

Second law-abiding-account of heat engines in active bath

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The second law of thermodynamics, quantitatively formalized through non-negative entropy production, defines the fundamental constraints on energy conversion, such as the Carnot limit for heat engines. While it has been suggested that active noises in heat baths might allow microscopic engines to surpass these bounds, such apparent violations often stem from an incomplete thermodynamic accounting of the active reservoir. Here, we demonstrate theoretically and experimentally that the classical second law remains valid when considering active noise as a particle under white Gaussian noise interacting with a Brownian bead, rather than relying on the concept of effective temperature. In this study, our illustrative model system consists of a two-dimensional Brownian gyrator with a bead immersed in temperature baths, combined with active noises. By treating active noise as explicit degrees of freedom, we analytically solve the Langevin equation by decoupling the energetics into two contributions: the interactions of the Brownian bead with the heat baths and with another particle subject to a different heat bath. We find that entropy production is always non-negative; the gyrator can operate as a heat engine in specific environments but cannot surpass the Carnot limit, and the thermodynamic uncertainty relation holds. This study offers a rigorous perspective on the relation between active noise and the second law, demonstrating that properly accounting for active noise as a distinct thermodynamic component preserves the fundamental principles of thermodynamics.

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I. INTRODUCTION

The second law of thermodynamics has been the cornerstone of thermodynamics for macroscopic physical systems for centuries [1]. This law, quantitatively formalized through the production of entropy, defines the fundamental constraints on energy conversion efficiency in physical systems, such as heat engines and refrigerators. It dictates that the minimal thermodynamic cost, defined using the concept of entropy production (EP) ΔS , for the transition between two states be non-negative ($\Delta S \geq 0$). This inequality further implies that, for heat engines operating between two heat reservoirs with different temperatures, the maximal efficiency cannot exceed the Carnot limit ($\eta \leq \eta_C$) [1]. In recent years, there has been an increasing focus on the thermodynamic inequality that quantifies a tradeoff: High-precision, steady-state currents cannot be achieved without a significant cost in energy dissipation (entropy production) [2–4]. This is known as the thermodynamic uncertainty relation (TUR), providing a tighter bound than the entropic one. The relation is expressed as an inequality of the following form:

$[\mathcal{D}(\Delta t)/\langle \dot{X}(\Delta t) \rangle^2] \dot{S}(\Delta t) \geq k_B$, where $\dot{S}$ is the production rate of entropy, $\mathcal{D}(\Delta t) \equiv \text{Var}[X(\Delta t)]/2\Delta t$ is the diffusion constant of X , $\text{Var}[X(\Delta t)]$ is the variance of the thermodynamic variable $X(\Delta t)$, and Δt is the sampling time duration. Here, $\dot{\cdot}$ denotes the time derivative. In addition to its stricter role as a thermodynamic bound of the second law, TUR has applications in inferring entropy production [5] and anomalous diffusion [6].

Assessing the thermodynamic performance and operational bounds of microscopic heat engines, specifically through entropy production, efficiency, and the TUR, has been challenging in nonequilibrium fluctuating environments. For the last three decades, the framework of fluctuation theorems [7] and stochastic thermodynamics [8] has been established, enabling the determination of energetics at the microscopic level of single trajectories [9–14]. Assisted by the manipulative technology for microscale systems [15], microscopic heat engines have been experimentally realized on the basis of a single Brownian bead [9–13,16]. In this minimal experimental setting, the second law has been shown to remain valid for colloidal heat engines in contact with thermal baths [11,12,16–19] as predicted by the theory [20–24].

More recently, active matter systems emerged as a promising substance, primarily due to their inherent ability to extract environmental energy and sustain out-of-equilibrium dynamics [25–27]. These active matter systems share an ability to exchange energy with the environment and transform it to sustain out-of-equilibrium motion, raising interesting questions about the effect of active matter on the thermodynamic performance of engines [28,29]. Recent experimental

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studies claimed that a cyclic colloidal heat engine can exceed the thermodynamic efficiency by using a bacterial suspension as active matter in the heat reservoir [30]. It is later established that a proper account of heat injection is critical to avoid the apparent violation of the second law in cyclic engines [31]. Some studies of an autonomous linear heat engine [32] as well as cyclic heat engines [33–35], coupled to Ornstein-Uhlenbeck active reservoirs [36], which likewise defines an effective temperature via an equipartition-type relation, demonstrate that the efficiency of the engine can substantially exceed the Carnot bound, therefore exposing the limitations of describing active baths by an effective temperature. Following these findings, several questions can be put forward. First, though counterintuitive, is the entropic second law also violated? If yes, what is the origin of the entropic violation? Conversely, if the entropic inequality $\Delta S \geq 0$ still holds, or the tighter TUR is valid, does the second law of thermodynamics need reassessment to regain a full account of the effects of active fluctuations?

To resolve this issue, Datta *et al.* instead proposed a renewed second law from the perspective of rectified excess EP [37]. Within this framework, the maximal achievable engine efficiency of an active heat engine is below the Carnot limit, which remains the upper bound for passive heat engines. Specifically, nonequilibrium systems subject to active baths cost extra EP to transit between nonequilibrium steady states associated with the active nature of the bath constituents. The authors argued that for an active heat engine, the conventional expression of the total EP, $\Delta S_{\text{tot}} = -Q_h/T_h - Q_c/T_c$, does not include all sources of EP. Here, h and c , respectively, denote high- and low-temperature (hot and cold) baths, Q is heat, and T is temperature. In addition to the EP associated with heat flow and output work, an information-theoretic term was found to contribute to the excess entropy. Consequently, an extra heat influx associated with the added term in excess entropy helps the Carnot efficiency regain its insurmountable status.

In this study, following the theoretical framework proposed by Datta *et al.* [37], we revisit the validity of the second law of thermodynamics and the tighter-bound TUR in the presence of active matter. The model system we used in this study is a Brownian gyrator under white noise baths at two different temperatures, first proposed by Filliger and Reimann [9] and subsequently investigated in several studies [19,38]. Furthermore, the Brownian gyrator under two active baths is employed here for detailed studies. To this end, we exploit the known dynamical structure of an active Ornstein-Uhlenbeck (OU) particle to investigate the effect of OU noise on the energetics of a Brownian gyrator. The active noise is assigned explicit degrees of freedom (DoF) to uncover the energetics arising from interactions between the observables and the active noise. The dynamics of the gyrator are analyzed from two perspectives as a comparison to highlight the criticality of the coupling energy between them and the OU active noises.

We organize this paper as follows. A theoretical model for the dynamics of a Brownian gyrator in an active bath with the OU noises is introduced in Sec. II. The derivation of the energetics of the gyrator in a two-dimensional (2D) hybrid bath is presented in Sec. III. In Sec. IV, we

provide a brief overview of the main analytical results from the four-dimensional (4D) perspective. Analytical expressions for energetics, efficiency, and the diffusion constant that characterize current fluctuations are presented in the subsections. Complete derivations for the two-dimensional active Ornstein-Uhlenbeck particle (2D AOUP) gyrator and 4D white noise gyrator can be found in the Appendixes B and C. In Sec. V, we experimentally record the trajectories of Brownian gyrators, calculate the thermodynamic quantities, and compare the experimental results with the exact solutions obtained from our theory. In Sec. VI, the conclusion and discussion are provided.

II. THEORETICAL MODEL OF A BROWNIAN GYRATOR IN THE ACTIVE BATH

Let us consider a two-dimensional Brownian gyrator as a model system, depicted in Fig. 1(a). A single colloidal bead is confined in a 2D harmonic potential of stiffnesses, κ_1 and κ_2 , and subject to the two temperatures, T_1 and T_2 . Here, 1 and 2 denote two orthogonal axes. Gaussian white noise $\eta_i(t)$ that comprises the thermal bath has zero mean and variance as $\overline{\eta_i(t)} = 0$ and $\overline{\eta_i(t)\eta_j(t')} = 2D_i\delta_{ij}\delta(t-t')$, respectively, with diffusion coefficients D_i and friction coefficients γ_i . δ_{ij} , and $\delta(t-t')$ represent the Kronecker delta and the delta function, respectively. The upper bar on variables denotes the ensemble average and $i = 1, 2$. We note that the variables in the main text are dimensionless as described in Appendix A. In addition to the thermal temperature, the Brownian gyrator is subjected to an additional nonequilibrium active bath independently in each of the two observed DoF x_1 and x_2 . The stochastic contribution of the active noises is modeled as noise generated by the OU processes, which are usually used to mimic the noise induced by the bacterial bath [33–35]. The thermal and OU noises form a hybrid bath, and the system is called a 2D AOUP gyrator. The mean and variance of the OU noises $\zeta_i(t)$ are defined as $\overline{\zeta_i(t)} = \overline{\zeta_i(t)\zeta_j(t')} = 0$ for $i \neq j$, $\overline{\zeta_1(t)\zeta_1(t')} = D_3/\tau_1 \exp(-|t-t'|/\tau_1)$, $\overline{\zeta_2(t)\zeta_2(t')} = D_4/\tau_2 \exp(-|t-t'|/\tau_2)$. Additional nonconservative forces (δx_2 and ϵx_1 in x_1 and x_2 directions, respectively) are applied to the bead to transfer the heat from the hot bath to the cold bath, where δ and ϵ are the coupling variables connecting x_1 to x_2 .

The two equations of motion of the colloidal bead describe dynamics in each of the observed observables x_1 and x_2 , such as the bead positions. Restoring force due to a harmonic potential, a nonconservative coupling force, and stochastic forces from a hybrid bath, all exerted on the bead, with the Langevin equation, takes the form of

$$\begin{aligned}\dot{x}_1 &= -\kappa_1 x_1 + \epsilon x_2 + \zeta_1(t) + \eta_1(t), \\ \dot{x}_2 &= -\kappa_2 x_2 + \delta x_1 + \zeta_2(t) + \eta_2(t),\end{aligned}\quad (1)$$

with the linear restoring coefficient κ_i and the two coupling constants ϵ and δ . We take advantage of the similarity between the generation of the OU noise via

$$\tau_i \dot{\zeta}_i = -\zeta_i + \psi_i(t), \quad (2)$$

and the Langevin equation in Eq. (1), and introduce the 4D vectors $\vec{X}^T \equiv (x_1, x_2, \zeta_1, \zeta_2)$ and

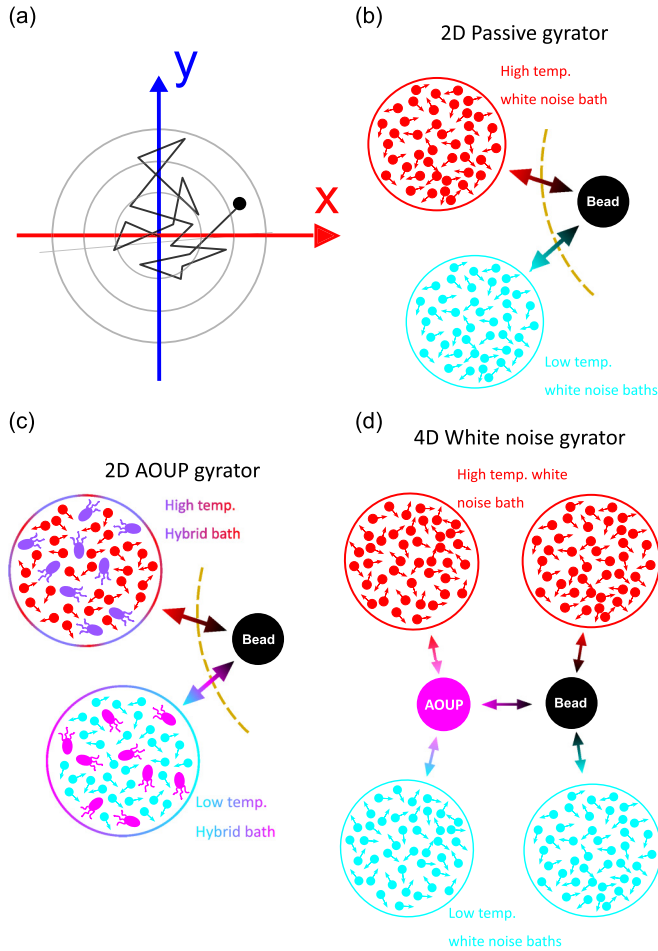


FIG. 1. (a) Schematics of a two-dimensional Brownian gyrator. The water-immersing bead (black solid circle) experiences stochastic motion (dark gray curve) in a 2D harmonic potential (light gray concentric potential contours). Two temperature baths, $T_1 > T_2$, are applied along the axes x_1 and x_2 , respectively. When the bead is further subject to coupling forces between two axes with proper conditions, the bead undergoes a steady-state gyration around the center of the harmonic potential. (b) Diagram of a 2D white noise gyrator without active noises. The bead, as the system of interest, exchanges energy with two white noise baths at different temperatures. (c) Diagram of a 2D AOUP gyrator. Active noise can be treated as part of the heat baths. (d) Diagram of a 4D white noise gyrator. The active noises can be treated as a particle immersed in two different white noise baths that act on the colloidal bead.

$\vec{\xi}^\top(t) = (\eta_1(t), \eta_2(t), \lambda_3\psi_1(t), \lambda_4\psi_2(t))$ to represent the stochastic dynamics in dimensionless form as follows:

$$\dot{\vec{X}} = -\mathcal{A}\vec{X} + \vec{\xi}(t), \quad \mathcal{A} = \begin{pmatrix} \kappa_1 & -\epsilon & -1 & 0 \\ -\delta & \kappa_2 & 0 & -1 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad (3)$$

where the superscript $\top$ stands for the transpose, $\lambda_3 = 1/\tau_1$ and $\lambda_4 = 1/\tau_2$. Here, $\psi_i(t)$ is the Gaussian white noise used in the generation of the OU noise. The mean and variance of

ψ_i are given by

$$\begin{aligned} \overline{\psi_1(t)} &= \overline{\psi_2(t)} = 0, \\ \overline{\psi_1(t)\psi_2(t')} &= 0, \\ \overline{\psi_1(t)\psi_1(t')} &= 2D_3\delta(t-t'), \\ \overline{\psi_2(t)\psi_2(t')} &= 2D_4\delta(t-t'). \end{aligned} \quad (4)$$

As x_i 's are the only available observables, the energies can be determined by the coupling forces between x_i 's and their infinitesimal displacement dx_i 's. The energy exchange is assumed to occur between the “system (bead)” and the “environment (temperature baths)” [Fig. 1(c)].

The second perspective utilizes the characteristics of the generating equation of the OU noise, in which the white noise (ψ_i) is present. The two OU noises that are applied separately to x_1 and x_2 are now regarded as a second “particle” with two-dimensional dynamics and are in contact with two other white noise baths [Fig. 1(d)]. The “environment” is a set of four independent white noise baths. Consequently, the original 2D dynamics of a single bead, $\vec{x}$, under the hybrid OU and white noises are equivalent to the 4D dynamics of a system consisting of one bead $\vec{x}$ and one AOUP $\vec{\zeta}$. In the setting, the energy exchange among the white noise baths, the “AOUP particle,” and the colloidal bead will become explicit. Hereafter, we denote a real Brownian particle as a bead and the active noise particles as AOUP.

III. A BROWNIAN GYRATOR IN HYBRID BATH

Let us first consider the model of the Brownian gyrator in the hybrid bath (2D AOUP gyrator), as depicted in Fig. 1(c). In this model, we treat the active force as an effective temperature that affects the bead, i.e., the energy exchange between the bead and the temperature baths. A full dynamical description is provided by Eq. (1), which can be put into the matrix form with $\vec{x} = (x_1, x_2)^\top$, $\vec{\zeta} = (\zeta_1, \zeta_2)^\top$, and $\vec{\eta} = (\eta_1, \eta_2)^\top$:

$$\dot{\vec{x}} = -\mathbf{A}\vec{x} + \vec{\zeta}(t) + \vec{\eta}(t), \quad \mathbf{A} = \begin{pmatrix} \kappa_1 & -\epsilon \\ -\delta & \kappa_2 \end{pmatrix}. \quad (5)$$

Here, the active OU noise $\vec{\zeta}$ is on the same footing as the thermal white noise $\vec{\eta}$. The total entropy production

$$\Delta S = \Delta S_{\text{sys}} + \Delta S_{\text{env}}, \quad (6)$$

where ΔS_{sys} represents the EP of the system and ΔS_{env} is the EP between the system and the environment. For steady states, ΔS_{sys} is unchanged, and thus $\Delta S = \Delta S_{\text{env}}$.

As for the stochastic energetics, the potential is given by $U(\vec{x}) = \frac{1}{4}\vec{x}^\top(\mathbf{A} + \mathbf{A}^\top)\vec{x} = \frac{1}{2}[\kappa_1 x_1^2 + \kappa_2 x_2^2 - (\epsilon + \delta)x_1 x_2]$, and the conservative force is then given by $\vec{f}_{\text{con}} = -\nabla U(\vec{x}) = (-\kappa_1 x_1 + \frac{\epsilon + \delta}{2}x_2, -\kappa_2 x_2 + \frac{\epsilon + \delta}{2}x_1)^\top$. Subtraction of the conservative force from the total force gives us the nonconservative force $\vec{f}_{\text{nc}} = \frac{\epsilon - \delta}{2}(x_2, -x_1)^\top$. The infinitesimal stochastic work associated with each DoF of the 2D system is then given by $dW_i = -(\vec{f}_{\text{nc}})_i \circ dx_i$ ($i = 1, 2$), where $\circ$ denotes the Stratonovich product.

Direct calculation of the stochastic heat and work for each DoF using $dQ_i = (\mathbf{A}\vec{x})_i \circ dx_i$ yields the following explicit

expressions:

$$\begin{aligned} dQ_1 &= (\kappa_1 x_1 - \epsilon x_2) \circ dx_1, \\ dQ_2 &= (\kappa_2 x_2 - \delta x_1) \circ dx_2, \\ dW_1 &= -\frac{(\epsilon - \delta)}{2} x_2 \circ dx_1, \\ dW_2 &= \frac{(\epsilon - \delta)}{2} x_1 \circ dx_2. \end{aligned} \quad (7)$$

The infinitesimal heat and work in this 2D system are, respectively, given by $dQ = dQ_1 + dQ_2$ and $dW = dW_1 + dW_2$. Inserting the above dimensionless stochastic heat dQ_i into the definition of EP, $dS = -dQ_1/D_1 - dQ_2/D_2$ [32,39] yields

$$\langle dS \rangle = \frac{\epsilon \langle x_2 \circ dx_1 \rangle}{D_1} + \frac{\delta \langle x_1 \circ dx_2 \rangle}{D_2}, \quad (8)$$

with the fact that $\langle x_i \circ dx_i \rangle = 0$. The details of the derivations are given in Appendix B, showing that stochastic and steady-state energetics and fluctuations can be computed from the couplings $x_1 \circ dx_2$ and $x_2 \circ dx_1$. For simplicity, we omit $\circ$ in the Stratonovich expression and use $\langle x_1 \dot{x}_2 \rangle = -\langle x_2 \dot{x}_1 \rangle$ to further show that the mean EP rate and work rate are given by

$$\langle \dot{S} \rangle = \left(\frac{\delta}{D_2} - \frac{\epsilon}{D_1} \right) \langle x_1 \dot{x}_2 \rangle, \quad (9)$$

$$\langle \dot{W} \rangle = (\epsilon - \delta) \langle x_1 \dot{x}_2 \rangle, \quad (10)$$

where the analytic expression for $\langle x_1 \dot{x}_2 \rangle$ is given by Eq. (B97).

Let us consider a specific case where there is no active force along axis-1, $\zeta_1 = 0$ (i.e., $D_3 = 0$, which is the scenario in our experiments), the mean work rate and EP rate can be simplified as

$$\langle \dot{W} \rangle = -\frac{\epsilon D_1 (\epsilon - \delta) (\delta - \delta_+) (\delta - \delta_-)}{(\kappa_1 + \kappa_2) [(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]}, \quad (11)$$

$$\langle \dot{S} \rangle = -\frac{\epsilon (\delta D_1 - \epsilon D_2) (\delta - \delta_+) (\delta - \delta_-)}{(\kappa_1 + \kappa_2) D_2 [(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]}, \quad (12)$$

where

$$\begin{aligned} \delta_{\pm} &\equiv \frac{1}{2D_1\epsilon} [\epsilon^2 D_2 + (\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) D_1 \\ &\pm \sqrt{[\epsilon^2 D_2 - (\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) D_1]^2 - 4\epsilon^2 \lambda_4^2 D_1 D_4}]. \end{aligned} \quad (13)$$

Equations (11) and (12) can help one to easily identify, respectively, the negative work or negative EP regimes in the parameter space (Fig. 2). Note that the EP rate depends on the relative strengths of the passive and active noises, D_2/D_1 and D_4/D_1 (Fig. 10). It is evident that across the range of the examined persistence times, the mean EP rate can assume negative values [see Fig. 3(b)].

One might argue that the negative EP rate results from a nonequilibrium steady state for a hybrid bath inside a harmonic potential, and that the effective elevated effective temperature should be used instead of a thermodynamic temperature. For the isolated hybrid bath 1, we have $\kappa_1 \langle x_1^2 \rangle = D_1 + D_3/(1 + \kappa_1 \tau_1) \equiv k_B T^e$, where $k_B T^e$ is defined as its effective temperature. Then, for the Brownian bead under two hybrid baths, one can define the EP using the effective

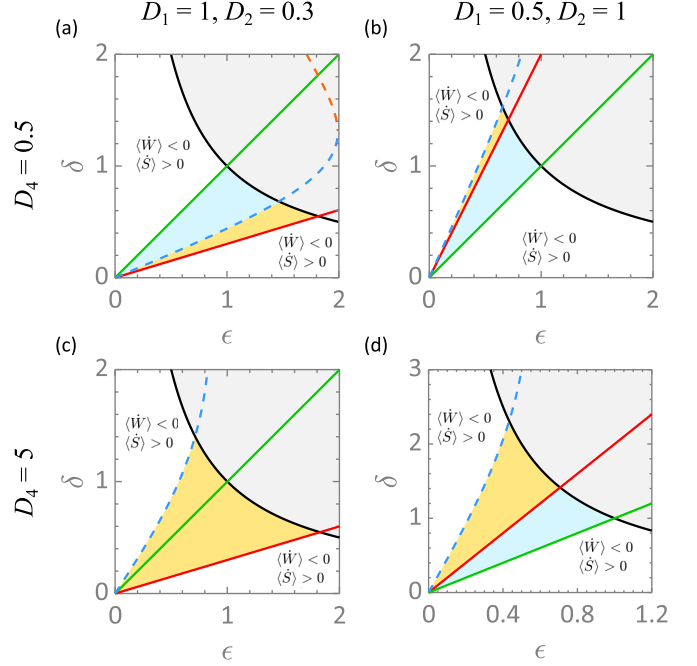


FIG. 2. The ϵ - δ parameter space for a 2D AOUP heat engine signifying regions of positive or negative $\langle \dot{W} \rangle$ and $\langle \dot{S} \rangle$. $D_3 = 0$, $\kappa_1 = 1$, $\kappa_2 = 1$, and $\lambda_4 = 1$ in all plots. (a) $D_1 = 1$, $D_2 = 0.3$, and $D_4 = 0.5$. (b) $D_1 = 0.5$, $D_2 = 1$, and $D_4 = 0.5$. (c) $D_1 = 1$, $D_2 = 0.3$, and $D_4 = 5$. (d) $D_1 = 0.5$, $D_2 = 1$, and $D_4 = 5$. The engine working region (sky blue shade) warrants a non-negative output work [$\langle \dot{W} \rangle \geq 0$; see Eq. (11)] and a non-negative entropy production [$\langle \dot{S} \rangle \geq 0$; see Eq. (12)]. This region is determined by four equations: $\epsilon \delta = \kappa_1 \kappa_2$ (solid black curve), $\delta = \delta_+$ (orange dashed curve), $\delta = \delta_-$ (blue dashed curve), and $\epsilon = \delta$ (green solid line). Light yellow shades in all figures denote the regime of entropic violation ($\langle \dot{S} \rangle < 0$) but $\langle \dot{W} \rangle \geq 0$. Light gray shades are nongyrotor regions.

temperatures of the two baths to obtain the mean EP rate:

$$\langle \dot{S} \rangle_{T^e} = \left(\frac{\delta}{D_2 + \frac{D_4}{1 + \kappa_2 \tau_2}} - \frac{\epsilon}{D_1 + \frac{D_3}{1 + \kappa_1 \tau_1}} \right) \langle x_1 \dot{x}_2 \rangle. \quad (14)$$

The mean EP rate can still become negative in certain parameter spaces even when the thermodynamic temperatures are replaced with effective temperatures [Fig. 3(c)].

IV. A 4D BROWNIAN GYRATOR IN THERMAL BATHS

In the previous section, we confirmed that the second law is still violated when the active noise is considered as part of the heat bath. This section presents a theoretical framework in which active noises are considered as explicit degrees of freedom and also treated as a particle (AOUP) interacting with a bead (the real Brownian particle), as depicted in Fig. 1(d). The main analytical results from the 4D white noise gyrotor analysis are presented below.

A. Energetics and the first law of thermodynamics

As the stochastic work $dW_i = -(\vec{f}_{nc})_i \circ dX_i$ ($i = 1, 2, 3, 4$) associated with each DoF of the 4D system requires the

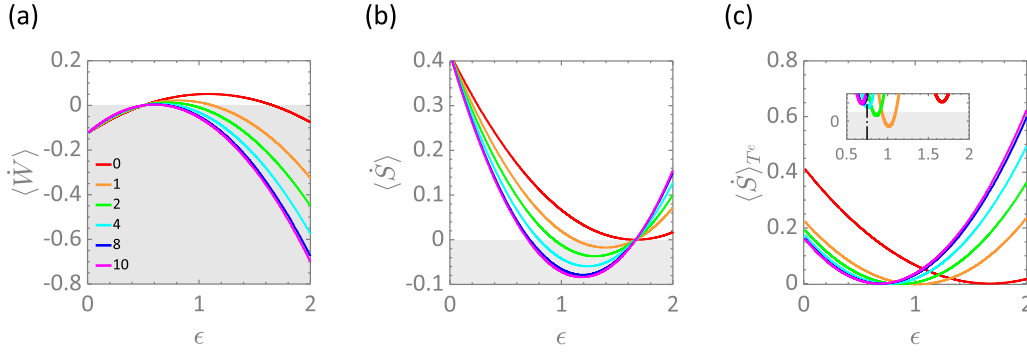


FIG. 3. Analytical results of (a) mean work rate $\langle \dot{W} \rangle$ [Eq. (11)], (b) mean EP rates $\langle \dot{S} \rangle$ [Eq. (12)], and (c) $\langle \dot{S} \rangle_{T^e}$ [Eq. (13)] for a 2D AOUP gyrotor for various values of λ_4 with $\kappa_1 = \kappa_2 = 1$, $\delta = 0.5$, $D_1 = 1$, $D_2 = 0.3$, $D_3 = 0$, and $D_4 = 0.5$. The stability condition $\epsilon\delta < \kappa_1\kappa_2$ is satisfied. The negative work rate and EP rate regions are shaded in grey. Inset in panel (c) highlights the regions of the entropic violation ($\langle \dot{S} \rangle_{T^e} < 0$).

knowledge of the nonconservative force $\vec{f}_{nc}$, we decompose the force exerted on the system into conservative and nonconservative parts. The conservative force can be computed by $\vec{f}_{con} = -\nabla\mathcal{U}(\vec{X})$ with the 4D potential:

$$\begin{aligned}\mathcal{U}(\vec{X}) &= \frac{1}{2}\vec{X}^T\mathcal{A}_{sym}\vec{X} \\ &= \frac{1}{2}[\kappa_1x_1^2 + \kappa_2x_2^2 - (\epsilon + \delta)x_1x_2 - x_1\zeta_1 \\ &\quad - x_2\zeta_2 + \lambda_3\zeta_1^2 + \lambda_4\zeta_2^2],\end{aligned}\quad (15)$$

constructed from the symmetrized matrix $\mathcal{A}_{sym} \equiv \frac{1}{2}(\mathcal{A} + \mathcal{A}^T)$ and $\mathcal{A}$ is given by Eq. (3). Straightforwardly, the nonconservative force

$$\vec{f}_{nc}^T = \frac{1}{2}[(\epsilon - \delta)x_2 + \zeta_1, -(\epsilon - \delta)x_1 + \zeta_2, -x_1, -x_2] \quad (16)$$

can be used to determine the stochastic heat $dQ_i = (\mathcal{A}\vec{X})_i \circ dx_i$ and work dW_i for each DoF, which take the following explicit expressions:

$$\begin{aligned}dQ_1 &= (\kappa_1x_1 - \epsilon x_2 - \zeta_1) \circ dx_1 = dQ_1 - \zeta_1 \circ dx_1, \\ dQ_2 &= (\kappa_2x_2 - \delta x_1 - \zeta_2) \circ dx_2 = dQ_2 - \zeta_2 \circ dx_2,\end{aligned}$$

$$\begin{aligned}dQ_3 &= \lambda_3\zeta_1 \circ d\zeta_1, \quad dQ_4 = \lambda_4\zeta_2 \circ d\zeta_2, \\ dW_1 &= -\frac{1}{2}[(\epsilon - \delta)x_2 + \zeta_1] \circ dx_1 = dW_1 - \frac{1}{2}\zeta_1 \circ dx_1, \\ dW_2 &= \frac{1}{2}[(\epsilon - \delta)x_1 - \zeta_2] \circ dx_2 = dW_2 - \frac{1}{2}\zeta_2 \circ dx_2, \\ dW_3 &= \frac{1}{2}x_1 \circ d\zeta_1, \quad dW_4 = \frac{1}{2}x_2 \circ d\zeta_2.\end{aligned}\quad (17)$$

The quantities dQ_i and dW_i are the heat and work, respectively, originating from x_i - x_j interactions in the previous section. The 4D stochastic total heat and work are simply $dQ = \sum_{i=1}^4 dQ_i$ and $dW = \sum_{i=1}^4 dW_i$. The expressions of 4D energetics indicate that a full description requires the x_i - ζ_j interactions. From Eq. (17), we note two key points. First, it is obvious that the equality $dU = dQ - dW = dQ - dW = dU$ holds true, exemplifying the validity of the first law of thermodynamics for both the 2D AOUP gyrotor and the 4D white noise gyrotor. In other words, the legitimacy of the thermodynamic first law is independent of the inclusion of the DoF for the active noise. Second, the bead of interest [black filled circle in Fig. 1(d)] exchanges energies not only with the associated white noises η_1 and η_2 , but also with the active noises ζ_1 and ζ_2 .

We derive the analytical solutions (Appendixes B and C) for the steady-state heat and work rates as

$$\begin{aligned}\langle \dot{Q} \rangle &= \langle \dot{W} \rangle = [(\epsilon - \delta)\langle x_1\dot{x}_2 \rangle - \langle \zeta_1\dot{x}_1 \rangle - \langle \zeta_2\dot{x}_2 \rangle] \\ &= \frac{1}{\kappa_1 + \kappa_2} \left\{ (\epsilon - \delta)(\delta D_1 - \epsilon D_2) - \frac{\lambda_3^2 D_3 [\delta^2 + (\kappa_1 + \kappa_2)(\kappa_2 + \lambda_3) - \epsilon\delta]}{(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta} - \frac{\lambda_4^2 D_4 [\epsilon^2 + (\kappa_1 + \kappa_2)(\kappa_1 + \lambda_4) - \epsilon\delta]}{(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta} \right\}.\end{aligned}\quad (18)$$

The equality $\langle \dot{Q} \rangle - \langle \dot{W} \rangle = \langle \dot{Q} \rangle - \langle \dot{W} \rangle = 0$ again confirms that the first law of thermodynamics holds for both the 4D white noise gyrotor and the 2D AOUP gyrotor representations.

B. Second law of thermodynamics

We next discuss the second law from the entropic viewpoint. Heat currents and the corresponding temperatures are needed in order to obtain the EP of the system. If only x_i is the observed DoF of the model system, heat currents $\dot{Q}_1$ and $\dot{Q}_2$ can be determined by computing $\langle x_1\dot{x}_2 \rangle$. The naive EP is

then given by $\langle S \rangle = -\langle Q_1 \rangle/D_1 - \langle Q_2 \rangle/D_2$, with D_1 and D_2 the corresponding thermal temperatures associated with the observed DoF. Alternatively, effective temperatures, T^e , have been used to obtain EP $\langle S \rangle_{T^e}$ due to the presence of active noise. However, the second law is challenged here in that the EP, considered above, can have negative values, as shown in Figs. 3(b) and 3(c) (also see Sec. III and Appendix A). On the other hand, if the DoF of the OU noise becomes explicit, the x - ζ coupling generates extra EP [Eq. (20)]. Additionally, the ζ - ζ coupling between the hidden DoFs accounts for yet another contribution to the total EP. The stochastic EP

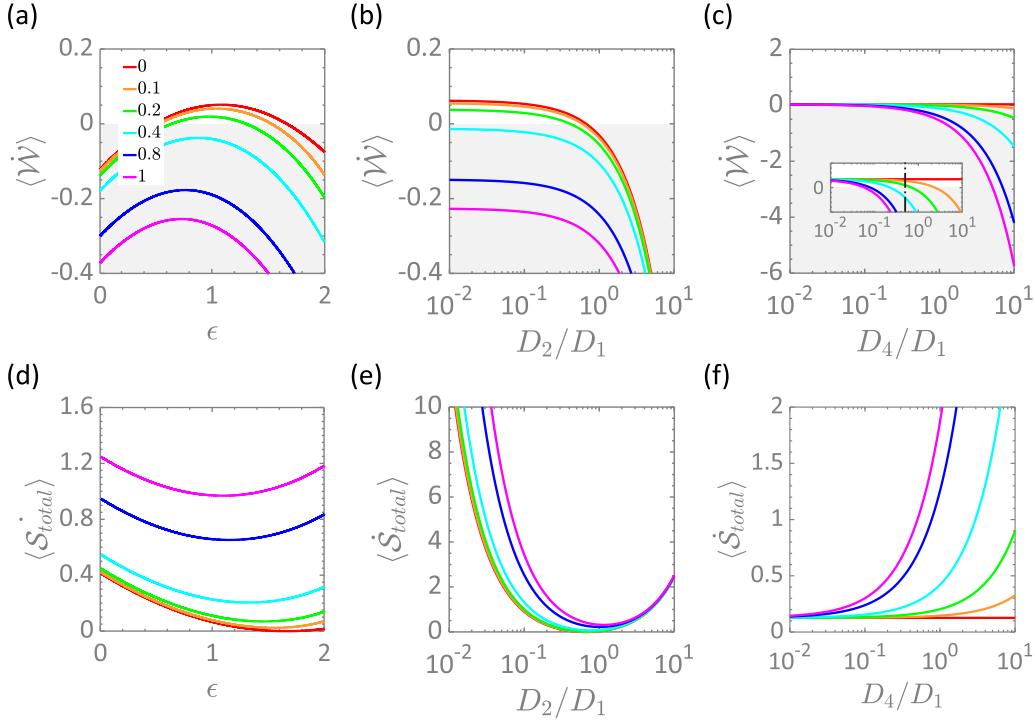


FIG. 4. Analytical results of the mean work rate $\langle \dot{W} \rangle$ and EP rate $\langle \dot{S}_{\text{total}} \rangle$ for a 4D white noise gyrator. $\langle \dot{W} \rangle$ is plotted as a function of (a) ϵ with $D_2 = 0.3$ and $D_4 = 0.5$, (b) D_2/D_1 with $\epsilon = 0.75$ and $D_4 = 0.5$, and (c) D_4/D_1 with $\epsilon = 0.75$ and $D_2 = 0.3$ at various $\lambda_4 = 0, 0.1, 0.2, 0.4, 0.8$, and 1 . The corresponding $\langle \dot{S}_{\text{total}} \rangle$ is, respectively, shown in panels (d)–(f). $\langle \dot{S}_{\text{total}} \rangle$ is non-negative for all parameter regimes examined. The stability condition $\epsilon\delta < \kappa_1\kappa_2$ is satisfied with $\kappa_1 = \kappa_2 = 1$. The shared parameters are $\delta = 0.5$, $D_1 = 1$, and $D_3 = 0$.

from each of the observed and hidden four DoF reads

$$dS_x = -\frac{dQ_1}{T_1} - \frac{dQ_2}{T_2}, \quad (19)$$

$$= dS + \frac{\zeta_1 \circ dx_1}{D_1} + \frac{\zeta_2 \circ dx_2}{D_2}, \quad (20)$$

$$dS_\zeta = -\frac{dQ_3}{T_3} - \frac{dQ_4}{T_4}, \quad (21)$$

$$= -\frac{\lambda_3 \zeta_1 \circ d\zeta_1}{D_3} - \frac{\lambda_4 \zeta_2 \circ d\zeta_2}{D_4}. \quad (22)$$

The EP linked to observed DoFs is denoted as dS_x while that from active noises is dS_ζ . The total stochastic EP from the surrounding environment (passive baths composed of white noise) is simply then $dS_{\text{total}} = dS_x + dS_\zeta$. In steady state, the EP rate $\langle \dot{S}_\zeta \rangle$ from ζ - ζ coupling further vanishes because $\langle \zeta_i \dot{\zeta}_i \rangle = 0$. Hence, the steady-state EP rate $\langle \dot{S}_{\text{total}} \rangle = \langle \dot{S}_x \rangle$ and can be analytically expressed by

$$\begin{aligned} \langle \dot{S}_{\text{total}} \rangle &= \langle \dot{S} \rangle + \frac{\langle \zeta_1 \dot{x}_1 \rangle}{D_1} + \frac{\langle \zeta_2 \dot{x}_2 \rangle}{D_2} \\ &= \frac{1}{\kappa_1 + \kappa_2} \left\{ \frac{(\delta D_1 - \epsilon D_2)^2}{D_1 D_2} + \frac{\lambda_4^2 D_4}{D_2} + \frac{\lambda_3^2 D_3}{D_1} \right. \end{aligned} \quad (23)$$

$$\begin{aligned} &+ \frac{\delta^2 \lambda_3^2 D_3}{D_1[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]} \\ &\left. + \frac{\epsilon^2 \lambda_4^2 D_4}{D_2[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]} \right\} \geq 0. \end{aligned} \quad (24)$$

All terms in Eq. (24) are conveniently found to be non-negative, suggesting the second law is obeyed in terms of the entropy (Fig. 4). We define the last two terms in Eq. (24) that occur due to the presence of active noise as $\langle \dot{S}_{\text{active}} \rangle \equiv \langle \zeta_1 \dot{x}_1 \rangle / D_1 + \langle \zeta_2 \dot{x}_2 \rangle / D_2$. In short, the paradoxical violation of the classical second law for active baths in terms of entropy arises from the lack of information about the coupling of active noises to the observed DoF. In other words, the second law of thermodynamics is fully respected if the information on the hidden degrees of freedom is properly taken into account.

It is noteworthy that in systems driven by active noise, entropy production can also be rigorously defined through the log ratio of the forward and time-reversed path probabilities under time reversal [40,41]. In this framework, the EP corresponds to a breaking of the time-reversal symmetry of particle trajectories. In our 4D Markovian embedding, the EP derived from our approach is mathematically consistent with the EP obtained from the path-probability ratio of 4D state-space trajectories.

C. Efficiency of heat engines

We begin the discussion of efficiency with a brief digression to revisit the efficiency determined solely by the observed DoF. Analysis of the 2D AOUP gyrator reveals regimes of positive work $\langle \dot{W} \rangle > 0$ while $\langle \dot{Q} \rangle > 0$ and warrants that the system can function as a heat engine. The energy conversion with pseudoefficiency is not bounded by the Carnot efficiency, as previously reported [37].

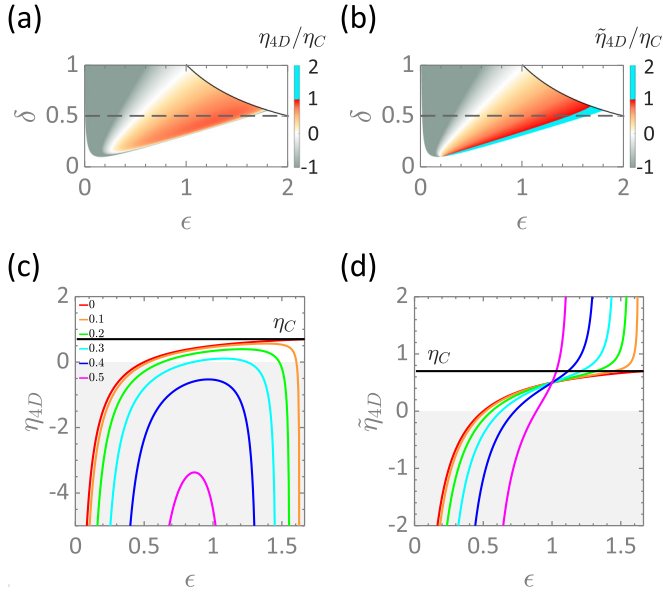


FIG. 5. The phase diagram of the relative efficiency (a) η_{4D}/η_C and (b) $\tilde{\eta}_{4D}/\eta_C$ in the $\epsilon - \delta$ plane for $\lambda_3 = 0.1$. These relative efficiencies are colored linearly in the range from -1 to 1 . Saturated gray is assigned to ratios below -1 , and cyan to those above 1 . The horizontal dashed line in panels (a) and (b) indicates the orange curve ($\delta = 0.5$) in panels (c) and (d). Analytical efficiency (c) η_{4D} and (d) $\tilde{\eta}_{4D}$ for a 4D white noise gyration are plotted against ϵ with $\kappa_1 = \kappa_2 = 1$, $\delta = 0.5$, $\lambda_4 = 0$, $D_1 = 1$, $D_2 = 0.3$, $D_3 = 0.5$, and $D_4 = 0$ at various $\lambda_3 = 0, 0.1, 0.2, 0.3, 0.4$, and 0.5 .

One can treat the system as a heat engine operating between two temperatures, with bath 1 being the hotter bath ($D_1 > D_2$). For passive baths, the efficiency is bound by the Carnot efficiency $\eta_C = 1 - D_2/D_1$. For 2D AOUP gyration analysis, the conventional definition of efficiency, the ratio of the extracted work to the injected heat, $\eta_{2D} = \langle \dot{W} \rangle / \langle \dot{Q}_1 \rangle$, still leads to the confusing result of $\eta > \eta_C$ in some ϵ regimes. In the 4D white noise perspective, we can use two work definitions in the determination of the efficiency: the total 4D work $\mathcal{W} = \sum_i^4 \mathcal{W}_i$ [Eq. (18)] or only the work extracted from observables x_1 and x_2 , $\mathcal{W}_1 + \mathcal{W}_2$ [Eq. (17)], as follows:

$$\eta_{4D} = \frac{\langle \dot{W} \rangle}{\langle \dot{Q}_1 \rangle} = 1 - \frac{\delta \langle x_1 \dot{x}_2 \rangle + \langle \zeta_2 \dot{x}_2 \rangle}{\epsilon \langle x_1 \dot{x}_2 \rangle - \langle \zeta_1 \dot{x}_1 \rangle}, \quad (25)$$

$$\tilde{\eta}_{4D} = \frac{\langle \dot{W}_1 \rangle + \langle \dot{W}_2 \rangle}{\langle \dot{Q}_1 \rangle} = \eta_{4D} - \frac{1}{2} (\langle \zeta_2 \dot{x}_2 \rangle + \langle \zeta_1 \dot{x}_1 \rangle). \quad (26)$$

As shown in Fig. 5, η_{4D} at $\lambda_3 = 0.1$ is bounded by the Carnot efficiency over the engine regime ($\langle \dot{Q}_1 \rangle > 0$) [see Fig. 5(a)], while $\tilde{\eta}_{4D}$ values can go beyond the Carnot limit [Fig. 5(b)]. This indicates that using only the bead's work output overestimates the engine efficiency. We also checked the dependence of η_{4D} on λ_3 . As λ_3 decreases, η_{4D} increases and approaches the Carnot efficiency at $\lambda_3 = 0$ and the given conditions in the caption in Fig. 5. On the contrary, $\tilde{\eta}_{4D}$ can be larger than η_C except for $\lambda_3 = 0$.

We note that our definition of efficiency, η_{4D} , satisfies the second law of thermodynamics. In addition, there is another

definition of efficiency deduced from the second-law-abiding entropic inequality, which is similar to that of Datta *et al.* [37]. This reformulated efficiency depends on the work and heat determined by the observed DoF. We notice that with the steady-state EP rate $\langle \dot{S} \rangle = -\langle \dot{Q}_1 \rangle / D_1 - \langle \dot{Q}_2 \rangle / D_2$ and the work $\langle \dot{W} \rangle = \langle \dot{Q}_1 \rangle + \langle \dot{Q}_2 \rangle$ from the 2D AOUP gyration analysis, $\langle \dot{S}_{\text{total}} \rangle$ can be rewritten as

$$\langle \dot{S}_{\text{total}} \rangle = \frac{\eta_C}{D_2} \langle \dot{Q}_1 \rangle - \frac{\langle \dot{W} \rangle}{D_2} + \langle \dot{S}_{\text{active}} \rangle \geq 0. \quad (27)$$

The second law in Eq. (27) motivated us to define the efficiency as

$$\begin{aligned} \eta &\equiv \frac{\langle \dot{W} \rangle}{\langle \dot{Q}_1 \rangle + \frac{D_1 D_2}{D_1 - D_2} \langle \dot{S}_{\text{active}} \rangle} \\ &= \eta_C \left(\frac{\langle \dot{W} \rangle}{\eta_C \langle \dot{Q}_1 \rangle + D_2 \langle \dot{S}_{\text{active}} \rangle} \right) \leq \eta_C. \end{aligned} \quad (28)$$

We note that our definition of the efficiency in Eq. (28) aligns with the one defined by Datta *et al.* [Eq. (29)] in Ref. [37] in which an information-theoretic contribution to the second law for an active cyclic heat engine, $\mathcal{I}$, was introduced. Moreover, Eq. (28) provides an explicit physical account of $\mathcal{I}$ associated with the explicit degrees of freedom in the active bath with $\mathcal{I} = \langle S_{\text{active}} \rangle$. The physical implication in the definition of efficiency in Eq. (28) is that the active entropy production should be taken into account as an additional heat source to drive the engine in addition to the heat rate $\langle \dot{Q}_1 \rangle$. The valid entropic inequality of the second law in Eq. (27) leads to a modified definition of efficiency that is upper-bounded by the classical Carnot efficiency: $\eta \leq \eta_C$, and no paradoxical violation will arise. This redefined formulation of efficiency could also apply to a heat engine subject to a fuel-containing heat source [42].

D. Thermodynamic uncertainty relation

In recent years, additional focus has shifted toward the TUR, which quantifies a trade-off between the precision of steady-state currents and the thermodynamic cost in terms of energy dissipation. The TUR addressed here, $[\mathcal{D}(\Delta t) / \langle \dot{X}(\Delta t) \rangle^2] \langle \dot{S}(\Delta t) \rangle \geq k_B$, is a standard steady-state version valid in overdamped autonomous Markovian systems. Unlike the second law, this TUR is not universally applicable to all thermodynamic processes. Specifically, the TUR can be violated in underdamped systems or in nonautonomous (cyclic) engines, even though the second law remains valid [43–45].

Here, we analytically examine whether TUR is still valid even in the presence of active noise. We use the diffusion coefficient $D_X = \lim_{\Delta t \rightarrow \infty} (\langle X(\Delta t)^2 \rangle - \langle X(\Delta t) \rangle^2) / 2\Delta t$ to characterize the fluctuation of a thermodynamic quantity X . The diffusion coefficient of the 4D total work can be calculated using the method of the cumulant generating function. See Eq. (C22) for the long general formula and details of its derivation. For the case of $D_3 = 0$ (in the absence of active

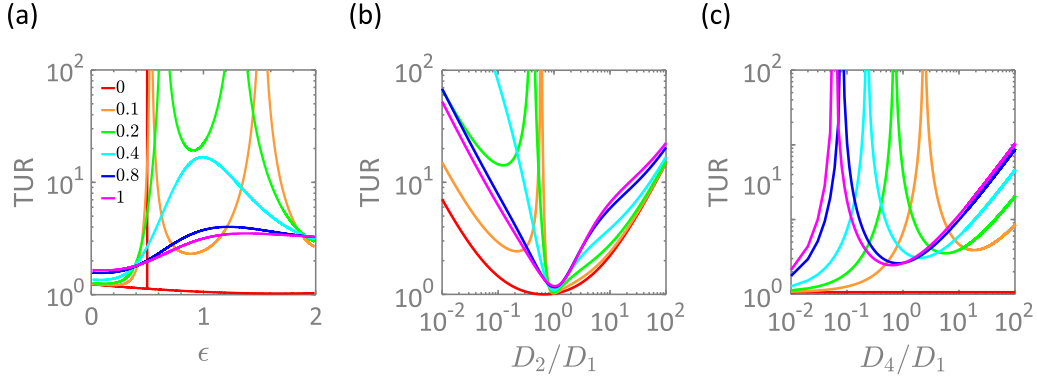


FIG. 6. Analytical TUR for a 4D white noise gyrotor as a function of (a) ϵ with $D_2 = 0.3$ and $D_4 = 0.5$, (b) D_2/D_1 with $\epsilon = 0.75$ and $D_4 = 0.5$, and (c) D_4/D_1 with $\epsilon = 0.75$ and $D_2 = 0.3$ for various values of λ_4 . $\kappa_1 = \kappa_2 = 1$, $\delta = 0.1$, $D_1 = 1$, and $D_3 = 0$ are the shared parameters. Note that TUR diverges when work rate approaches zero [see Figs. 4(a)–4(c)]. Nonmonotonic behavior of TUR originates from the diffusion of work rate $\mathcal{D}_{\mathcal{W}}$.

noise along x_1), $\mathcal{D}_{\mathcal{W}}$ reads

$$\mathcal{D}_{\mathcal{W}} = \frac{1}{(\kappa_1 + \kappa_2)^3} \left\{ (\epsilon - \delta)^2 (\delta^2 D_1^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon\delta] D_1 D_2 + \epsilon^2 D_2^2) \right. \\ \left. + \frac{\Phi_{14} \lambda_4^2 D_1 D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^2} + \frac{\Phi_{24} \lambda_4^2 D_2 D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^2} + \frac{\Phi_{44} \lambda_4^4 D_4^2}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^3} \right\}, \quad (29)$$

where Φ_{14} , Φ_{24} , and Φ_{44} are given by Eqs. (C25), (C26), and (C28). Using Eqs. (18), (24), and (29), straightforward calculations reveal that the TUR holds for the 4D stochastic white noise system, as follows:

$$\text{TUR} = \frac{\mathcal{D}_{\mathcal{W}} \langle \dot{S}_{\text{total}} \rangle}{\langle \dot{W} \rangle^2} \geq 1. \quad (30)$$

Figure 6 illustrates how TUR depends on ϵ and the ratios of D_2/D_1 and D_4/D_1 . TUR remains larger than unity for all conditions examined. In addition, TUR diverges at the parameter boundary that separates positive and negative work rates (see curve with $\lambda_4 < 0.4$).

V. EXPERIMENTAL VERIFICATION

A. Experimental model and setup

As an illustrative model to experimentally test the above theoretical relations, we realize a Brownian gyrotor in a temperature bath in the presence of active noises. In real experiments, the thermal noise cannot be avoided. To build the gyrotor system, two-laser-based optical feedback traps are implemented to exert the necessary forces on the bead [46–48]. Experimental details on the demonstration of the gyrotor can be found in Ref. [19]. Briefly, a water-immersed 1 μm polystyrene bead is trapped in a symmetric two-dimensional harmonic potential with stiffness $\kappa_1 = \kappa_2 = 30 \text{ pN}/\mu\text{m}$ using one laser, which is also used for the bead position detection. Nonconservative and stochastic forces are applied to the bead by rapid shift of the second laser (feedback laser) at intervals of 10 μm . The temperature of the surrounding medium is set at $T_r = 300 \text{ K}$ by a temperature controller. The high-temperature

bath is constituted only by a passive noise $\eta_1(t)$ applied along the x_1 axis and set to $T_1 = 800 \text{ K}$. The low-temperature bath along x_2 axis remains a hybrid one that includes a thermal noise $\eta_2(t)$ ($T_2 = 300 \text{ K}$) and an OU noise $\zeta_2(t)$. We applied the external OU noise to the Brownian bead and thus overall the effective temperature is $T_2^e \equiv \kappa_2 \langle x^2 \rangle / k_B = T_r + T_4^e = 400 \text{ K}$ along the x_2 axis, where the active effective temperature $T_4^e \equiv T_4 / [1 + \kappa_2 / (\lambda_4 \gamma)] = 100 \text{ K}$ stems from the variance of the bead caused by active noises. As the surrounding aqueous environment already contributes 300 K to each axis, the feedback laser is responsible for generating additional passive noise equivalent to 500 K along x_1 axis to fulfill T_1 , and OU noise with $T_4^e = 100 \text{ K}$ along x_2 axis, which corresponds to a noise strength of $T_4 = 480 \text{ K}$, generated according to Eq. (2). In this experiment, we used two persistence times, $1/\lambda_4 = \tau_2 = 1$ and 10 ms. The active noise in length unit is recovered using the relation $\gamma(dx/dt) = \kappa_t x_t$, where γ is the friction coefficient and independent of temperature in our experiments, dt is the updating time 10 μm , κ_t is the stiffness of the feedback laser, and x_t is the measured displacement of the feedback laser within the updating time. Note that we record both the bead position and the strength of the active noise, which cannot be obtained in a real bacterial bath.

For simplicity, we fix the coupling constant δ at 10 $\text{pN}/\mu\text{m}$ and vary ϵ between 10 and 26 $\text{pN}/\mu\text{m}$ to cover the alleged working range of the 2D AOUP engine. Influenced by the stochastic forces, the confining harmonic potential, and nonconservative forces, the fluctuating bead undergoes steady-state gyration inside the potential well.

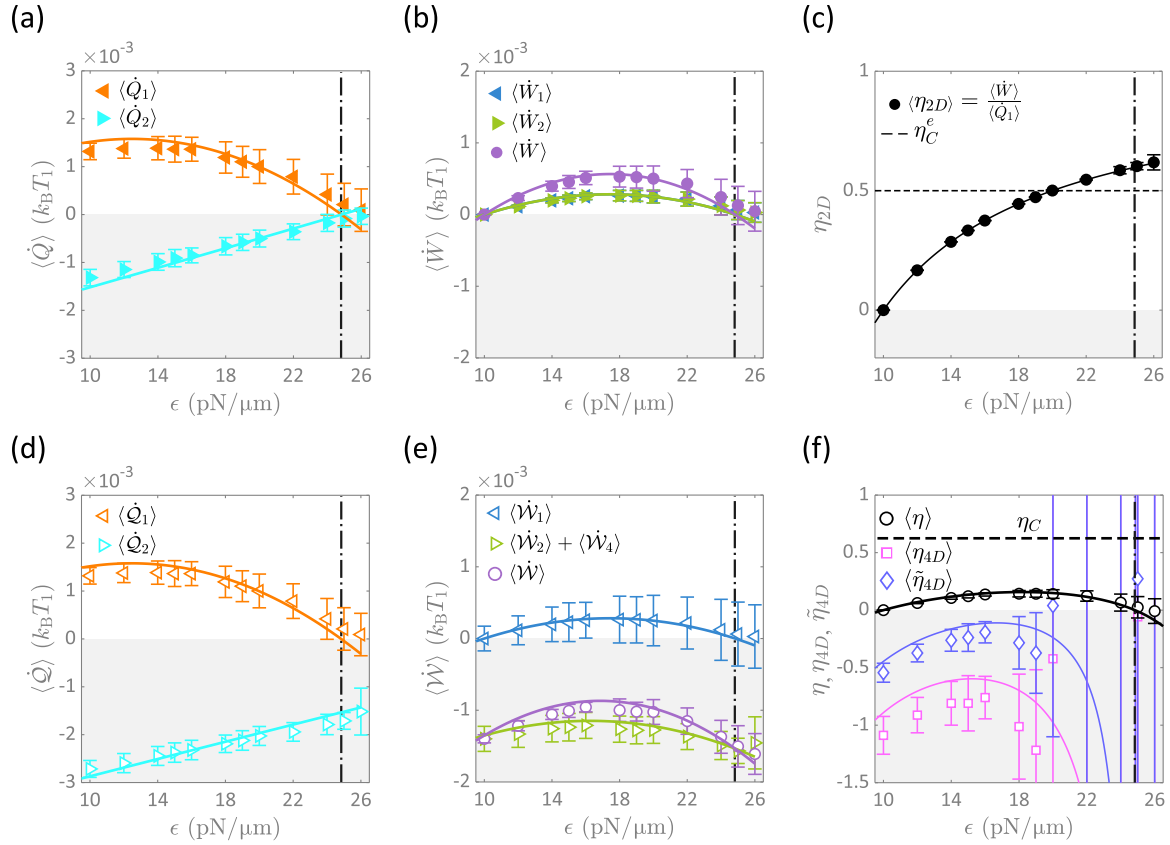


FIG. 7. Experimentally measured energetics of a gyator in the hybrid bath with $\tau_2 = 1$ ms. (a)–(c) Energetics from 2D AOUP gyator analyses. The efficiency η_{2D} apparently violates the second law by surpassing the effective Carnot limit $\eta_C^e = 1 - T_c^e/T_h^e$ based on effective temperature for $\epsilon > 20$ pN/ μ m. (d)–(f) Energetics from 4D white noise gyator analyses. While η_{4D} and $\tilde{\eta}_{4D}$ are negative over the range of $\langle \dot{Q}_1 \rangle > 0$, the modified efficiency, η [Eq. (28)], is positive. Here, ensemble-averaged values (symbol) and standard deviations, calculated by taking error propagation into account, are obtained from 10^4 bootstrapped samples. Analytic results are displayed as solid curves. The vertical dot-dashed line at $\epsilon \approx 24.6$ pN/ μ m in each figure indicates the maximal ϵ that the gyator can operate as an engine from the 2D AOUP perspective. All energies are in units of $k_B T_1$.

B. Thermodynamic currents and efficiency

Energetics are calculated from every single trajectory of a Brownian bead based on the framework of stochastic thermodynamics [8,49] and compared with our theoretical derivations (see Secs. III and IV and Appendixes B and C). Each trajectory is at least 10 s long to ensure robust statistics. The energetics from 10 experimental trajectories are used to generate 10^4 bootstrapped samples and calculate ensemble means and standard deviations via error propagation.

We first compare the energetics of the system obtained from the 2D AOUP gyator analysis with those from 4D white noise analysis for $1/\lambda_4 = \tau_2 = 1$ ms. The explicit forms of $\langle \dot{Q}_i \rangle$ and $\langle \dot{W}_i \rangle$ can be conveniently obtained from Eqs. (7) and (B97). As shown in Fig. 7(a), the heat influx current $\langle \dot{Q}_1 \rangle$ is positive for $\epsilon < \epsilon_m$ with the maximum value $\epsilon_m \approx 24.6$ pN/ μ m for the gyator working as an engine from the 2D AOUP perspective. On the other hand, the dissipative heat current $\langle \dot{Q}_2 \rangle$ increases linearly with ϵ but is negative. In the figure, the vertical dot-dashed line indicates ϵ_m . The heat difference between influx and efflux is used by the gyator to output work [Fig. 7(b)]. Non-negative work indicates that the gyator works as a heat engine in the purported working ϵ regime as claimed previously. The overall work current $\langle \dot{W} \rangle$

agrees with the 2D AOUP analysis to be evenly $(\langle \dot{W}_1 \rangle = \langle \dot{W}_2 \rangle)$ output to both axes [Fig. 7(b)]. The efficiency $\eta_{2D} \equiv \langle \dot{W} \rangle / \langle \dot{Q}_1 \rangle$ is allowed to surpass the corresponding Carnot bound calculated using the effective temperature, $\eta_C^e = 1 - T_c^e/T_h^e$, for $\epsilon > 20$ pN/ μ m [Fig. 7(c)], indicating the paradoxical violation of the second law.

On the other hand, when the OU noise is explicitly considered, the 4D white noise analysis gives us $\langle \dot{Q}_1 \rangle = \langle \dot{Q}_1 \rangle$ and a larger heat dissipative current $\langle \dot{Q}_2 \rangle$ (more negative) due to the ζ_2 - x_2 interaction compared to the 2D dissipative heat $\langle \dot{Q}_2 \rangle$ [Fig. 7(d)]. Consequently, less work can be extracted, as suggested by the first law of thermodynamics [Fig. 7(e)]. The absence of ζ_1 ensures that $\langle \dot{W}_1 \rangle = \langle \dot{W}_1 \rangle$, while the presence of ζ_2 greatly reduces $\langle \dot{W}_2 \rangle$ by an amount of $\langle \zeta_2 \dot{x}_2 \rangle / 2$, causing $\langle \dot{W}_2 \rangle = \langle \dot{W}_2 \rangle + \langle \dot{W}_4 \rangle$ to become negative [Figs. 7(b) and 7(e)]. The corresponding efficiencies of the gyator are presented in Fig. 7(f). For our given experimental condition, $\langle \eta_{4D} \rangle$ and $\langle \tilde{\eta}_{4D} \rangle$ are less than η_C , but negative, indicating that they do not work as engines, although the second law still holds. On the contrary, η obtained from the entropy consideration in Eq. (28) is non-negative in the engine regime. It is important to note that the parameter regimes where $\eta_{4D} < 0$ in Figs. 5 and 7 signify that the device is operating as a heat

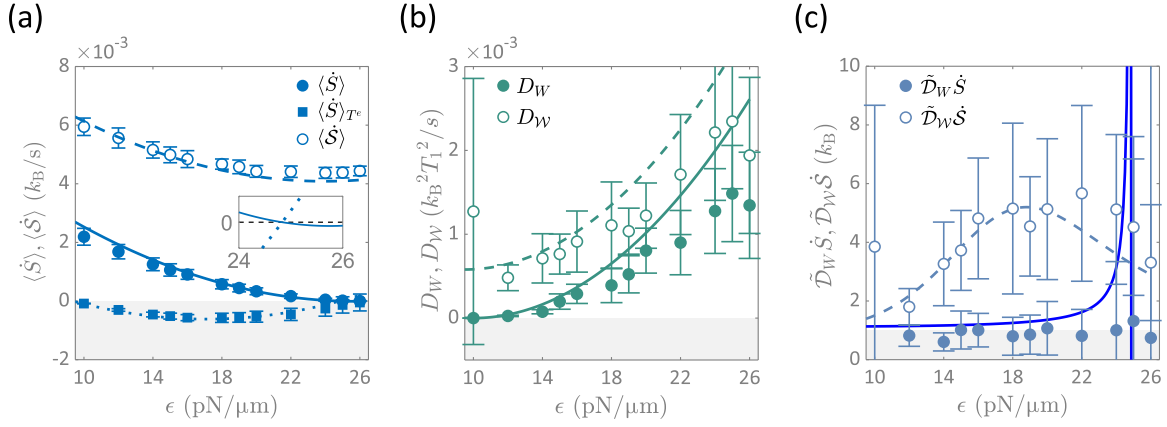


FIG. 8. TUR of the gyator in the hybrid bath with $\tau_2 = 1$ ms. (a) The EP rates and corresponding analytics. Two EP rates, $\langle \dot{S} \rangle$ and $\langle \dot{S} \rangle_{T^e}$, are obtained from the 2D AOUP analysis. The inset plot shows the negative EP at a certain range of ϵ . The 4D EP rates, $\langle \dot{S} \rangle$, are displayed as open circles. (b) Work diffusion constants D_W and $D_{\mathcal{W}}$ computed from 2D and 4D analysis, respectively. (c) The TUR for W and $\mathcal{W}$. The theoretical TUR for W is colored in blue to highlight its violation ($< k_B$) at $\epsilon > \epsilon_m$. Here, the shaded area indicates the area forbidden by the conventional second law and TUR. In all figures, error bars were obtained from 10^4 bootstrapped samples.

pump rather than a heat engine. In these regions, the mean work rate $\langle \dot{W} \rangle$ is negative, indicating that the active noise coupling performs work on the system to drive heat transfer against a thermal gradient.

Here, we demonstrate the crucial role of the coupling energy $\langle \zeta_2 \dot{x}_2 \rangle$ in the accurate interpretation of the energetics. Specifically, we show that the OU noise ζ_2 interacts with x_2 by infusing energy in the form of work into the bead, but the same amount of energy is discarded by the bead as dissipative heat to the bath. The energetics computed from experimental trajectories are in line with the analytics predicted by Eq. (17).

C. Entropy production and thermodynamic uncertainty relation

Figure 8(a) shows that the experimental EP rate $\langle \dot{S} \rangle$ for the 2D AOUP gyator, calculated based on $\langle \dot{Q}_1 \rangle$ and $\langle \dot{Q}_2 \rangle$, appears to be positive within the explored ϵ range. However, the analytical $\langle \dot{S} \rangle$ bears a negative value at $\epsilon > 24.6$ pN/ μ m [inset in Fig. 8(a)]. We confirmed that the second-law-violating negative entropy production is not rescued by replacing D_2 with an effective one, $D_2^e = D_2 + D_4^e = D_2 + D_4/(1 + \kappa_2 \tau_2)$, as shown by $\langle \dot{S} \rangle_{T^e}$ [Eq. (14)] in Fig. 8(a), highlighting the limitation of analyzing the 2D AOUP gyator merely with the observables. Conversely, the 4D EP rate $\langle \dot{S} \rangle$ includes the contribution from the coupling energy $\langle \zeta_2 \dot{x}_2 \rangle$ leading to an additional thermodynamic cost $\langle S_{\text{active}} \rangle$ for the system, making $\langle \dot{S} \rangle > \langle \dot{S} \rangle_{T^e}$ and rescues the second law, $\langle \dot{S} \rangle \geq 0$ [Fig. 8(a)].

Up to this point, our experimental results align with the 4D white noise analysis, providing a second-law-abiding description, particularly for the efficiency and EP rate, of a Brownian gyator in a hybrid bath containing active noise. It is then straightforward to further examine the validity of TUR. We first show that the computed ensemble-averaged work diffusion coefficients, $D_W = \text{Var}[W]/2\Delta t$ for the 2D AOUP gyator and $D_{\mathcal{W}} = \text{Var}[\mathcal{W}]/2\Delta t$ for the 4D gyator, are not far from each other, suggesting that this quantity is not so sensitive to the presence of the active noise [Fig. 8(b)].

Both work diffusion coefficients, D_W and $D_{\mathcal{W}}$, agree well with the analytical formula in Eqs. (B95) and (29), showing a monotonic increase with ϵ . With this, we proceed to examine the validity of $\text{TUR} \equiv \tilde{D}_W \dot{S} > k_B$, with $\tilde{D}_W = D_W / \langle \dot{W} \rangle^2$. As anticipated, $\tilde{D}_W \dot{S}$ becomes negative for $\epsilon > \epsilon_m$ since $\dot{S} < 0$. The 4D TUR, $\tilde{D}_{\mathcal{W}} \dot{S}$ (where $\tilde{D}_{\mathcal{W}} \equiv D_{\mathcal{W}} / \langle \dot{\mathcal{W}} \rangle^2$), is still above k_B despite a nonmonotonic relation with ϵ [Fig. 8(c)], indicating that TUR is also validated for the 4D white noise interpretation.

D. An AOUP gyator with a longer persistence time

We showed that for $\tau_2 = 1$ ms, the Brownian gyator is unable to extract work from the environment as shown in Fig. 7(e), meaning that it cannot operate as an engine. In fact, Eq. (18) manifests that decreasing D_4 or λ_4 ($1/\tau_2$) helps $\langle \zeta_2 \dot{x}_2 \rangle$ regain work output [Fig. 5(c)]. In our experimental configuration setting $\tau_1 = 0$, the switch of the work rate from negative to positive occurs at $\tau_2 \approx 1.8$ ms. To provide a clearer visual illustration, we conducted experiments with a 10-fold larger $\tau_2 = 10$ ms and an unchanged D_4 . The overall work current $\langle \dot{W} \rangle$ increases markedly above zero, indicating the gyator can operate as a heat engine [Fig. 9(a)]. The injected heat $\langle \dot{Q}_1 \rangle$ also increases and asymptotically approaches the limited value [Fig. 9(b)].

Our 4D theoretical framework demonstrates that the efficiency η_{4D} is highly sensitive to the persistence time (τ) of the active fluctuations. Specifically, as shown in Fig. 9(c), the efficiency markedly increases when transitioning from $\tau = 1$ ms to $\tau = 10$ ms. The engine efficiency increases with the coupling force ϵ up to approximately 20 pN/ μ m and then sharply declines [Fig. 9(c)]. The maximum efficiency asymptotically approaches the Carnot limit for $\tau_2 \rightarrow \infty$ (dotted lines), as shown in Eq. (28). The discrepancy between the peak efficiency observed in our experiments (≈ 0.35 at $\tau_2 = 10$ ms) and the calculated effective Carnot limit ($\eta_C \approx 0.63$) does not represent a fundamental limitation of the linear gyator model itself, but rather reflects the constraints of the

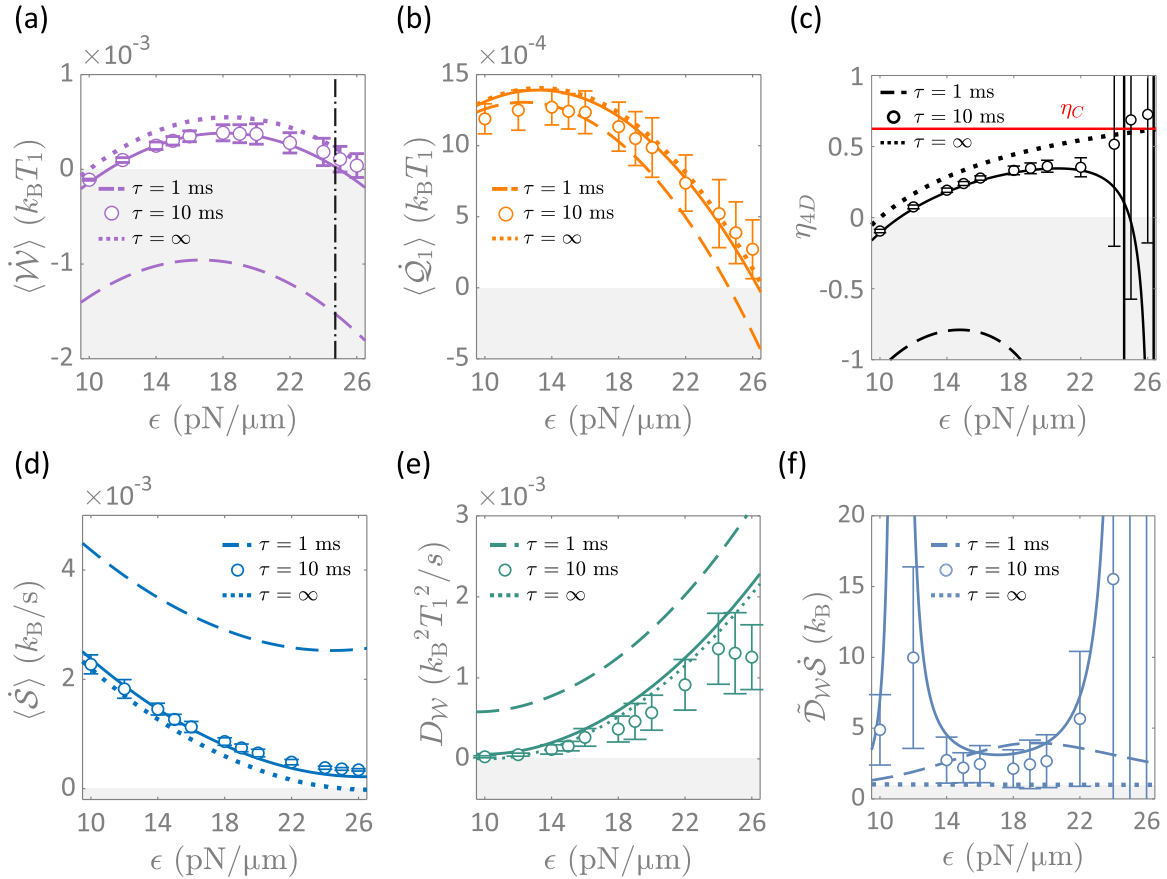


FIG. 9. The energetics of the 4D white noise gyration with $\tau_2 = 10$ ms. (a) Total work rate $\langle \dot{W} \rangle$. (b) Heat rate $\langle \dot{Q}_1 \rangle$. (c) The efficiency $\langle \eta_{4D} \rangle$. (d) The EP rates $\langle \dot{S} \rangle$. (e) The work diffusion constant D_W . (f) The TUR. The light gray shade denotes the forbidden regime constrained by the TUR. In each figure, theoretical curves for $\tau = 1$ ms (dashed line), 10 ms (solid line), and $\tau \rightarrow \infty$ (dotted line) are shown as a visual guide to their τ -dependent evolution. Error bars are calculated from 10^4 bootstrapped samples of 17 experimental trajectories.

specific parameters and finite persistence times employed in our experimental setup. Analytically, the system recovers the insurmountable status of the Carnot bound only in the limit of infinite persistence time ($\tau \rightarrow \infty$), where the active noise is effectively converted into a stable thermodynamic drive.

The EP rate $\langle \dot{S} \rangle$ monotonically decreases with ϵ and remains well above zero [Fig. 9(d)], concurrent with a monotonic increase in the work fluctuations, quantified by the work diffusion D_W [Fig. 9(e)]. Our experimental TUR, calculated from the EP rate and work fluctuation, agrees with the theory, showing values well above unity [Fig. 9(f)]. We confirm that the gyration can work as a heat engine in the presence of active noises.

VI. CONCLUSION AND DISCUSSION

The temperature bath involving active noises can seemingly drive stochastic heat engines to surpass the efficiency upper bound imposed by the second law of thermodynamics [30]. This apparent violation has been a subject of significant interest and debate. Our study has systematically addressed this issue by employing a Brownian gyration as an exactly solvable model system. We conducted experimental investigations and theoretical modeling using the Ornstein-Uhlenbeck active noise framework.

Our central finding is that by explicitly accounting for the degrees of freedom of the active noises, the classical framework of thermodynamics remains fully valid. In contrast to approaches that rely on effective temperatures, our 4D white noise gyration model integrates the noise dynamics directly, which reveals that the energetics of the system can be decoupled into contributions of a bead in the heat bath and of an active particle subject to its own heat baths [Fig. 1(d)]. Our analytical and experimental results consistently show that the total entropy production is always non-negative, efficiency is upper-bounded by η_C when it works as a heat engine, and the conventional thermodynamic uncertainty relation holds, reaffirming that the classical second law is not violated.

The framework discussed here could encourage reevaluating previous experimental and theoretical results that reported efficiencies exceeding the Carnot limit. This would help to determine whether explicitly taking active degrees of freedom into account can align those findings with classical thermodynamics. Our study highlights the limitations of relying solely on effective temperature for analyzing the energetics and entropy production of heat engines. Therefore, future research should focus on models that explicitly incorporate the dynamics of active components.

The conclusions drawn from our 4D white noise analysis are expected to hold beyond the linear, Gaussian

framework of the current model. While the harmonic potential allows for an exactly solvable treatment, this approach can be applied to more complex active matter systems beyond the Ornstein-Uhlenbeck noise model, which includes noises involving various microswimmers or dense biological suspensions [27,50,51]. The restoration of the second law and the Carnot limit depends on the physical resolution of the energy pathways rather than the specific form of the potential. In non-linear systems or those driven by non-Gaussian active baths (such as bacterial suspensions), the 4D Markovian embedding still provides a rigorous basis for calculating the total entropy production. As long as the work influx from the active fluctuations is explicitly accounted for as a thermodynamic cost, the underlying consistency with the second law remains intact, even in strong-coupling regimes where the power-efficiency trade-off becomes more complex. By thoroughly understanding the energy pathways involved, we can develop design principles for microscopic machines that function in active environments by optimizing performance within the boundaries of the second law of thermodynamics rather than attempting to violate it. Ultimately, this research encourages a deeper understanding of energy and entropy in the far-from-equilibrium landscape of active matter.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [52].

APPENDIX A: LANGEVIN EQUATIONS OF AN AOU GYRATOR

Let us consider at time t , the position $\tilde{x}_i(t)$ ($i = 1, 2$) of the gyrator of a Brownian bead is governed by force fluctuations. We shall use $\dot{\cdot}$ to denote the time derivative. Assuming the bead is subjected to both white and OU noises that act independently on $\tilde{x}_1$ and $\tilde{x}_2$, the equations of motion can be put into the form

$$\dot{\tilde{x}}_1 = -\frac{\tilde{\kappa}_1}{\gamma_1}\tilde{x}_1 + \frac{\tilde{\epsilon}}{\gamma_1}\tilde{x}_2 + \nu_0\tilde{\zeta}_1(t) + \tilde{\eta}_1(t), \quad (\text{A1})$$

$$\dot{\tilde{x}}_2 = -\frac{\tilde{\kappa}_2}{\gamma_2}\tilde{x}_2 + \frac{\tilde{\delta}}{\gamma_2}\tilde{x}_1 + \nu_0\tilde{\zeta}_2(t) + \tilde{\eta}_2(t), \quad (\text{A2})$$

where γ_i are the drag coefficients, $\tilde{\kappa}_i$ are the stiffnesses of the harmonic potential, and $\tilde{\epsilon}$ and $\tilde{\delta}$ are the coupling force constants between $\tilde{x}_1$ and $\tilde{x}_2$. Here, $\tilde{\eta}_i(t)$ and $\nu_0\tilde{\zeta}_i(t)$ are the zero-mean Gaussian white velocity fluctuations and the temporally correlated OU noise whose respective variances and

correlations are given by

$$\overline{\tilde{\eta}_i(t)} = 0, \quad \overline{\tilde{\eta}_i(t)\tilde{\eta}_j(t')} = 2\tilde{D}_i\delta_{ij}\delta(t-t'), \quad (\text{A3})$$

$$\tilde{D}_i \equiv \frac{k_B T_i}{\gamma_i}, \quad i = 1, 2, \quad (\text{A4})$$

and

$$\overline{\tilde{\zeta}_1(t)} = \overline{\tilde{\zeta}_2(t)} = 0, \quad (\text{A5})$$

$$\overline{\tilde{\zeta}_1(t)\tilde{\zeta}_2(t')} = 0, \quad (\text{A6})$$

$$\nu_0^2 \overline{\tilde{\zeta}_1(t)\tilde{\zeta}_1(t')} = \frac{\tilde{D}_3}{\tilde{\tau}_1} e^{-\frac{|t-t'|}{\tilde{\tau}_1}}, \quad (\text{A7})$$

$$\nu_0^2 \overline{\tilde{\zeta}_2(t)\tilde{\zeta}_2(t')} = \frac{\tilde{D}_4}{\tilde{\tau}_2} e^{-\frac{|t-t'|}{\tilde{\tau}_2}}, \quad (\text{A8})$$

where ν_0^{-1} is some fixed time unit introduced for convenience such that $\tilde{\zeta}_i$ also has the unit of length as $\tilde{x}_i$. In order to simplify the analysis of Eq. (A2), it is convenient to introduce dimensionless variables defined as $x_i = \tilde{x}_i/\tilde{x}_0$, where $\tilde{x}_0 = \sqrt{\tilde{D}_1/\nu_0}$, $\kappa_i = (\tilde{\kappa}_i/\gamma_i)/\nu_0$, $\delta = (\tilde{\delta}/\gamma_i)/\nu_0$, $\epsilon = (\tilde{\epsilon}/\gamma_i)/\nu_0$, $D_i = \tilde{D}_i/\tilde{D}_{\text{RT}}$, and $\tau_i = \tilde{\tau}_i\nu_0$. Here, $\tilde{D}_{\text{RT}}$ is the diffusion constant of the Brownian bead at room temperature. Then, Eq. (A2) can be rewritten as

$$\dot{x}_1 = -\kappa_1 x_1 + \epsilon x_2 + \zeta_1(t) + \eta_1(t), \quad (\text{A9})$$

$$\dot{x}_2 = -\kappa_2 x_2 + \delta x_1 + \zeta_2(t) + \eta_2(t). \quad (\text{A10})$$

We shall carry out theoretical calculations and simulations with all quantities in dimensionless forms.

By introducing the 4D vector $\vec{X}^\top \equiv (x_1, x_2, \zeta_1, \zeta_2)$, the stochastic dynamics is governed by

$$\dot{\vec{X}} = -\mathcal{A}\vec{X} + \vec{\xi}(t) \quad (\text{A11})$$

and

$$\mathcal{A} = \begin{pmatrix} \kappa_1 & -\epsilon & -1 & 0 \\ -\delta & \kappa_2 & 0 & -1 \\ 0 & 0 & \frac{1}{\tau_1} & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_2} \end{pmatrix}, \quad \vec{\xi}(t) = \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \\ \frac{\psi_1(t)}{\tau_1} \\ \frac{\psi_2(t)}{\tau_2} \end{pmatrix}, \quad (\text{A12})$$

where η_i is defined as in Eq. (A3), and $\psi_i(t)$ is the Gaussian white noise that acts on ζ_i via $\tau_i\dot{\zeta}_i = -\zeta_i + \psi_i(t)$, whose mean and variance are given by

$$\overline{\psi_1(t)} = \overline{\psi_2(t)} = \overline{\psi_1(t)\psi_2(t')} = 0, \\ \overline{\psi_1(t)\psi_1(t')} = 2D_3\delta(t-t'), \quad \overline{\psi_2(t)\psi_2(t')} = 2D_4\delta(t-t'). \quad (\text{A13})$$

All D_1, D_2, D_3 , and D_4 have the same unit of diffusion coefficients. For convenience, we define $\lambda_3 \equiv 1/\tau_1$ and $\lambda_4 \equiv 1/\tau_2$.

APPENDIX B: ANALYTIC EXPRESSIONS FOR THE 4D LINEAR GYRATOR UNDER WHITE NOISES DERIVED BY THE SPECTRAL DECOMPOSITION METHOD

From the equation of motion (A12) and assuming $\mathcal{A}$ is diagonalizable, i.e., $\mathbf{P}^{-1}\mathcal{A}\mathbf{P} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_i are the eigenvalues and $\mathbf{P}$ is the matrix of eigenvectors:

$\mathbf{P} = (\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4)$, $\mathcal{A}\vec{u}_i = \lambda_i\vec{u}_i$. The eigenvalues and eigenvectors of $\mathcal{A}$ can be directly calculated to give

$$\lambda_1 = \lambda_+, \quad \lambda_2 = \lambda_-, \quad \lambda_3 = \frac{1}{\tau_1}, \quad \lambda_4 = \frac{1}{\tau_2}, \quad \lambda_{\pm} \equiv \frac{1}{2}(\kappa_1 + \kappa_2 \pm \sqrt{4\epsilon\delta + (\kappa_1 - \kappa_2)^2}), \quad (\text{B1})$$

$$\vec{u}_1^T = \frac{1}{\mathcal{N}_1} \left(\frac{\kappa_2 - \lambda_1}{\delta}, 1, 0, 0 \right) \equiv (u_{11}, u_{12}, 0, 0), \quad (\text{B2})$$

$$\vec{u}_2^T = \frac{1}{\mathcal{N}_2} \left(\frac{\kappa_2 - \lambda_2}{\delta}, 1, 0, 0 \right) \equiv (u_{21}, u_{22}, 0, 0), \quad (\text{B3})$$

$$\vec{u}_3^T = \frac{1}{\mathcal{N}_3} \left(\frac{\kappa_2 - \lambda_3}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta}, \frac{\delta}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta}, 1, 0 \right) \equiv \left(u_{31}, u_{32}, \frac{1}{\mathcal{N}_3}, 0 \right), \quad (\text{B4})$$

$$\vec{u}_4^T = \frac{1}{\mathcal{N}_4} \left(\frac{\epsilon}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta}, \frac{\kappa_1 - \lambda_4}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta}, 0, 1 \right) \equiv \left(u_{41}, u_{42}, 0, \frac{1}{\mathcal{N}_4} \right), \quad (\text{B5})$$

where $\mathcal{N}_i$ are the normalization constants for the eigenvectors. Assuming the deterministic part of Eq. (A12) is stable, i.e., all eigenvalues of $\mathcal{A}$ are non-negative, which is given by the condition $\epsilon\delta \leq \kappa_1\kappa_2$, which is the same stability criterion as in the 2D passive Brownian gyrator case. Then, the long-time solution is given by

$$\vec{X}(t) = \int^t e^{-\mathcal{A}(t-t')} \vec{\xi}(t') dt'. \quad (\text{B6})$$

Higher-order correlation functions can be conveniently calculated analytically by utilizing the spectral decomposition of $\mathcal{A}$ in terms of its eigenvalues and the matrices $\mathbf{M}^{(\ell)}$ formed by the matrix of eigenvector $\mathbf{P}$ and its inverse, as given by $\mathbf{M}^{(\ell)} \equiv \vec{u}_\ell \vec{v}_\ell^T$, $\ell = 1, 2, 3, 4$, where $\vec{v}_\ell$ are the vectors forming $\mathbf{P}^{-T}$: $\mathbf{P}^{-T} = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ and $\vec{v}_i^T \vec{u}_j = \delta_{ij}$, $\sum_{j=1}^4 \vec{v}_j \vec{u}_j^T = \mathbf{I}$:

$$\mathbf{P} = \begin{pmatrix} u_{11} & u_{21} & u_{31} & u_{41} \\ u_{12} & u_{22} & u_{32} & u_{42} \\ 0 & 0 & \frac{1}{\mathcal{N}_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathcal{N}_4} \end{pmatrix}, \quad (\text{B7})$$

and its inverse can be directly calculated to give

$$\mathbf{P}^{-1} = \begin{pmatrix} \frac{u_{22}}{\Delta} & -\frac{u_{21}}{\Delta} & \frac{\mathcal{N}_3}{\Delta}(u_{21}u_{32} - u_{22}u_{31}) & \frac{\mathcal{N}_4}{\Delta}(u_{21}u_{42} - u_{22}u_{41}) \\ -\frac{u_{12}}{\Delta} & \frac{u_{11}}{\Delta} & \frac{\mathcal{N}_3}{\Delta}(u_{12}u_{31} - u_{11}u_{32}) & \frac{\mathcal{N}_4}{\Delta}(u_{12}u_{41} - u_{11}u_{42}) \\ 0 & 0 & \mathcal{N}_3 & 0 \\ 0 & 0 & 0 & \mathcal{N}_4 \end{pmatrix}, \quad \Delta \equiv u_{11}u_{22} - u_{12}u_{21}. \quad (\text{B8})$$

The vectors $\vec{v}_i$ can be read off from the rows of $\mathbf{P}^{-1}$. In terms of $\mathbf{P}$ and its inverse, the elements of $\mathbf{M}$ are given by

$$M_{ij}^{(\ell)} = P_{i\ell} P_{\ell j}^{-1}. \quad (\text{B9})$$

Then, the matrix $\mathcal{A}$ can be decomposed as $\mathcal{A} = \sum_{\ell=1}^4 \lambda_\ell \mathbf{M}^{(\ell)}$. Apply the spectral decomposition to the matrix $\mathcal{A}$ in Eq. (A12), the solution can be written as

$$\vec{X}(t) = \sum_{\ell=1}^4 e^{-t\lambda_\ell} \int^t dt' e^{t'\lambda_\ell} \mathbf{M}^{(\ell)} \vec{\xi}(t'). \quad (\text{B10})$$

From the special structure of $\mathbf{P}$, the matrices $\mathbf{M}^{(\ell)}$ can be obtained explicitly as

$$\mathbf{M}^{(1)} = \begin{pmatrix} \frac{u_{11}u_{22}}{\Delta} & -\frac{u_{11}u_{21}}{\Delta} & P_{11}P_{13}^{-1} & P_{11}P_{14}^{-1} \\ \frac{u_{12}u_{22}}{\Delta} & -\frac{u_{12}u_{21}}{\Delta} & P_{21}P_{13}^{-1} & P_{21}P_{14}^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{(3)} = \begin{pmatrix} 0 & 0 & \frac{\kappa_2 - \lambda_3}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & 0 \\ 0 & 0 & \frac{\delta}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B11})$$

$$\mathbf{M}^{(2)} = \begin{pmatrix} -\frac{u_{12}u_{21}}{\Delta} & \frac{u_{11}u_{21}}{\Delta} & P_{12}P_{23}^{-1} & P_{12}P_{24}^{-1} \\ -\frac{u_{12}u_{22}}{\Delta} & \frac{u_{11}u_{22}}{\Delta} & P_{22}P_{23}^{-1} & P_{22}P_{24}^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}^{(4)} = \begin{pmatrix} 0 & 0 & 0 & \frac{\epsilon}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ 0 & 0 & 0 & \frac{\kappa_1 - \lambda_4}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{B12})$$

Note that $\text{Tr}\mathbf{M}^{(\ell)} = 1$. Explicit expressions for $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ can be obtained as

$$\mathbf{M}^{(1)} = \frac{1}{\sqrt{(\kappa_1 - \kappa_2)^2 + 4\epsilon\delta}} \begin{pmatrix} -(\kappa_2 - \lambda_1) & -\epsilon & \frac{(\kappa_2 - \lambda_1)(\lambda_2 - \lambda_3)}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & -\frac{\kappa_2 - \lambda_1}{\delta} \frac{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_2) - \epsilon\delta}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ -\delta & \kappa_2 - \lambda_2 & \frac{\delta(\lambda_2 - \lambda_3)}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & -\frac{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_2) - \epsilon\delta}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B13})$$

$$\mathbf{M}^{(2)} = \frac{1}{\sqrt{(\kappa_1 - \kappa_2)^2 + 4\epsilon\delta}} \begin{pmatrix} \kappa_2 - \lambda_2 & \epsilon & -\frac{(\kappa_2 - \lambda_2)(\lambda_1 - \lambda_3)}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & \frac{\kappa_2 - \lambda_2}{\delta} \frac{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_1) - \epsilon\delta}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ \delta & -(\kappa_2 - \lambda_1) & -\frac{\delta(\lambda_1 - \lambda_3)}{(\kappa_1 - \lambda_3)(\kappa_2 - \lambda_3) - \epsilon\delta} & \frac{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_1) - \epsilon\delta}{(\kappa_1 - \lambda_4)(\kappa_2 - \lambda_4) - \epsilon\delta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B14})$$

1. Solution of the 4D Langevin equation

Hence, from Eq. (B10), the long-time limit solutions are given by

$$x_i(t) = a_i(t) + b_i(t) + c_i(t) + d_i(t), \quad i = 1, 2, \quad (\text{B15})$$

where

$$\begin{aligned} a_i(t) &\equiv \sum_{\ell=1}^4 M_{i1}^{(\ell)} \int_0^t dt' e^{(t'-t)\lambda_{\ell}} \eta_1(t'), & b_i(t) &\equiv \sum_{\ell=1}^4 M_{i2}^{(\ell)} \int_0^t dt' e^{(t'-t)\lambda_{\ell}} \eta_2(t'), \\ c_i(t) &\equiv \lambda_3 \sum_{\ell=1}^4 M_{i3}^{(\ell)} \int_0^t dt' e^{(t'-t)\lambda_{\ell}} \psi_1(t'), & d_i(t) &\equiv \lambda_4 \sum_{\ell=1}^4 M_{i4}^{(\ell)} \int_0^t dt' e^{(t'-t)\lambda_{\ell}} \psi_2(t'), \\ \zeta_1(t) &= \lambda_3 \int_0^t dt' e^{(t'-t)\lambda_3} \psi_1(t'), & \zeta_2(t) &= \lambda_4 \int_0^t dt' e^{(t'-t)\lambda_4} \psi_2(t'), \end{aligned} \quad (\text{B16})$$

and higher-order correlations can then be analytically calculated. One can show that

$$\langle r(t)s(t)u(t')v(t') \rangle = \langle r(t)s(t) \rangle \langle u(t')v(t') \rangle + \langle r(t)u(t') \rangle \langle s(t)v(t') \rangle + \langle r(t)v(t') \rangle \langle s(t)u(t') \rangle, \quad (\text{B17})$$

$$\langle r(t)s(t)u(t')\eta(t') \rangle = \langle r(t)s(t) \rangle \langle u(t')\eta(t') \rangle + \langle r(t)u(t') \rangle \langle s(t)\eta(t') \rangle + \langle r(t)\eta(t') \rangle \langle s(t)u(t') \rangle, \quad (\text{B18})$$

$$\langle r(t)\eta(t)u(t')\eta(t') \rangle = \langle r(t)\eta(t) \rangle \langle u(t')\eta(t') \rangle + \langle r(t)u(t') \rangle \langle \eta(t)\eta(t') \rangle + \langle r(t)\eta(t') \rangle \langle s(t)u(t') \rangle, \quad (\text{B19})$$

where r, s, u, v can be x_i, a_i, b_i, c_i, d_i , or ζ_i . For instance,

$$\langle a_i^2(t)a_j^2(t') \rangle = \langle a_i^2 \rangle \langle a_j^2 \rangle + 2\langle a_i(t)a_j(t') \rangle^2. \quad (\text{B20})$$

For further calculations, it will be convenient to define the matrices

$$\mathbf{Q}^{(m,n)} \equiv \mathbf{M}^{(m)} \mathbf{D} \mathbf{M}^{(n)\top}, \quad m, n = 1, 2, 3, 4, \quad (\text{B21})$$

$$\text{where } \mathbf{D} \equiv \text{diag}(D_1, D_2, D_3, D_4), \quad (\text{B22})$$

$$\text{i.e., } Q_{ij}^{(m,n)} \equiv D_1 M_{i1}^{(m)} M_{j1}^{(n)} + D_2 M_{i2}^{(m)} M_{j2}^{(n)} + \lambda_3^2 D_3 M_{i3}^{(m)} M_{j3}^{(n)} + \lambda_4^2 D_4 M_{i4}^{(m)} M_{j4}^{(n)}. \quad (\text{B23})$$

2. Averages and correlation functions

Using Eqs. (B15) and (B16), one gets the equal-time averages/correlations

$$\langle x_i^2 \rangle = \langle a_i^2 \rangle + \langle b_i^2 \rangle + \langle c_i^2 \rangle + \langle d_i^2 \rangle, \quad i = 1, 2 \quad \text{with} \quad (\text{B24})$$

$$\langle a_i^2 \rangle = 2D_1 \sum_{m,n=1}^4 \frac{M_{i1}^{(m)} M_{i1}^{(n)}}{\lambda_m + \lambda_n}, \quad \langle b_i^2 \rangle = 2D_2 \sum_{m,n=1}^4 \frac{M_{i2}^{(m)} M_{i2}^{(n)}}{\lambda_m + \lambda_n}, \quad (\text{B25})$$

$$\langle c_i^2 \rangle = 2\lambda_3^2 D_3 \sum_{m,n=1}^4 \frac{M_{i3}^{(m)} M_{i3}^{(n)}}{\lambda_m + \lambda_n}, \quad \langle d_i^2 \rangle = 2\lambda_4^2 D_4 \sum_{m,n=1}^4 \frac{M_{i4}^{(m)} M_{i4}^{(n)}}{\lambda_m + \lambda_n}, \quad (\text{B26})$$

$$\langle \zeta_1^2 \rangle = \lambda_3 D_3, \quad \langle \zeta_2^2 \rangle = \lambda_4 D_4, \quad (\text{B27})$$

$$\langle a_1 a_2 \rangle = 2D_1 \sum_{m,n=1}^4 \frac{M_{11}^{(m)} M_{21}^{(n)}}{\lambda_m + \lambda_n}, \quad \langle b_1 b_2 \rangle = 2D_2 \sum_{m,n=1}^4 \frac{M_{12}^{(m)} M_{22}^{(n)}}{\lambda_m + \lambda_n}, \quad (\text{B28})$$

$$\langle c_1 c_2 \rangle = 2\lambda_3^2 D_3 \sum_{m,n=1}^4 \frac{M_{13}^{(m)} M_{23}^{(n)}}{\lambda_m + \lambda_n}, \quad \langle d_1 d_2 \rangle = 2\lambda_4^2 D_4 \sum_{m,n=1}^4 \frac{M_{14}^{(m)} M_{24}^{(n)}}{\lambda_m + \lambda_n}, \quad (\text{B29})$$

$$\langle x_i x_j \rangle = 2 \sum_{m,n=1}^4 \frac{Q_{ij}^{(m,n)}}{\lambda_m + \lambda_n}, \quad (\text{B30})$$

$$\langle x_1 \zeta_1 \rangle = \langle c_1 \zeta_1 \rangle = 2\lambda_3^2 D_3 \sum_{m=1}^4 \frac{M_{13}^{(m)}}{\lambda_m + \lambda_3} = \frac{\lambda_3(\kappa_2 + \lambda_3)D_3}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]}, \quad (\text{B31})$$

$$\langle x_1 \zeta_2 \rangle = \langle d_1 \zeta_2 \rangle = 2\lambda_4^2 D_4 \sum_{m=1}^4 \frac{M_{14}^{(m)}}{\lambda_m + \lambda_4} = \frac{\epsilon\lambda_4 D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]}, \quad (\text{B32})$$

$$\langle x_2 \zeta_1 \rangle = \langle c_2 \zeta_1 \rangle = 2\lambda_3^2 D_3 \sum_{m=1}^4 \frac{M_{23}^{(m)}}{\lambda_m + \lambda_3} = \frac{\delta\lambda_3 D_3}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]}, \quad (\text{B33})$$

$$\langle x_2 \zeta_2 \rangle = \langle d_2 \zeta_2 \rangle = 2\lambda_4^2 D_4 \sum_{m=1}^4 \frac{M_{24}^{(m)}}{\lambda_m + \lambda_4} = \frac{\lambda_4(\kappa_1 + \lambda_4)D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]}, \quad (\text{B34})$$

$$\langle x_1 \eta_2 \rangle = \langle b_1 \eta_2 \rangle = D_2 \sum_{m=1}^4 M_{12}^{(m)} = 0, \quad \langle x_2 \eta_1 \rangle = \langle a_2 \eta_1 \rangle = D_1 \sum_{m=1}^2 M_{21}^{(m)} = 0, \quad (\text{B35})$$

$$\langle x_1 \eta_1 \rangle = \langle a_1 \eta_1 \rangle = D_1 \sum_{m=1}^4 M_{11}^{(m)} = D_1, \quad \langle x_2 \eta_2 \rangle = \langle b_2 \eta_2 \rangle = D_2 \sum_{m=1}^4 M_{22}^{(m)} = D_2, \quad (\text{B36})$$

$$\langle \zeta_1 \psi_1 \rangle = \lambda_3 D_3, \quad \langle \zeta_2 \psi_1 \rangle = \lambda_4 D_4, \quad (\text{B37})$$

$$(\kappa_1 + \kappa_2) \langle x_1^2 \rangle = D_1 + \frac{\epsilon^2 D_2 + \kappa_2^2 D_1}{\kappa_1 \kappa_2 - \epsilon\delta} + \frac{[\kappa_2^2(\kappa_1 + \kappa_2) + \lambda_3(\kappa_2(\kappa_1 + \kappa_2) - \epsilon\delta)]\lambda_3 D_3}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]} \quad (\text{B38})$$

$$+ \frac{\epsilon^2(\kappa_1 + \kappa_2 + \lambda_4)\lambda_4 D_4}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]}, \quad (\text{B39})$$

$$(\kappa_1 + \kappa_2) \langle x_2^2 \rangle = D_2 + \frac{\delta^2 D_1 + \kappa_1^2 D_2}{\kappa_1 \kappa_2 - \epsilon\delta} + \frac{\delta^2(\kappa_1 + \kappa_2 + \lambda_3)\lambda_3 D_3}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]} \quad (\text{B40})$$

$$+ \frac{[\kappa_1^2(\kappa_1 + \kappa_2) + \lambda_4(\kappa_1(\kappa_1 + \kappa_2) - \epsilon\delta)]\lambda_4 D_4}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]}, \quad (\text{B41})$$

$$(\kappa_1 + \kappa_2) \langle x_1 x_2 \rangle = \frac{\kappa_2 \delta D_1 + \kappa_1 \epsilon D_2}{\kappa_1 \kappa_2 - \epsilon\delta} + \frac{\delta \kappa_2(\kappa_1 + \kappa_2 + \lambda_3)\lambda_3 D_3}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]} \quad (\text{B42})$$

$$+ \frac{\epsilon \kappa_1(\kappa_1 + \kappa_2 + \lambda_4)\lambda_4 D_4}{(\kappa_1 \kappa_2 - \epsilon\delta)[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]}, \quad (\text{B43})$$

$$\langle \dot{x}_1 \zeta_1 \rangle = -\kappa_1 \langle x_1 \zeta_1 \rangle + \epsilon \langle x_2 \zeta_1 \rangle + \langle \zeta_1^2 \rangle = \lambda_3 D_3 \left(1 - \frac{\kappa_1 \kappa_2 - \epsilon\delta + \kappa_1 \lambda_3}{(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta} \right), \quad (\text{B44})$$

$$\langle \dot{x}_2 \zeta_2 \rangle = -\kappa_2 \langle x_2 \zeta_2 \rangle + \delta \langle x_1 \zeta_2 \rangle + \langle \zeta_2^2 \rangle = \lambda_4 D_4 \left(1 - \frac{\kappa_1 \kappa_2 - \epsilon\delta + \kappa_2 \lambda_4}{(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta} \right), \quad (\text{B45})$$

$$\langle \zeta_i \dot{\zeta}_i \rangle = (-\langle \zeta_i^2 \rangle + \langle \zeta_i \psi_i \rangle) / \tau_i = 0. \quad (\text{B46})$$

The time-lag correlations can be calculated to be

$$\langle a_i(t) a_j(t') \rangle = 2D_1 \sum_{m,n=1}^2 \frac{M_{i1}^{(m)} M_{j1}^{(n)}}{\lambda_m + \lambda_n} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_n |t-t'|}, & t < t', \end{cases} \quad (\text{B47})$$

$$\langle b_i(t) b_j(t') \rangle = 2D_2 \sum_{m,n=1}^2 \frac{M_{i2}^{(m)} M_{j2}^{(n)}}{\lambda_m + \lambda_n} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_n |t-t'|}, & t < t', \end{cases} \quad (\text{B48})$$

$$\langle c_i(t)c_j(t') \rangle = 2\lambda_3^2 D_3 \sum_{m,n=1}^4 \frac{M_{i3}^{(m)} M_{j3}^{(n)}}{\lambda_m + \lambda_n} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_n |t-t'|}, & t < t', \end{cases} \quad (\text{B49})$$

$$\langle d_i(t)d_j(t') \rangle = 2\lambda_4^2 D_4 \sum_{m,n=1}^4 \frac{M_{i4}^{(m)} M_{j4}^{(n)}}{\lambda_m + \lambda_n} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_n |t-t'|}, & t < t', \end{cases} \quad (\text{B50})$$

$$\langle x_i(t)\zeta_2(t') \rangle = \langle d_i(t)\zeta_2(t') \rangle = 2\lambda_4^2 D_4 \sum_{m=1}^4 \frac{M_{i4}^{(m)}}{\lambda_m + \lambda_4} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_4 |t-t'|}, & t < t', \end{cases} \quad (\text{B51})$$

$$\langle x_i(t)\eta_1(t') \rangle = \langle a_i(t)\eta_1(t') \rangle = 2D_1 \sum_{m=1}^2 M_{i1}^{(m)} e^{-\lambda_m |t-t'|} \Theta(t-t'), \quad (\text{B52})$$

$$\langle x_i(t)\eta_2(t') \rangle = \langle b_i(t)\eta_2(t') \rangle = 2D_2 \sum_{m=1}^2 M_{i2}^{(m)} e^{-\lambda_m |t-t'|} \Theta(t-t'), \quad (\text{B53})$$

$$\begin{aligned} \langle x_i(t)x_j(t') \rangle &= \langle a_i(t)a_j(t') \rangle + \langle b_i(t)b_j(t') \rangle + \langle c_i(t)c_j(t') \rangle + \langle d_i(t)d_j(t') \rangle, \quad i, j = 1, 2 \\ &= 2 \sum_{m,n=1}^4 \frac{Q_{ij}^{(m,n)}}{\lambda_m + \lambda_n} \begin{cases} e^{-\lambda_m |t-t'|}, & t > t', \\ e^{-\lambda_n |t-t'|}, & t < t'. \end{cases} \end{aligned} \quad (\text{B54})$$

The time-lag correlations are functions of $|t - t'|$ only. Equal time correlation can be obtained from the above equations by putting $t = t'$ and is time independent (as expected because of the steady state).

3. Energetics and work diffusion coefficient of the 2D AOUP gyrotor

We first calculate the mean power of the AOUP gyrotor under steady condition using $P(t) = \dot{W}_1(t) + \dot{W}_2(t) = (\frac{\epsilon - \delta}{2})(x_1 \dot{x}_2 - x_2 \dot{x}_1)$, together with $\dot{x}_1 = -(\mathcal{A}\vec{X})_1 = \epsilon x_2 - \kappa_1 x_1 + \zeta_1$ and $\dot{x}_2 = -(\mathcal{A}\vec{X})_2 = \delta x_1 - \kappa_2 x_2 + \zeta_2$ from Eq. (A12); one gets the steady-state work for a duration τ as

$$W(\tau) = \left(\frac{\epsilon - \delta}{2} \right) \int_0^\tau [\delta x_1^2 - \epsilon x_2^2 + (\kappa_1 - \kappa_2)x_1 x_2 + x_1(\eta_2 + \zeta_2) - x_2(\eta_1 + \zeta_1)] dt. \quad (\text{B55})$$

The mean power or work rate can be calculated to give

$$\langle \dot{W} \rangle = (\epsilon - \delta) \langle x_1 \dot{x}_2 \rangle \quad (\text{B56})$$

$$= (\epsilon - \delta) (\delta \langle x_1^2 \rangle - \kappa_2 \langle x_1 x_2 \rangle + \langle x_1 \zeta_2 \rangle). \quad (\text{B57})$$

Hence, the steady-state mean work for a duration of τ is

$$\langle W(\tau) \rangle = (\epsilon - \delta) [\delta \langle x_1^2 \rangle - \kappa_2 \langle x_1 x_2 \rangle + \langle x_1 \zeta_2 \rangle] \tau. \quad (\text{B58})$$

One needs to compute $\langle W^2(\tau) \rangle$ for the work variance. $\langle W^2(\tau) \rangle$ can be expressed in terms of higher-order correlations of x_1 , x_2 , η_2 , and ζ_2 .

$\langle W^2(\tau) \rangle$ can be expressed in terms of higher-order correlations of x_1 , x_2 , η_2 , and ζ_2 ,

$$\begin{aligned} \frac{4\langle W^2(\tau) \rangle}{(\epsilon - \delta)^2} &= \int_0^\tau \int_0^\tau dt dt' \{ \delta^2 \langle x_1^2(t)x_1^2(t') \rangle + \epsilon^2 \langle x_2^2(t)x_2^2(t') \rangle + (\kappa_1 - \kappa_2)^2 \langle x_1(t)x_2(t)x_1(t')x_2(t') \rangle \\ &\quad + \langle x_1(t)\eta_2(t)x_1(t')\eta_2(t') \rangle + \langle x_1(t)\zeta_2(t)x_1(t')\zeta_2(t') \rangle + \langle x_1(t)\eta_2(t)x_1(t')\zeta_2(t') \rangle + \langle x_1(t)\zeta_2(t)x_1(t')\eta_2(t') \rangle \\ &\quad + \langle x_2(t)\eta_1(t)x_2(t')\eta_1(t') \rangle + \langle x_2(t)\zeta_1(t)x_2(t')\zeta_1(t') \rangle + \langle x_2(t)\eta_1(t)x_2(t')\zeta_1(t') \rangle + \langle x_2(t)\zeta_1(t)x_2(t')\eta_1(t') \rangle \\ &\quad - 2\epsilon\delta \langle x_1^2(t)x_2^2(t') \rangle + 2(\kappa_1 - \kappa_2) [\delta \langle x_1^2(t)x_1(t')x_2(t') \rangle - \epsilon \langle x_2^2(t)x_1(t')x_2(t') \rangle] \\ &\quad + 2\delta [\langle x_1^2(t)x_1(t')\eta_2(t') \rangle + \langle x_1^2(t)x_1(t')\zeta_2(t') \rangle] + 2\epsilon [\langle x_2^2(t)x_2(t')\eta_1(t') \rangle + \langle x_2^2(t)x_2(t')\zeta_1(t') \rangle] \\ &\quad - 2\delta [\langle x_1^2(t)x_2(t')\eta_1(t') \rangle + \langle x_1^2(t)x_2(t')\zeta_1(t') \rangle] - 2\epsilon [\langle x_2^2(t)x_1(t')\eta_2(t') \rangle + \langle x_2^2(t)x_1(t')\zeta_2(t') \rangle] \\ &\quad + 2(\kappa_1 - \kappa_2) [\langle x_1(t)x_2(t)x_1(t')\zeta_2(t') \rangle + \langle x_1(t)x_2(t)x_1(t')\eta_2(t') \rangle - \langle x_1(t)x_2(t)x_2(t')\eta_1(t') \rangle \\ &\quad - \langle x_1(t)x_2(t)x_2(t')\zeta_1(t') \rangle] - 2\langle x_1(t)\eta_2(t)x_2(t')\eta_1(t') \rangle - 2\langle x_1(t)\zeta_2(t)x_2(t')\eta_1(t') \rangle \\ &\quad - 2\langle x_1(t)\eta_2(t)x_2(t')\zeta_1(t') \rangle - 2\langle x_1(t)\zeta_2(t)x_2(t')\zeta_1(t') \rangle \}. \end{aligned}$$

The higher-order correlations can be expressed in terms of second-order correlations using Eqs. (B17)–(B19) to give

$$\langle x_1^2(t)x_1^2(t') \rangle = \langle x_1^2 \rangle^2 + 2\langle x_1(t)x_1(t') \rangle^2, \quad (\text{B59})$$

$$\langle x_1(t)x_2(t)x_1(t')x_2(t') \rangle = \langle x_1x_2 \rangle^2 + \langle x_1(t)x_1(t') \rangle \langle x_2(t)x_2(t') \rangle + \langle x_1(t)x_2(t') \rangle \langle x_1(t')x_2(t) \rangle, \quad (\text{B60})$$

$$\langle x_1(t)\zeta_2(t)x_1(t')\zeta_2(t') \rangle = \langle x_1\zeta_2 \rangle^2 + \langle x_1(t)x_1(t') \rangle \langle \zeta_2(t)\zeta_2(t') \rangle + \langle x_1(t)\zeta_2(t') \rangle \langle x_1(t')\zeta_2(t) \rangle, \quad (\text{B61})$$

$$\langle x_1(t)\eta_2(t)x_1(t')\eta_2(t') \rangle = \langle x_1(t)x_1(t') \rangle \langle \eta_2(t)\eta_2(t') \rangle + \langle x_1(t)\eta_2(t') \rangle \langle x_1(t')\eta_2(t) \rangle, \quad (\text{B62})$$

$$\langle x_1(t)\zeta_2(t)x_1(t')\eta_2(t') \rangle = \langle x_1(t)\eta_2(t') \rangle \langle x_1(t')\zeta_2(t) \rangle, \quad (\text{B63})$$

$$\langle x_1^2(t)x_1(t')\zeta_2(t') \rangle = \langle x_1^2 \rangle \langle x_1\zeta_2 \rangle + 2\langle x_1(t)x_1(t') \rangle \langle x_1(t)\zeta_2(t') \rangle, \quad (\text{B64})$$

$$\langle x_1^2(t)x_1(t')\eta_2(t') \rangle = 2\langle x_1(t)x_1(t') \rangle \langle x_1(t)\eta_2(t') \rangle, \quad (\text{B65})$$

$$\langle x_1^2(t)x_1(t')x_2(t') \rangle = \langle x_1^2 \rangle \langle x_1x_2 \rangle + 2\langle x_1(t)x_1(t') \rangle \langle x_1(t)x_2(t') \rangle, \quad (\text{B66})$$

$$\langle x_1(t)x_2(t)x_1(t')\eta_2(t') \rangle = \langle x_1(t)x_1(t') \rangle \langle x_2(t)\eta_2(t') \rangle + \langle x_1(t)\eta_2(t') \rangle \langle x_1(t')x_2(t) \rangle,$$

$$\langle x_1(t)x_2(t)x_1(t')\zeta_2(t') \rangle = \langle x_1x_2 \rangle \langle x_1\zeta_2 \rangle + \langle x_1(t)x_1(t') \rangle \langle x_2(t)\zeta_2(t') \rangle + \langle x_1(t)\zeta_2(t') \rangle \langle x_1(t')x_2(t) \rangle. \quad (\text{B67})$$

Using the above identities, the τ^2 terms can be shown to cancel out exactly in $\langle W(\tau)^2 \rangle - \langle W(\tau) \rangle^2$ and the work variance is linear in τ , explicitly demonstrating the diffusive behavior of the steady-state work. For example, the coefficient of the δ^2 terms in $\langle W^2(\tau) \rangle - \langle W(\tau) \rangle^2$ is $\int_0^\tau \int_0^\tau dt dt' [\langle x_1^2(t)x_1^2(t') \rangle - \langle x_1^2 \rangle^2] = \int_0^\tau \int_0^\tau dt dt' 2\langle x_1(t)x_1(t') \rangle^2 \propto \tau$. After some algebra [note that $\langle x_1(t)\eta_2(t') \rangle \langle x_1(t')\eta_2(t) \rangle \propto \Theta(t-t')\Theta(t'-t) = 0$ in Eq. (B62)], one can obtain

$$\begin{aligned} \langle W(\tau)^2 \rangle - \langle W(\tau) \rangle^2 &= \frac{(\epsilon - \delta)^2}{4} \int_0^\tau \int_0^\tau dt dt' \{ 2\delta^2 \langle x_1(t)x_1(t') \rangle^2 + 2\epsilon^2 \langle x_2(t)x_2(t') \rangle^2 - 4\epsilon\delta \langle x_1(t)x_2(t') \rangle^2 \\ &\quad + (\kappa_1 - \kappa_2)^2 [\langle x_1(t)x_1(t') \rangle \langle x_2(t)x_2(t') \rangle + \langle x_1(t)x_2(t') \rangle \langle x_2(t)x_1(t') \rangle] \\ &\quad + 2D_2 \langle x_1^2(t) \rangle \delta(t-t') + D_4 \tau_2 \langle x_1(t)x_1(t') \rangle e^{-\lambda_4|t-t'|} + \langle x_1(t)\zeta_2(t') \rangle \langle x_1(t')\zeta_2(t) \rangle \\ &\quad + 2D_1 \langle x_2^2(t) \rangle \delta(t-t') + D_3 \tau_1 \langle x_2(t)x_2(t') \rangle e^{-\lambda_3|t-t'|} + \langle x_2(t)\zeta_1(t') \rangle \langle x_2(t')\zeta_1(t) \rangle \\ &\quad + 2\langle x_1(t)\zeta_2(t') \rangle \langle x_1(t')\eta_2(t) \rangle + 2\langle x_2(t)\zeta_1(t') \rangle \langle x_2(t')\eta_1(t) \rangle + 4(\kappa_1 - \kappa_2) [\delta \langle x_1(t)x_1(t') \rangle \langle x_1(t)x_2(t') \rangle \\ &\quad - \epsilon \langle x_2(t)x_1(t') \rangle \langle x_2(t)x_2(t') \rangle] + 4\delta [\langle x_1(t)x_1(t') \rangle \langle x_1(t)\eta_2(t') \rangle + \langle x_1(t)x_1(t') \rangle \langle x_1(t)\zeta_2(t') \rangle] \\ &\quad + 4\epsilon [\langle x_2(t)x_2(t') \rangle \langle x_2(t)\eta_1(t') \rangle + \langle x_2(t)x_2(t') \rangle \langle x_2(t)\zeta_1(t') \rangle] - 4\delta [\langle x_1(t)x_2(t') \rangle \langle x_1(t)\eta_1(t') \rangle \\ &\quad + \langle x_1(t)x_2(t') \rangle \langle x_1(t)\zeta_1(t') \rangle] - 4\epsilon [\langle x_2(t)x_1(t') \rangle \langle x_2(t)\eta_2(t') \rangle + \langle x_2(t)x_1(t') \rangle \langle x_2(t)\zeta_2(t') \rangle] \\ &\quad + 2(\kappa_1 - \kappa_2) [\langle x_1(t)x_1(t') \rangle \langle x_2(t)\zeta_2(t') \rangle + \langle x_2(t)x_1(t') \rangle \langle x_1(t)\zeta_2(t') \rangle + \langle x_1(t)x_1(t') \rangle \langle x_2(t)\eta_2(t') \rangle \\ &\quad + \langle x_1(t)\eta_2(t') \rangle \langle x_2(t)x_1(t') \rangle - \langle x_1(t)x_2(t') \rangle \langle x_2(t)\eta_1(t') \rangle - \langle x_2(t)x_2(t') \rangle \langle x_1(t)\eta_1(t') \rangle \\ &\quad - \langle x_1(t)x_2(t') \rangle \langle x_2(t)\zeta_1(t') \rangle - \langle x_2(t)x_2(t') \rangle \langle x_1(t)\zeta_1(t') \rangle] - 2\langle x_1(t)\eta_1(t') \rangle \langle x_2(t')\eta_2(t) \rangle \\ &\quad - 2\langle x_1(t)\eta_1(t') \rangle \langle x_2(t')\zeta_2(t) \rangle - 2\langle x_1(t)\zeta_1(t') \rangle \langle x_2(t')\eta_2(t) \rangle - 2\langle x_1(t)\zeta_1(t') \rangle \langle x_2(t')\zeta_2(t) \rangle \}. \end{aligned} \quad (\text{B68})$$

Since all the correlation functions contain terms of the form $e^{-\lambda|t-t'|}$, the double integral is proportional to τ . The exact expression for $\langle W^2(\tau) \rangle - \langle W(\tau) \rangle^2$ can be derived upon explicit integration using Eqs. (B47)–(B54).

Here, we will show analytically that the variance $\langle W(\tau)^2 \rangle - \langle W(\tau) \rangle^2$ is indeed $\propto \tau$ and calculate $\mathcal{D}_W$ analytically. To compute the correlations, it is useful to use the following formula for the integrals:

$$\int_0^\tau dt e^{-\lambda t} = \frac{1 - e^{-\lambda\tau}}{\lambda} \xrightarrow{\tau\lambda \rightarrow \infty} \frac{1}{\lambda}, \quad (\text{B69})$$

$$\int_0^\tau dt \int_0^\tau dt' e^{-\lambda|t-t'|} = \frac{2}{\lambda^2} (\lambda\tau + e^{-\lambda\tau} - 1) \xrightarrow{\tau\lambda \rightarrow \infty} \frac{2\tau}{\lambda}, \quad (\text{B70})$$

$$\int_0^t du \int_0^{t'} dv e^{u\lambda_m + v\lambda_n} \langle \eta_i(u) \eta_j(v) \rangle = 2D_i \delta_{ij} \frac{e^{(\lambda_m + \lambda_n) \min(t, t')} - 1}{\lambda_m + \lambda_n}, \quad (\text{B71})$$

$$\langle \eta(u) \eta(v) \eta(u') \eta(v') \rangle = \langle \eta(u) \eta(v) \rangle \langle \eta(u') \eta(v') \rangle + \langle \eta(u) \eta(u') \rangle \langle \eta(v) \eta(v') \rangle \quad (\text{B72})$$

$$+ \langle \eta(u) \eta(v') \rangle \langle \eta(u') \eta(v) \rangle, \quad (\text{B73})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle a_i(t) a_j(t') \rangle^2 = 4D_1^2 \tau \sum_{m,n,p,q=1}^4 \frac{M_{i1}^{(m)} M_{j1}^{(n)} M_{i1}^{(p)} M_{i1}^{(q)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_p} \frac{1}{\lambda_n + \lambda_q} \right\}, \quad (\text{B74})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle a_i(t) a_j(t') \rangle \langle b_k(t) b_\ell(t') \rangle = 4D_1 D_2 \tau \sum_{m,n,p,q=1}^4 \frac{M_{i1}^{(m)} M_{j1}^{(n)} M_{k2}^{(p)} M_{\ell 2}^{(q)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_q} \right\}, \quad (\text{B75})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle a_i(t) a_j(t') \rangle \langle \zeta_2(t) \zeta_2(t') \rangle = 2D_1 D_4 \lambda_4 \tau \sum_{m,n=1}^4 \frac{M_{i1}^{(m)} M_{j1}^{(n)}}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)}, \quad (\text{B76})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle d_i(t) \zeta_2(t') \rangle \langle d_j(t') \zeta_2(t) \rangle = 4\lambda_4^4 D_4^2 \tau \sum_{m,n=1}^4 \frac{M_{i4}^{(m)} M_{j4}^{(n)}}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)} \left\{ \frac{1}{\lambda_m + \lambda_4} + \frac{1}{\lambda_n + \lambda_4} \right\}, \quad (\text{B77})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle b_i(t) \eta_2(t') \rangle \langle d_j(t) \zeta_2(t') \rangle = 4D_2 D_4 \lambda_4^2 \tau \sum_{m,n=1}^4 \frac{M_{i2}^{(m)} M_{j4}^{(n)}}{(\lambda_n + \lambda_4)(\lambda_m + \lambda_n)}, \quad (\text{B78})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle a_i(t) a_j(t') \rangle \langle d_k(t) \zeta_2(t') \rangle = 4D_1 D_4 \lambda_4^2 \tau \sum_{m,n,p=1}^4 \frac{M_{i1}^{(m)} M_{j1}^{(n)} M_{k4}^{(p)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_4)} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_4} \right\}, \quad (\text{B79})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle a_i(t) a_j(t') \rangle \langle b_k(t) \eta_2(t') \rangle = 4D_1 D_2 \tau \sum_{m,n,p=1}^4 \frac{M_{i1}^{(m)} M_{j1}^{(n)} M_{k2}^{(p)}}{(\lambda_m + \lambda_n)(\lambda_m + \lambda_p)}. \quad (\text{B80})$$

Using Eqs. (B74) and (B75) (with similar expressions for integrals involving combinations of $\langle c_i(t) c_j(t') \rangle$ and $\langle d_i(t) d_j(t') \rangle$), one obtains

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle^2 = 4\tau \sum_{m,n,p,q=1}^4 \frac{Q_{11}^{(m,n)} Q_{11}^{(p,q)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_q} \right\}, \quad (\text{B81})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_2(t) x_2(t') \rangle = 4\tau \sum_{m,n,p,q=1}^4 \frac{Q_{11}^{(m,n)} Q_{22}^{(p,q)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_q} \right\}, \quad (\text{B82})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_2(t') \rangle \langle x_1(t') x_2(t) \rangle = 4\tau \sum_{m,n,p,q=1}^4 \frac{Q_{12}^{(m,n)} Q_{21}^{(p,q)}}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_q} + \frac{1}{\lambda_n + \lambda_p} \right\}, \quad (\text{B83})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle \zeta_2(t) \zeta_2(t') \rangle = 2D_4 \lambda_4 \tau \sum_{m,n=1}^4 \frac{Q_{11}^{(m,n)}}{(\lambda_m + \lambda_n)} \left\{ \frac{1}{\lambda_m + \lambda_4} + \frac{1}{\lambda_n + \lambda_4} \right\}, \quad (\text{B84})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) \zeta_2(t') \rangle \langle x_1(t') \zeta_2(t) \rangle = 4D_4^2 \lambda_4^4 \tau \sum_{m,n=1}^4 \frac{M_{14}^{(m)} M_{14}^{(n)}}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)} \left\{ \frac{1}{\lambda_m + \lambda_4} + \frac{1}{\lambda_n + \lambda_4} \right\},$$

$$2D_2 \int_0^\tau \int_0^\tau dt' dt \langle x_1^2 \rangle \delta(t - t') = 4D_2 \tau \sum_{m,n=1}^4 \frac{Q_{11}^{(m,n)}}{\lambda_m + \lambda_n}, \quad (\text{B85})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) \eta_2(t') \rangle \langle x_1(t') \zeta_2(t) \rangle = 4D_2 D_4 \lambda_4^2 \tau \sum_{m,n=1}^4 \frac{M_{12}^{(m)} M_{14}^{(n)}}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)}, \quad (\text{B86})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_1(t) \zeta_2(t') \rangle = 4D_4 \lambda_4^2 \tau \sum_{m,n,p=1}^4 \frac{Q_{11}^{(m,n)} M_{14}^{(p)}}{\lambda_m + \lambda_n \lambda_p + \lambda_4} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_4} \right\}, \quad (\text{B87})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_1(t) \eta_2(t') \rangle = 4D_2 \tau \sum_{m,n,p=1}^4 \frac{Q_{11}^{(m,n)} M_{12}^{(p)}}{(\lambda_m + \lambda_n)(\lambda_m + \lambda_p)}, \quad (\text{B88})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_1(t) x_2(t') \rangle = 4\tau \sum_{m,n,p,q=1}^4 \frac{Q_{11}^{(m,n)}}{(\lambda_m + \lambda_n)} \frac{Q_{12}^{(p,q)}}{(\lambda_p + \lambda_q)} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_q} \right\}, \quad (\text{B89})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_2(t) \eta_2(t') \rangle = 4D_2\tau \sum_{m,n,p=1}^4 \frac{Q_{11}^{(m,n)} M_{22}^{(p)}}{(\lambda_m + \lambda_n)(\lambda_m + \lambda_p)}, \quad (\text{B90})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) \eta_2(t') \rangle \langle x_1(t') x_2(t) \rangle = 4D_2\tau \sum_{m,n,p=1}^4 \frac{Q_{21}^{(m,n)} M_{12}^{(p)}}{(\lambda_m + \lambda_n)(\lambda_n + \lambda_p)}, \quad (\text{B91})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_1(t') \rangle \langle x_2(t) \zeta_2(t') \rangle = 4D_4\lambda_4^2\tau \sum_{m,n,p=1}^4 \frac{Q_{11}^{(m,n)} M_{24}^{(p)}}{\lambda_m + \lambda_n \lambda_p + \lambda_4} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_4} \right\}, \quad (\text{B92})$$

$$\int_0^\tau \int_0^\tau dt' dt \langle x_1(t) x_2(t') \rangle \langle x_1(t) \zeta_2(t') \rangle = 4D_4\lambda_4^2\tau \sum_{m,n,p=1}^4 \frac{Q_{12}^{(m,n)} M_{14}^{(p)}}{\lambda_m + \lambda_n \lambda_p + \lambda_4} \left\{ \frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_4} \right\}. \quad (\text{B93})$$

The τ^2 term will cancel with the corresponding term from $\langle W \rangle^2$, proving the diffusive behavior of $W(\tau)$. Evaluating all integrals in Eq. (B68), we obtain the analytic result for $\mathcal{D}_W$:

$$\begin{aligned} \mathcal{D}_W = & \frac{(\epsilon - \delta)^2}{4} \times \left\{ \sum_{m,n=1}^4 \frac{2D_1 Q_{22}^{(m,n)} + 2D_2 Q_{11}^{(m,n)}}{\lambda_m + \lambda_n} \right. \\ & + \sum_{m,n,p=1}^4 [8\delta(D_2 Q_{11}^{(m,n)} M_{12}^{(p)} - D_1 Q_{12}^{(m,n)} M_{11}^{(p)}) + 8\epsilon(D_1 Q_{22}^{(m,n)} M_{21}^{(p)} - D_2 Q_{21}^{(m,n)} M_{22}^{(p)}) \\ & + 4(\kappa_1 - \kappa_2)[D_2(Q_{11}^{(m,n)} M_{22}^{(p)} + Q_{21}^{(m,n)} M_{12}^{(p)}) - D_1(Q_{12}^{(m,n)} M_{21}^{(p)} + Q_{22}^{(m,n)} M_{11}^{(p)})]] \frac{1}{(\lambda_m + \lambda_n)(\lambda_m + \lambda_p)} \\ & + \sum_{m,n,p,q=1}^4 [4\delta^2 Q_{11}^{(m,n)} Q_{11}^{(p,q)} + 4\epsilon^2 Q_{22}^{(m,n)} Q_{22}^{(p,q)} - 8\epsilon\delta Q_{12}^{(m,n)} Q_{12}^{(p,q)} + 2(\kappa_1 - \kappa_2)^2 [Q_{11}^{(m,n)} Q_{22}^{(p,q)} + Q_{12}^{(m,n)} Q_{21}^{(p,q)}] \\ & + 8(\kappa_1 - \kappa_2)[\delta Q_{11}^{(m,n)} Q_{12}^{(p,q)} - \epsilon Q_{21}^{(m,n)} Q_{22}^{(p,q)}]] \frac{1}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_q)} \left(\frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_q} \right) \\ & + \sum_{m,n=1}^4 \left[\frac{\lambda_3^2 D_3 Q_{22}^{(m,n)}}{\lambda_m + \lambda_n} \left(\frac{1}{\lambda_m + \lambda_3} + \frac{1}{\lambda_n + \lambda_3} \right) + \frac{\lambda_4^2 D_4 Q_{11}^{(m,n)}}{\lambda_m + \lambda_n} \left(\frac{1}{\lambda_m + \lambda_4} + \frac{1}{\lambda_n + \lambda_4} \right) \right. \\ & + \frac{4\lambda_3^2 D_3 (D_1 M_{23}^{(m)} M_{21}^{(n)} - D_2 M_{22}^{(m)} M_{13}^{(n)})}{(\lambda_m + \lambda_3)(\lambda_n + \lambda_3)} + \frac{4\lambda_4^2 D_4 (D_2 M_{14}^{(m)} M_{12}^{(n)} - D_1 M_{11}^{(m)} M_{24}^{(n)})}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)} \\ & + \frac{2\lambda_3^4 D_3^2 M_{23}^{(m)} M_{23}^{(n)}}{(\lambda_m + \lambda_3)(\lambda_n + \lambda_3)} \left(\frac{1}{\lambda_m + \lambda_3} + \frac{1}{\lambda_n + \lambda_3} \right) + \frac{2\lambda_4^4 D_4^2 M_{14}^{(m)} M_{14}^{(n)}}{(\lambda_m + \lambda_4)(\lambda_n + \lambda_4)} \left(\frac{1}{\lambda_m + \lambda_4} + \frac{1}{\lambda_n + \lambda_4} \right) \Big] \\ & + \sum_{m,n,p=1}^4 \left[[2(\epsilon Q_{22}^{(m,n)} M_{23}^{(p)} - \delta Q_{12}^{(m,n)} M_{13}^{(p)} - (\kappa_1 - \kappa_2)(Q_{12}^{(m,n)} M_{23}^{(p)} + Q_{22}^{(m,n)} M_{13}^{(p)})] \frac{4\lambda_3^2 D_3}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_3)} \right. \\ & \times \left(\frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_3} \right) + [2(\delta Q_{11}^{(m,n)} M_{14}^{(p)} - \epsilon Q_{21}^{(m,n)} M_{24}^{(p)} + (\kappa_1 - \kappa_2)(Q_{11}^{(m,n)} M_{24}^{(p)} + Q_{21}^{(m,n)} M_{14}^{(p)})] \\ & \times \left. \frac{4\lambda_4^2 D_4}{(\lambda_m + \lambda_n)(\lambda_p + \lambda_4)} \left(\frac{1}{\lambda_m + \lambda_p} + \frac{1}{\lambda_n + \lambda_4} \right) \right] \Big\}. \quad (\text{B94}) \end{aligned}$$

Substituting the expressions for $\mathbf{Q}$ and $\mathbf{M}$ and after some very lengthy algebra, an explicit formula for $\mathcal{D}_W$ in terms of the parameters of the system can be derived to give

$$\mathcal{D}_W = \frac{(\epsilon - \delta)^2}{(\kappa_1 + \kappa_2)^3} \left\{ \delta^2 D_1^2 + \epsilon^2 D_2^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon\delta] D_1 D_2 \right.$$

$$\begin{aligned}
& + \frac{\epsilon^2 \lambda_4^4 D_4^2 [(\kappa_1 + \kappa_2 + \lambda_4)^2 + (\kappa_1 + \kappa_2) \lambda_4 + \kappa_1 \kappa_2 - \epsilon \delta]}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]^3} + \frac{2 \lambda_4^2 \epsilon^2 D_2 D_4 [\lambda_4^2 + 2(\kappa_1 + \kappa_2) \lambda_4 + \kappa_1 \kappa_2 - \epsilon \delta]}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]^2} \\
& + \frac{\lambda_4^2 D_1 D_4 [(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta][(\kappa_1 + \kappa_2)^2 - 2\epsilon \delta] - 2(\kappa_1 + \kappa_2) \lambda_4 \epsilon \delta}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]^2} \\
& + \frac{\delta^2 \lambda_3^4 D_3^2 [(\kappa_1 + \kappa_2 + \lambda_3)^2 + (\kappa_1 + \kappa_2) \lambda_3 + \kappa_1 \kappa_2 - \epsilon \delta]}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta]^3} + \frac{2 \delta^2 \lambda_3^2 D_1 D_3 [\lambda_3^2 + 2(\kappa_1 + \kappa_2) \lambda_3 + \kappa_1 \kappa_2 - \epsilon \delta]}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta]^2} \\
& + \frac{\lambda_3^2 D_2 D_3 [(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta][(\kappa_1 + \kappa_2)^2 - 2\epsilon \delta] - 2(\kappa_1 + \kappa_2) \lambda_3 \epsilon \delta}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta]^2} \\
& + \frac{\lambda_3^2 \lambda_4^2 D_3 D_4}{(\lambda_3 + \lambda_4)[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta]^2 [(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta]^2} \\
& \times [(\kappa_1 + \kappa_2)^2 (\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3)(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4)(\kappa_1 + \kappa_2 + \lambda_3 + \lambda_4) - 2\epsilon^3 \delta^3 (\lambda_3 + \lambda_4) \\
& + \epsilon^2 \delta^2 [(\kappa_1 + \kappa_2)^3 + (\lambda_3 + \lambda_4)[2(\kappa_1 + \kappa_2 + \lambda_3 + \lambda_4)^2 + (\kappa_1 + \kappa_2)^2 + (\kappa_1 \kappa_2 - \lambda_3 \lambda_4)]] \\
& - \epsilon \delta [2(\kappa_1 + \kappa_2)[\kappa_1 \kappa_2 (\kappa_1 + \kappa_2)^2 + (\lambda_3 + \lambda_4)^2 (\kappa_1^2 + \kappa_2^2 + 4\kappa_1 \kappa_2) + \lambda_3 \lambda_4 (\kappa_1 + \kappa_2)^2] \\
& + (\lambda_3 + \lambda_4)[2\lambda_3^2 \lambda_4^2 + 4\lambda_3 \lambda_4 (\lambda_3 + \lambda_4)(\kappa_1 + \kappa_2) + (\kappa_1 + \kappa_2)^4 + 2\kappa_1 \kappa_2 (2\kappa_1 + \kappa_2)(\kappa_1 + 2\kappa_2) \\
& + (\lambda_3 + \lambda_4)^2 (\kappa_1^2 + \kappa_2^2 + 4\kappa_1 \kappa_2) + 2\lambda_3 \lambda_4 (3\kappa_1^2 + 3\kappa_2^2 + 4\kappa_1 \kappa_2)]]]. \tag{B95}
\end{aligned}$$

Similarly, $\langle x_1 \dot{x}_2 \rangle$ can be represented with $\mathbf{Q}$ and $\mathbf{M}$ as

$$\langle x_1 \dot{x}_2 \rangle = 2 \left(\sum_{m,n=1}^4 \frac{\delta Q_{11}^{(m,n)} - \kappa_2 Q_{12}^{(m,n)}}{\lambda_m + \lambda_n} + \lambda_4^2 D_4 \sum_{m=1}^4 \frac{M_{14}^{(m)}}{\lambda_m + \lambda_4} \right). \tag{B96}$$

Substituting system parameters into Eq. (B97) yields

$$\langle x_1 \dot{x}_2 \rangle = \frac{1}{\kappa_1 + \kappa_2} \left(\delta D_1 - \epsilon D_2 + \frac{\delta \lambda_3^2 D_3}{(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon \delta} - \frac{\epsilon \lambda_4^2 D_4}{(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon \delta} \right). \tag{B97}$$

And the EP rate is

$$\langle \dot{S} \rangle = \left(\frac{\delta}{D_2} - \frac{\epsilon}{D_1} \right) \langle x_1 \dot{x}_2 \rangle. \tag{B98}$$

APPENDIX C: ANALYTIC EXPRESSIONS FOR THE ENERGETICS AND WORK DIFFUSION COEFFICIENT OF THE 4D WHITE NOISE GYRATOR DERIVED BY THE CULMULANT GENERATING FUNCTION

1. 2D white noise gyrator case

First, we review the 2D Gaussian white noise case. The mean and fluctuation of the heat can be derived using the cumulant generating function (as outlined in the Supplemental Material in Ref. [38])

$$\Lambda(z) = \lim_{t \rightarrow \infty} \frac{1}{t} \langle e^{z Q_1(t)} \rangle = \langle \dot{Q}_1 \rangle z + D_{Q_1} z^2 + \mathcal{O}(z^3), \tag{C1}$$

which is the largest eigenvalue of the tilted adjoint Fokker-Planck operator. Using the Gaussian ansatz for the eigenfunction $e^{-\vec{x}^\top \mathbf{G}(z) \vec{x}/2}$, and comparing the coefficients of the quadratic form gives the equations for Λ and $\mathbf{G}$, with $\mathbf{A} = \begin{pmatrix} \kappa_1 & -\epsilon \\ -\delta & \kappa_2 \end{pmatrix}$, $\mathbf{Q} \equiv \text{diag}(1, 0)$, and $\mathbf{D} \equiv \text{diag}(D_1, D_2)$:

$$\Lambda(z) = -\text{Tr}(\mathbf{D}\mathbf{G}) - z \text{Tr}(\mathbf{D}\mathbf{Q}\mathbf{A}), \tag{C2}$$

$$(\mathbf{G}\mathbf{A})_{\text{sym}} + \mathbf{G}\mathbf{D}\mathbf{G} + z\mathbf{A}^\top \mathbf{Q}\mathbf{A} + 2z(\mathbf{G}\mathbf{D}\mathbf{Q}\mathbf{A})_{\text{sym}} + z^2 \mathbf{A}^\top \mathbf{Q}\mathbf{D}\mathbf{Q}\mathbf{A} = 0, \tag{C3}$$

which can be solved up to order z^2 by expanding $\mathbf{G}(z)$ as

$$\mathbf{G}(z) = z\mathbf{G}_1 + z^2\mathbf{G}_2 + \mathcal{O}(z^3). \tag{C4}$$

The first and second orders in z give

$$\langle \dot{Q}_1 \rangle = \gamma \text{Tr}(\mathbf{D}\mathbf{G}_1 + \mathbf{D}\mathbf{Q}\mathbf{A}), \tag{C5}$$

$$(\mathbf{G}_1 \mathbf{A})_{\text{sym}} = -\mathbf{A}^\top \mathbf{Q} \mathbf{A}, \quad (\text{C6})$$

$$D_{Q_1} = -\gamma^2 \text{Tr}(\mathbf{D} \mathbf{G}_2), \quad (\text{C7})$$

$$(\mathbf{G}_2 \mathbf{A})_{\text{sym}} = -\mathbf{G}_1 \mathbf{D} \mathbf{G}_1 - 2(\mathbf{G}_1 \mathbf{D} \mathbf{Q} \mathbf{A})_{\text{sym}} - \mathbf{A}^\top \mathbf{Q} \mathbf{D} \mathbf{Q} \mathbf{A}. \quad (\text{C8})$$

Finally, one obtains

$$\mathbf{G}_1 = \frac{1}{\kappa_1 + \kappa_2} \begin{pmatrix} \epsilon \delta - \kappa_1(\kappa_1 + \kappa_2) & \epsilon \kappa_1 \\ \epsilon \kappa_1 & -\epsilon^2 \end{pmatrix}, \quad (\text{C9})$$

$$\mathbf{G}_2 = \frac{\epsilon^2}{(\kappa_1 + \kappa_2)^2} \begin{pmatrix} \delta(\epsilon D_2 - \delta D_1) - \kappa_1(\kappa_1 + \kappa_2) D_2 & \kappa_2 \delta D_1 + \kappa_1 \epsilon D_2 \\ \kappa_2 \delta D_1 + \kappa_1 \epsilon D_2 & \epsilon(\delta D_1 - \epsilon D_2) - \kappa_2(\kappa_1 + \kappa_2) D_1 \end{pmatrix},$$

and hence

$$\langle \dot{Q}_1 \rangle = \frac{\gamma \epsilon (\delta D_1 - \epsilon D_2)}{\kappa_1 + \kappa_2} \Rightarrow \langle \dot{W} \rangle = \langle \dot{Q}_1 \rangle + \langle \dot{Q}_2 \rangle = \left(1 - \frac{\delta}{\epsilon}\right) \langle \dot{Q}_1 \rangle = \frac{\gamma(\epsilon - \delta)(\delta D_1 - \epsilon D_2)}{\kappa_1 + \kappa_2}, \quad (\text{C10})$$

$$D_{Q_1} = \frac{\gamma^2 \epsilon^2}{(\kappa_1 + \kappa_2)^3} (\delta^2 D_1^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon \delta] D_1 D_2 + \epsilon^2 D_2^2) \quad (\text{C11})$$

$$\Rightarrow \mathcal{D}_W = \frac{\gamma^2 (\epsilon - \delta)^2}{(\kappa_1 + \kappa_2)^3} (\delta^2 D_1^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon \delta] D_1 D_2 + \epsilon^2 D_2^2). \quad (\text{C12})$$

The steady-state total EP is given by

$$\langle \dot{S} \rangle = -\frac{\langle \dot{Q}_1 \rangle}{D_1} - \frac{\langle \dot{Q}_2 \rangle}{D_2} = \frac{(\delta D_1 - \epsilon D_2)^2}{(\kappa_1 + \kappa_2) D_1 D_2} \geq 0. \quad (\text{C13})$$

Figure 10 displays the analytic results of various mean work and EP rates as functions of D_2/D_1 or D_4/D_1 .

The above method can be easily modified to compute directly $\langle \dot{W} \rangle$ and $\mathcal{D}_W$ simply by putting $\mathbf{Q} = \mathbf{I}$ and repeating the calculations. In this case, we get

$$\mathbf{G}_1 = \frac{1}{\kappa_1 + \kappa_2} \begin{pmatrix} \delta(\epsilon - \delta) - \kappa_1(\kappa_1 + \kappa_2) & \epsilon \kappa_1 + \delta \kappa_2 \\ \epsilon \kappa_1 + \delta \kappa_2 & -\epsilon(\epsilon - \delta) - \kappa_2(\kappa_1 + \kappa_2) \end{pmatrix}, \quad (\text{C14})$$

$$\mathbf{G}_2 = \frac{(\epsilon - \delta)^2}{(\kappa_1 + \kappa_2)^2} \begin{pmatrix} \delta(\epsilon D_2 - \delta D_1) - \kappa_1(\kappa_1 + \kappa_2) D_2 & \kappa_2 \delta D_1 + \kappa_1 \epsilon D_2 \\ \kappa_2 \delta D_1 + \kappa_1 \epsilon D_2 & \epsilon(\delta D_1 - \epsilon D_2) - \kappa_2(\kappa_1 + \kappa_2) D_1 \end{pmatrix},$$

and

$$\langle \dot{W} \rangle = \gamma \text{Tr}(\mathbf{D} \mathbf{G}_1 + \mathbf{D} \mathbf{A}) = \frac{\gamma(\epsilon - \delta)(\delta D_1 - \epsilon D_2)}{\kappa_1 + \kappa_2}, \quad (\text{C15})$$

$$\mathcal{D}_W = -\gamma^2 \text{Tr}(\mathbf{D} \mathbf{G}_2) = \frac{\gamma^2 (\epsilon - \delta)^2}{(\kappa_1 + \kappa_2)^3} (\delta^2 D_1^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon \delta] D_1 D_2 + \epsilon^2 D_2^2). \quad (\text{C16})$$

2. The 4D white noise gyrator

For the 4D stochastic white noise gyrator, we carry out similar calculations for $\langle \dot{W} \rangle$ and $D_{\mathcal{W}}$. In this case, the first and second-order generating functions, $\mathcal{G}_1$ and $\mathcal{G}_2$ can be calculated (in dimensionless form)

$$\mathcal{G}_1 = \frac{1}{\kappa_1 + \kappa_2} \times \begin{bmatrix} \delta(\epsilon - \delta) - \kappa_1(\kappa_1 + \kappa_2) & \epsilon \kappa_1 + \delta \kappa_2 & \frac{(\kappa_1 + \kappa_2)\phi(0, \lambda_3) + \lambda_3 \delta(\epsilon - \delta)}{\phi(\lambda_3, \lambda_3)} & \frac{(\epsilon - \delta)\phi(0, \lambda_4) - \lambda_4 \delta(\kappa_1 + \kappa_2)}{\phi(\lambda_4, \lambda_4)} \\ \epsilon \kappa_1 + \delta \kappa_2 & -\epsilon(\epsilon - \delta) - \kappa_2(\kappa_1 + \kappa_2) & -\frac{(\epsilon - \delta)\phi(\lambda_3, 0) + \lambda_3 \epsilon(\kappa_1 + \kappa_2)}{\phi(\lambda_3, \lambda_3)} & \frac{(\kappa_1 + \kappa_2)\phi(\lambda_4, 0) - \lambda_4 \epsilon(\epsilon - \delta)}{\phi(\lambda_4, \lambda_4)} \\ \frac{(\kappa_1 + \kappa_2)\phi(0, \lambda_3) + \lambda_3 \delta(\epsilon - \delta)}{\phi(\lambda_3, \lambda_3)} & -\frac{(\epsilon - \delta)\phi(\lambda_3, 0) + \lambda_3 \epsilon(\kappa_1 + \kappa_2)}{\phi(\lambda_3, \lambda_3)} & -(\kappa_1 + \kappa_2)\lambda_3 - \frac{\delta(\delta - \epsilon) + (\kappa_1 + \kappa_2)(\kappa_2 + \lambda_3)}{\phi(\lambda_3, \lambda_3)} & \frac{\mu}{(\lambda_3 + \lambda_4)\phi(\lambda_3, \lambda_3)\phi(\lambda_4, \lambda_4)} \\ \frac{(\epsilon - \delta)\phi(0, \lambda_4) - \lambda_4 \delta(\kappa_1 + \kappa_2)}{\phi(\lambda_4, \lambda_4)} & \frac{(\kappa_1 + \kappa_2)\phi(\lambda_4, 0) - \lambda_4 \epsilon(\epsilon - \delta)}{\phi(\lambda_4, \lambda_4)} & \frac{\mu}{(\lambda_3 + \lambda_4)\phi(\lambda_3, \lambda_3)\phi(\lambda_4, \lambda_4)} & -(\kappa_1 + \kappa_2)\lambda_4 - \frac{\epsilon(\epsilon - \delta) + (\kappa_1 + \kappa_2)(\kappa_1 + \lambda_4)}{\phi(\lambda_4, \lambda_4)} \end{bmatrix},$$

where

$$\phi(x, y) \equiv (\kappa_1 + x)(\kappa_2 + y) - \epsilon \delta, \quad (\text{C17})$$

$$\mu \equiv \lambda_3 \lambda_4 \{(\epsilon - \delta)(\lambda_3 \kappa_1 - \lambda_4 \kappa_2) - (\kappa_1 + \kappa_2)[\delta(\lambda_3 + 2\kappa_1) + \epsilon(\lambda_4 + 2\kappa_2)]\}$$

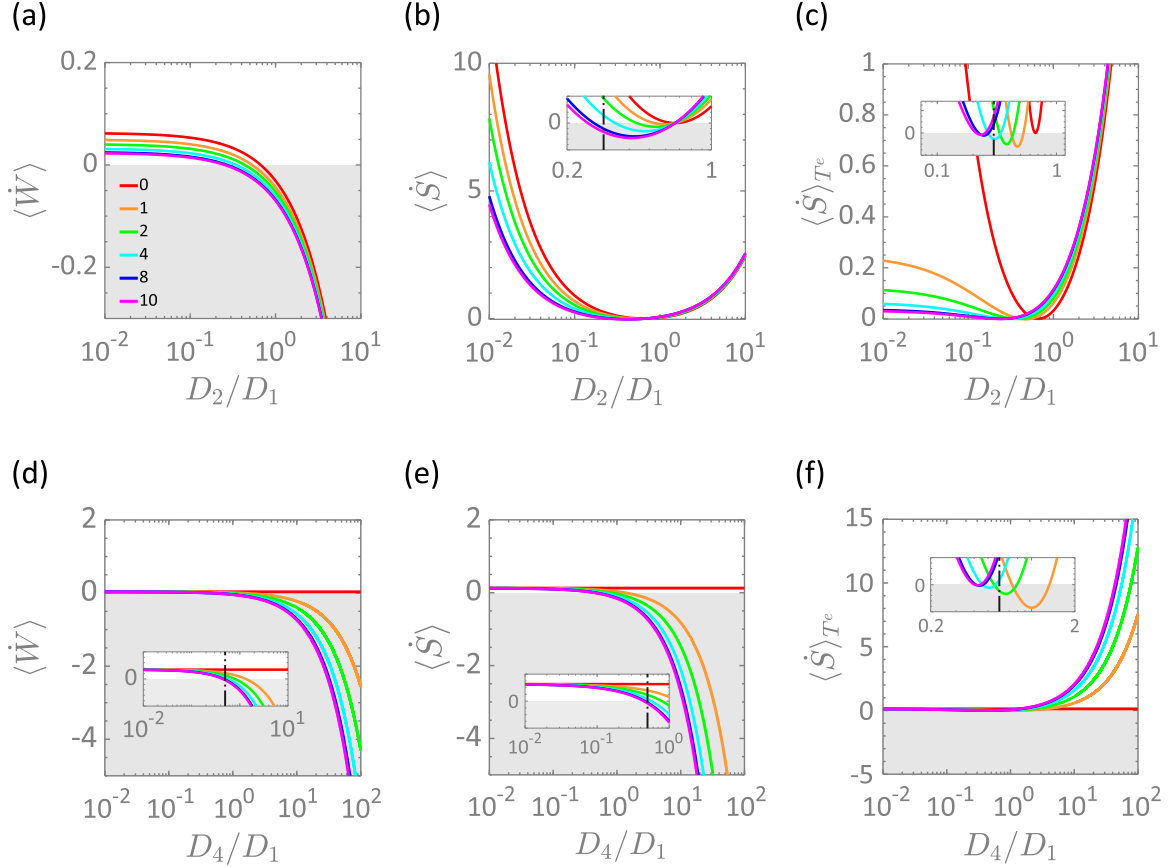


FIG. 10. Analytical results of mean work rate $\langle \dot{W} \rangle$ [Eq. (11)], mean EP rates $\langle \dot{S} \rangle$ [Eq. (12)], and (c), (f) $\langle \dot{S} \rangle_{Te}$ [Eq. (13)] for a 2D AOUP gyrotor as a function of (a)–(c) D_2/D_1 and (d)–(f) D_4/D_1 at various λ_4 with $\kappa_1 = \kappa_2 = 1$, $\delta = 0.5$, $\lambda_3 = 0$, $D_1 = 1$, $D_2 = 0.3$, $D_3 = 0$, and $D_4 = 0.5$. The stability condition $\epsilon\delta < \kappa_1\kappa_2$ is satisfied. Insets in panel (c) highlight the regions of the entropic violation ($\langle \dot{S} \rangle < 0$ or $\langle \dot{S} \rangle_{Te} < 0$). The vertical dot-dashed line in each figure denotes the same parameter line.

$$+ (\lambda_3 + \lambda_4)(\kappa_1\kappa_2 - \epsilon\delta)[\epsilon(\lambda_3 - \lambda_4 - \kappa_2) + \delta(\lambda_3 - \lambda_4 - \kappa_1)]. \quad (\text{C18})$$

The explicit expression for $\mathcal{G}_2$ is too long and will not be listed here. Thus, the work current can then be derived to give

$$\langle \dot{W} \rangle = \text{Tr}(\mathbf{D}\mathcal{G}_1 + \mathbf{D}\mathcal{A}) \quad (\text{C19})$$

$$= \frac{1}{\kappa_1 + \kappa_2} \left\{ (\epsilon - \delta)(\delta D_1 - \epsilon D_2) - \frac{\lambda_3^2 D_3 [\delta^2 + (\kappa_1 + \kappa_2)(\kappa_2 + \lambda_3) - \epsilon\delta]}{(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta} \right. \quad (\text{C20})$$

$$\left. - \frac{\lambda_4^2 D_4 [\epsilon^2 + (\kappa_1 + \kappa_2)(\kappa_1 + \lambda_4) - \epsilon\delta]}{(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta} \right\}, \quad (\text{C21})$$

which agrees with that in Eq. (18).

The work fluctuations of the 4D system are characterized by the diffusion coefficient $D_{\mathcal{W}} = \lim_{\tau \rightarrow \infty} \frac{\langle \mathcal{W}(\tau)^2 \rangle - \langle \mathcal{W}(\tau) \rangle^2}{2\tau}$, which can be calculated using the method of the cumulant generating function as in the 2D case. After some tedious calculations, one obtains

$$D_{\mathcal{W}} = -\text{Tr}(\mathbf{D}\mathcal{G}_2) = \frac{(\epsilon - \delta)^2}{(\kappa_1 + \kappa_2)^3} \left\{ \delta^2 D_1^2 + [(\kappa_1 + \kappa_2)^2 - 2\epsilon\delta] D_1 D_2 + \epsilon^2 D_2^2 \right. \\ + \frac{\Phi_{13} \lambda_3^2 D_1 D_3}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]^2} + \frac{\Phi_{23} \lambda_3^2 D_2 D_3}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]^2} + \frac{\Phi_{14} \lambda_4^2 D_1 D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^2} \\ + \frac{\Phi_{24} \lambda_4^2 D_2 D_4}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^2} + \frac{\Phi_{33} \lambda_3^4 D_3^2}{[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]^3} + \frac{\Phi_{44} \lambda_4^4 D_4^2}{[(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^3} \\ \left. + \frac{\Phi_{34} \lambda_3^2 \lambda_4^2 D_3 D_4}{(\lambda_3 + \lambda_4)[(\kappa_1 + \lambda_3)(\kappa_2 + \lambda_3) - \epsilon\delta]^2 [(\kappa_1 + \lambda_4)(\kappa_2 + \lambda_4) - \epsilon\delta]^2} \right\}, \quad (\text{C22})$$

where

$$\Phi_{13} \equiv (\kappa_1 \kappa_2 - \epsilon \delta) \{ \delta^2 (\kappa_1 + \kappa_2)^2 - 2\epsilon(\epsilon - \delta) [\delta(\epsilon - \delta) - \kappa_2(\kappa_1 + \kappa_2)] \} + \lambda_3 (\kappa_1 + \kappa_2) \{ \delta^2 (\kappa_1 + \kappa_2)^2 - 4\epsilon(\epsilon - \delta) [\delta(\epsilon - \delta) - \kappa_2(\kappa_1 + \kappa_2)] \} + \lambda_3^2 \{ \delta^2 (\kappa_1 + \kappa_2)^2 - 2\epsilon(\epsilon - \delta) [\delta(\epsilon - \delta) - (\kappa_1 + \kappa_2)(\kappa_1 + 2\kappa_2)] \}, \quad (C23)$$

$$\Phi_{23} \equiv (\kappa_1 \kappa_2 - \epsilon \delta) \{ 2\delta^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [\kappa_2 [2\delta^2 + (\kappa_1 + \kappa_2)^2] - \epsilon \delta (\kappa_1 + 3\kappa_2)] \} + \lambda_3 (\kappa_1 + \kappa_2) \{ 4\delta^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [\kappa_2 [4\delta^2 + (\kappa_1 + \kappa_2)(2\kappa_1 + \kappa_2)] - 2\epsilon \delta (\kappa_1 + 3\kappa_2)] \} + \lambda_3^2 \{ 2\delta^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(\kappa_1 + 2\kappa_2)(2\delta^2 + (\kappa_1 + \kappa_2)^2) - \epsilon \delta (3\kappa_1 + 5\kappa_2)] \} + \lambda_3^3 (\kappa_1 + \kappa_2)^3, \quad (C24)$$

$$\Phi_{14} \equiv (\kappa_1 \kappa_2 - \epsilon \delta) \{ \epsilon^2 (\kappa_1 + \kappa_2)^2 - 2\delta(\epsilon - \delta) [\epsilon(\epsilon - \delta) + \kappa_1(\kappa_1 + \kappa_2)] \} + \lambda_4 (\kappa_1 + \kappa_2) \{ \epsilon^2 (\kappa_1 + \kappa_2)^2 - 4\delta(\epsilon - \delta) [\epsilon(\epsilon - \delta) + \kappa_1(\kappa_1 + \kappa_2)] \} + \lambda_4^2 \{ \epsilon^2 (\kappa_1 + \kappa_2)^2 - 2\delta(\epsilon - \delta) [\epsilon(\epsilon - \delta) + (\kappa_1 + \kappa_2)(2\kappa_1 + \kappa_2)] \}, \quad (C25)$$

$$\Phi_{24} \equiv (\kappa_1 \kappa_2 - \epsilon \delta) \{ 2\epsilon^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [\kappa_1 [2\epsilon^2 + (\kappa_1 + \kappa_2)^2] - \epsilon \delta (3\kappa_1 + \kappa_2)] \} + \lambda_4 (\kappa_1 + \kappa_2) \{ 4\epsilon^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [\kappa_1 [4\epsilon^2 + (\kappa_1 + \kappa_2)(\kappa_1 + 2\kappa_2)] - 2\epsilon \delta (3\kappa_1 + \kappa_2)] \} + \lambda_4^2 \{ 2\epsilon^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(2\kappa_1 + \kappa_2)(2\epsilon^2 + (\kappa_1 + \kappa_2)^2) - \epsilon \delta (5\kappa_1 + 3\kappa_2)] \} + \lambda_4^3 (\kappa_1 + \kappa_2)^3, \quad (C26)$$

$$\Phi_{33} \equiv (\kappa_1 + \kappa_2)^3 (\kappa_2 + \lambda_3)^3 + \delta \{ \delta^3 (\kappa_1^2 + 3\kappa_1 \kappa_2 + \kappa_2^2) - \epsilon \delta^2 (\epsilon - \delta)^2 + 2\delta \kappa_2 (2\kappa_1 + \kappa_2) [\kappa_2 (\kappa_1 + \kappa_2) - \epsilon \delta] - \epsilon \kappa_2 [\kappa_2 (\kappa_1 + \kappa_2) (3\kappa_1 + \kappa_2) - \epsilon \delta (3\kappa_1 + 2\kappa_2)] + 3\lambda_3 (\kappa_1 + \kappa_2) (\epsilon - \delta) [\delta(\epsilon - \delta) - 2\kappa_2 (\kappa_1 + \kappa_2)] + \lambda_3^2 [\delta(\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(4\delta - 5\epsilon)\kappa_2 + (2\delta - 3\epsilon)\kappa_1]] \}, \quad (C27)$$

$$\Phi_{44} \equiv (\kappa_1 + \kappa_2)^3 (\kappa_1 + \lambda_4)^3 + \epsilon \{ \epsilon^3 (\kappa_1^2 + 3\kappa_1 \kappa_2 + \kappa_2^2) - \epsilon^2 \delta (\epsilon - \delta)^2 + 2\epsilon \kappa_1 (\kappa_1 + 2\kappa_2) [\kappa_1 (\kappa_1 + \kappa_2) - \epsilon \delta] - \delta \kappa_1 [\kappa_1 (\kappa_1 + \kappa_2) (\kappa_1 + 3\kappa_2) - \epsilon \delta (2\kappa_1 + 3\kappa_2)] + 3\lambda_4 (\kappa_1 + \kappa_2) (\epsilon - \delta) [\epsilon(\epsilon - \delta) + 2\kappa_1 (\kappa_1 + \kappa_2)] + \lambda_4^2 [\epsilon(\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(4\epsilon - 5\delta)\kappa_1 + (2\epsilon - 3\delta)\kappa_2]] \}, \quad (C28)$$

$$\begin{aligned} \frac{1}{2} \Phi_{34} \equiv & (\kappa_1 \kappa_2 - \epsilon \delta) \{ \epsilon^2 \delta^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(2\kappa_1 + \kappa_2) \epsilon^2 \kappa_2 + (\kappa_1 + 2\kappa_2) \delta^2 \kappa_1] - \epsilon \delta [(\epsilon \kappa_2 - \delta \kappa_1)^2 - 2\kappa_1 \kappa_2 ((\kappa_1 + \kappa_2)^2 - \epsilon^2 - \delta^2)] \} (\lambda_3 + \lambda_4) + 2(\kappa_1 + \kappa_2) (\kappa_1 \kappa_2 - \epsilon \delta) [(\kappa_1 + \kappa_2) (\epsilon^2 \kappa_2 + \delta^2 \kappa_1) - \epsilon \delta (\epsilon - \delta)^2] (\lambda_3^2 + \lambda_4^2) \\ & + 2(\kappa_1 + \kappa_2) \{ 2\epsilon^2 \delta^2 (\epsilon - \delta)^2 + (\kappa_1 + \kappa_2) [(3\kappa_1 + \kappa_2) \epsilon^2 \kappa_2 + (\kappa_1 + 3\kappa_2) \delta^2 \kappa_1] - 2\epsilon \delta [(\epsilon \kappa_2 - \delta \kappa_1)^2 - \kappa_1 \kappa_2 ((\kappa_1 + \kappa_2)^2 - 2\epsilon^2 - 2\delta^2)] \} \lambda_3 \lambda_4 + (\kappa_1 \kappa_2 - \epsilon \delta) [(\kappa_1 + \kappa_2) (\epsilon^2 \kappa_2 + \delta^2 \kappa_1) - \epsilon \delta (\epsilon - \delta)^2] (\lambda_3^3 + \lambda_4^3) \\ & + [4(\kappa_1 + \kappa_2)^2 + \kappa_1 \kappa_2 - \epsilon \delta] [(\kappa_1 + \kappa_2) (\epsilon^2 \kappa_2 + \delta^2 \kappa_1) - \epsilon \delta (\epsilon - \delta)^2] (\lambda_3^2 \lambda_4 + \lambda_3 \lambda_4^2) \\ & + 2(\kappa_1 + \kappa_2) \{ (\kappa_1 + \kappa_2) [(\kappa_1 + \kappa_2) (\epsilon - \delta)^2 + 2(\epsilon^2 \kappa_2 + \delta^2 \kappa_1)] - 2\epsilon \delta (\epsilon - \delta)^2 \} \lambda_3^2 \lambda_4^2 \\ & + 2(\kappa_1 + \kappa_2) [(\kappa_1 + \kappa_2) (\epsilon^2 \kappa_2 + \delta^2 \kappa_1) - \epsilon \delta (\epsilon - \delta)^2] (\lambda_3^3 \lambda_4 + \lambda_3 \lambda_4^3). \end{aligned} \quad (C29)$$

The TUR will hold for the 4D stochastic white noise system with

$$\frac{\mathcal{D}_{\mathcal{W}} \langle \dot{\mathcal{S}}_{\text{total}} \rangle}{\langle \dot{\mathcal{W}} \rangle^2} \geq 1. \quad (C30)$$

For the case of $D_3 = 0$, $\mathcal{D}_{\mathcal{W}}$ reduces to Eq. (29).

APPENDIX D: ANALYTIC RESULTS VERIFIED BY LANGEVIN DYNAMICS SIMULATIONS

We also performed numerical simulations of the Langevin equation to verify our theoretical results. The dimensionless Langevin equations for the Brownian bead in a hybrid bath governed by Eqs. (1) and (2) are solved numerically using the Euler-Maruyama scheme with a time step of $\delta t = 5 \times 10^{-4}$. After a sufficient time for achieving the steady state, a stochastic trajectory of a duration of 20 time units is generated for further measurement of various observables and stochastic energetic quantities. Ensemble averages are computed using up to 10 000 stochastic trajectories. For simulations, we consider the more general case of the active noises acting on both the x_1 and x_2 components ($D_3 \neq 0$ and $D_4 \neq 0$).

In the simulation, we first verified that the mean measured work over a period of τ in the steady state of the 2D AOUP

gyrator, $\langle W \rangle$, is proportional to τ , indicating constant power in the steady state. The work variance is also measured from the simulations and confirmed to display a diffusive behavior that is proportional to τ . As shown in Fig. 11(a), the measured mean work, heat, and work diffusion coefficient (symbols) for the 2D AOUP gyrator all agree well with the theoretical formulas (curves). Figure 11(b) displays the measured mean EP rate, which indicates that the EP can be negative in some regimes, as predicted by the theoretical formula. The key quantity in the TUR, $\frac{\mathcal{D}_{\mathcal{W}} \langle \dot{\mathcal{S}} \rangle}{\langle \dot{\mathcal{W}} \rangle^2}$, can be less than unity and can even be negative (in the negative EP regime), violating the TUR.

On the other hand, Fig. 12 shows the results from the 4D white noise gyrator point of view. As shown in Fig. 12(a), the measured mean work, heat, and work diffusion coefficients for the 4D white noise gyrator agree well with the theoretical

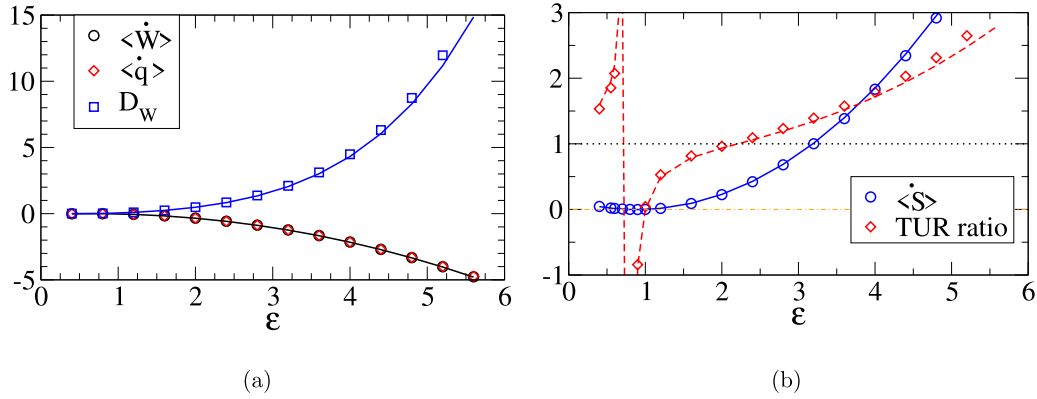


FIG. 11. Langevin simulations results for the 2D AOUP gyrotor with $\kappa_1 = 2$, $\kappa_2 = 1.5$, $\tau_1 = 1$, $\tau_2 = 0.5$, $\delta = 0.5$, $D_1 = 1$, $D_2 = 0.5$, $D_3 = 0.1$, and $D_4 = 0.5$. Symbols: simulation results. The corresponding theoretical results are denoted by the curves. (a) $\langle \dot{W} \rangle$, $\langle \dot{Q} \rangle$, and D_W plotted as a function of ϵ . (b) $\langle \dot{S} \rangle$ and the TUR ratio $D_W \langle \dot{S} \rangle / \langle \dot{W} \rangle^2$ plotted as a function of ϵ . The horizontal dotted and dot-dashed lines mark the values of unity and zero.

formulas. Figure 12(b) displays the measured mean total EP rate, which indicates that $\langle \dot{S}_{\text{total}} \rangle$ is positive, agreeing well with the theoretical formula (24). The key quantity in the TUR,

$D_W \langle \dot{S} \rangle / \langle \dot{W} \rangle^2$, always exceeds unity, indicating the validity of the TUR when hidden degrees of freedom are properly taken into account.

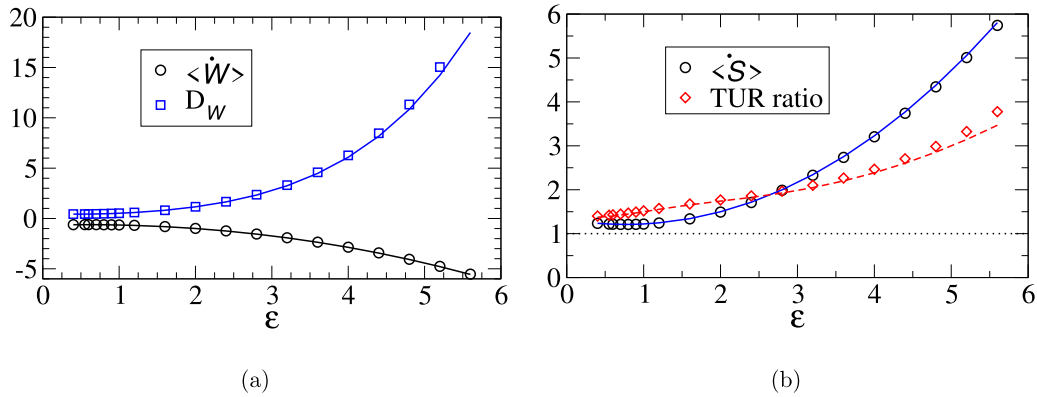


FIG. 12. Langevin simulations results for the 2D AOUP gyrotor as a 4D white noise gyrotor with $\kappa_1 = 2$, $\kappa_1 = 1.5$, $\tau_1 = 1$, $\tau_2 = 0.5$, $\delta = 0.5$, $D_1 = 1$, $D_2 = 0.5$, $D_3 = 0.1$, and $D_4 = 0.5$. Symbols: simulation results. The corresponding theoretical results are denoted by the curves. (a) $\langle \dot{W} \rangle$ and D_W plotted as a function of ϵ . (b) $\langle \dot{S} \rangle$ and the TUR ratio $D_W \langle \dot{S} \rangle / \langle \dot{W} \rangle^2$ plotted as a function of ϵ . The corresponding theoretical predictions are shown by the solid and dashed curves. The horizontal dotted line marks the value of unity.

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