected at all times,

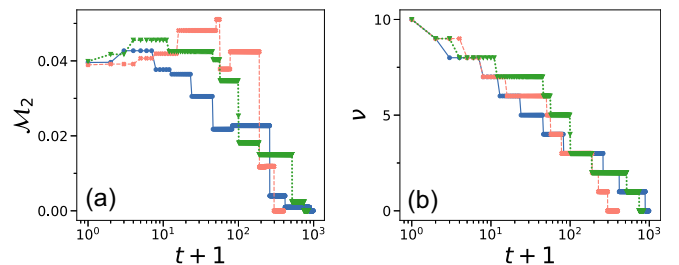


FIG. 16. (a) $\mathcal{M}_2$ vs measurement $t + 1$ computed for three different trajectories starting from $|\psi\rangle_{\text{GUE}}$, for $N = 10$ and $\theta_M = 0$. (b) ν vs measurement $t + 1$ computed for the same trajectories as in panel (a).

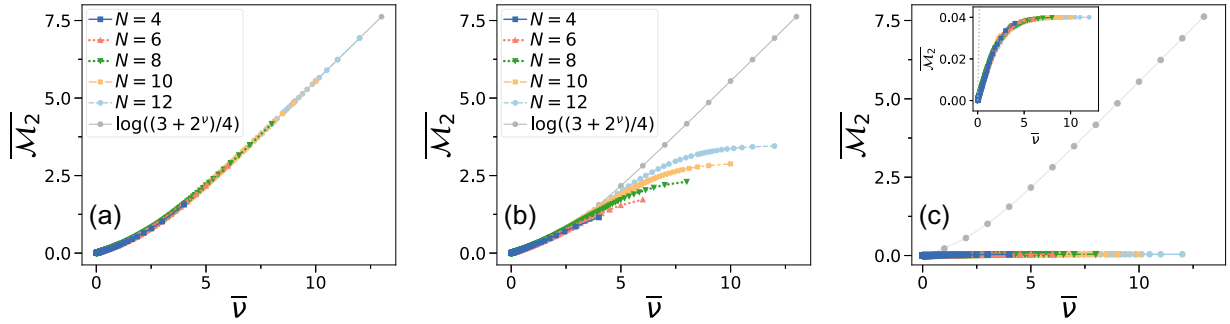


FIG. 17. Protocol with $\theta_M = 0$. (a) $\mathcal{M}_2$ against ν for initial Haar-random states for different N , plotted together with the analytical prediction in grey dotted line. (b) $\mathcal{M}_2$ against ν for initial $|T\rangle^{\otimes N}$ for different N , plotted together with the analytical prediction in grey dotted line. (c) $\mathcal{M}_2$ against ν for initial $|\psi\rangle_{\text{GUE}}$ state for different N . Inset: Enlarged view of the main plot. In this case, the legend was removed for clarity.

verifying that this protocol sends Haar-random states still in Haar-random states.

We note that this relationship only works for initial Haar-random states. For other types of states such as $|T\rangle^{\otimes N}$ and $|\psi\rangle_{\text{GUE}}$, we observe distinct dynamics. In particular, we can see that the relation between the two magic quantities for $|T\rangle^{\otimes N}$ as shown in Fig. 17(b) is respected for small values of the nullity, while the magic deviates from the predicted Haar value. Meanwhile, comparing the analytical prediction with the numerical results for $|\psi\rangle_{\text{GUE}}$, the dynamics is far from the values expected from Haar-random states.

APPENDIX F: NUMBER OF MEASUREMENTS REQUIRED TO CONVERGENCE FOR SRE

In this Appendix, we discuss the time needed to reach the steady state under our dynamics. In general, determining whether the steady state is reached is difficult. Here, the convergence of magic to its steady-state value serves as a necessary but not sufficient criterion for the system to reach the steady state.

In our results, we observe that the time needed to reach the steady state depends on θ_M , with $\theta_M = 0$ requiring exponential time (in qubit number), while large θ_M only requires polynomial time. For Haar-random states, we can numerically give a qualitative and a quantitative explanation for this behavior. One possible qualitative explanation for the exponential scaling with N at small angle parameters is the continuous change of the SRE steady state as $\theta_M \neq 0$. For $\theta_M = 0$, we proved that the number of measurements required for the state to reach a random stabilizer—hence vanishing nullity and SRE—grows exponentially with the system size. Since for small θ_M the SRE dynamics closely resemble those in the computational-basis case, it is reasonable to expect that the number of measurements needed to reach the SRE steady state remains exponential in N .

In Fig. 18(a), we show the time to reach the steady state of SRE for initial random Haar states with increasing system sizes N for different θ_M . As one can see from the log scale on the y axis, while for angle parameter $\theta_M = 1$ the time to convergence is comparable to N , for $\theta_M = 0.01$ the time to convergence $\bar{t}$ increases exponentially with N . For intermediate values of θ_M , the required number of measurements to

reach the steady-state value $\bar{m}_2^{\text{ss}}$ is in between the two limiting behaviors.

We estimate the time to convergence $\bar{t}$ numerically by computing steady-state SRE

$$\bar{m}_2^{\text{ss}} = \frac{1}{T} \int_0^{t_f} dt \bar{m}_2, \quad (\text{F1})$$

and then identifying the first time $\bar{t}$ when the $\bar{m}_2(t)$ intersects with $\bar{m}_2^{\text{ss}}$. Although this procedure provides an approximate measure of the scaling of the convergence time, it is sufficient to capture the qualitative behavior across different parameter regimes.

The same procedure also works for $|0\rangle^{\otimes N}$; independently on the size of the system and the angle parameter, the $\bar{m}_2$ is initially zero, and it always grows to an asymptotic value $\bar{m}_2^{\text{ss}} > 0$, defining a clear convergence time for the magic. In this case, the numerical procedure yields the results compatible with those obtained for the Haar-random states shown in Fig. 18(a); hence, we only show one case.

As previously stated, defining the convergence time for the steady state via the convergence of the magic alone is not a sufficient condition. In Fig. 18(b), we show the dynamics of the magic density $\bar{m}_2$ for $\theta_M = 0.3$ starting from $|T\rangle^{\otimes N}$ as an example. In this case, one can see that the initial value of magic is already very close to the asymptotic value. We find similar behavior also for $N = 12$ and $\theta_M = 0.1$ in Fig. 5(b).

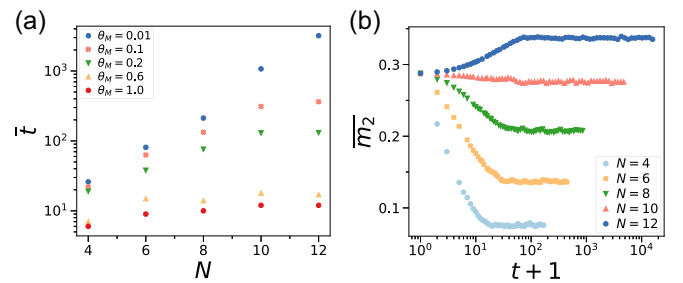


FIG. 18. (a) Time to convergence $\bar{t}$ to the steady-state value of the SRE $\bar{m}_2^{\text{ss}}$ as a function of the system size N for different fixed values of the angle parameter for initial Haar-random states. (b) Average magic density $\bar{m}_2 = \mathcal{M}_2/N$ over $N_r \approx 1000$ trajectories as a function of measurement steps for $\theta_M = 0.3$ for different N , starting from $|T\rangle^{\otimes N}$.

APPENDIX G: QUBIT NUMBER DEPENDENCE OF STEADY STATE OF SRES

We now study how the steady state of the SRE in our circuit depends on qubit number N . For $\theta_M = 0$, the steady state of the SRE is trivially zero as the final state is a random stabilizer state. In particular, in Fig. 19 we show the steady state of the SRE $\overline{\mathcal{M}}_2^{\text{ss}}$ against N for $\theta_M = 1$. We find as expected that asymptotically, the SRE increases linearly with N . As a reference, we also plot the average SRE of Haar-random states for $N - 2$ qubits. Curiously, we find that the steady-state SRE for N qubits nearly matches the SRE of Haar-random states of $N - 2$. While the steady state does not have the characteristics of a Haar-random state, we leave as an open question whether this relationship between steady state and Haar-random magic has a deeper physical meaning.

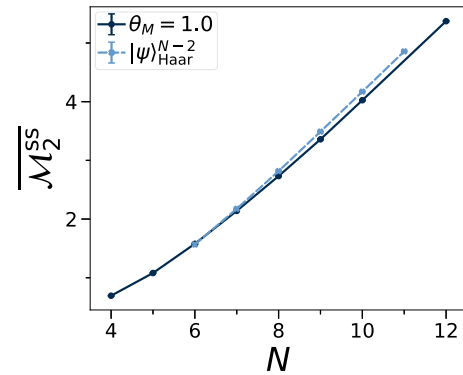


FIG. 19. Steady state of the SRE $\overline{\mathcal{M}}_2$ against N with $\theta_M = 1$ are shown as dots, while the average SRE of Haar-random states of $N - 2$ qubits is shown as a solid line.

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