

Local ergotropy dynamically witnesses many-body localized phases

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Many-body localization is a dynamical phenomenon characteristic of strongly interacting and disordered many-body quantum systems that fail to achieve thermal equilibrium. From a quantum information perspective, one of the most relevant fingerprints of this phenomenon is the logarithmic growth of the entanglement entropy over time. We perform intensive numerical simulations, applied to a paradigmatic model system, showing that the local ergotropy, the maximum extractable work via local unitary operations on a small subsystem in the presence of Hamiltonian coupling, dynamically witnesses the change from ergodic to localized phases. Within the many-body localized phase, both the local ergotropy and its quantum fluctuations slowly vary over time with a characteristic logarithmic law analogous to the behavior of entanglement entropy. This showcases how directly leveraging local control, instead of local observables or entropies analyzed in previous works, provides a thermodynamic marker of localization phenomena based on the locally extractable work.

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I. INTRODUCTION

Numerous studies have investigated dynamical localization phenomena, in which quantum many-body systems fail to reach thermodynamic equilibrium [1–4]. Many-body localization (MBL), characteristic of strongly interacting and disordered many-body systems, is qualitatively different from localization of noninteracting systems, called Anderson localization (AL). The hallmark of MBL is the development of nonlocal correlations, which, however, lead only to a logarithmic growth of entanglement entropy over time [5–7]. Directly measuring the entanglement entropy is notoriously challenging: In Ref. [7], the authors use the sum of number entropy and configurational entropy, the latter requiring access to quantum coherences across many states, to infer qualitative insights about entanglement. To characterize MBL, which still has many elusive aspects [4], alternative experimental markers have been used [8–11]. Many studies, such as Ref. [12], report measurements of the imbalance, a global quantity, that, however, is unable to distinguish between MBL and AL phases.

The properties of MBL can be relevant not only for fundamental research, but also for applications. One promising field of application is that of quantum batteries [13–15]. Recent works have focused not only on the charging process and energy storage [16–21], but also on work extraction [22,23],

where the main investigated quantity is the ergotropy [24,25], defined as the maximum amount of work extractable from an isolated quantum system via cyclic unitary transformations [26,27]. Assuming $\hat{H}$ to be the system Hamiltonian and $\hat{\rho}$ the statistical operator describing the quantum state, the ergotropy $\mathcal{E}$ is a functional of $\hat{\rho}$ and $\hat{H}$:

$$\mathcal{E}[\hat{\rho}, \hat{H}] := \max_{\hat{U} \in \mathcal{U}(d)} \text{Tr} \{ \hat{H}(\hat{\rho} - \hat{U} \hat{\rho} \hat{U}^\dagger) \}, \quad (1)$$

where Tr denotes the trace operation. The maximization has to be carried out over the set $\mathcal{U}$ of all available unitary operators $\hat{U}$ acting on the system d -dimensional Hilbert space $\mathcal{H}$, yielding a quantity that does not vary over time for an isolated system [13,26,28,29].

It was shown in Ref. [30] that the dynamical behavior of the ergotropy of subsets of quantum cells of a quantum spin chain distinguishes AL, MBL, and ergodic (ERG) phases. Here, quantum cells refer to the basic units of a quantum battery that store energy and allow, in principle, work extraction via unitary operations [13,30,31]. The low level of entanglement in MBL and AL phases ensures significantly superior work extraction capabilities compared to ERG phase. We remark that the analyses presented in Ref. [30] and partly in Ref. [32] focus on the charging process, analyzing the stored energy and the ergotropy extractable from an isolated subsystem for various charging times, along with the mean energy temporal fluctuations.

In this paper, we analyze the one-dimensional disordered XXZ Heisenberg model, which is one of the most commonly used to investigate localization effects [2,4], and which is potentially realizable in laboratory [33]. Our results are obtained using numerical techniques based on matrix product state (MPS) representation of quantum many-body states and operators [34–36]. We employ the time-dependent variational principle (TDVP) to compute the time evolution of several

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quantities [37,38], among them the entanglement entropy, relevant to characterize the MBL phase.

Following the strategy adopted in previous works [30,32], in this paper, we consider a thermodynamic quantity that serves as an alternative probe of the MBL phase aiming to build a bridge between two research fields: localization in strongly correlated systems and quantum thermodynamics. In particular, we use a local work extraction protocol and its associated thermodynamic measure, the local ergotropy, recently introduced in the context of quantum thermodynamics [39]. In this paper, we focus on the local ergotropy and its quantum fluctuations for a small subsystem within the spin chain. Actually, the subsystem can be interpreted as an open quantum battery experiencing interaction with its surroundings (the rest of the chain) under nonweak coupling conditions. Therefore, the results presented in this paper substantially differ from the analyses in Refs. [30,32]. Moreover, we remark that the work extraction protocol differs from alternative perspectives where full chains are considered as quantum batteries [40] and forms of nonlocal control are typically required to extract work. In order to evaluate the maximum extractable work via local operations, in this paper, we combine TDVP and a Bayesian optimization algorithm (BOA) [41]. We stress that these tools have been previously applied to open systems showing particular sensitivity to quantum phase transitions [42].

In this paper, we have found that, over time, in the ERG case, the local ergotropy rapidly drops to zero; in the AL phase, it stabilizes to a constant value; and in the MBL phase, it slowly decreases following a characteristic logarithmic law. Moreover, the quantum fluctuations exhibit a similar logarithmic time behavior that, like for entanglement entropy, turns out to be a robust feature. Indeed, we have found similar behaviors for three very different initial states: the factorized Néel state with alternating up and down spins, a factorized state with a random extracted configuration of up and down spins, and an entangled state with spins grouped into pairs of singlets. Our results pave the way for the direct extraction of local work in the presence of Hamiltonian coupling to characterize various localized phases in many-body systems.

The paper is organized as follows: Sec. II deals with model and numerical methods, Sec. III exposes results, and Sec. IV concludes the paper.

II. MODEL AND NUMERICAL METHODS

We consider a one-dimensional disordered XXZ Heisenberg model with nearest-neighbor interactions between 1/2 spins, which can exhibit MBL, AL, and ERG phases. The Hamiltonian for N spins reads as

$$\hat{H} = \frac{J_{\perp}}{2} \sum_{i=1}^{N-1} \{\hat{S}_i^+ \hat{S}_{i+1}^- + \hat{S}_i^- \hat{S}_{i+1}^+\} + J_z \sum_{i=1}^{N-1} \hat{S}_i^z \hat{S}_{i+1}^z + \sum_{i=1}^N h_i \hat{S}_i^z. \quad (2)$$

For each chain site i , we have the Pauli operator $\hat{S}_i^z = \frac{1}{2} \hat{\sigma}_i^z = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)$ and the ladder operators $\hat{S}_i^{\pm} = |\downarrow\rangle\langle\uparrow|$

and $\hat{S}_i^+ = \hat{S}_i^{-\dagger}$. $J_{\perp}$ is the spin-flip transverse coupling energy, J_z is the longitudinal coupling energy, and h_i are on-site random energy values, related to an effective magnetic field, extracted by a flat probability distribution of width $2W$, such that

$$h_i \in [-W, W]. \quad (3)$$

In the following, the parameter R will indicate the number of disorder realizations employed to calculate several quantities of the system. Hard-wall boundary conditions are imposed for simplicity on a chain with size N in the subspace $S_{\text{tot}}^z = 0$, $S_{\text{tot}}^z = \sum_{i=1}^N S_i^z$ being the z component of the total spin. This model can be mapped onto a one-dimensional disordered spinless electron model with nearest-neighbor particle-particle interactions through the Jordan-Wigner transformation (see the Supplemental Material [43]). In the following, we will use $J_{\perp}$ as reference energy scale and $\hbar = 1$.

We shall consider a scenario in which the isolated quantum system is partitioned into two subsystems S and E . Hence, the total Hamiltonian $\hat{H}$ reads

$$\hat{H} = \hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{V}_{SE}, \quad (4)$$

where $\hat{H}_S$ and $\hat{H}_E$ are the Hamiltonians for the subsystems S and E , respectively, while $\hat{V}_{SE}$ is the interaction operator.

Being the full system SE always in a pure state, its ergotropy in Eq. (1) (later on referred as *global* ergotropy) coincides with its excitation energy (constant in time) [43]. However, having a unitary operator acting on the whole Hilbert space means that an *experimenter* has full control over the whole isolated system. This is sometimes an unrealistic assumption.

In the following, we will assume that the experimenter has control only over the subsystem S , having the d_S -dimensional Hilbert space $\mathcal{H}_S$. This leads us to consider S as an open system, in contact with E , which plays the role of an environment. The system SE is initially in a high-energy state (a charged state of the potential battery) that is not an eigenstate of the Hamiltonian $\hat{H}$, such that $\hat{\rho}(t)$ is the resulting time-evolving statistical operator. Indeed, AL and MBL are dynamical phenomena, where nonstationary quantities are used as markers for localization [2,4].

We preferably choose the first two spins of the chain ($i = 1, 2$) as subsystem S , with the remaining $N - 2$ spins to be the environment E . The interaction energy between S and E is then related to the bond between the spins at $i = 2$ and $i = 3$ [see panel (a) of Fig. 1]. The main aim is to investigate whether localization effects can be inferred by actions on just two spins, representing the minimal subsystem with interacting degrees of freedom [44]. This, for instance, differs from the analytical study of local ergotropy of Ref. [39] on the same model, where the subsystem S was identified with the first spin of the chain, under some approximations and without focusing on the different phases of the model.

We perform numerical simulations using MPS techniques to represent the quantum states and operators from ITensor library [45]. For the first simulations presented in this work, the system is initialized in the Néel state: $|\psi\rangle_N = |\uparrow\downarrow\uparrow\downarrow\cdots\uparrow\downarrow\rangle$, typically used for MBL dynamics [2,4]. Indeed, Néel state has high mean energy in comparison

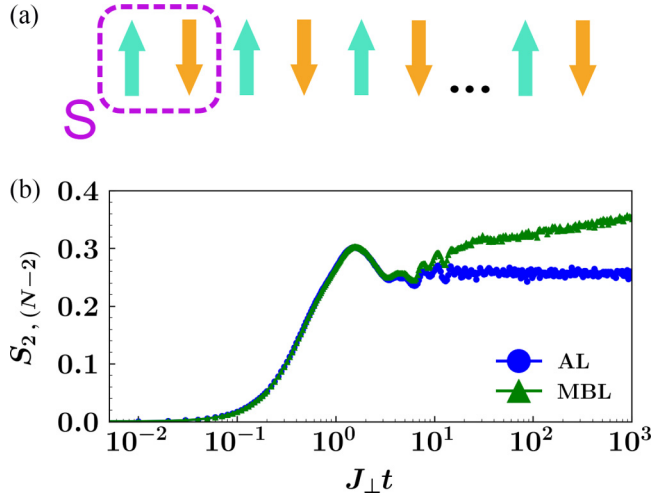


FIG. 1. (a) Quantum spin chain divided into the subsystem S (first two spins) and environment E (remaining spins). The initial Néel state is depicted. (b) Entanglement entropy as a function of time, for the $2, (N-2)$ partition, comparing the MBL phase ($J_z/J_\perp = 0.2$, green triangle) and AL phase ($J_z/J_\perp = 0.0$, blue circle). Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$ and disorder strength $W/J_\perp = 5$.

with the ground state and it is a product state in the spin basis. Adopting the 2-TDVP [37,38], we compute the time-evolved quantum state. Since the model is disordered, every simulation starts from a different stochastically extracted configuration of disorder. The quantities at each time step t are obtained by averaging over all the realizations R of the disorder configurations. A closed formula for the local ergotropy is known only for a subsystem made of a single spin-1/2 [39]. Therefore, for our subsystem with two 1/2 spins, we approach the problem numerically. For each realization of disorder and time step, we use a BOA to approximate the optimal local operator for the work extraction task [43]. We remark that exact diagonalization techniques can be used for the system sizes considered in this work. However, the calculation of local ergotropy requires an additional optimization step, made through the use of a BOA, which significantly increases simulation times. For instance, in the case $J_z/J_\perp = 0.2$ and $N = 8$, it takes about 19 h for a simulation with a single disorder realization when implementing BOA, compared to about 46 min without BOA. The optimization procedure becomes more tractable by employing faster MPS methods. Indeed, the approach used in this paper mimics in spirit the one of Ref. [42].

III. RESULTS

A. Entanglement entropy

The entanglement entropy for an isolated system partitioned into two interacting subsystems S and E is defined as the von Neumann entropy of one of these two:

$$S_{\text{ent}}(SE) = -\text{Tr}_S\{\hat{\rho}_S(t) \ln \hat{\rho}_S(t)\} = S_{vN}(t), \quad (5)$$

with $\hat{\rho}_S(t) = \text{Tr}_E\{\hat{\rho}(t)\}$ being the subsystem S density operator. It is well known in the literature [2,5,6] that, for a quantum spin chain partitioned into two equal halves ($N/2, N/2$), the time evolution of entanglement entropy is a marker for MBL

and AL phases. Localized systems show a time behavior logarithmically dependent on the strength of the interactions: $S_{\text{ent}}(t) \sim \ln(J_z t/\hbar)$ [2,5,46–48]. Conversely, for ergodic systems, a ballistic growth over time is observed: $S_{\text{ent}}(t) \sim vt$. Further details on time-dependent entanglement entropy for this partition are provided in Appendix A, where the ERG case is analyzed, finding perfect agreement with established results.

The following numerical results analyze the case in which the partition is different. As depicted in panel (a) of Fig. 1, our subsystem now consists of two spins, while the remaining $N-2$ sites form the environment. In panel (b) of Fig. 1, we find a qualitative agreement between the entropy dynamics of this two-spin partition and the standard $N/2, N/2$ case, which is included for comparison in Appendix A. At initial time, the Néel state has zero entanglement entropy. Then, there is an initial quick rise of the entanglement, until $J_\perp t \simeq 1$, corresponding to the expansion of wave packets to a size comparable to the localization length. After this expansion, in the AL phase ($J_z = 0$), the entanglement entropy saturates to a constant, but in the MBL phase, it increases logarithmically for any nonzero interaction strength ($J_z \neq 0$). As discussed in Appendix B, these results are robust upon increasing the chain size N . The behavior of the entanglement entropy shows some analogies with that of the local ergotropy defined in the next section.

B. Local ergotropy

We show now how local ergotropy can be used as a direct probe for discerning ergodic and localized phases. One can consider a unitary quench on the subsystem S while this is still interacting with the environment. Upon optimization, this approach leads to the recently investigated [42,49] *local ergotropy* introduced in Ref. [39], which reads as

$$\mathcal{E}_S[\hat{\rho}(t), \hat{H}] := \max_{\hat{U}_S \in \mathcal{U}(d_S)} \text{Tr}\{\hat{H}(\hat{\rho}(t) - (\hat{U}_S \otimes \hat{I}_E)\hat{\rho}(t)(\hat{U}_S^\dagger \otimes \hat{I}_E))\}, \quad (6)$$

and is a time-dependent quantity (see also Refs. [50,51] for more recent extended versions).

In Fig. 2, we compare the local ergotropy as a function of time in the three different phases: AL, MBL, and ERG. Since the system is initially prepared in the high-energy Néel state, we can interpret the subsystem as a charged battery. The postwork extraction state depends on the specific procedure responsible for work extraction. We have mostly simulated the system in the regime $W/J_\perp \gg 1$, which ensures that the system is localized and the disorder energy scale dominates the dynamics.

Figure 2 shows that in the ERG case ($W = 0$, no disorder), after a time transient analogous to that observed in the entanglement entropy ($J_\perp t \simeq 1$), the local ergotropy rapidly decreases to zero. This decay is characterized by small oscillations due to finite-size effects (more on this in Appendix B and the Supplemental Material [43]). Consequently, ergodic batteries undergo a rapid discharge, rendering them incapable of sustaining work extraction over long durations. In contrast, within localized phases, local ergotropy remains much higher. The order of magnitude of the local ergotropy is about $W/2$ in the regime $W/J_\perp \gg 1$ (see Appendix D). It can be concluded

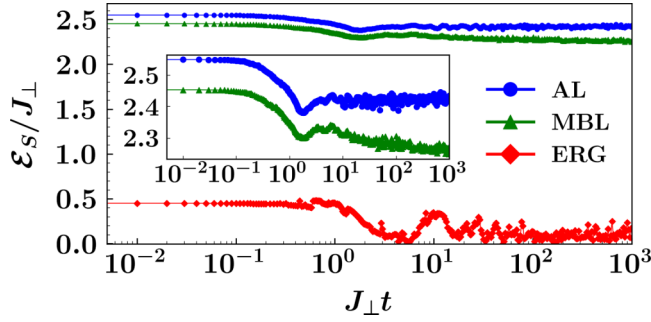


FIG. 2. Local ergotropy as a function of time in the three phases: AL (blue circle) for $J_z/J_\perp = 0.0$ and $W/J_\perp = 5$, MBL (green triangle) for $J_z/J_\perp = 0.2$ and $W/J_\perp = 5$, and ERG (red diamond) for $J_z = 0.2$ and $W = 0$. The inset provides an enlarged view of the local ergotropy as a function of time for the two localized phases. Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$.

that, within the observed timescales, localized batteries facilitate the energy storage for a much longer duration and the extraction of a greater amount of work in comparison with ergodic batteries.

Focusing now specifically on the difference between the two localized phases, it can be observed that after the initial transient, the spins entangle only weakly with each other, as characteristic of localization. From this point on, a clear distinction emerges between the AL and MBL phases: In the AL phase, local ergotropy stabilizes at a constant value, just like the entanglement entropy. On the other hand, in the MBL case, local ergotropy shows a logarithmic decrease over time, in a manner analogous yet opposite to the entanglement entropy, which instead logarithmically increases. It is thus evident that the time behavior of local ergotropy can be used as witness for localization phenomena (see also Appendixes B and E and the Supplemental Material [43]). Indeed,

$$\mathcal{E}_S = (e_S^c + e_{\text{int}}^c + e_E^c) - (e_S^f + e_{\text{int}}^f + e_E^f), \quad (7)$$

where e_S^c (e_S^f) represents the energy of the subsystem S , e_E^c (e_E^f) is the energy of the environment E , and e_{int}^c (e_{int}^f) is the interaction energy between S and E , for the global charged (postwork extraction) state. While the difference $e_E^c - e_E^f$ can be analytically checked to be zero [39], in localized phases also the difference $e_{\text{int}}^c - e_{\text{int}}^f$ is expected to be small. Therefore, in localized phases, the local ergotropy is controlled by the difference $e_S^c - e_S^f$ derived at each time step from the global system evolution. Nevertheless, local ergotropy turns out to be a dynamical witness of the system phases. Thus, it can serve as a marker of localization dynamics, directly leveraging local control rather than local projective measurements or entropies. Indeed, Fig. 2 summarizes one of the most relevant original results of this work showing that the local ergotropy, a quantum thermodynamic quantity, is capable of clearly distinguishing between AL, MBL, and ERG phases.

Next, we have examined the scaling of local ergotropy with respect to the chain length N and its dependence on the disorder distribution width W . Both analyses are reported in Fig. 3. In panel (a), as the environment size grows (we refer

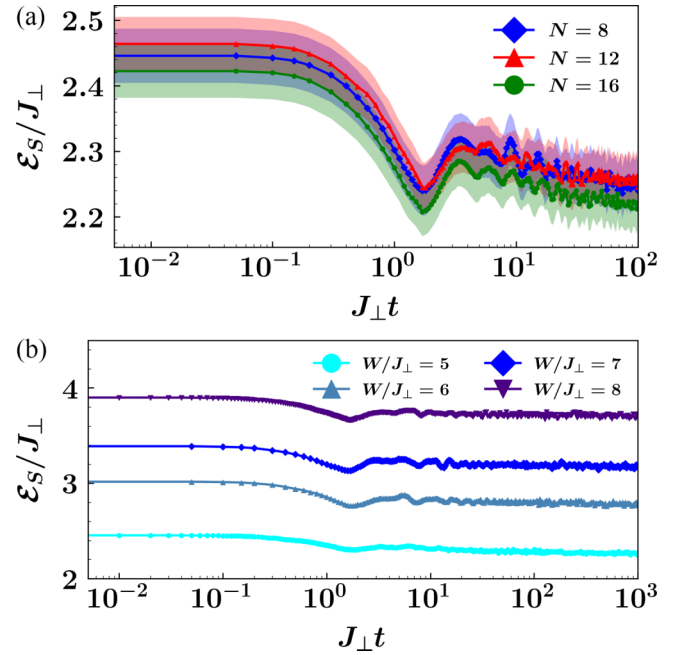


FIG. 3. (a) Local ergotropy as a function of time for various chain lengths N . Results are evaluated in the MBL phase averaging over $R = 3000$ disorder realizations for $J_z/J_\perp = 0.2$ and $W/J_\perp = 5$, without using the BOA, in the cases $N = 8$ (blue diamonds), $N = 12$ (red triangles), and $N = 16$ (green circles). The shaded areas represent the error margins determined by the disorder fluctuations $\sigma_{\mathcal{E}_S}^{\text{dis}}$ [see Eq. (B2)]. (b) Local ergotropy as a function of time for various values of disorder strength W . Results are obtained in the MBL phase, at $J_z/J_\perp = 0.2$ and $N = 8$, adopting the BOA, in the cases $W/J_\perp = 5$ (cyan circle), $W/J_\perp = 6$ (steelblue triangle), $W/J_\perp = 7$ (blue diamond), and $W/J_\perp = 8$ (indigo reverse triangle).

to environment rather than system size since the subsystem S size is always fixed at two spins for any chain length), the work extractable by acting locally on S does not show an appreciable variation, as expected for localized systems. For simplicity, we report the numerical results obtained without BOA, since, for the MBL analyzed in this paper, the optimization procedure allows a gain of only 1% on the local ergotropy value [43]. Actually, it is the number R of disorder configurations that plays a primary role. As explained in Appendix B, the magnitude of ergotropy fluctuations is predominantly determined by the number R rather than the system length N . We stress that, for the simulated values of N , the local ergotropy value attains convergence within the disorder fluctuations margin, as shown in panel (a) of Fig. 3 by the shaded area.

Panel (b) of Fig. 3 shows that the local ergotropy values get larger as the disorder strength W increases. This is not surprising, since it is always $J_z, J_\perp \ll W$ in the considered parameter regimes. Thus, the disorder strength W is expected to provide the dominant energy contribution (see Appendix D for further analysis on the ergotropy order of magnitude and its dependence on W).

Another procedure for work extraction involves disconnecting the subsystem from the environment before extracting work from the subsystem S . This procedure has the energy

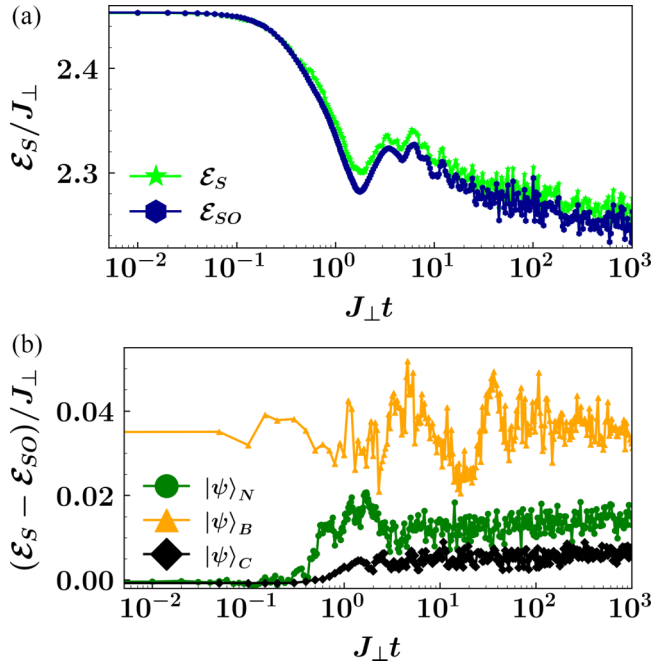


FIG. 4. (a) Local ergotropy $\mathcal{E}_S$ (lime star) and switch-off ergotropy $\mathcal{E}_{SO}$ (dark-blue hexagon), both in units of $J_\perp$, as functions of time for the system in MBL phase, in the case the initial state is the Néel state. Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 12$, disorder strength $W/J_\perp = 5$, and $J_z/J_\perp = 0.2$. (b) Difference between local and switch-off ergotropy, both in MBL phase at $W/J_\perp = 5$ and $J_z/J_\perp = 0.2$, as a function of time, in the case the initial state is a random factorized one $|\psi\rangle_C$ (black diamond), an entangled state $|\psi\rangle_B$ (orange triangle), or the Néel state $|\psi\rangle_N$ (green circle). Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$, using the BOA.

cost

$$\Delta_{SO}(t) = -\text{Tr}_{SE}\{\hat{V}_{SE}\hat{\rho}(t)\}. \quad (8)$$

The subsystem ergotropy is the maximum amount of work extractable via unitary operations $\hat{U}_S$ acting only on the subsystem S , which is now isolated:

$$\mathcal{E}_{SS}[\hat{\rho}_S(t), \hat{H}_S] := \max_{\hat{U}_S \in \mathcal{U}(d_S)} \text{Tr}_S \hat{H}_S(\hat{\rho}_S(t) - \hat{U}_S \hat{\rho}_S(t) \hat{U}_S^\dagger). \quad (9)$$

This is also a time-dependent quantity. Finally, the total amount of work extractable from S , including the previous disconnection, is defined as *switch-off ergotropy* [39]:

$$\mathcal{E}_{SO}[\hat{\rho}(t), \hat{H}] := \mathcal{E}_{SS}[\hat{\rho}_S(t), \hat{H}_S] - \Delta_{SO}(t). \quad (10)$$

One important consideration is that turning off the interactions implicitly requires control over E as well.

We compare the two aforementioned protocols for local and switch-off ergotropy. The plot in panel (a) of Fig. 4 for the switch-off ergotropy can be related to the results previously found in Ref. [30]. Moreover, it reveals that the switch-off ergotropy is smaller than the local ergotropy, albeit slightly. This is because the energy term Δ_{SO} in Eq. (8) is small in localized phases. Since the interactions are weak compared to the other energy scales ($J_z, J_\perp \ll W$), and the coupling between the system and environment occurs only through the interactions present on a single bond, the difference between the local and

the switch-off ergotropy remains small in localized phases. We also perform work extraction without BOA, observing that it is equivalent to the switch-off protocol, allowing a gain of 1% on the local ergotropy value [43].

Panel (b) of Fig. 4 shows the difference between local and switch-off ergotropy in the case of different initial states. In addition to the usual Néel state $|\psi\rangle_N$, we consider a factorized state corresponding to a random extracted spin configuration. Specifically, in the case $N = 8$, we get

$$|\psi\rangle_C = |\uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \downarrow \uparrow\rangle. \quad (11)$$

Finally, we consider an initial state that is not a product state, but rather has nonzero entanglement, allowing to explore whether entanglement can serve as a resource for extracting more work from the subsystem S . For the purpose of this investigation, we adopt a state where the first and last spins of the chain do not exhibit entanglement with their neighbors. The remaining spins are grouped into pairs of singlets. Thus, this initial state is given by

$$|\psi\rangle_B = |\uparrow, \Psi^-, \Psi^-, \dots, \Psi^-, \downarrow\rangle, \quad (12)$$

where $|\Psi^-\rangle$ is the Bell state:

$$|\Psi^-\rangle = \frac{|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle}{\sqrt{2}}. \quad (13)$$

We observe that for the two factorized states, the difference between local and switch-off ergotropy is quite similar. Values for both are, in fact, on the order of 1% of the ergotropy value. Furthermore, $\mathcal{E}_S - \mathcal{E}_{SO}$ shows a weak monotonic increase as a function of time, apart from small oscillations. In the case of the entangled state $|\psi\rangle_B$, such difference is a factor of two larger compared to that of the factorized states and is more oscillating. However, as shown in Fig. 11, the local ergotropy value for the entangled state is the smallest (reduced by a factor around two) among the three analyzed states (for more information, see Appendix E). Indeed, this confirms that low levels of entanglement both in the initial state and in the resulting dynamics characteristic of AL/MBL phases favor larger values of local ergotropy. Finally, from the comparison of the results shown in panel (b) of Fig. 4 and in Fig. 11, it seems that there is a trade-off between the magnitude of the local ergotropy and its difference with the switch-off ergotropy.

We stress that the switch-off protocol introduces a form of nonlocal control and requires an energy cost for turning off the interaction between the subsystems. Previous works, like Ref. [30], focused on the subsystem ergotropy, without taking into account this energy price (in MBL phases analyzed in this paper, it is actually small). In contrast, our study focuses on identifying dynamical signatures of the AL and MBL phases using local ergotropy [Eq. (6)], without needing to disable any Hamiltonian coupling [39].

C. Quantum fluctuations

In order to highlight additional differences between AL and MBL phases, we have finally computed quantum fluctuations of the local ergotropy following Ref. [52]:

$$\sigma^2(t) = \text{Tr}[(\hat{H}' - \hat{H})^2 \hat{\rho}(t)] - \{\text{Tr}[(\hat{H}' - \hat{H}) \hat{\rho}(t)]\}^2, \quad (14)$$

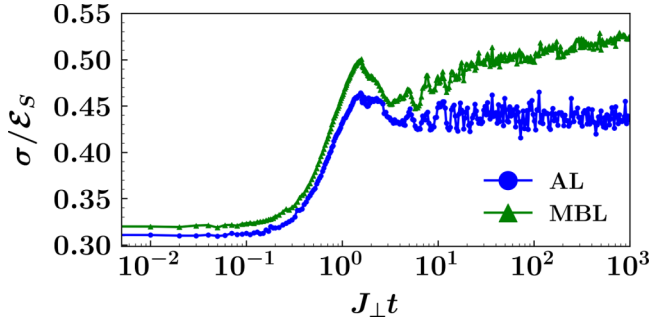


FIG. 5. Relative quantum fluctuations of the local ergotropy as a function of time in the MBL ($J_z/J_\perp = 0.2$, green triangle) and AL phases ($J_z/J_\perp = 0.0$, blue circle). Results are obtained by averaging over 10^3 disorder realizations for the AL case and approximately 2.6×10^3 disorder realizations for the MBL case, both with chain length $N = 8$ and disorder strength $W/J_\perp = 5$.

in which $\hat{H}' = (\hat{U}_S^\dagger \otimes \hat{I}_E) \hat{H} (\hat{U}_S \otimes \hat{I}_E)$. Numerical results in Fig. 5 show that the fluctuations' behavior over time also represents a marker for MBL and AL phases. In analogy with the behavior of the entanglement entropy, after a transient of ballistic growth, the fluctuations saturate to a constant value in the AL case, while they grow logarithmically in the MBL case. For both phases, the short-time value is due to the quantum fluctuations of the initial state and the disorder strength. In Appendix C, we show that increasing the disorder strength W enhances the value of local ergotropy while, remarkably, reducing relative quantum fluctuations. Disorder-induced fluctuations turn out to be smaller than quantum fluctuations [43]. We find that disorder fluctuations are larger in the AL phase than in the MBL phase, consistent with previous results in the literature [30]. Moreover, while disorder fluctuations increase with W , the relative disorder fluctuations remain nearly constant, unlike quantum fluctuations.

IV. CONCLUSIONS

We have demonstrated that local ergotropy and its quantum fluctuations can serve as thermodynamic markers of localization phenomena. Specifically, in the ERG phase, the extractable work rapidly drops to zero; in the AL phase, it reaches a stationary constant value; and in the MBL case, it decreases logarithmically over time. A summary of the different dynamical signatures of localization phenomena is reported in Table I. We have shown that the main results of this work are robust against the change of the initial state, stressing the differences between factorized and entangled ones. Our findings open the way to the characterization of

TABLE I. Dynamical signatures of localization phenomena (first row for established previous results).

	ERG	AL	MBL
$S_{(N/2, N/2)}$	Ballistic increase	Constant	Logarithmic increase
$S_{2, (N-2)}$	Ballistic increase	Constant	Logarithmic increase
$\mathcal{E}_S$	Ballistic decrease	Constant	Logarithmic decrease
σ	Ballistic increase	Constant	Logarithmic increase

different localized phases in many-body systems via direct local work extraction in the presence of Hamiltonian coupling.

We stress that, in our approach, the local ergotropy protocol, applied to a system with nearest-neighbor interactions, requires only local measurements on one pair of spins that are, however, able to distinguish AL, MBL, and ERG phases depending on system's disorder configurations. For small subsystems, the optimization of local ergotropy can be efficiently performed via random sampling of unitaries. We expect that, as the size of the subsystem increases, the optimization procedure becomes computationally more demanding.

The analysis presented in this work can be extended to MBL systems in the presence of an electric field without disorder [53] and generalized to other models, such as the Sachdev-Ye-Kitaev one [13,32,54] and open ones [55,56].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIXES

We detail some points relevant for the analysis of the disordered XXZ Heisenberg chain. In particular, we provide additional information on the entanglement entropy for the chain partition $(N/2, N/2)$. In the case when the subsystem S consists of two spins, we give further details on the scaling behavior of the physical quantities with increasing chain size N , on the quantum fluctuations of local ergotropy, and on the order of magnitude of the ergotropy. We also discuss the role played by the initial state.

APPENDIX A: ENTANGLEMENT ENTROPY FOR THE PARTITION $(N/2, N/2)$

Here, we report the analysis on the time behavior of the entanglement entropy in the case of the chain partitioned into two exact halves, $(N/2, N/2)$, to highlight the differences

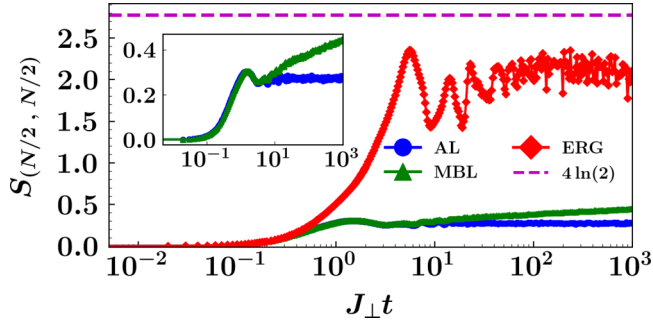


FIG. 6. Entanglement entropy for the $(N/2, N/2)$ partition as a function of time in the three phases: AL (blue circle, $J_z/J_{\perp} = 0$, $W/J_{\perp} = 5$), MBL (green triangle, $J_z/J_{\perp} = 0.2$, $W/J_{\perp} = 5$), and ERG phases (red diamond, $J_z/J_{\perp} = 0.2$, $W = 0$). The dotted line indicates the saturation value of entanglement entropy for the system. The inset provides an enlarged view of the local ergotropy as a function of time in the two localized cases. Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$.

between the three phases: MBL, AL, and ERG. These results are qualitatively identical to those obtained in the 2, $(N - 2)$ case reported in Fig. 1, with only small differences at long times. In fact, considering a subsystem length of $N/2$, the finite-size saturation value for $N \geq 3$ is $S_{N/2, N/2}^{\max} = (N/2) \ln 2 > S_{2, (N-2)}^{\max} = 2 \ln 2$.

In each of the three phases, the entropy starts from zero because of the initial Néel state. For a first transient of time, each phase exhibits a ballistic growth. After that, in the ergodic phase (red diamond in Fig. 6), entropy keeps growing toward the saturation value (dotted line in Fig. 6). In the two localized phases, the entropy growth halts around $J_{\perp} t \sim 1$. In the MBL phase, entropy grows logarithmically, while in the AL phase, it remains constant. These results are fully consistent with previous theoretical and experimental results.

APPENDIX B: FURTHER SCALING ANALYSIS OF PHYSICAL QUANTITIES

Here, we examine how several physical quantities scale with chain lengths N , as robustness to size increase is a key feature of MBL phenomenology [4].

In panel (a) of Fig. 7, we report the entanglement entropy, in the case of $N/2, N/2$ partition, as a function of time for different chain sizes. Moreover, we show the time behavior of the imbalance defined as the normalized charge difference between odd and even sites that, in the spin representation, reads $\mathcal{I} = (\langle S_o^z \rangle - \langle S_e^z \rangle) / (\langle S_o^z \rangle + \langle S_e^z \rangle + N/2)$ (where sites' first index is 1; see Refs. [12,44]). From $N = 8$ to $N = 12$, the plots of both quantities show only minor differences.

In panel (b) of Fig. 7, we report the entanglement entropy, in the case of 2, $(N - 2)$ chain partition, as a function of time for different sizes. In analogy with the entanglement entropy for the $N/2, N/2$ chain partition, very small differences are obtained with increasing N .

In addition to the quantum fluctuations of the local ergotropy, we have also computed its fluctuations over disorder realizations to distinguish the purely quantum effects. One expects that the small oscillations shown by the numerical

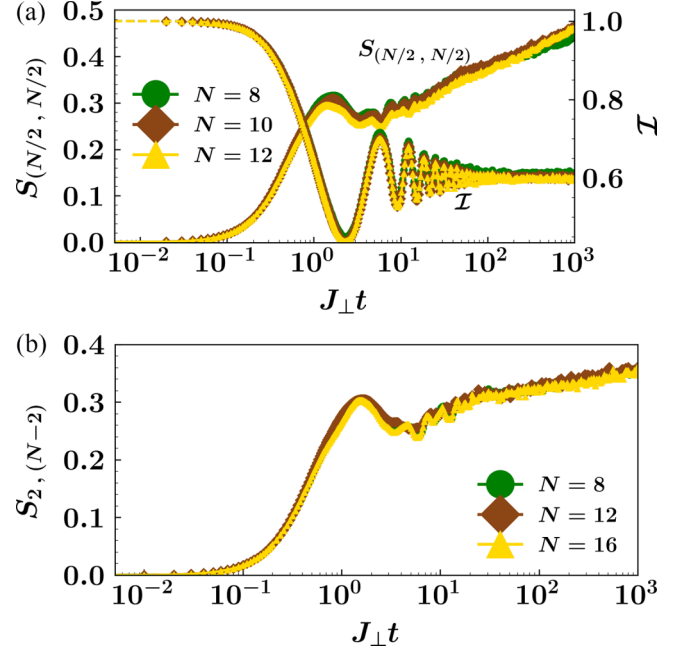


FIG. 7. (a) Entanglement entropy and imbalance $\mathcal{I}$ for $(N/2, N/2)$ partition as a function of time for different sizes: $N = 8$ (green circle), $N = 10$ (brown diamond), and $N = 12$ (gold triangle). Results are obtained by averaging over 1.6×10^3 disorder realizations, with coupling $J_z/J_{\perp} = 0.1$ and disorder strength $W/J_{\perp} = 5$. (b) Entanglement entropy for 2, $(N - 2)$ partition as a function of time in the MBL phase ($J_z/J_{\perp} = 0.2$, $W/J_{\perp} = 5$) for various lengths: $N = 8$ (green circle), $N = 12$ (brown diamond), and $N = 16$ (gold triangle). Results are obtained by averaging over 10^3 disorder realizations for $N = 8, 12$ and around 6.6×10^2 for $N = 16$.

results of local ergotropy at long times arise not only from finite-size effects but also from averaging over a finite large number of disorder realizations.

To evaluate the fluctuations for a single local ergotropy realization at time t , one can use the standard deviation:

$$\sigma^{\text{dis}}(t) = \sqrt{\frac{1}{R-1} \sum_{i=1}^R (\mathcal{E}_S^{(i)}(t) - \overline{\mathcal{E}}_S(t))^2}, \quad (\text{B1})$$

where R is the number of the disorder realizations over which we average, $\mathcal{E}_i(t)$ is the single ergotropy realization at time t , and $\overline{\mathcal{E}}(t)$ is the disorder-averaged local ergotropy at time t .

To quantify the fluctuations for disorder-averaged ergotropy at time t , we can use the mean standard deviation:

$$\sigma_{\mathcal{E}_S}^{\text{dis}}(t) = \sigma^{\text{dis}}(t) / \sqrt{R}. \quad (\text{B2})$$

We make a deep investigation of the scaling properties of the local ergotropy as a function of the chain length N and the number R of disorder realizations. As mentioned in the main text, the analysis shows that the most important quantity to get the convergence of the local ergotropy in the MBL phase is not N , but rather R . Panel (a) of Fig. 8 reveals how local ergotropy disorder fluctuations at time $t = 0$ scale with R . For

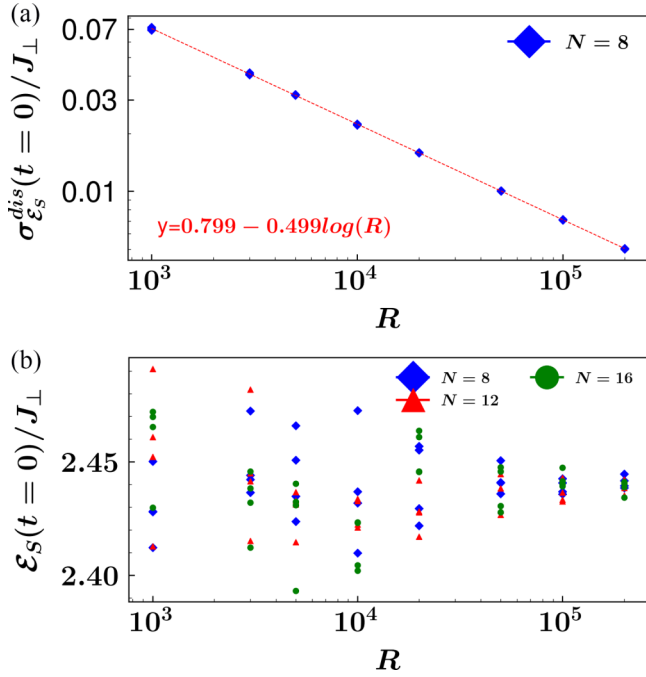


FIG. 8. (a) Disorder fluctuations of the local ergotropy evaluated at the initial time $t = 0$ as a function of the number R of disorder configurations. The dotted line denotes the fit to the data. Results are obtained for a chain length $N = 8$ in the MBL phase, $J_z/J_\perp = 0.2$ and $W/J_\perp = 5$, and presented in log-log scale. (b) Local ergotropy evaluated at the initial time $t = 0$ as a function of the number R of disorder configurations for different values of chain length: $N = 8$ (blue diamond), $N = 12$ (red triangle), and $N = 16$ (green circle). For fixed R , four points are reported, each of them being an independent average over R realizations. Local ergotropy data are presented in linear-log scale.

$R = 2 \times 10^5$, we find $\sigma_{\mathcal{E}_S}^{\text{dis}} \approx 5 \times 10^{-3}$, a value much smaller than unity, which is the typical order of magnitude of the ergotropy calculated in this paper (all these quantities are in units of $J_\perp$). For $R = 10^3$, we find $\sigma_{\mathcal{E}_S}^{\text{dis}} \approx 0.07$, a value still quite smaller than unity. This suggests that the value $R = 10^3$, employed in the main text, is enough to get accurate values of the ergotropy.

The reported equation in panel (a) of Fig. 8 is the best fit obtained using the ergotropy values as a function of the number of realizations. The slope of the fit curve, close to -0.5 , clearly reveals that a relevant contribution to the scaling is provided by the square root of R in Eq. (B2).

Next, we have analyzed the combined role of the chain length N and the number R of disorder realizations [see Fig. 8(b)]. The analysis confirms the primary role of R over N for the ergotropy convergence. In fact, for each R , the values of local ergotropy for three distinct values of N are reported, and a clear overlap between them is observed. The convergence is then determined by R , not N within the MBL phase. For all the values of N , the averages over different R realizations are consistent within the $\sigma_{\mathcal{E}_S}^{\text{dis}}$ margin, similarly to what is observed in Fig. 3(a).

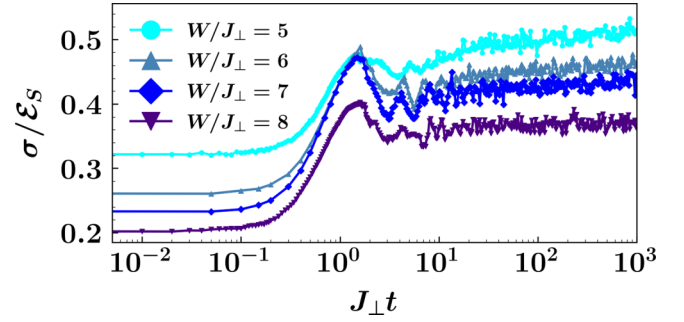


FIG. 9. Local ergotropy relative quantum fluctuations as functions of time for different values of the disorder strength W in the MBL phase ($J_z/J_\perp = 0.2$): $W/J_\perp = 5$ (cyan circle), $W/J_\perp = 6$ (steel-blue triangle), $W/J_\perp = 7$ (blue diamond), and $W/J_\perp = 8$ (indigo reverse triangle). Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$.

APPENDIX C: FURTHER ANALYSIS ON QUANTUM FLUCTUATIONS

We report here details relative to quantum fluctuations of local ergotropy additional to those presented in Fig. 5. In Fig. 9, the relative quantum fluctuations decrease with increasing W , underscoring the robustness of localization.

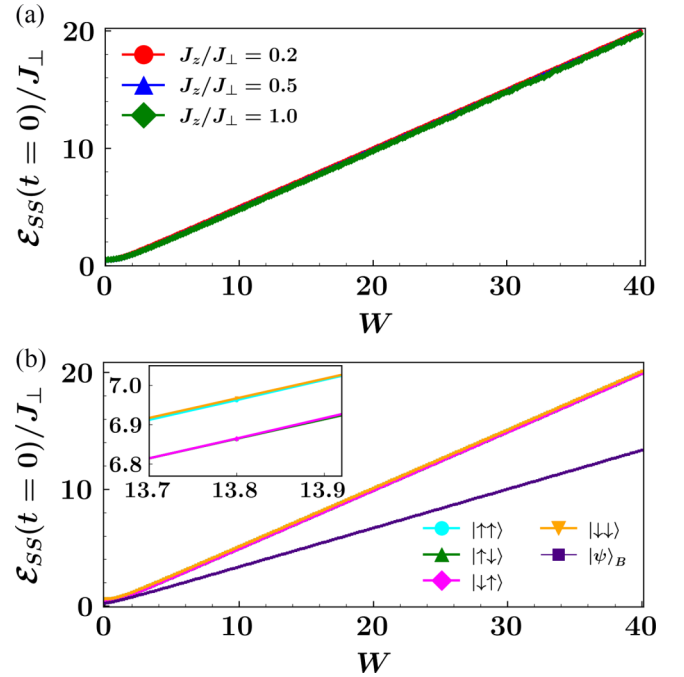


FIG. 10. (a) Subsystem ergotropy at $t = 0$ as a function of the disorder strength W , in the case of initial Néel state, for various longitudinal interaction strength values: $J_z/J_\perp = 0.2$ (red circle), $J_z/J_\perp = 0.5$ (blue triangle), and $J_z/J_\perp = 1.0$ (green diamond). (b) Subsystem ergotropy at $t = 0$ as a function of the disorder strength W at $J_z/J_\perp = 0.2$ in the case of all possible factorized initial states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$ and finally in the case of entangled initial state $|\psi\rangle_B$. In both panels, results are obtained by averaging over 10^5 disorder realizations.

APPENDIX D: ERGOTROPY ORDER OF MAGNITUDE

Since the values of the interaction strength considered in the entire work are small compared to the disorder strength ($J_z, J_\perp \ll W$), the order of magnitude of the local ergotropy is expected to be dominated by W . This expectation is confirmed by numerical simulations. In particular, for the Néel initial state (see Fig. 2), the local ergotropy is found to be on the order of $W/4$ per each subsystem site, implying that the subsystem ergotropy for two spins scales as approximately $W/2$ in the parameter regime with large disorder.

This scaling behavior is further corroborated by the results shown in Fig. 10. Panel (a) displays the dependence on W of the subsystem ergotropy evaluated at the initial time $t = 0$ for the Néel state. For $W \gtrsim 1$, the scaling is clearly linear with a slope close to $1/2$ for any value of J_z . For an initial factorized state, the switch-off and the local ergotropy are equal at the initial time (the interaction energy is negligible for the order of magnitude analysis). Since localized systems preserve memory of the initial state indefinitely, the time-evolved quantum state is expected to remain close to the initial state for the timescales under consideration. This is the reason why, even if the plot regards solely the subsystem ergotropy at the initial time $t = 0$, local ergotropy is expected to maintain the same order of magnitude for $t > 0$.

Panel (b) of Fig. 10 reports the initial subsystem ergotropy (at fixed J_z) for various initial states. The first four curves correspond to all possible factorized spin configurations, while the fifth refers to the subsystem density operator obtained by tracing over the remaining $N - 2$ spins in the entangled state $|\psi\rangle_B$ introduced in Eq. (12). In all the cases, the ergotropy exhibits a linear dependence on W . For product states, the slope is approximately $1/2$, with minor deviations due to interaction effects. As expected, the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ yield comparable results due to their energetic degeneracy, and similarly for $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$. In contrast, for the entangled state $|\psi\rangle_B$, the scaling is slightly reduced, with $\mathcal{E}_{SS} \propto W/3$. Therefore, the ratio of the ergotropy between the entangled and the factorized case is on the order of $2/3$. Actually, this is roughly similar to what is observed in Fig. 11.

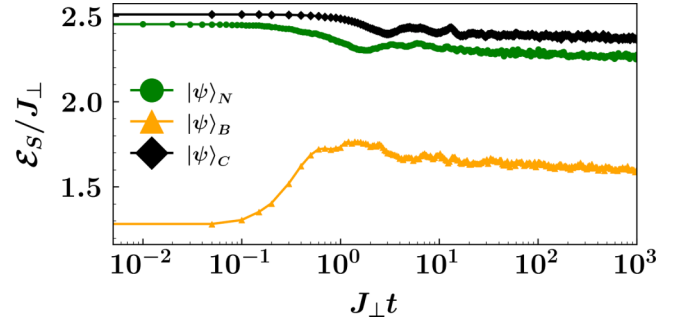


FIG. 11. Local ergotropy as a function of time in the MBL phase, for $J_z/J_\perp = 0.2$ and $W/J_\perp = 5$, for three different initial states: $|\psi\rangle_N$ (green circle), $|\psi\rangle_B$ (orange triangle), and $|\psi\rangle_C$ (black diamond). Results are obtained by averaging over 10^3 disorder realizations, with chain length $N = 8$, using the BOA.

APPENDIX E: FURTHER ANALYSIS ON DIFFERENT INITIAL STATES

In addition to the Néel state $|\psi\rangle_N$, in this section, we present an investigation by varying the initial states, in order to complete the analysis exposed in panel (b) of Fig. 4.

Figure 11 presents a comparative analysis of the local ergotropy behavior over time in three cases: the initial state is $|\psi\rangle_N$, $|\psi\rangle_B$, or $|\psi\rangle_C$. Beyond the Néel state, the second case is the state $|\psi\rangle_B = |\uparrow, \Psi^-, \Psi^-, \dots, \Psi^-, \downarrow\rangle$ introduced in Eq. (12), with $|\Psi^-\rangle$ defined in Eq. (13), which is not a product state having nonzero entanglement. In such way, we explore whether entanglement can serve as a resource for extracting more work from the subsystem S .

For the last case, we adopted the random factorized state $|\psi\rangle_C = |\uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \downarrow \uparrow\rangle$ defined in Eq. (11).

After an initial transient period, during which the ergotropy values differ depending on the presence of entanglement, all the three cases exhibit a logarithmic decrease over time, as expected in the MBL phase. Besides the trend, the amount of extractable work for the $|\psi\rangle_B$ state is lower compared to the two initial factorized states, due to a greater portion of the system's energy being stored in correlations.

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