

Discovering quasiorder parameters in the Potts model: A bridge between machine learning and critical phenomena

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Machine-learning (ML) models trained on Ising spin configurations have demonstrated surprising effectiveness in classifying phases of Potts models, even when processing severely reduced representations that retain only two spin states. To unravel this remarkable capability, we identify a family of alternative order parameters for the $q = 3$ and $q = 4$ Potts models on a square lattice, constructed from the occupancies of secondary and minimal spin states rather than the conventional dominant-state order parameter. Through systematic finite-size scaling analyses, we demonstrate that these quantities, along with a magnetizationlike quantity derived from a reduced spin representation, accurately capture critical behavior, yielding critical temperatures and exponents consistent with established theoretical predictions and numerical benchmarks. Furthermore, we rigorously establish the fundamental relationships between these alternative (quasi)order parameters, demonstrating how they collectively encode criticality through different aspects of spin configurations. Our results clarify, within this specific setting, how reduced spin representations can retain the essential thermodynamic information needed for identifying critical behavior. Taken together, this work establishes a concrete bridge between Ising-trained ML models and critical phenomena in Potts systems by showing that Potts criticality can be encoded in more compact, nontraditional forms, thereby opening avenues for discovering analogous order parameters in broader spin systems.

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I. INTRODUCTION

Recent advances in machine learning (ML) have profoundly impacted the scientific studies [1–3]. In condensed matter and statistical physics, ML has enabled tasks such as phase classification [4–14], topological feature identification [12,13,15–18], ground-state searching [19–22], and accelerated Monte Carlo sampling [23–27]. Central to these applications is the ability of artificial neural networks (ANNs) to recognize patterns in high-dimensional data, such as spin configurations generated via Monte Carlo simulations. Both supervised [8,9,28] and unsupervised [4–7,29] approaches have proven successful in analyzing such simulation data. The Ising model has emerged as a key test bed for these ML techniques due to its simplicity and well-characterized critical behavior, with ANNs successfully determining its critical temperature and exponents [5,8,30–32], establishing a paradigm for data-driven investigations of phase transitions.

Despite these successes, the “black-box” nature of ANNs poses a significant challenge: Their decision-making processes and learned features often lack interpretability, limiting both reliability and generalizability. Recent work has begun addressing these limitations by examining the underlying mechanisms behind critical temperature determination [33] and exploring generalization in ML applications [34,35]. These concerns become particularly salient when extending ML models to systems beyond their training data, such as from the Ising model to Potts model—a generalization of the former from $Z(2)$ to $Z(q)$ symmetry. The Potts model serves as an ideal framework to explore the applicability [36–41] and generalizability [42–47] of the trained networks across physical systems, as its q -component spin configurations differ fundamentally from the binary Ising spins. Furthermore, the universality classes of the Potts model also govern the phase transition of $SU(N)$ gauge theories, so this line of study also has direct impact to high-energy physics.

This fundamental difference necessitates careful reconciliation between Ising and Potts spin representations when studying generalizability. Recent work has proposed various procedures to bridge this gap [42–44,46]. In Ref. [46], a mapping procedure of spin configuration from q components to first two components, i.e., $\{1, 2, 3, \dots, q\} \mapsto \{-1, 1, 0, \dots, 0\}$, is employed without loss of generality. Remarkably, ANNs trained on Ising data can classify phases and extract critical exponents from these mapped Potts configurations for different q value and lattice geometry [46],

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suggesting that critical properties persist in these simplified spin representations. Yet the fundamental reasons for this success, specifically what physical quantities the networks actually learn from reduced configurations, remain unresolved.

In this work, we address this gap by identifying alternative (quasi)order parameters for the Potts model that encode the model's critical behavior. Building on established evidence that ANNs trained on Ising model configurations effectively learn magnetizationlike features [8,33], we hypothesize that when applied to reduced Potts model representations, these networks similarly extract quasiorder parameters analogous to magnetization, specifically, quantities proportional to the sum of all spin values. Furthermore, we propose a family of quantities derived not only from dominant spin state (conventional order parameter) but also from secondary and minimal occupancies, revealing a hidden encoding of critical properties. Through finite-size scaling (FSS) analyses of $q = 3$ and $q = 4$ Potts models on square lattices, we demonstrate that these alternative (quasi)order parameters—including a magnetization analog matching the form of Z_5 clock model order parameters [48] and representing a static version of dynamic Potts order parameters [49]—accurately reproduce established critical temperatures and exponents. Our results illustrate how the mapping procedure preserves the relevant thermodynamic information needed for identifying criticality in the Potts model, thereby explaining why machine-learning models trained on the Ising model can succeed in this setting. This demonstrates that critical behavior can remain encoded in more compact, nontraditional forms of representation.

This paper is organized as follows: Sec. II gives a brief introduction to the Ising and Potts models, detailing their conventional order parameters and Monte Carlo simulation methodology. In Sec. III, we propose alternative (quasi)order parameters, rigorously validate their critical behavior through finite-size scaling, and elucidate their connection to ML-compatible mappings of spin configurations. Finally, in Sec. IV we summarize our key findings and discuss their implications for both statistical physics and machine-learning applications. Complementary finite-size scaling analyses supporting our main results are presented in the Appendix.

II. THE ISING AND POTTS MODELS

The Hamiltonian of the ferromagnetic nearest-neighbor Ising model with vanishing external magnetic field reads

$$H_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \quad (1)$$

where $\sigma \in \{-1, 1\}$ denotes the spin configurations, $\langle i, j \rangle$ stands for the nearest neighboring sites i and j , and the coupling J is set to unity. This system undergoes a second-order phase transition at the critical temperature $T_c = 2/\ln(1 + \sqrt{2}) \approx 2.269$. The order parameter is the magnetization per site: $M = |\sum_i \sigma_i|/N$, where N is the total number of spins. Below T_c , the system is ordered with $\langle M \rangle > 0$, while above T_c , the system is disordered with $\langle M \rangle = 0$.

The Potts model is a generalization of the Ising model from two to q states. The Hamiltonian of ferromagnetic Potts model

reads

$$H_{\text{Potts}} = -J \sum_{\langle i,j \rangle} \delta_{\text{Kr}}(\sigma_i, \sigma_j), \quad (2)$$

where $\sigma \in \{1, \dots, q\}$ and the Kronecker delta δ_{Kr} evaluates to 1 if $\sigma_i = \sigma_j$ and 0 otherwise. For $q = 2$, this reduces to the Ising model up to a factor of 2 in the coupling constant. The Ising model and Potts model can be defined for different lattice geometries and dimensions. For two-dimensional ($d = 2$) q -state Potts model on a square lattice, the critical temperature $T_c = 1/\ln(1 + \sqrt{q})$ [50], with second-order phase transitions for $q \leq 4$ [51] and first order for $q \geq 5$ [52]; see also the review [53] and references therein. In this work, we restrict our attention to $q \leq 4$ Potts model on a square lattice in two dimensions ($d = 2$). The order parameter for q -state Potts model is usually defined as

$$M_{\text{Potts}} = \frac{q \frac{\max\{N_i\}}{N} - 1}{q - 1}, \quad (3)$$

where N_i ($i = 1, 2, \dots, q$) is the number of spins in state i in the configuration of the Potts model and is referred to as the multiplicity of state i . $N = \sum_i N_i$ is the total number of spins and equals to L^d with L the lattice size. This definition actually makes use of the occupancy of the most numerous state in each configuration $\max\{N_i\}/N$. For $q = 2$, it is equivalent to the conventional order parameter of the Ising model, i.e., the global magnetization

$$M_{\text{Ising}} = \frac{|N_1 - N_2|}{N}, \quad (4)$$

where N_1 and N_2 are referred to as the number of spins -1 and $+1$, respectively. N_1/N and N_2/N are not independent and we actually have

$$\frac{\max\{N_i\}}{N} = \frac{|N_1 - N_2|}{2N} + \frac{1}{2}. \quad (5)$$

Since the definition of M_{Potts} in Eq. (3) contains the maximal occupancy, in the following we use M_{max} to label M_{Potts} to distinguish the alternative (quasi)order parameters we will propose for the Potts model.

In our Monte Carlo simulations, we use the Swendsen-Wang algorithm [54] by the package in GitHub [55] to generate spin configurations with periodic boundary conditions. Each lattice is first equilibrated for 1000 cluster updates, after which spin configurations are sampled every k cluster updates, where k is the number of updates to decorrelate the current update with previous ones for each lattice size setup. A total of 64 sampling temperatures T are used in Monte Carlo simulations with an interval $0.002J$, and the critical temperature T_c is covered. For lattice of size $L \times L$ with $L = \{16, 32, 48, 64, 80, 96\}$, 100 000 samples are generated for each temperature.

Taking $q = 3$ Potts model on a square lattice ($d = 2$) as an example, we show the simulation results of the order parameter, i.e., the ensemble average of $M_{\text{max}}(T)$ for different lattice size L in Fig. 3 in the Appendix. Theoretically, $\bar{M}_{\text{max}}(T)$ satisfies that $\bar{M}_{\text{max}} = 0$ when $T > T_c$ and $\bar{M}_{\text{max}} > 0$ when $T < T_c$. The Monte Carlo lattice simulation shows the finite-size effect for this picture in the left panel. Near T_c , the order parameter $\bar{M}$ in general satisfies $\bar{M}(L, T) = L^{-\Delta_\sigma} \tilde{m}(tL^{1/\nu})$, where

$t = (T - T_c)/J$, $\nu = 5/6$, and $\Delta_\sigma = 2/15$ are the critical exponent and scaling dimension [53]. With the finite-size scaling procedure, the plot $\bar{M}(L, T)L^{\Delta_\sigma} - T$ with different lattice size L shows a crossing point at T_c and the plot $\bar{M}(L, T)L^{\Delta_\sigma} - tL^{1/\nu}$ with different L presents a data collapse on top of each other, as shown in the middle and right panels of Fig. 3 in the Appendix, respectively.

III. ALTERNATIVE (QUASI)ORDER PARAMETERS FOR POTTS MODEL

Reference [46] demonstrated a remarkable capability of ANNs in classifying phases across different spin models. ANNs were first trained using Ising model spin configurations across various temperatures to classify phases. Then, these Ising-trained ANNs were successfully applied to classify phases in Potts models of the same lattice size. To bridge the fundamental difference in spin representations between these models (binary spins in Ising versus q -state spins in Potts), an elegant mapping procedure $\{1, 2, 3, \dots, q\} \mapsto \{-1, 1, 0, \dots, 0\}$ was introduced for $q \leq 7$, where only first two spin states are preserved while others are set to zero. This mapping maintained all essential physics while providing compatibility with the Ising-trained networks. Remarkably, these ANNs not only accurately classified Potts model phases but also correctly extracted critical exponents for $q \leq 4$ cases. The success of this mapping procedure inspired our current investigation into alternative order parameters that might underlie this surprising generalizability.

A. $q = 3$ Potts model on a square lattice

For the $q = 3$ case, the mapping of three spin components takes the form

$$\{1, 2, 3\} \mapsto \{-1, 1, 0\}, \quad (6)$$

where we keep the first two components and ignore the third one by setting it as zero. Building on the established understanding that ANNs trained on Ising models learn to detect magnetization patterns, i.e., the sum of all spin values, we define a corresponding quantity for the mapped Potts model:

$$M_{\text{map}} = \frac{|N_1 - N_2|}{N}. \quad (7)$$

M_{map} represents a natural generalization of order parameter for the Ising model. For $q = 2$ Potts model, M_{map} is equivalent to M_{max} in Eq. (3), as demonstrated by Eq. (5). Our finite-size scaling analysis (shown in Fig. 4 in the Appendix) demonstrates that M_{map} successfully captures critical behavior using the known critical exponents ν and Δ_σ , confirming its validity as an order parameter. In this work, we are concerned with the static critical properties of Potts model in equilibrium and notably M_{map} could be viewed as a static version of the alternative order parameter for Potts model at an early stage of the time evolution of dynamics at nonequilibrium [49]. It is also worth noting that M_{map} has the same form of the order parameter for Z_5 model, *aka* five-state clock model [48].

To relate M_{max} and M_{map} theoretically, we can make use of the minimal state occupancy in each configuration $\min\{N_i\}/N$. As we know, $\min\{N_i\}/N$ also tends to $1/q$ when $T > T_c$. While at low T , this occupancy is suppressed by the dominant

state. So, here we construct a quantity M_{min} by replacing the occupancy of the most state in each configuration $\max\{N_i\}/N$ in Eq. (3) with the least one, $\min\{N_i\}/N$:

$$M_{\text{min}} = \frac{q^{\frac{\min\{N_i\}}{N}} - 1}{q - 1}. \quad (8)$$

We assume that in one configuration, the occupancy of three states descends as a, b, c with $a + b + c = 1$. With the $Z(3)$ symmetry, we could have $A_3^3 = 6$ cases of how these three occupancies are assigned to these three states. After smearing a specific state of these three ones with the mapping procedure, the average M_{map} of these six configurations equals $2(a - c)/3$. Since $M_{\text{max}} = (3a - 1)/2$ and $M_{\text{min}} = (3c - 1)/2$, as in Eqs. (3) and (8), the equilibrium averages satisfy the fundamental relationship

$$\bar{M}_{\text{map}} = \frac{4}{9}(\bar{M}_{\text{max}} - \bar{M}_{\text{min}}), \quad (9)$$

where the overline signifies the ensemble average. This is also confirmed numerically in the upper panel of Fig. 1, where M_{max} , M_{min} , and M_{map} as functions of temperature T with lattice size $L = 96$ are shown for visual comparison and the curve of M_{map} labeled with red triangles and that of $\frac{4}{9}(M_{\text{max}} - M_{\text{min}})$ labeled with gray dashed line collapse on top of each other. In the middle and lower panels of Fig. 1, we show the standard error of mean (SEM) and relative standard error (RSE), i.e., the ratio of SEM over mean, of these quantities, respectively. Due to the high statistics in this work (100 000 samples), the standard errors of mean are too small to be visualized along with their mean values in the upper panel. One can observe that below T_c the fluctuations of M_{map} are much larger than those of other quantities, particularly $\frac{4}{9}(M_{\text{max}} - M_{\text{min}})$, which are mainly due to the nonthermal fluctuations between those aforementioned six symmetric configurations for M_{map} arising from the different treatment on all Potts states.

One can easily understand these nonthermal fluctuations in M_{map} at $T = 0$ limit, where one expects full occupation of one of the Potts states and zero occupation of the other states, i.e., $a = 1$ and $b = c = 0$. For $q = 3$, there are $A_3^3/A_2^2 = 3$ equally likely possibilities,

$$(N_1, N_2, N_3)/N = (1, 0, 0), (0, 1, 0), (0, 0, 1). \quad (10)$$

For these three configurations, M_{map} takes the values 1, 1, and 0, despite that all three configurations are ordered to the same extent. This leads to a mean $\bar{M}_{\text{map}} = 2/3$, a standard deviation (SD) $M_{\text{map}}^{\text{SD}} = \sqrt{2}/3$, and a standard error of the mean $M_{\text{map}}^{\text{SEM}} = M_{\text{map}}^{\text{SD}}/\sqrt{100\,000} = 0.00149$. Note that these predictions for $T = 0$ are quite compatible with the simulation results in the upper two panels of Fig. 1. For general q states, $\bar{M}_{\text{map}} = 2/q$, $M_{\text{map}}^{\text{SD}} = \frac{\sqrt{2(q-2)}}{q}$, and $M_{\text{map}}^{\text{SEM}} = M_{\text{map}}^{\text{SD}}/\sqrt{n}$, where n is the total sample number.

Note that the fluctuations of M_{map} , quantified by its SEM, do not behave like a susceptibility with a clear peak near T_c for finite system size. In this sense, M_{map} can only be regarded as a quasiorder parameter. However, as T approaches T_c from above in the second panel of Fig. 1, the SEM for M_{map} rises steeply, as if approaching a peak, similar to the behavior of M_{max} and M_{min} . We interpret this as a susceptibility like

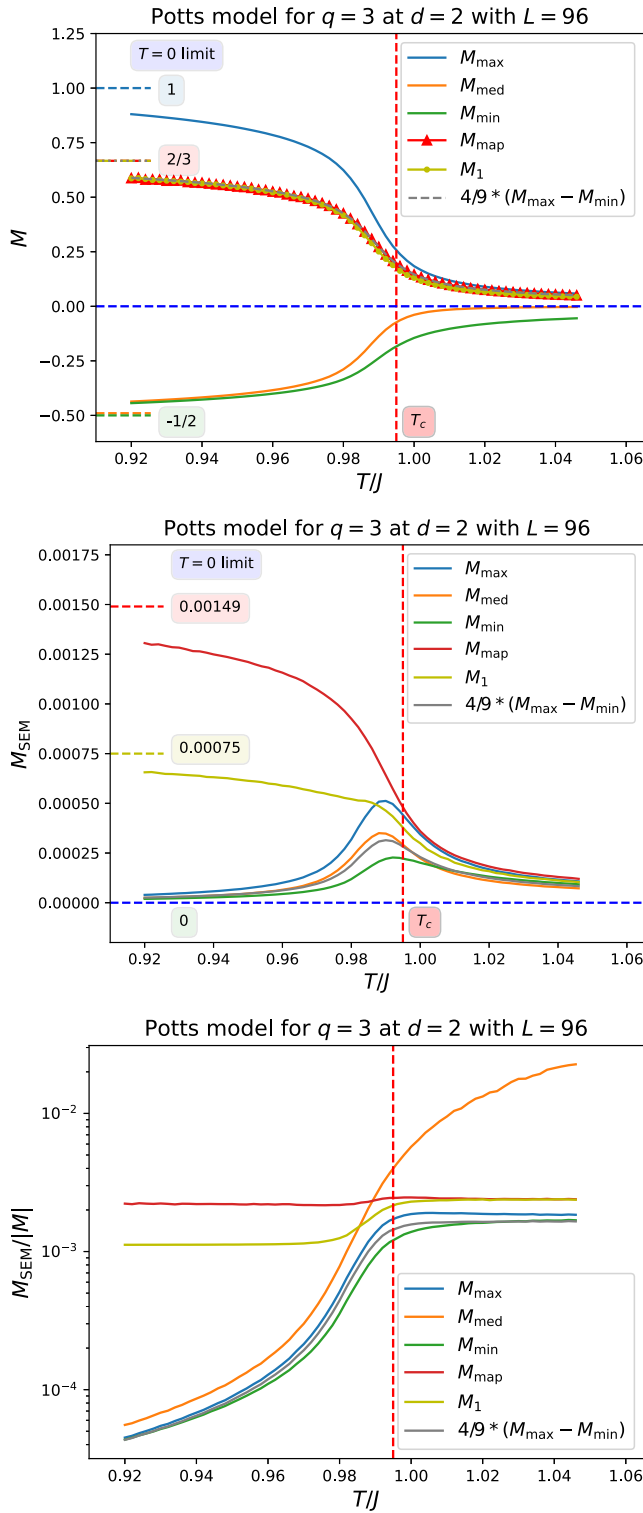


FIG. 1. The mean (upper), SEM (middle), and RSE (lower) of alternative order parameters $M_{\max}$, M_{med} , and $M_{\min}$ and quasioorder parameter M_{map} as functions of temperature T for the three-state Potts model at $d = 2$, simulated on a square lattice of size $L = 96$. The relation $M_{\text{map}} = \frac{4}{9}(M_{\max} - M_{\min})$ is confirmed by the overlap of the curves in the upper panel. The blue horizontal dashed lines mark zero to indicate the sign, while the red vertical dashed lines denote the theoretical critical temperature T_c .

enhancement that, however, is difficult to resolve for $T < T_c$ because of additional nonthermal fluctuations inherent to the definition of M_{map} , which increase with decreasing temperature, as the system orders increasingly, and become strongest at $T = 0$.

An even simpler quantity than M_{map} is

$$M_1 = \frac{|q \frac{N_1}{N} - 1|}{q - 1}. \quad (11)$$

It reduces to the Ising order parameter when $q = 2$ due to $|\frac{2N_1}{N} - 1| = \frac{|N_1 - N_2|}{N}$ and is expected to behave as an order parameter in general for q -state Potts model. However, it does not treat all Potts states equally and therefore introduces nonthermal fluctuations as M_{map} . This quantity is of interest since it only involves one of the Potts states and provides motivation for ANN studies to extract Potts critical properties from simplified configurations that only retain the occupancy of one of the q states. We include the corresponding results for M_1 in Fig. 1. Interestingly, the mean values of M_1 coincide with M_{map} and its $T = 0$ limit can be obtained analytically as $2/q$, consistent with that of M_{map} as well. The SEM of M_1 , with the limit $\frac{q-2}{q\sqrt{q-1}} \frac{1}{\sqrt{n}} = 0.00075$ at $T = 0$, is smaller than that of M_{map} . More generally, we find that any quantity, such as M_{map} and M_1 , which treats ordering in a subset of Potts states differently from the others, will have nonthermal fluctuations that grow as T decreases and may obscure the susceptibility peak associated with the thermal fluctuations near the critical point.

With Eq. (9), we can conjecture that $M_{\min}$ could also serve as an alternative order parameter. For $q = 3$, we also have the medium state for each configuration and therefore define

$$M_{\text{med}} = \frac{q \frac{\text{medium}\{N_i\}}{N} - 1}{q - 1}. \quad (12)$$

Due to the constraint that $M_{\max} + M_{\text{med}} + M_{\min} = 1$, only two of them are independent. We therefore conjecture that M_{med} could serve as an alternative order parameter as well. The temperature dependence of $|M_{\min}|$ and $|M_{\text{med}}|$ and their associated finite-size scaled results are shown, respectively, in Figs. 5 and 6 in the Appendix with the known values of ν and Δ_σ , which validates that $|M_{\min}|$ and $|M_{\text{med}}|$ could also serve as alternative order parameters. There we take the absolute value of $M_{\min}$ and M_{med} so that they increase monotonically as temperature decreases and reach a maximum at the ground state with $T = 0$, similar to the traditional order parameter $M_{\max}$. Any order parameter that is symmetric in the occupancies of the q states, such as $M_{\max}$, $M_{\min}$, or M_{med} , takes the same value for each of the three ground-state configurations, and therefore exhibits no fluctuations at $T = 0$. Specifically, at this limit one finds $M_{\max} = 1$, $M_{\min} = M_{\text{med}} = -1/2$, with both the SD and SEM vanishing. These predictions labeled by the short dashed lines are also quite compatible with the simulation results in the upper two panels of Fig. 1. From the lower panel of Fig. 1, it is obvious that although $M_{\max}$, M_{med} , and $M_{\min}$ all could serve as the order parameters of Potts model, which is consistent with the model's universality class and symmetry, they are not completely equivalent in the sense of their relative

TABLE I. Critical properties of the three-state Potts model at $d = 2$ on a square lattice, obtained using the alternative order parameters $M_{\max}$, M_{med} , and $M_{\min}$ and quasiorder parameter M_{map} . The critical temperature T_c , scaling dimension Δ_σ , and critical exponent ν , along with their standard errors, are estimated via finite-size scaling within the interval $T \in (0.920, 1.046)$ with step size $\Delta T = 0.002$. Theoretical values from the literature are included for comparison [53].

(Quasi)order parameter	T_c	Δ_σ	ν
$M_{\max}$	0.9935(1)	0.137(6)	0.84(2)
$M_{\min}$	0.9931(1)	0.139(4)	0.83(6)
M_{med}	0.9940(2)	0.138(16)	0.84(5)
M_{map}	0.9942(4)	0.134(16)	0.84(10)
Theoretical value	$1/\ln(1+\sqrt{3})$ ~ 0.9950	$2/15$ ~ 0.1333	$5/6$ ~ 0.8333

standard error, and $M_{\min}$ exhibits the minimal relative standard error for the whole critical temperature interval of interest.

With the FSS procedure [56], we locate the critical temperature and extract the critical exponents for all these order parameters $M_{\max}$, M_{med} , and $M_{\min}$ and quasiorder parameter M_{map} as detailed in Table I. They agree with the theoretical values very well and we conclude that all occupancies of these states along with the magnetizationlike quantity M_{map} can serve as (quasi)order parameters for $q = 3$ Potts model. In terms of quantifying order, the suppression of secondary or minimally occupied spin states is directly complementary to the dominance of the majority spin type. This suppression reflects the onset of symmetry breaking and forms the conceptual basis for proposing these quantities as alternative order parameters. While these quantities may be mathematically related to $M_{\max}$, they represent a distinct and meaningful perspective, i.e., focusing on the elimination of minority components rather than the rise of the dominant one. This conceptual shift provides complementary insight into the nature of order in the Potts models and helps to elucidate the features learned by machine-learning models that implicitly capture phase transitions.

It would be interesting to check whether the above observations and conclusion are valid for other q -state Potts models. In Ising model, we have two occupancies and only one is independent, which is a trivial case. So, we will turn to $q = 4$ Potts model in next subsection.

B. $q = 4$ Potts model on a square lattice

For the $q = 4$ case, besides the most and least states, we also have the second most and second least states for each configuration; therefore, we define

$$M_{\max}^{\text{sec}} = \frac{q \frac{\text{secmax}[N_i]}{N} - 1}{q - 1}, \quad (13)$$

$$M_{\min}^{\text{sec}} = \frac{q \frac{\text{secmin}[N_i]}{N} - 1}{q - 1}. \quad (14)$$

In the mapping procedure for $q = 4$ case [46], only first two states are kept in the configuration without loss of generality,

$$\{1, 2, 3, 4\} \mapsto \{-1, 1, 0, 0\}, \quad (15)$$

and M_{map} is defined the same as in Eq. (7). As the analysis for three-state Potts model, we assume that in one configuration, the occupancy of four states descends as a, b, c, d with $a + b + c + d = 1$. With the $Z(4)$ symmetry, we could have $A_4^4 = 24$ cases of how these four occupancies are assigned to these four states. After smearing two specific states of these four ones with the mapping procedure, the average M_{map} of these 24 configurations equals $(3(a - d) + (b - c))/6$. Since $M_{\max} = (4a - 1)/3$, $M_{\min} = (4d - 1)/3$, $M_{\max}^{\text{sec}} = (4b - 1)/3$, and $M_{\min}^{\text{sec}} = (4c - 1)/3$, as in Eqs. (3), (8), (13), and (14), the equilibrium averages satisfy the fundamental relationship

$$\bar{M}_{\text{map}} = \frac{1}{8}(3(\bar{M}_{\max} - \bar{M}_{\min}) + \bar{M}_{\max}^{\text{sec}} - \bar{M}_{\min}^{\text{sec}}), \quad (16)$$

where the overline signifies the ensemble average. This is also confirmed numerically in the upper panel of Fig. 2, where $M_{\max}$, $M_{\max}^{\text{sec}}$, $M_{\min}^{\text{sec}}$, $M_{\min}$, M_{map} , and M_1 as functions of temperature T with lattice size $L = 96$ are shown for visual comparison and the curves of M_{map} labeled with red triangles, M_1 labeled with yellow dots, and that of $\frac{1}{8}(3(M_{\max} - M_{\min}) + M_{\max}^{\text{sec}} - M_{\min}^{\text{sec}})$ labeled with gray dashed line collapse on top of each other. In the middle and lower panels of Fig. 2, we show the SEM and RSE of these quantities, respectively. Similar to $q = 3$ case, one can observe that below T_c the fluctuations of M_{map} and M_1 are much larger than those of other quantities, particularly $\frac{1}{8}(3(M_{\max} - M_{\min}) + M_{\max}^{\text{sec}} - M_{\min}^{\text{sec}})$, which are mainly due to the nonthermal fluctuations between those aforementioned 24 symmetric configurations for M_{map} arising from the different treatment on all Potts states.

We estimate the nonthermal fluctuations in M_{map} and M_1 at $T = 0$ limit where $a = 1$ and $b = c = d = 0$. For $q = 4$, there are $A_4^4/A_3^3 = 4$ equally likely possibilities,

$$(N_1, N_2, N_3, N_4)/N \\ = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1). \quad (17)$$

For these four configurations, M_{map} takes the values 1, 1, 0, and 0, despite that all four configurations are ordered to the same extent. This yields $\bar{M}_{\text{map}} = 1/2$, $M_{\text{map}}^{\text{SD}} = 1/2$, and $M_{\text{map}}^{\text{SEM}} = M_{\text{map}}^{\text{SD}}/\sqrt{100000} = 0.00158$. However, M_1 takes the values 1, 1/3, 1/3, and 1/3, leading to $\bar{M}_1 = 1/2$, $M_1^{\text{SD}} = \sqrt{1/12}$, and $M_1^{\text{SEM}} = M_1^{\text{SD}}/\sqrt{100000} = 0.00091$. These $T = 0$ predictions are fully consistent with the general q -state results discussed in the previous subsection and are quite compatible with the simulation data shown in the upper two panels of Fig. 2. Again, the fluctuations of M_{map} and M_1 , quantified by their SEM, do not behave like a susceptibility with a peak near the T_c for finite system size. In this sense, M_{map} and M_1 can only be regarded as quasiorder parameters.

Order parameters that are symmetric in the occupancies of the q states, such as $M_{\max}$, $M_{\min}$, $M_{\max}^{\text{sec}}$, or $M_{\min}^{\text{sec}}$, take the same value for each of the four ground-state configurations, and therefore exhibit no fluctuations at $T = 0$. Specifically, at this limit one finds $M_{\max} = 1$, $M_{\min} = M_{\max}^{\text{sec}} = M_{\min}^{\text{sec}} = -1/3$, with both the SD and SEM vanishing. These predictions labeled by the short dashed lines are also quite compatible with the simulation results in the upper two panels of Fig. 2.

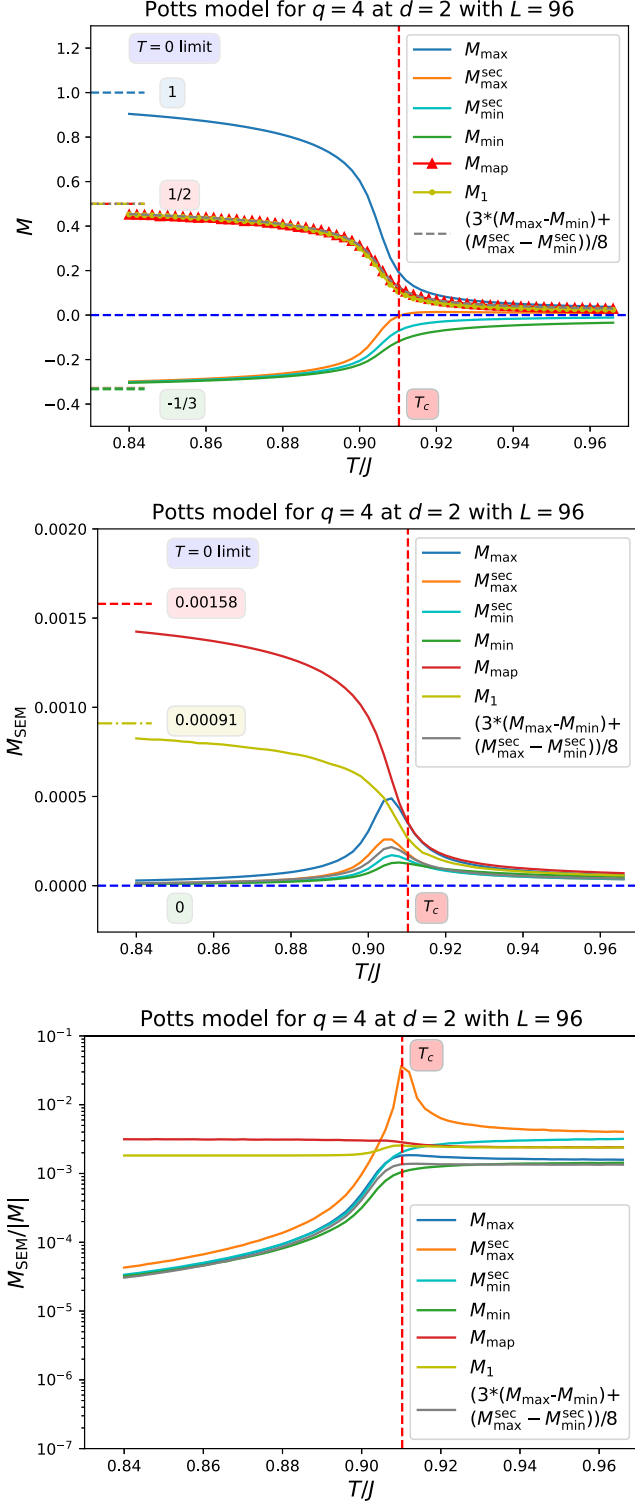


FIG. 2. The mean (upper), SEM (middle), and RSE (lower) of the alternative order parameters $M_{\max}$, $M_{\max}^{\text{sec}}$, $M_{\min}^{\text{sec}}$, $M_{\min}$, and quasiorder parameter M_{map} as functions of temperature T for the four-state Potts model at $d = 2$, simulated on a square lattice of size $L = 96$. The relation $M_{\text{map}} = \frac{1}{8}(3(M_{\max} - M_{\min}) + M_{\max}^{\text{sec}} - M_{\min}^{\text{sec}})$ is confirmed by the overlap of the curves in the upper panel. The blue horizontal dashed lines mark zero to indicate the sign, while the red vertical dashed lines denote the theoretical critical temperature T_c .

TABLE II. Critical properties of the four-state Potts model at $d = 2$ on a square lattice, obtained using the alternative order parameters $M_{\max}$, $M_{\max}^{\text{sec}}$, $M_{\min}^{\text{sec}}$, and $M_{\min}$ and quasiorder parameter M_{map} . The critical temperature T_c , scaling dimension Δ_σ , and critical exponent ν , along with their standard errors, are estimated via finite-size scaling within the interval $T \in (0.840, 0.966)$ with step size $\Delta T = 0.002$. For comparison, we include Monte Carlo numerical estimates of Δ_σ and ν that neglect higher-order corrections and mark them with * [57], along with the theoretical values from the literature [53].

(Quasi)order parameter	T_c	Δ_σ	ν
$M_{\max}$	0.909 48(8)	0.128(12)	0.723(27)
$M_{\max}^{\text{sec}}$	0.909 65(8)	0.128(42)	0.724(5)
$M_{\min}^{\text{sec}}$	0.909 45(8)	0.128(15)	0.721(3)
$M_{\min}$	0.909 16(8)	0.129(8)	0.72812(1)
M_{map}	0.9094(2)	0.128(22)	0.722(13)
Numerical estimate*	$1/\ln(1 + \sqrt{4})$ ~ 0.9102	0.128*	0.722*
Theoretical value	$1/\ln(1 + \sqrt{4})$ ~ 0.9102	$1/8 = 0.125$	$2/3$ ~ 0.667

The temperature dependence of $M_{\max}$, $-M_{\max}^{\text{sec}}$, $|M_{\min}^{\text{sec}}|$, $|M_{\min}|$, and M_{map} and their associated finite-size scaled results are shown, respectively, in Figs. 7–11 in the Appendix with the known Monte Carlo numerical estimates of $\nu = 0.722$ and $\Delta_\sigma = 0.128$ [57], which validates that all of these quantities could serve as alternative (quasi)order parameters. There we take the absolute value or reverse the sign of $M_{\min}$, $M_{\min}^{\text{sec}}$, and $M_{\max}^{\text{sec}}$ so that they increase monotonically as temperature decreases and reach a maximum at the ground state with $T = 0$, similar to the traditional order parameter $M_{\max}$. From the lower panel of Fig. 2, it is obvious that although $M_{\max}$, $M_{\max}^{\text{sec}}$, $M_{\min}^{\text{sec}}$, and $M_{\min}$ all could serve as the order parameters of Potts model, they are not completely equivalent in the sense of their relative standard error, and same as $q = 3$ case, $M_{\min}$ exhibits the minimal relative standard error for the whole critical temperature interval of interest. With the FSS procedure [56], we locate the critical temperature and extract the critical exponents for all these (quasi)order parameters as detailed in Table II. They agree with the existing numerical estimates of $\nu = 0.722$ and $\Delta_\sigma = 0.128$ [57] very well. The deviation from the theoretical values of $\nu = 2/3$ and $\Delta_\sigma = 1/8$ [53] arises from the omission of the higher-order corrections. We conclude that all occupancies of these states along with the magnetizationlike quantity M_{map} can serve as alternative (quasi)order parameters for $q = 4$ Potts model as well.

IV. CONCLUSION

Through systematic finite-size scaling analyses, we have discovered and validated a family of alternative (quasi)order parameters for the square-lattice $q = 3$ and $q = 4$ Potts models, grounded in the underlying principles of universality and symmetry. These include (1) the conventional order parameter based on dominant spin state occupancy, (2) order parameters derived from secondary and minimal spin states, and

(3) magnetizationlike quasiorder parameter emerging from reduced representations with only two spin components. We have rigorously established the intrinsic relationships between these alternative (quasi)order parameters, demonstrating that they collectively provide multiple consistent characterizations of critical behavior associated with different level of standard uncertainty. This unified framework explains the remarkable generalizability of Ising-trained machine-learning models to Potts systems. The models remain effective because criticality is redundantly encoded in these various (quasi)order parameters, persisting even when explicit spin state information is partially lost through representation reduction.

Motivated by these findings, we further note that other quantities, such as $|N_1/N - 1/q|$, may also serve as (quasi)order parameters. This corresponds to an even more drastic mapping in which all except one of the Potts states are mapped to zero. We find that any quantity, such as M_{map} and $M_1 \propto |N_1/N - 1/q|$, that treats ordering in a subset of the Potts states differently from ordering in the other Potts states will have nonthermal fluctuations that increase as T decreases and may obscure a peak in the thermal fluctuations near the critical point.

Our findings suggest that these (quasi)order parameters generalize to other q -state Potts models, different lattice geometries (e.g., triangular or 3D), and systems with extended interactions (antiferromagnetic couplings, next-nearest neighbors). They may further enable studies of real-time dynamics or nonequilibrium phase transitions. More broadly, this work provides a concrete example of how reduced spin representations can preserve the essential thermodynamic information

required for identifying criticality, highlighting that conventional order parameters are not unique descriptors of critical phenomena and pointing toward the possibility that analogous quasiorder parameters, and the representations implicitly used by machine-learning models, may arise in a wider class of interacting spin systems.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: FINITE-SIZE SCALING OF ALTERNATIVE (QUASI)ORDER PARAMETERS FOR $q = 3$ AND $q = 4$ POTTS MODEL ON A SQUARE LATTICE

In this Appendix, we present additional finite-size scaling analyses for the alternative (quasi)order parameters introduced in the main text for the $q = 3$ and $q = 4$ Potts models on a square lattice. These results complement the main discussion by providing detailed numerical evidence for the scaling behavior with known theoretical or numerical critical temperatures and critical exponents, thereby supporting the robustness of our conclusions.

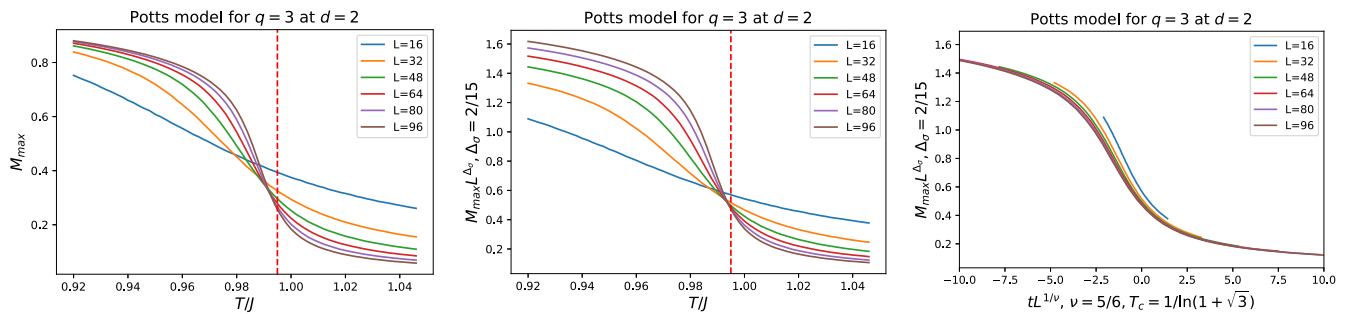


FIG. 3. (Left) Order parameter M_{max} as a function of temperature T for the $q = 3$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\text{max}} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\text{max}} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

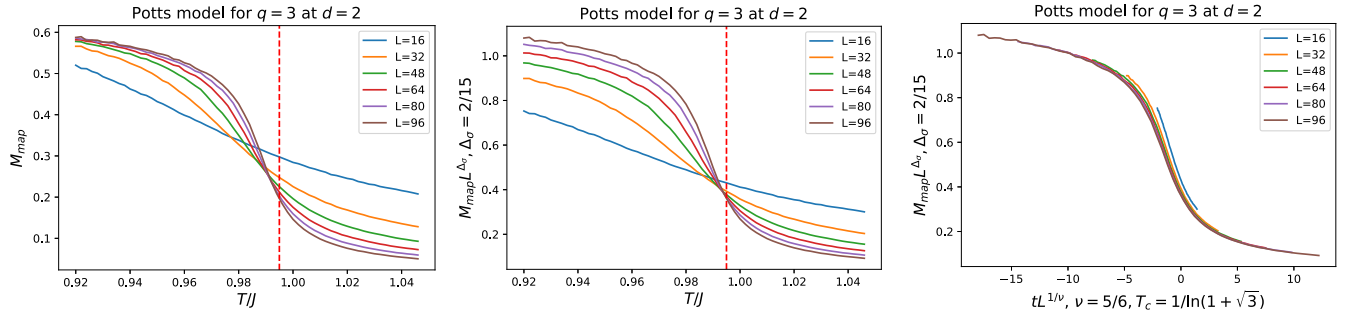


FIG. 4. (Left) Quasiorder parameter M_{map} as a function of temperature T for the $q = 3$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled quasiorder parameter $M_{\text{map}}L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled quasiorder parameter $M_{\text{map}}L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

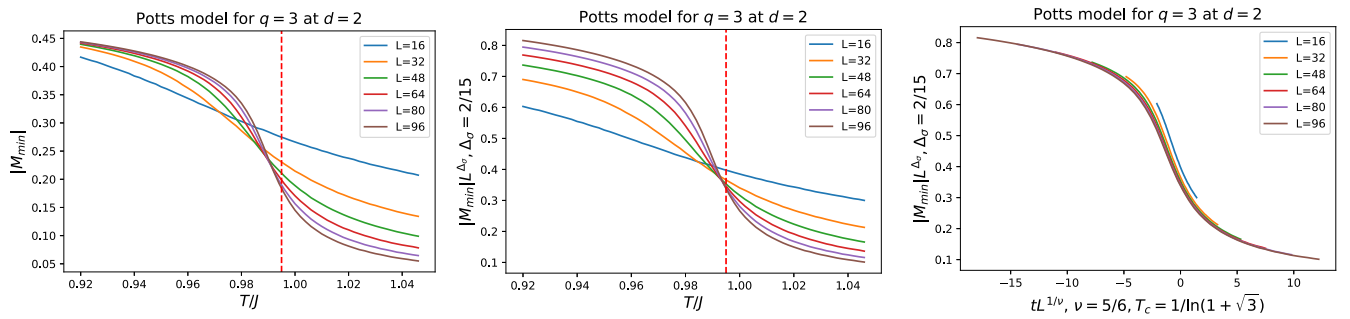


FIG. 5. (Left) Order parameter M_{min} as a function of temperature T for the $q = 3$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\text{min}}L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\text{min}}L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

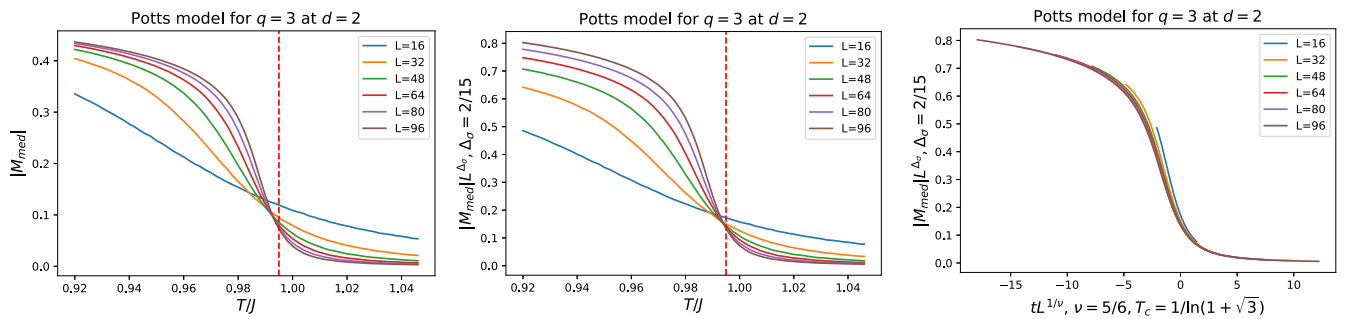


FIG. 6. (Left) Order parameter M_{med} as a function of temperature T for the $q = 3$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\text{med}}L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\text{med}}L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

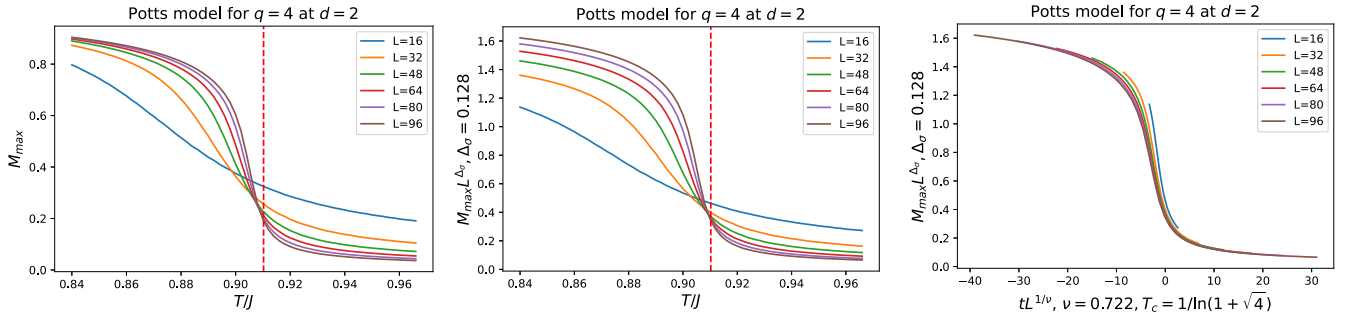


FIG. 7. (Left) Order parameter $M_{\max}$ as a function of temperature T for the $q = 4$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\max} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\max} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

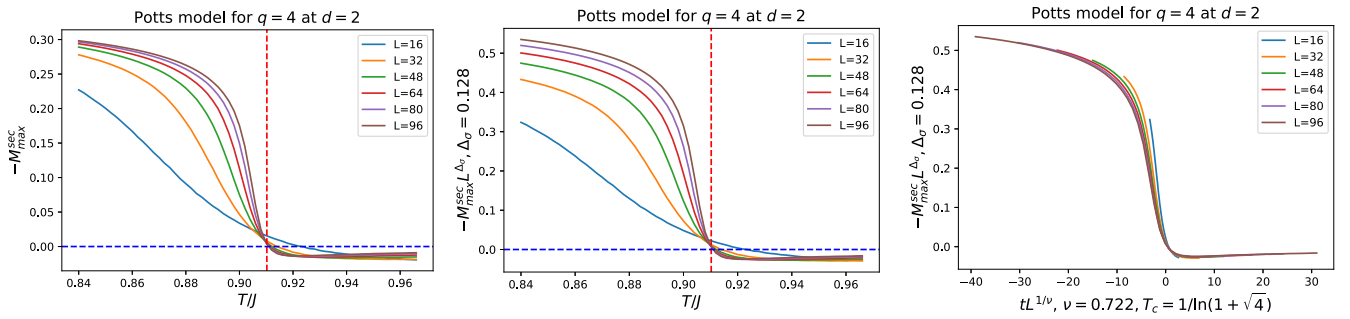


FIG. 8. (Left) Order parameter $M_{\max}^{\text{sec}}$ as a function of temperature T for the $q = 4$ Potts model on a square lattice with six different system sizes L . The blue horizontal dashed line marks zero to highlight the sign change. (Middle) Scaled order parameter $M_{\max}^{\text{sec}} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\max}^{\text{sec}} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

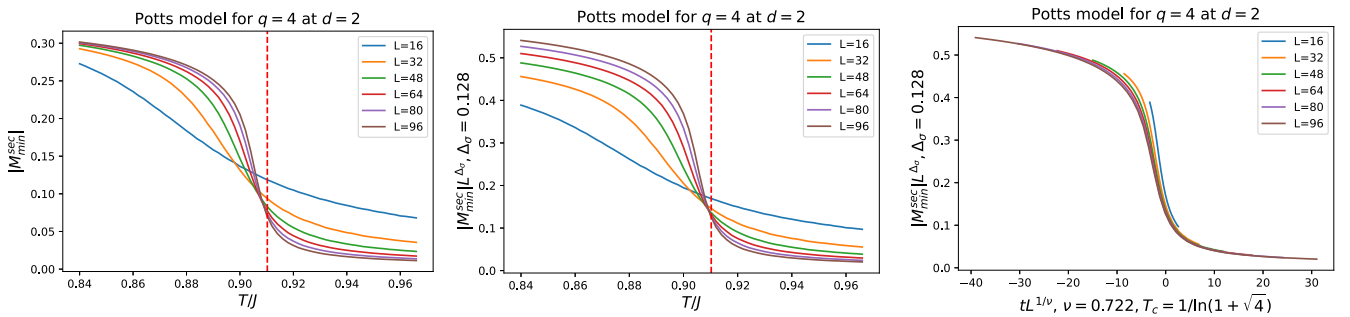


FIG. 9. (Left) Order parameter $M_{\min}^{\text{sec}}$ as a function of temperature T for the $q = 4$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\min}^{\text{sec}} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\min}^{\text{sec}} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

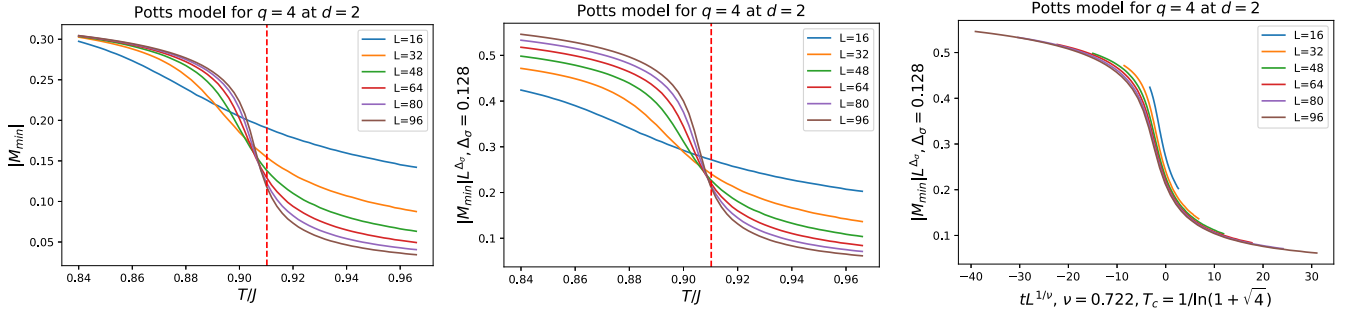


FIG. 10. (Left) Order parameter $M_{\min}$ as a function of temperature T for the $q = 4$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled order parameter $M_{\min} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled order parameter $M_{\min} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

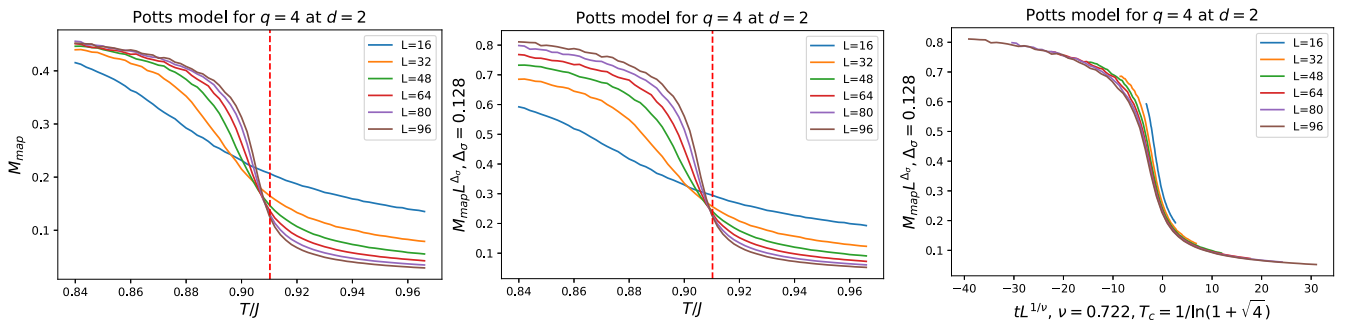


FIG. 11. (Left) Quasiorder parameter M_{map} as a function of temperature T for the $q = 4$ Potts model on a square lattice with six different system sizes L . (Middle) Scaled quasiorder parameter $M_{\text{map}} L^{\Delta_\sigma}$ vs T , with curves intersecting near the theoretical critical temperature T_c , marked by the red vertical dashed line. (Right) Scaled quasiorder parameter $M_{\text{map}} L^{\Delta_\sigma}$ against the rescaled temperature $tL^{1/\nu}$, demonstrating data collapse consistent with finite-size scaling.

- [1] M. I. Jordan and T. M. Mitchell, Machine learning: Trends, perspectives, and prospects, *Science* **349**, 255 (2015).
- [2] P. Mehta, M. Bukov, C.-H. Wang, A. G. Day, C. Richardson, C. K. Fisher, and D. J. Schwab, A high-bias, low-variance introduction to machine learning for physicists, *Phys. Rep.* **810**, 1 (2019).
- [3] G. Carleo, I. Cirac, K. Cranmer, L. Daudet, M. Schuld, N. Tishby, L. Vogt-Maranto, and L. Zdeborová, Machine learning and the physical sciences, *Rev. Mod. Phys.* **91**, 045002 (2019).
- [4] L. Wang, Discovering phase transitions with unsupervised learning, *Phys. Rev. B* **94**, 195105 (2016).
- [5] E. P. Van Nieuwenburg, Y.-H. Liu, and S. D. Huber, Learning phase transitions by confusion, *Nat. Phys.* **13**, 435 (2017).
- [6] S. J. Wetzel, Unsupervised learning of phase transitions: From principal component analysis to variational autoencoders, *Phys. Rev. E* **96**, 022140 (2017).
- [7] W. Hu, R. R. Singh, and R. T. Scalettar, Discovering phases, phase transitions, and crossovers through unsupervised machine learning: A critical examination, *Phys. Rev. E* **95**, 062122 (2017).
- [8] J. Carrasquilla and R. G. Melko, Machine learning phases of matter, *Nat. Phys.* **13**, 431 (2017).
- [9] S. J. Wetzel and M. Scherzer, Machine learning of explicit order parameters: From the Ising model to SU(2) lattice gauge theory, *Phys. Rev. B* **96**, 184410 (2017).
- [10] B. S. Rem, N. Käming, M. Tarnowski, L. Asteria, N. Fläschner, C. Becker, K. Sengstock, and C. Weitenberg, Identifying quantum phase transitions using artificial neural networks on experimental data, *Nat. Phys.* **15**, 917 (2019).
- [11] X.-Y. Dong, F. Pollmann, and X.-F. Zhang, Machine learning of quantum phase transitions, *Phys. Rev. B* **99**, 121104(R) (2019).
- [12] W. Zhang, J. Liu, and T.-C. Wei, Machine learning of phase transitions in the percolation and XY models, *Phys. Rev. E* **99**, 032142 (2019).
- [13] Y. Zhang, P. Ginsparg, and E.-A. Kim, Interpreting machine learning of topological quantum phase transitions, *Phys. Rev. Res.* **2**, 023283 (2020).
- [14] D. L. Boyda, M. N. Chernodub, N. V. Gerasimeniuk, V. A. Goy, S. D. Liubimov, and A. V. Molochkov, Finding the deconfinement temperature in lattice Yang-Mills theories from outside

- the scaling window with machine learning, *Phys. Rev. D* **103**, 014509 (2021).
- [15] Y. Zhang and E.-A. Kim, Quantum loop topography for machine learning, *Phys. Rev. Lett.* **118**, 216401 (2017).
- [16] P. Zhang, H. Shen, and H. Zhai, Machine learning topological invariants with neural networks, *Phys. Rev. Lett.* **120**, 066401 (2018).
- [17] H. Araki, T. Mizoguchi, and Y. Hatsugai, Phase diagram of a disordered higher-order topological insulator: A machine learning study, *Phys. Rev. B* **99**, 085406 (2019).
- [18] J. F. Rodriguez-Nieva and M. S. Scheurer, Identifying topological order through unsupervised machine learning, *Nat. Phys.* **15**, 790 (2019).
- [19] G. Carleo and M. Troyer, Solving the quantum many-body problem with artificial neural networks, *Science* **355**, 602 (2017).
- [20] K. Mills, P. Ronagh, and I. Tamblyn, Finding the ground state of spin Hamiltonians with reinforcement learning, *Nat. Mach. Intell.* **2**, 509 (2020).
- [21] C. Fan, M. Shen, Z. Nussinov, Z. Liu, Y. Sun, and Y.-Y. Liu, Searching for spin glass ground states through deep reinforcement learning, *Nat. Commun.* **14**, 725 (2023).
- [22] T. Duric, J. H. Chung, B. Yang, and P. Sengupta, Spin-1/2 kagome Heisenberg antiferromagnet: Machine learning discovery of the spinon pair-density-wave ground state, *Phys. Rev. X* **15**, 011047 (2025).
- [23] L. Huang and L. Wang, Accelerated Monte Carlo simulations with restricted Boltzmann machines, *Phys. Rev. B* **95**, 035105 (2017).
- [24] L. Wang, Exploring cluster Monte Carlo updates with Boltzmann machines, *Phys. Rev. E* **96**, 051301(R) (2017).
- [25] T. Song and H. Lee, Accelerated continuous time quantum Monte Carlo method with machine learning, *Phys. Rev. B* **100**, 045153 (2019).
- [26] S. Li, P. M. Dee, E. Khatami, and S. Johnston, Accelerating lattice quantum Monte Carlo simulations using artificial neural networks: Application to the Holstein model, *Phys. Rev. B* **100**, 020302(R) (2019).
- [27] B. McNaughton, M. V. Milosević, A. Perali, and S. Pilati, Boosting Monte Carlo simulations of spin glasses using autoregressive neural networks, *Phys. Rev. E* **101**, 053312 (2020).
- [28] Z. Tian, S. Zhang, and G.-W. Chern, Machine learning for structure-property mapping of Ising models: Scalability and limitations, *Phys. Rev. E* **108**, 065304 (2023).
- [29] S. S. Funai and D. Giataganas, Thermodynamics and feature extraction by machine learning, *Phys. Rev. Res.* **2**, 033415 (2020).
- [30] K. Kashiwa, Y. Kikuchi, and A. Tomiya, Phase transition encoded in neural network, *Prog. Theor. Exp. Phys.* **2019**, 083A04 (2019).
- [31] Z. Li, M. Luo, and X. Wan, Extracting critical exponents by finite-size scaling with convolutional neural networks, *Phys. Rev. B* **99**, 075418 (2019).
- [32] A. Abuali, D. A. Clarke, M. Hjorth-Jensen, I. Konstantinidis, C. Ratti, and J. Yang, Deep learning of phase transitions with minimal examples, *Phys. Rev. E* **112**, 035315 (2025).
- [33] D. Kim and D.-H. Kim, Smallest neural network to learn the Ising criticality, *Phys. Rev. E* **98**, 022138 (2018).
- [34] T. Westerhout, N. Astrakhantsev, K. S. Tikhonov, M. I. Katsnelson, and A. A. Bagrov, Generalization properties of neural network approximations to frustrated magnet ground states, *Nat. Commun.* **11**, 1593 (2020).
- [35] I. Corte, S. Acevedo, M. Arlego, and C. Lamas, Exploring neural network training strategies to determine phase transitions in frustrated magnetic models, *Comput. Mater. Sci.* **198**, 110702 (2021).
- [36] C.-D. Li, D.-R. Tan, and F.-J. Jiang, Applications of neural networks to the studies of phase transitions of two-dimensional Potts models, *Ann. Phys. (NY)* **391**, 312 (2018).
- [37] X. Zhao and L. Fu, Machine learning phase transition: An iterative proposal, *Ann. Phys. (NY)* **410**, 167938 (2019).
- [38] D.-R. Tan, C. Li, W.-P. Zhu, and F.-J. Jiang, A comprehensive neural networks study of the phase transitions of Potts model, *New J. Phys.* **22**, 063016 (2020).
- [39] A. Tirelli, D. O. Carvalho, L. A. Oliveira, J. P. de Lima, N. C. Costa, and R. R. dos Santos, Unsupervised machine learning approaches to the q -state Potts model, *Eur. Phys. J. B* **95**, 189 (2022).
- [40] X. Chen, F. Liu, S. Chen, J. Shen, W. Deng, G. Papp, W. Li, and C. Yang, Study of phase transition of Potts model with domain adversarial neural network, *Phys. A* **617**, 128666 (2023).
- [41] X. Chen, F. Liu, W. Deng, S. Chen, J. Shen, G. Papp, W. Li, and C. Yang, Applications of domain adversarial neural network in phase transition of 3D Potts model, *Physica A* **637**, 129533 (2024).
- [42] K. Shiina, H. Mori, Y. Okabe, and H. K. Lee, Machine-learning studies on spin models, *Sci. Rep.* **10**, 2177 (2020).
- [43] D. Bachtis, G. Aarts, and B. Lucini, Mapping distinct phase transitions to a neural network, *Phys. Rev. E* **102**, 053306 (2020).
- [44] K. Fukushima and K. Sakai, Can a CNN trained on the Ising model detect the phase transition of the q -state Potts model? *Prog. Theor. Exp. Phys.* **2021**, 061A01 (2021).
- [45] D. Giataganas, C.-Y. Huang, and F.-L. Lin, Neural network flows of low q -state Potts and clock models, *New J. Phys.* **24**, 043040 (2022).
- [46] H. M. Yau and N. Su, On the generalizability of artificial neural networks in spin models, *SciPost Phys. Core* **5**, 032 (2022).
- [47] Y.-H. Tseng and F.-J. Jiang, Learning the phase transitions of two-dimensional Potts model with a pre-trained one-dimensional neural network, *Results Phys.* **56**, 107264 (2024).
- [48] C. Vanderzande, Bulk and surface critical behaviour at the conformal invariant point in the Z_5 model, *J. Phys. A* **20**, L549 (1987).
- [49] H. A. Fernandes, E. Arashiro, J. D. de Felício, and A. Caparica, An alternative order parameter for the 4-state Potts model, *Phys. A* **366**, 255 (2006).
- [50] V. Beffara and H. Duminil-Copin, The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$, *Probab. Theory Relat. Fields* **153**, 511 (2012).
- [51] H. Duminil-Copin, V. Sidoravicius, and V. Tassion, Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$, *Commun. Math. Phys.* **349**, 47 (2017).
- [52] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion, Discontinuity of the phase transition for

- the planar random-cluster and Potts models with $q > 4$, [arXiv:1611.09877](#).
- [53] F.-Y. Wu, The Potts model, *Rev. Mod. Phys.* **54**, 235 (1982).
- [54] R. H. Swendsen and J.-S. Wang, Nonuniversal critical dynamics in Monte Carlo simulations, *Phys. Rev. Lett.* **58**, 86 (1987).
- [55] D. J. Sharpe, Swendsen-Wang algorithm for Potts model, 2018, https://github.com/danieljsharp/swendsen_wang_potts.
- [56] O. Melchert, autoScale.py—A program for automatic finite-size scaling analyses: A user’s guide, [arXiv:0910.5403](#).
- [57] J. Salas and A. D. Sokal, Logarithmic corrections and finite-size scaling in the two-dimensional 4-state Potts model, *J. Stat. Phys.* **88**, 567 (1997).