

Exact nonequilibrium steady states of boundary-driven circuit with XYZ gates

Xin Zhang^{1,*}, Tomaž Prosen^{2,3} and Vladislav Popkov^{2,4}

¹Beijing National Laboratory for Condensed Matter Physics, *Institute of Physics, Chinese Academy of Sciences*, Beijing 100190, China

²Faculty of Mathematics and Physics, *University of Ljubljana*, Jadranska 19, SI-1000 Ljubljana, Slovenia

³*Institute of Mathematics, Physics and Mechanics*, Jadranska 19, SI-1000 Ljubljana, Slovenia

⁴Department of Physics, *University of Wuppertal*, Gausstraße 20, 42119 Wuppertal, Germany



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We obtain the exact many-body density operator of a boundary-driven XXZ quantum circuit via a spatially inhomogeneous matrix product ansatz. The ansatz has formally infinite bond dimension and generalizes authors' previous construction [Popkov and Prosen, *Phys. Rev. Lett.* **135**, 070401 (2025)] for the XXZ interactions. The boundary qubits are coupled to reset quantum channels that project them toward arbitrary pure target states. We find and describe a family of relatively robust separable chiral nonequilibrium steady states, which are elliptic analogs of spin helices for the circuit, and which are particularly attractive from an experimental perspective.

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I. INTRODUCTION

In today's era of noisy intermediate-scale quantum (NISQ) computers, the theory of open quantum systems [1–3] is essential for building useful quantum simulations. In this respect, exactly solvable models play a special role since they allow to test, control, and benchmark NISQ devices [4–7]. One powerful example is a trotterized integrable XXZ quantum spin chain realized as a brickwork quantum circuit and driven by specific Kraus dissipators at its ends, the nonequilibrium steady state (NESS) of which can be expressed as an infinite bond-dimension matrix product ansatz (MPA) [8,9]. The XXZ brickwork quantum circuit protocol was implemented on an actual NISQ device [5,7]. Beyond its mathematical beauty, this setup reveals surprising and often exotic physics: for instance, fractal-like behavior of transport coefficients (Drude weights) [10,11], insulating behavior far from equilibrium, and even negative differential conductance, where increasing the driving force actually reduces the current [12,13].

In this communication, we are treating the most general, fully anisotropic scenario in which the principal qubit gates constituting the XYZ-type qubit circuit do not satisfy the ice rule, i.e., the local constraint requiring two of the four qubits incident to each vertex to be in one computational-basis state and the remaining two to be in the other computational-basis state, corresponding to the standard “two arrows in-two arrows out” condition. Namely, we express the NESS of the XYZ boundary-driven qubit circuit in terms of MPA. After Ref. [9], our solution gives the second and more general

example of the two-replica MPA that has nontrivially coupled auxiliary spaces, meaning that the NESS is not factorizable in terms of a product of two Cholesky factors, as in all previously solved cases [14]. Analyzing the MPA, we find a set of remarkably simple chiral separable states, which, being a circuit extensions of spin-helix states [15–18], can be named elliptic brickwork spin helices. The Hamiltonian (continuous-time) limit of elliptic spin-helix eigenstates is well known [19,20] even beyond one-dimensional lattices [21,22]. The XXZ counterparts of elliptic helices turned out to be prominent in cold-atom experiments, where the exceptional robustness of the helices allowed to use them for controlling and benchmarking purposes [23,24]. Likewise, we expect elliptic brickwork spin helices to be useful benchmarking tool in potential future experiments with qubit gates not satisfying the ice rule [i.e., breaking $U(1)$ symmetry].

II. BOUNDARY RESET-DRIVEN XYZ CIRCUITS

We consider a chain of $N+2$ qubits labeled by $0, 1, \dots, N, N+1$. An XYZ gate is defined as

$$U(u, \eta) = \left[\frac{\tilde{\theta}_1(\eta)\tilde{\theta}_1(\eta)}{\tilde{\theta}_1(\eta+u)\tilde{\theta}_1(\eta-u)} \right]^{\frac{1}{2}} \tilde{U}(u, \eta), \quad (1)$$

$$\tilde{U}(u, \eta) = \begin{pmatrix} r_1 & 0 & 0 & r_4 \\ 0 & r_3 & r_2 & 0 \\ 0 & r_2 & r_3 & 0 \\ r_4 & 0 & 0 & r_1 \end{pmatrix}, \quad (2)$$

$$r_1 = \frac{\theta_4(u)\theta_1(u+\eta)}{\theta_4(0)\theta_1(\eta)}, \quad r_2 = \frac{\theta_1(u)\theta_4(u+\eta)}{\theta_4(0)\theta_1(\eta)}, \quad (3)$$

$$r_3 = \frac{\theta_4(u)\theta_4(u+\eta)}{\theta_4(0)\theta_4(\eta)}, \quad r_4 = \frac{\theta_1(u)\theta_1(u+\eta)}{\theta_4(0)\theta_4(\eta)}, \quad (4)$$

where $\theta_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{2i\pi\tau})$ and $\tilde{\theta}_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{i\pi\tau})$ are standard elliptic theta functions (see Appendix A). The U matrix is parametrized by the anisotropic parameter η , modulus of the elliptic theta function τ [$\text{Im}(\tau) > 0$], and a spectral

*Contact author: xinzhang@iphy.ac.cn

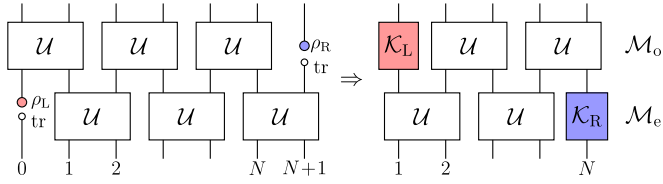


FIG. 1. Two-step reset-driven XYZ circuit in folded notation. Empty circles represent local traces, while red/blue disks represent arbitrary pure boundary states. The right scheme corresponds to an equivalent reduced circuit where squares represent boundary Kraus maps (see details in the main text).

parameter u . The matrix $P\tilde{U}$, where P is the permutation operator, coincides with the standard parametrization of the R -matrix for eight-vertex model [25]. The operator U satisfies a symmetry relation

$$U(u, \eta) = U(-u, -\eta), \quad (5)$$

and the unitarity relation

$$U(u, \eta)U(-u, \eta) = U(u, \eta)U(u, -\eta) = I. \quad (6)$$

For small $u \ll 1$, we have

$$\tilde{U}(u, \eta) = I + u \frac{\tilde{\theta}'_1(0)}{2\tilde{\theta}_1(\eta)} \left(h + \frac{\tilde{\theta}'_1(\eta)}{\tilde{\theta}'_1(0)} I \right) + O(u^2), \quad (7)$$

where h is the energy density of the paradigmatic XYZ Hamiltonian

$$h_{n,n+1} = \sum_{\alpha=x,y,z} J_\alpha \sigma_n^\alpha \sigma_{n+1}^\alpha. \quad (8)$$

Here, $\{\sigma_n^\alpha\}$ are Pauli matrices acting on n th spin in the chain, and the couplings $\{J_\alpha\}$ read

$$J_x = \frac{\tilde{\theta}_4(\eta)}{\tilde{\theta}_4(0)}, \quad J_y = \frac{\tilde{\theta}_3(\eta)}{\tilde{\theta}_3(0)}, \quad J_z = \frac{\tilde{\theta}_2(\eta)}{\tilde{\theta}_2(0)}. \quad (9)$$

The couplings J_α in Eq. (9) are real if the modular parameter τ of the elliptic theta function is purely imaginary, $\tau \in i\mathbb{R}$, and the anisotropy parameter satisfies $\eta \in \mathbb{R}$ or $\eta \in i\mathbb{R}$. These two choices, after performing the XXZ reduction

$$\lim_{\tau \rightarrow +i\infty} J_{x,y} = 1, \quad \lim_{\tau \rightarrow +i\infty} J_z = \cos(\pi\eta),$$

correspond to the easy plane $|J_z/J_x| < 1$ and easy axis $|J_z/J_x| > 1$ regime, respectively. The U gate (1) is unitary if (1) $u \in \mathbb{R}$, $\eta \in i\mathbb{R}$ or (2) $u \in i\mathbb{R}$, $\eta \in \mathbb{R}$.

The gate U acts on a pair of neighboring qubits via so-called folded map as $\mathcal{U}(\rho) = U\rho U^\dagger$. We consider a chain of $N+2$ qubits labeled as $0, 1, \dots, N+1$. Assuming N to be odd (similar construction can be given for N even), we define a dynamical protocol consisting of two—even and odd—steps (see Fig. 1). During the even step, we apply a string of $\mathcal{U}$ gates to all neighboring pairs of qubits $(1, 2), (3, 4), \dots, (N, N+1)$, while qubit 0 is traced out and reset to an arbitrary pure state $\rho_L = |\psi_L\rangle\langle\psi_L|$. During the odd step, we similarly apply $\mathcal{U}$ gates to qubits $(0, 1), (2, 3), \dots, (N-1, N)$ and reset the qubit $N+1$ to arbitrary pure state $\rho_R = |\psi_R\rangle\langle\psi_R|$. This defines a many-body locally interacting (trotterized XYZ)

quantum channel, targeting predefined local spin states at the ends. Here, u can be interpreted as the Trotter time step, so $u \rightarrow 0$ corresponds to continuous-time limit, governed by the paradigmatic XYZ spin chain Hamiltonian [25] (8) in the bulk, while the reset channels (Fig. 1) correspond to Zeno limit of instantaneous relaxation of boundary spins toward targeted states ρ_L and ρ_R .

Ignoring the trivially separable state of the boundary qubit (left/right in even/odd step), we can write an equivalent effective brickwork dynamical quantum channel acting on interior N qubits, labeled $1, 2, \dots, N$, as

$$\rho_{t+1} = \mathcal{M}(\rho_t), \quad \mathcal{M} = \mathcal{M}_o \mathcal{M}_e, \\ \mathcal{M}_o = \mathcal{K}_L \otimes \mathcal{U}^{\otimes(N-1)/2}, \quad \mathcal{M}_e = \mathcal{U}^{\otimes(N-1)/2} \otimes \mathcal{K}_R, \quad (10)$$

where $\mathcal{K}_{L/R} : \text{End}(\mathbb{C}^2) \rightarrow \text{End}(\mathbb{C}^2)$ are the local boundary channels

$$\mathcal{K}_L(\rho) = \text{tr}_1(U(\rho_L \otimes \rho)U^\dagger), \\ \mathcal{K}_R(\rho) = \text{tr}_2(U(\rho \otimes \rho_R)U^\dagger). \quad (11)$$

The boundary channels $\mathcal{K}_{L/R}(\rho)$ can be expressed explicitly in terms of pairs of Kraus matrices $\mathcal{K}_{L/R}(\rho) = \sum_{\mu=1}^2 K_{L/R}^{[\mu]} \rho K_{L/R}^{[\mu]\dagger}$. For our further proofs, an explicit form of Kraus matrices $K_{L/R}^{[\mu]}$ is not needed; for completeness, we give them in Appendix B.

III. INHOMOGENEOUS YANG-BAXTER EQUATION

Let us denote the physical space of N qubits as $\mathfrak{h} = (\mathbb{C}^2)^{\otimes N}$ and label local operators over $\mathfrak{h}$ that act nontrivially only on n th tensor factor (qubit) by subscript n , e.g., Pauli operators $\sigma_n^{x,y,z}$. In addition, we consider an infinite-dimensional auxiliary space $\mathfrak{a}$ spanned by basis vectors $\{|j\rangle_a; j = 0, 1, \dots\}$. We now define a special type of inhomogeneous Lax operators, and their conjugates, $L_{a,n}^\pm, L_{a,n}^{\pm*} \in \text{End}(\mathfrak{a} \otimes \mathfrak{h})$ as

$$L_{a,n}^+ = \sum_{j=0}^{\infty} [|j\rangle\langle j|_a L_n(n-2j) + |j\rangle\langle j+1|_a X'_n(n-2j)], \\ L_{a,n}^- = \sum_{j=0}^{\infty} [|j\rangle\langle j|_a L'_n(n-2j) + |j\rangle\langle j+1|_a X_n(n-2j)], \\ L_{a,n}^{+*} = \sum_{j=0}^{\infty} [|j\rangle\langle j|_a L_n(n-2j)^\dagger + |j\rangle\langle j+1|_a X'_n(n-2j)^\dagger], \\ L_{a,n}^{-*} = \sum_{j=0}^{\infty} [|j\rangle\langle j|_a L'_n(n-2j)^\dagger + |j\rangle\langle j+1|_a X_n(n-2j)^\dagger], \quad (12)$$

where $L_n(m), L'_n(m), X_n(m), X'_n(m)$ are rank-1 explicit site-dependent matrices, acting on the n th qubit.

Defining two unnormalized orthogonal qubit states as

$$|\psi(u)\rangle = \begin{pmatrix} \theta_1(u) \\ \theta_4(u) \end{pmatrix}, \quad \langle\xi(u)| = (\theta_4(u), -\theta_1(u)), \quad (13)$$

we postulate

$$\begin{aligned} L(m) &= \frac{1}{C(m)} |\psi(m\eta + a)\rangle \langle \xi(-u - m\eta - a)|, \\ L'(m) &= \frac{1}{C(m)} |\psi(u + m\eta + a)\rangle \langle \xi(-m\eta - a)|, \\ X(m) &= \frac{1}{C(m)} |\psi(m\eta - u + a)\rangle \langle \xi(-m\eta - a)|, \\ X'(m) &= \frac{1}{C(m)} |\psi(m\eta + a)\rangle \langle \xi(u - m\eta - a)|, \\ C(m) &= \tilde{\theta}_2(a + m\eta), \end{aligned} \quad (14)$$

where $a \in \mathbb{C}$ is an arbitrary complex parameter.

The following Yang-Baxter-type (or RLL) identities play a key role in the following:

$$\begin{aligned} U_{n,n+1} L_{a,n}^+ L_{a,n+1}^- &= L_{a,n}^- L_{a,n+1}^+ U_{n,n+1}, \\ U_{n,n+1} L_{a,n}^{+*} L_{a,n+1}^{-*} &= L_{a,n}^{-*} L_{a,n+1}^{+*} U_{n,n+1}, \end{aligned} \quad (15)$$

where we interpret the gate $U_{n,n+1}$ as an R -matrix of a unitary eight-vertex model. Contrary to the usual situation, the RLL relations (a form of Yang-Baxter relations intertwining a pair of Lax operators with the R -matrix) (15) are inhomogeneous, since our Lax operators $L_{a,n}^+, \dots$ (12) have explicit n dependence (while the gate U remains the same). A proof of the identities (15) is given in Appendix C.

IV. EXPLICIT INHOMOGENEOUS MATRIX PRODUCTNESS FOR XYZ CIRCUIT

Similarly to Ref. [9], we formulate an ansatz for the NESS $\rho_\infty, \rho'_\infty$ appearing at the even and odd steps of the qubit circuit evolution

$$\mathcal{M}_o \mathcal{M}_e \rho_\infty = \rho_\infty, \quad (16)$$

$$\mathcal{M}_e \mathcal{M}_o \rho'_\infty = \rho'_\infty, \quad (17)$$

as follows:

$$\begin{aligned} \rho_\infty &= \langle L | \mathbb{L}_{ab,1}^+ \mathbb{L}_{ab,2}^- \mathbb{L}_{ab,3}^+ \cdots \mathbb{L}_{ab,N-1}^- \mathbb{L}_{ab,N}^+ | R \rangle, \\ \rho'_\infty &= \langle L | \mathbb{L}_{ab,1}^- \mathbb{L}_{ab,2}^+ \mathbb{L}_{ab,3}^- \cdots \mathbb{L}_{ab,N-1}^+ \mathbb{L}_{ab,N}^- | R \rangle, \end{aligned} \quad (18)$$

$$\mathbb{L}_{ab,n}^\pm = L_{a,n}^\pm L_{b,n}^{\pm*},$$

where $\langle L |, |R\rangle$ are auxiliary vectors in a product of two formally unbounded auxiliary spaces $\mathfrak{a} \otimes \mathfrak{b}$, i.e., $\langle L | = \sum_{j,j'=0}^\infty l_{j,j'} \langle j |_{\mathfrak{a}} \langle j' |_{\mathfrak{b}}$, and $|R\rangle = \sum_{j,j'=0}^\infty r_{j,j'} |j\rangle_{\mathfrak{a}} |j'\rangle_{\mathfrak{b}}$. However, our solution for $\langle L |$ has just few nonzero components $l_{j,j'}$ [see Eqs. (23) and (30)], and therefore, generic Eq. (18) always contains finite number of terms; see more details after Eq. (29).

Consistency of the ansatz (18) in the bulk is guaranteed by Eq. (15), while the boundaries yield extra conditions to determine $\langle L |$ and $|R\rangle$:

$$\langle L | (\mathbb{L}_{ab,1}^+ - \mathcal{K}_L(\mathbb{L}_{ab,1}^-)) = 0, \quad (19)$$

$$(\mathbb{L}_{ab,N}^- - \mathcal{K}_R(\mathbb{L}_{ab,N}^+)) |R\rangle = 0. \quad (20)$$

Like Eq. (15), the conditions (19) and (20) are overdetermined. Solutions of these equations depend on the reset

channels (11) that will be parametrized with two complex numbers α_L and α_R ,

$$\begin{aligned} \rho_L &= |\psi_L\rangle \langle \psi_L|, \quad \rho_R = |\psi_R\rangle \langle \psi_R|, \\ |\psi_L\rangle &= \frac{1}{\sqrt{\tilde{\theta}_4(\text{Re}(\alpha_L)) \tilde{\theta}_3(i\text{Re}(\alpha_L))}} \begin{pmatrix} \theta_1(\alpha_L) \\ \theta_4(\alpha_L) \end{pmatrix}, \\ |\psi_R\rangle &= \frac{1}{\sqrt{\tilde{\theta}_4(\text{Re}(\alpha_R)) \tilde{\theta}_3(i\text{Re}(\alpha_R))}} \begin{pmatrix} \theta_1(\alpha_R) \\ \theta_4(\alpha_R) \end{pmatrix}. \end{aligned} \quad (21)$$

The U gate (1) is unitary if either $u \in \mathbb{R}, \eta \in i\mathbb{R}$ (case (A)) or $u \in i\mathbb{R}, \eta \in \mathbb{R}$ (case (B)). The two cases are treated separately below. The proofs of following Eqs. (22)–(31) are given in Appendixes D and E.

Case $\eta \in \mathbb{R}, u \in i\mathbb{R}$. The NESS of the XYZ qubit circuit in Fig. 1 is given by Eq. (18) with local entries of Lax matrix given by Eq. (14), where $a \equiv \alpha_L = w_0 + iw_1$, and the auxiliary vector $\langle L |$ has just four nonzero components,

$$\begin{aligned} \langle L | &= \sum_{j,j'=0}^1 l_{j,j'} \langle j |_{\mathfrak{a}} \langle j' |_{\mathfrak{b}}, \quad (22) \\ l_{0,0} &= 1, \quad l_{1,1} = \frac{\tilde{\theta}_3(u - iw_1)}{\tilde{\theta}_3(u + iw_1)}, \\ l_{0,1} &= \frac{\tilde{\theta}_3(iw_1) \tilde{\theta}_4(u + w_0)}{\tilde{\theta}_3(u + iw_1) \tilde{\theta}_4(w_0)}, \\ l_{1,0} &= l_{0,1}^*, \quad w_0 = \text{Re}[\alpha_L], \quad w_1 = \text{Im}[\alpha_L]. \end{aligned} \quad (23)$$

The auxiliary vector $|R\rangle$ is unbounded and has components

$$|R\rangle = \sum_{j,j'=0}^\infty r_{j,j'} |j\rangle_{\mathfrak{a}} |j'\rangle_{\mathfrak{b}}, \quad (24)$$

$$r_{j,j'} = C_{j,j'} \prod_{k=0}^{j-1} b_k \prod_{k'=0}^{j'-1} b_{k'}^*, \quad (25)$$

$$C_{j,j'} = \frac{\tilde{\theta}_3((j-j')\eta - i\Gamma_1) \tilde{\theta}_4((j+j')\eta - \Gamma_0)}{\tilde{\theta}_3(i\Gamma_1) \tilde{\theta}_4(\Gamma_0)}, \quad (26)$$

$$b_n = -\frac{\tilde{\theta}_1(n\eta + v) \tilde{\theta}_2(u + \Gamma - n\eta + v)}{\tilde{\theta}_1(u + (n+1)\eta + v) \tilde{\theta}_2(\Gamma - (n+1)\eta + v)}, \quad (27)$$

$$\Gamma = \alpha_L + (N+1)\eta, \quad \Gamma_0 = \text{Re}[\Gamma], \quad \Gamma_1 = \text{Im}[\Gamma], \quad (28)$$

$$v = \frac{1}{2}(\alpha_R - \alpha_L - u - (N+1)\eta). \quad (29)$$

Even though the vector $|R\rangle$ is formally unbounded, for a system of size N , only the finite subset of coefficients $r_{j,j'}$, with $j, j' \leq N+2$, matters. Indeed, due to the band structure of Lax operators, the expression in Eq. (18) $\langle L | \mathbb{L}_{ab,1}^+ \mathbb{L}_{ab,2}^- \cdots$ can be seen as a walk on two-dimensional graph with coordinates $j, j' \geq 0$ starting from a 2×2 seed (22) at the origin. Every application of $\mathbb{L}_{ab,n}^\pm$ increases the walk range by at most one unit $j \rightarrow j, j+1, j' \rightarrow j', j'+1$. After N steps, the walk can only reach the points $j, j' \leq N+2$ on the graph, making the elements of $r_{j,j'}$ outside the walk range obsolete.

Case $\eta \in i\mathbb{R}, u \in \mathbb{R}$. The NESS of the XYZ qubit circuit in Fig. 1 is given by Eq. (18) with Lax matrices (14), where

$a \equiv \alpha_L = w_0 + iw_1$, and the auxiliary vector $\langle L|$ (22) with

$$\begin{aligned} l_{0,0} &= 1, \quad l_{1,1} = \frac{\tilde{\theta}_4(u - w_0)}{\tilde{\theta}_4(u + w_0)}, \\ l_{0,1} &= \frac{\tilde{\theta}_3(u + iw_1)\tilde{\theta}_4(w_0)}{\tilde{\theta}_3(iw_1)\tilde{\theta}_4(u + w_0)}, \quad l_{1,0} = l_{0,1}^*, \end{aligned} \quad (30)$$

while the right boundary vector $|R\rangle$ has the form (24) and (25), with b_n given by the expression (27) and $C_{j,j'}$ given by

$$C_{j,j'} = \frac{\tilde{\theta}_4((j - j')\eta - \Gamma_0)\tilde{\theta}_3((j + j')\eta - i\Gamma_1)}{\tilde{\theta}_4(\Gamma_0)\tilde{\theta}_3(i\Gamma_1)}. \quad (31)$$

Notably, both left and right auxiliary vectors generally cannot be recast in a product form, $\langle L|_{ab} \neq \langle v|_a \langle \tilde{v}|_b$, $|R\rangle_{ab} \neq |w\rangle_a |\tilde{w}\rangle_b$. Consequently, the NESS of a driven qubit circuit (18) cannot be factorized into a product of two Cholesky-type factors as it was in all previous solvable instances in models with continuous-time evolution [14]. Apparently, in the absence of $U(1)$ symmetry, it is the time discreteness, i.e., a finite Trotter time step $u \neq 0$ in Eq. (1), which makes the coupling between auxiliary spaces a, b a necessary ingredient of a theory. In special cases, factorization can still happen, e.g., for $u = 0$ or for $u_L = \frac{1}{2}$ the left auxiliary vector $\langle L|$ is factorized, while a generic infinite-dimensional $|R\rangle$ is factorized only in a trivial case $\eta = 0$. Generic Schmidt rank for both auxiliary vectors is equal to 2. However, for specially tuned boundary conditions, the vector $|R\rangle$ has a finite support, leading to special class of robust circuit states—brickwork helices and their descendants—which are described below.

V. ELLIPTIC SPIN HELICES IN XYZ BRICKWORK CIRCUITS

An analysis of Eqs. (24) and (27) reveals an especially simple class of brickwork NESS (16): It corresponds to zeros and poles of b_n . Namely, setting $v = 0$, or

$$\alpha_R = \alpha_L + (N + 1)\eta + u, \quad (32)$$

leads to $b_0 = 0$ and consequently $|R\rangle = |0\rangle_a |0\rangle_b$. Here, Eq. (32) is understood modulo 2 and 2τ . The same convention applies to other analogous cases below. Due to upper triangular structure of the Lax matrices (12), the NESS (18) then simplifies to

$$\begin{aligned} \rho_\infty &= L(1)L(1)^\dagger \otimes L'(2)L'(2)^\dagger \otimes \cdots \otimes L(N)L(N)^\dagger \\ &= A|\Psi_+\rangle\langle\Psi_+|, \\ |\Psi_+\rangle &= \psi_1(0) \otimes \psi_2(u) \otimes \psi_3(0) \otimes \psi_4(u) \\ &\quad \cdots \otimes \psi_{N-1}(u) \otimes \psi_N(0), \end{aligned} \quad (33)$$

$$\begin{aligned} \rho'_\infty &= A'|\Psi'_+\rangle\langle\Psi'_+|, \\ |\Psi'_+\rangle &= \psi_1(u) \otimes \psi_2(0) \otimes \psi_3(u) \otimes \psi_4(0) \\ &\quad \cdots \otimes \psi_{N-1}(0) \otimes \psi_N(u), \end{aligned} \quad (34)$$

where A and A' are normalization constants and

$$\psi_n(u) = \begin{pmatrix} \theta_1(\alpha_L + n\eta + u) \\ \theta_4(\alpha_L + n\eta + u) \end{pmatrix}. \quad (35)$$

The state (33) is periodic in space (with lattice period $2/\eta$ or $2\tau/\eta$, depending on whether the anisotropy η is real or imaginary), with qubit polarization vector following an elliptic

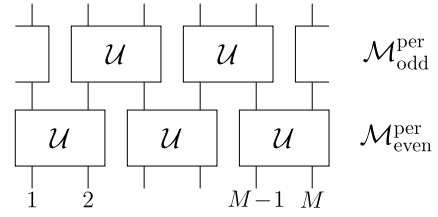


FIG. 2. Two-step reset-driven periodic XYZ circuit in folded notation.

helicoidal spiral. The argument of elliptic function grows linearly by unit η with spin position, with an additional even/odd site staggering due to additional argument increment “+ u ” for even sites. Typical local magnetization profiles for elliptic helices are shown in Figs. 3 and 4. For odd steps of evolution, the NESS is still fully factorized, as in Eq. (34), but the increment “+ u ” occurs at odd sites, which results in different magnetization profiles at odd and even steps of qubit circuit.

Significance of the factorized helix state (33) goes beyond just being a special fine-tuned solution of the steady-state equation (16). A particularity of helix (33) is that it is locally reproduced in the bulk after every full step $\mathcal{M}_o\mathcal{M}_e$. This

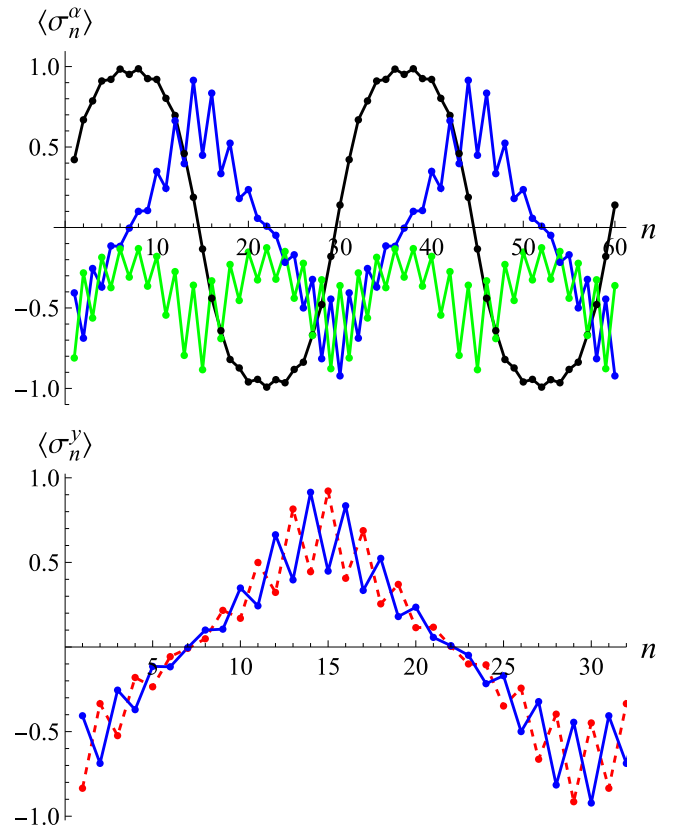


FIG. 3. Average local magnetization profiles for elliptic brickwork helix (48) for real η . Top panel: black, blue, and green points connected with full lines correspond to $\langle\sigma_n^x\rangle$, $\langle\sigma_n^y\rangle$, and $\langle\sigma_n^z\rangle$, respectively. Bottom panel: Blowup of $\langle\sigma_n^y\rangle$ for even/odd steps of evolution (50) and (51) ($|\Psi_+\rangle$ and $|\Psi'_+\rangle$) is shown with full blue and dashed red lines, respectively. Parameters: $M = 60$, $u = 0.15i$, $\tau = 0.65i$, $\alpha_L = 0.095 + 0.1i$, and $\eta = 4/M$.

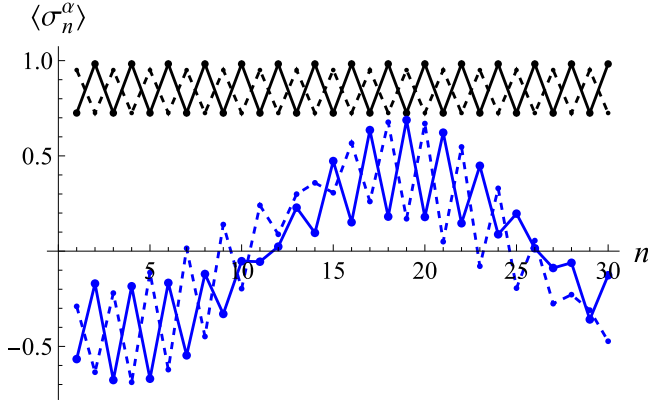


FIG. 4. Average local magnetization profiles for elliptic brickwork helix (48) for imaginary η . Black and blue points connected with full lines correspond to $\langle \sigma_n^x \rangle$ and $\langle \sigma_n^y \rangle$, respectively. Dashed lines show the same quantity at odd steps of evolution (51) ($|\tilde{\Psi}_+\rangle$). Parameters: $M = 30$, $u = 0.15$, $\tau = 0.325i$, $\alpha_L = 0.095 + 0.1i$, and $\eta = 2\tau/M$.

renders it particularly robust also in open circuits of size N , with arbitrary boundary conditions, where spin-helix lifetime would be of order N , while a lifetime of a generic state is of order $O(1)$. In the XXZ limit $\text{Im}(\tau) \rightarrow \infty$ and for real η , the state (33) describes winding of the polarization vector along the chain in harmonic fashion, with even-odd site alternation (see Ref. [9]). The usefulness of the long-lived helices for calibration purposes has been demonstrated in cold-atom experiments [23]. Further, we shall refer to the state (33) as an elliptic (XYZ) brickwork spin helix or simply a brickwork helix. Note that periodic in-space helix (33) with imaginary η and lattice period $2\tau/\eta$ does not have a trigonometric XXZ analog. Brickwork helix state also appears as a special degenerate site-factorized stationary state for periodic circuits [see Eqs. (48) and (49)].

Notably, slight generalization of Eq. (32),

$$\begin{aligned}\alpha_R &= \alpha_L + (N + 1 - 2m_k)\eta + u, \\ m_k &= 0, 1, 2, \dots,\end{aligned}\quad (36)$$

with integer $m_k = 0, 1, 2, \dots$, leads to $b_{m_k} = 0$ and cropping of $|\mathbb{R}\rangle$ to dimension $(m_k + 1) \times (m_k + 1)$, i.e., $r_{j,j'} = 0$ if $j > m_k$ or $j' > m_k$. The respective NESS appears to be no longer pure [like Eq. (33)], but it still has a low rank, and it can be seen as consisting of imperfect helices with m_k kinks, in analogy to Ref. [18]. The NESS ranks of ρ_∞ for the cases $m_k = 0, 1, 2$ are 1, $N + 1$, and $\binom{N+1}{2} + 1$, respectively.

An elliptic brickwork helix NESS with a winding in the opposite direction,

$$|\Psi_-\rangle = \psi_{-1}(0) \otimes \psi_{-2}(-u) \otimes \psi_{-3}(0) \otimes \psi_{-4}(-u) \cdots \otimes \psi_{-N+1}(-u) \otimes \psi_{-N}(0), \quad (37)$$

appears for a choice of parameters

$$\alpha_R = \alpha_L - (N + 1)\eta - u, \quad (38)$$

[compare to Eq. (32)], and it corresponds to a pole $1/b_N = 0$ in Eq. (27). Then, a renormalized right auxiliary vector $|\mathbb{R}\rangle$ has just one nonzero component, $|\mathbb{R}\rangle = |N + 1\rangle_a |N + 1\rangle_b$. The

NESS (18) then becomes

$$\begin{aligned}\rho_\infty &= l_{1,1} |1\rangle_a \langle 1|_b \mathbb{L}_{ab,1}^+ \mathbb{L}_{ab,2}^- \cdots |N + 1\rangle_a \langle N + 1|_b \\ &= l_{1,1} X'(-1) X'(-1)^\dagger \otimes X(-2) X(-2)^\dagger \\ &\quad \otimes \cdots \otimes X(1 - N) X(1 - N)^\dagger \otimes X'(-N) X'(-N)^\dagger \\ &= A |\Psi_-\rangle \langle \Psi_-|,\end{aligned}\quad (39)$$

where A is a normalization constant and $|\Psi_-\rangle$ is given by Eq. (37). A generalized condition

$$\alpha_R = \alpha_L - (N + 1 - 2m_k)\eta - u, \quad (40)$$

[compare to Eq. (36)] corresponds to a pole $1/b_{N-m_k} = 0$ and yields NESS in the form of imperfect helix with m_k kinks.

Divergence condition. Bulk stationarity of elliptic brickwork helices (33) and (37) also follows in a simple way, via two Yang-Baxter RLL-type relations,

$$\begin{aligned}U |\psi_n(0)\rangle \langle \psi_n(0)| \otimes |\psi_{n+1}(u)\rangle \langle \psi_{n+1}(u)| \\ = |\psi_n(u)\rangle \langle \psi_n(u)| \otimes |\psi_{n+1}(0)\rangle \langle \psi_{n+1}(0)| U,\end{aligned}\quad (41)$$

$$\begin{aligned}U |\psi_n(0)\rangle \langle \psi_n(0)| \otimes |\psi_{n-1}(-u)\rangle \langle \psi_{n-1}(-u)| \\ = |\psi_n(-u)\rangle \langle \psi_n(-u)| \otimes |\psi_{n-1}(0)\rangle \langle \psi_{n-1}(0)| U,\end{aligned}\quad (42)$$

which can be derived from Eq. (C1). The RLL-type relations (41) and (42) guarantee the stationarity of XYZ brickwork circuits even in the absence of any dissipative Krauss maps. Consider a space-periodic circuit of even size M , $M + 1 \equiv 1$ with fully coherent discrete time evolution dynamics

$$\rho_{t+1} = \mathcal{M}^{\text{per}}(\rho_t) = \mathcal{M}_{\text{odd}}^{\text{per}} \mathcal{M}_{\text{even}}^{\text{per}} \rho_t, \quad (43)$$

$$\mathcal{M}_{\text{odd}}^{\text{per}} = \mathcal{U}_{M,1} \mathcal{U}_{2,3} \cdots \mathcal{U}_{M-2,M-1}, \quad (44)$$

$$\mathcal{M}_{\text{even}}^{\text{per}} = \mathcal{U}_{1,2} \mathcal{U}_{3,4} \cdots \mathcal{U}_{M-1,M}, \quad (45)$$

(see Fig. 2).

If a periodic closure condition is satisfied,

$$\eta M = 0 \pmod{2}, \quad \text{if } \eta \in \mathbb{R}, \quad (46)$$

$$\eta M = 0 \pmod{2\tau}, \quad \text{if } \eta \in i\mathbb{R}, \quad (47)$$

then an even-size version of Eqs. (33) and (37),

$$|\Psi_+\rangle = \psi_1(0) \otimes \psi_2(u) \otimes \psi_3(0) \otimes \psi_4(u) \cdots \otimes \psi_{M-1}(0) \otimes \psi_M(u), \quad (48)$$

$$|\Psi_-\rangle = \psi_{-1}(0) \otimes \psi_{-2}(-u) \otimes \psi_{-3}(0) \otimes \psi_{-4}(-u) \cdots \otimes \psi_{-M+1}(0) \otimes \psi_{-M}(-u), \quad (49)$$

satisfy stationarity conditions

$$\mathcal{M}_{\text{odd}}^{\text{per}} \mathcal{M}_{\text{even}}^{\text{per}} (|\Psi_\pm\rangle \langle \Psi_\pm|) = |\Psi_\pm\rangle \langle \Psi_\pm|, \quad (50)$$

$$\begin{aligned}\mathcal{M}_{\text{even}}^{\text{per}} \mathcal{M}_{\text{odd}}^{\text{per}} (|\tilde{\Psi}_\pm\rangle \langle \tilde{\Psi}_\pm|) &= |\tilde{\Psi}_\pm\rangle \langle \tilde{\Psi}_\pm|, \\ |\tilde{\Psi}_\pm\rangle \langle \tilde{\Psi}_\pm| &= \mathcal{M}_{\text{even}}^{\text{per}} |\Psi_\pm\rangle \langle \Psi_\pm|,\end{aligned}\quad (51)$$

where $|\tilde{\Psi}_\pm\rangle$ are obtained from $|\Psi_\pm\rangle$ by swapping the arguments $0 \leftrightarrow \pm u$ in Eqs. (48) and (49). Equations (50) and (51) follow straightforwardly from Eqs. (41) and (42). Stationary helices (48) and (49) in the periodic setup (43) are degenerate,

since the choice of the overall phase α_L in Eq. (35) is arbitrary. Based on numerical investigations, we propose that the degeneracy of helices in a periodic system of M sites (see Fig. 2), including both chirality signs, is equal to $\text{deg} = 2M$. Helices (48) and (49) for $\eta \in i\mathbb{R}$ do not have analogs in XXZ-type brickwork circuits [18]. Figures 3 and 4 show examples of magnetization profiles for the periodic elliptic helices (48) for both $\eta \in \mathbb{R}$ and $\eta \in i\mathbb{R}$ cases.

Visualization of elliptic spin helices via measuring one-point correlations. Coming back to the open circuit with boundary resets (Fig. 1), we propose to observe the elliptic brickwork helices in a potential experiment, where we change the anisotropy and measure a one-point correlation at a boundary.

To this end, let us fix the left/right boundary resets as in Eq. (21), with arbitrary α_L and $\alpha_R = \alpha_L + u$. Keeping the resets fixed, we adiabatically change the anisotropy η . Condition (32) predicts pure helix NESS for the anisotropy satisfying

$$(N+1)\eta = 2n\tau, \quad n = 1, 2, \dots, N, \quad \text{for imaginary } \eta, \quad (52)$$

$$(N+1)\eta = 2n, \quad n = 1, 2, \dots, N, \quad \text{for real } \eta. \quad (53)$$

For experimental detection of helices, we propose to measure the one-point steady-state correlations of the leftmost spin: $a = \langle \sigma_1^+ \rangle$ and $b = \langle \sigma_1^z \rangle$, from which the reduced density matrix of the leftmost spin $\rho_1 = \text{tr}_{2,3,\dots,N}(\rho_\infty)$ can be constructed. Next, we define two helix indicators:

$$f_1 = 1 - \text{tr}(\rho_1^2) = \frac{1}{2} - \frac{b^2}{2} - 2aa^*, \quad (54)$$

$$f_2(x) = \frac{1}{2} \text{tr}|\rho_1 - \rho(\alpha_L + x)| = \frac{1}{2} \sum_{j=1}^2 |\lambda_j|, \quad (55)$$

$$\rho(x) = \frac{1}{|\theta_1(x)|^2 + |\theta_4(x)|^2} \begin{pmatrix} \theta_1(x) \\ \theta_4(x) \end{pmatrix} \begin{pmatrix} \theta_1(x^*) \\ \theta_4(x^*) \end{pmatrix}. \quad (56)$$

The indicator f_1 simply measures a purity of a qubit state. The indicator $f_2(\eta)$ is a trace distance between the actual state of the first qubit ρ_1 and $\rho(\alpha_L + \eta)$, which is the state of the first qubit in the helix (33). Analogously, $f_2(-\eta)$ measures a distance between ρ_1 and respective state $\rho(\alpha_L - \eta)$ in the helix of opposite chirality (37). In Eq. (55), $|\lambda_j|$ are absolute values of the eigenvalues of the difference $\rho_1 - \rho(\alpha_L + x)$. By definition, f_1, f_2 have range $0 \leq f_1 \leq 1/2$ and $0 \leq f_2 \leq 1$. Whenever the NESS becomes an elliptic helix state (33), both indicators vanish, $f_1 = f_2(\eta) = 0$. This is expected to happen whenever the condition (52) or (53) is satisfied. Otherwise, $f_1, f_2(\pm\eta)$ serve as a measure of a purity and of a distance to an elliptic helices (33) and (37), respectively.

Figure 5 shows that indicators $f_1, f_2(\eta)$ are very sensitive not only to the presence of the helices [at positions (52) and (53), as expected], but they also show very clear, less deep minima (indicated by green dashed lines). Positions of the extra minima can be readily identified as location of helix descendants, i.e., helices with kinks, according to Eq. (36). For the case $N = 7$, we see clear minima of f_1, f_2 for helix descendants with $m_k = 1$. For an increasing system size, more and more helix descendants $m_k \geq 1$ can be identified (see Fig. 6).

In Fig. 5, only helices with the same winding direction appear, the helices of type (33). However, with a specific choice

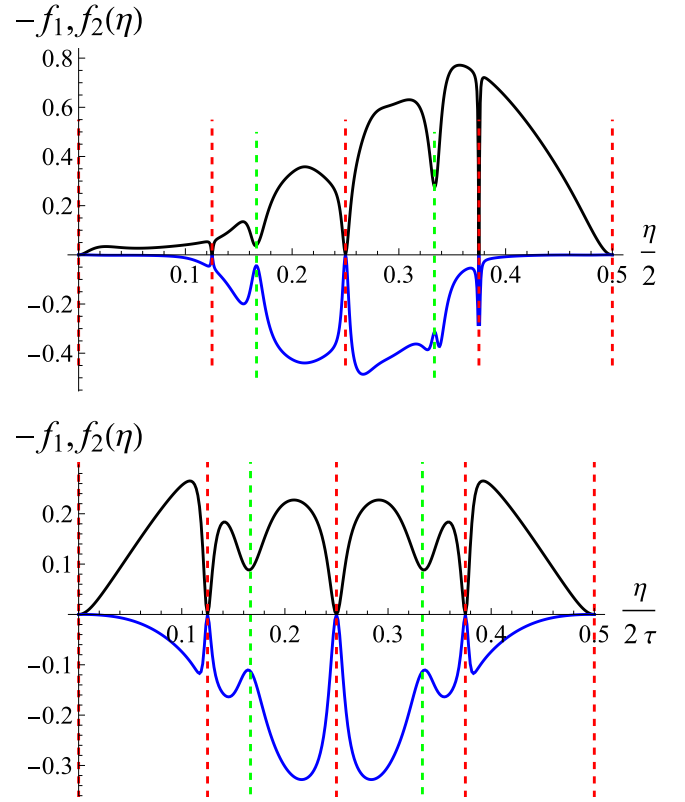


FIG. 5. Helix indicators $-f_1$ (solid blue curve) and $f_2(\eta)$ (solid black curve), given by Eqs. (54) and (55), respectively, and measured in the NESS, plotted vs the rescaled anisotropy for a system of $N = 7$ sites. Top panel: $u = 0.185i$, $\tau = 0.65i$, $\alpha_L = 0.165 + 0.13i$, $\alpha_R = \alpha_L + u$. Bottom panel: $u = 0.185$, $\tau = 0.65i$, $\alpha_L = 0$, $\alpha_R = u = 0.185$. Zeros of the indicators coincide with the pure helix condition $(N+1)\eta = 2p + 2q\tau$, $p, q \in \mathbb{Z}$ (red dashed lines), while less pronounced minima (green dashed lines) occur at anisotropies leading to helix with integer number of kinks $m_k = 1$ (helix descendants) $(N+1 - 2m_k)\eta = 2p + 2q\tau$, $p, q \in \mathbb{Z}$, $m_k = 1$.

of boundary condition α_L, α_R , and only in XYZ circuits, by varying anisotropy, one can generate NESS with any winding direction, i.e., both of Eq. (33) type and of Eq. (37) type. This becomes possible due to a specific property of theta functions θ_1, θ_4 parametrizing our reset channels, namely,

$$\begin{pmatrix} \theta_1(\alpha) \\ \theta_4(\alpha) \end{pmatrix} = \begin{pmatrix} \theta_1(1 - \alpha) \\ \theta_4(1 - \alpha) \end{pmatrix}. \quad (57)$$

A scenario featuring steady helices of both chiralities is illustrated in Fig. 6 for real η . Namely, choosing the left and right reset as $\alpha_L = 0, \alpha_R = u$, we generate NESS—helices of type (33)—when condition (32), equivalent to $(N+1)\eta = 0 \pmod{2}$, is satisfied. On the other hand, due to Eq. (57), the choice $\alpha_R = u$ is equivalent to $\alpha_R = 1 - u$, also enabling helices of reversed chirality (37) according to Eq. (38), at anisotropies $(N+1)\eta = 1 \pmod{2}$ (see bottom panel of Fig. 6).

VI. DISCUSSION

We have presented analytic expression for a unique steady state of a boundary-driven XYZ brickwork circuit, via a spatially inhomogeneous matrix product ansatz with formally

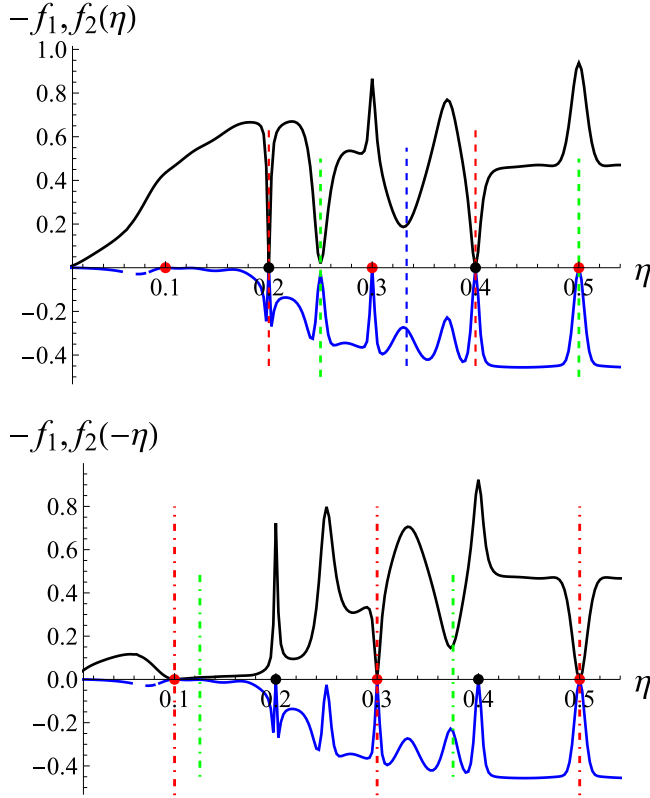


FIG. 6. Helix indicators $-f_1$ (blue curve) and $f_2(\pm\eta)$ (black curve) given by Eqs. (54) and (55), vs the anisotropy, for a system with $N = 9$ sites. Zeros of f_1 are marked by black and red points. Parameters: $u = 0.185i$, $\tau = 0.65i$, $\alpha_L = 0$, $\alpha_R = \alpha_L + u$. Top panel: A subset of zeros of f_1 (black points) coincides with the pure helix condition $(N + 1)\eta = 0 \pmod{2}$ and corresponds to helices of type (33) (red dashed lines). Helix descendants with one/two kinks correspond to $(N + 1 - 2m_k)\eta = 0 \pmod{2}$, with $m_k = 1, 2$, and are marked with green/blue dashed lines. Red points mark locations of helices with opposite helicity of the form (37), appearing at $(N + 1)\eta = 1 \pmod{2}$; see text for more explanations. Bottom panel: Remaining zeros of f_1 (red points) correspond to helices of reversed chirality (37) and their descendants and appear at positions $(N + 1 - 2m_k)\eta = 1 \pmod{2}$, with $m_k = 0, 1$, marked with red and green dot-dashed lines.

infinite bond dimension. Our analysis reveals the existence of especially simple NESS types—pure factorized chiral qubit configurations (elliptic spin helices) and their descendants, helices with kinks. We find conditions for the existence of helices both in open dissipative setup and for fully coherent periodic brickwork qubit system. Helices are characterized by a periodic modulation of qubit polarization, with wavelength inversely proportional to the XYZ gate anisotropy η , both for “easy-plane-like” case (real η) and for “easy-axis-like” case (imaginary η). Elliptic “easy-plane-like” helices reduce to conventional trigonometric ones with harmonic modulations of qubits’ polarization in the XXZ limit. Periodic “easy-axis-like” helices do not have trigonometric counterparts. Our findings show that the bulk qubit dynamics in systems without $U(1)$ symmetry is amenable to analytic treatment. In addition, it exhibits a rich family of robust, chiral, and easily creatable steady states—elliptic helices that are potentially important for experimental applications.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: THETA FUNCTIONS

We introduce the following Jacobi theta functions $\vartheta_\alpha(u, q)$ [26]:

$$\begin{aligned}\vartheta_1(u, q) &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin[(2n+1)u], \\ \vartheta_2(u, q) &= 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos[(2n+1)u], \\ \vartheta_3(u, q) &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nu), \\ \vartheta_4(u, q) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nu).\end{aligned}\quad (\text{A1})$$

For convenience, we use the following shorthand notations $\theta_\alpha(u)$, $\tilde{\theta}_\alpha(u)$, $\alpha = 1, 2, 3, 4$:

$$\theta_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{2i\pi\tau}), \quad \tilde{\theta}_\alpha(u) \equiv \vartheta_\alpha(\pi u, e^{i\pi\tau}), \quad (\text{A2})$$

where τ is a complex number with a positive imaginary part.

APPENDIX B: EFFECTIVE KRAUSS OPERATORS

Here, we show that local boundary channels

$$\begin{aligned}\mathcal{K}_L(\rho) &= \text{tr}_1[U(\rho_L \otimes \rho)U^\dagger], \\ \mathcal{K}_R(\rho) &= \text{tr}_2[U(\rho \otimes \rho_R)U^\dagger],\end{aligned}\quad (\text{B1})$$

with pure $\rho_{L/R} = |\psi_{L/R}\rangle\langle\psi_{L/R}|$, can be expressed in terms of a pair of Krauss matrices $K_{L/R}^{[1]}, K_{L/R}^{[2]}$ as $\mathcal{K}_{L/R}(\rho) = \sum_{\mu=1}^2 K_{L/R}^{[\mu]\dagger} \rho K_{L/R}^{[\mu]}$. Note that we do not need an explicit form of the Krauss matrices $K_{L/R}^{[\mu]}$ for the proof of our MPA (18). Here, we give the derivation of the Krauss matrices for completeness and for possible use in numerical algorithms.

Let us introduce a unitary matrix V such that $|\psi_L\rangle = V \binom{1}{0}$ and define a trace-orthonormal basis of 2×2 matrices φ_j as

$$\begin{aligned}\varphi_0 &= V \frac{I + \sigma^z}{2} V^\dagger \equiv |\psi_L\rangle\langle\psi_L| = \rho_L, \\ \varphi_1 &= V \frac{I - \sigma^z}{2} V^\dagger, \quad \varphi_2 = V \sigma^+ V^\dagger, \quad \varphi_3 = V \sigma^- V^\dagger,\end{aligned}$$

satisfying

$$\text{tr}(\varphi_j \varphi_k^\dagger) = \delta_{j,k}.$$

Using the basis, we can expand the XYZ gate U from Eq. (1) as

$$U = \sum_{k=0}^3 \varphi_k \otimes u_k, \quad U^\dagger = \sum_{k=0}^3 \varphi_k^\dagger \otimes u_k^\dagger, \quad u_k = \text{tr}_1((\varphi_k^\dagger \otimes I)U).$$

Now, for arbitrary 2×2 matrix A , we have

$$\begin{aligned} \mathcal{K}_L(A) &= \text{tr}_1[U(\rho_L \otimes A)U^\dagger] \\ &= \sum_{k,j=0}^3 \text{tr}_1[(\varphi_k \otimes u_k)(\varphi_0 \otimes A)(\varphi_j^\dagger \otimes u_j^\dagger)] \\ &= \sum_{k,j=0}^3 \text{tr}(\varphi_k \varphi_0 \varphi_j^\dagger) (u_k A u_j^\dagger) = \sum_{k,j=0}^3 C_{j,k} (u_k A u_j^\dagger), \end{aligned} \quad (\text{B2})$$

where $C_{j,k} = \text{tr}(\varphi_k \varphi_0 \varphi_j^\dagger)$. Only nonzero $C_{j,k}$ coefficients are $C_{0,0} = C_{3,3} = 1$. Denoting $K_L^{[1]} \equiv u_0$, $K_L^{[2]} \equiv u_3$, we finally obtain

$$\mathcal{K}_L(A) = \sum_{\mu=1}^2 K_L^{[\mu]} A K_L^{[\mu]\dagger}, \quad (\text{B3})$$

$$K_L^{[1]} = \text{tr}_1((\varphi_0^\dagger \otimes I)U), \quad (\text{B4})$$

$$K_L^{[2]} = \text{tr}_1((\varphi_3^\dagger \otimes I)U). \quad (\text{B5})$$

Taking U in the form

$$U(u, \eta) = \frac{1}{\sqrt{Y}} \begin{pmatrix} r_1 & 0 & 0 & r_4 \\ 0 & r_3 & r_2 & 0 \\ 0 & r_2 & r_3 & 0 \\ r_4 & 0 & 0 & r_1 \end{pmatrix}, \quad (\text{B6})$$

$$r_1 = \frac{\theta_4(u)\theta_1(u+\eta)}{\theta_4(0)\theta_1(\eta)}, \quad r_2 = \frac{\theta_1(u)\theta_4(u+\eta)}{\theta_4(0)\theta_1(\eta)}, \quad (\text{B7})$$

$$\begin{aligned} r_3 &= \frac{\theta_4(u)\theta_4(u+\eta)}{\theta_4(0)\theta_4(\eta)}, \quad r_4 = \frac{\theta_1(u)\theta_1(u+\eta)}{\theta_4(0)\theta_4(\eta)}, \\ Y &= \frac{\tilde{\theta}_1(\eta+u)\tilde{\theta}_1(\eta-u)}{\tilde{\theta}_1(\eta)\tilde{\theta}_1(\eta)}, \end{aligned} \quad (\text{B8})$$

and parametrizing $|\psi_L\rangle$ via a polar angle θ_L and azimuthal angle φ_L , we have

$$|\psi_L\rangle = \begin{pmatrix} e^{-i\varphi_L/2} \cos \frac{\theta_L}{2} \\ e^{i\varphi_L/2} \sin \frac{\theta_L}{2} \end{pmatrix} = V \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\text{B9})$$

$$V = \exp\left(-i\varphi_L \frac{\sigma^z}{2}\right) \begin{pmatrix} \cos \frac{\theta_L}{2} & -\sin \frac{\theta_L}{2} \\ \sin \frac{\theta_L}{2} & \cos \frac{\theta_L}{2} \end{pmatrix}. \quad (\text{B10})$$

Calculating $K_L^{[1]}$, $K_L^{[2]}$ from Eqs. (B4) and (B5), we obtain

$$K_L^{[1]} = \frac{1}{\sqrt{Y}} \begin{pmatrix} f_L(r_1, r_3) & g_L(r_2, r_4)F_L \\ g_L(r_4, r_2)F_L & f_L(r_3, r_1) \end{pmatrix}, \quad (\text{B11})$$

$$K_L^{[2]} = \frac{1}{\sqrt{Y}} \begin{pmatrix} (r_3 - r_1)F_L & \tilde{f}_L(r_2, r_4) \\ \tilde{f}_L(r_4, r_2) & (r_1 - r_3)F_L \end{pmatrix}, \quad (\text{B12})$$

where

$$\begin{aligned} F_L &= \frac{\sin \theta_L}{2}, \quad \tilde{f}_L(x, y) = x e^{-i\varphi_L} \cos^2 \frac{\theta_L}{2} - y e^{i\varphi_L} \sin^2 \frac{\theta_L}{2}, \\ f_L(x, y) &= x \cos^2 \frac{\theta_L}{2} + y \sin^2 \frac{\theta_L}{2}, \\ g_L(x, y) &= e^{-i\varphi_L} x + e^{i\varphi_L} y. \end{aligned} \quad (\text{B13})$$

Completely analogously, we treat the right boundary. We then obtain

$$\mathcal{K}_R(\rho) = \text{tr}_2[U(\rho \otimes \rho_R)U^\dagger] = \sum_{\mu=1}^2 K_R^{[\mu]} \rho K_R^{[\mu]\dagger}, \quad (\text{B14})$$

where $K_R^{[\mu]}$ is given by exactly the same expressions as for $K_L^{[\mu]}$, with the substitutions of all subscripts $L \rightarrow R$ in all the expressions in Eqs. (B12) and (B13).

APPENDIX C: PROOF OF EQ. (15)

Let us introduce some useful identities [27,28]

$$\tilde{U}|\psi(x \mp u)\rangle \otimes |\psi(x \pm \eta)\rangle = \frac{\tilde{\theta}_1(u+\eta)}{\tilde{\theta}_1(\eta)} |\psi(x)\rangle \otimes |\psi(x \mp u \pm \eta)\rangle, \quad (\text{C1})$$

$$\tilde{U}|\psi(x \pm \eta)\rangle \otimes |\psi(y \mp u)\rangle = \frac{\tilde{\theta}_2(\frac{x+y}{2} \mp u)}{\tilde{\theta}_2(\frac{x+y}{2})} |\psi(x \pm u \pm \eta)\rangle \otimes |\psi(y)\rangle + \frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\frac{x+y}{2} \pm \eta)}{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\frac{x+y}{2})} |\psi(y \mp u \mp \eta)\rangle \otimes |\psi(x)\rangle, \quad (\text{C2})$$

$$\langle \xi(x \mp u) | \otimes \langle \xi(x \pm \eta) | \tilde{U} = \frac{\tilde{\theta}_1(u+\eta)}{\tilde{\theta}_1(\eta)} \langle \psi(x) | \otimes \langle \psi(x \mp u \pm \eta) |, \quad (\text{C3})$$

$$\langle \xi(x \pm \eta) | \otimes \langle \xi(y \mp u) | \tilde{U} = \frac{\tilde{\theta}_2(\frac{x+y}{2} \mp u)}{\tilde{\theta}_2(\frac{x+y}{2})} \langle \xi(x \pm u \pm \eta) | \otimes \langle \xi(y) | + \frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\frac{x+y}{2} \pm \eta)}{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\frac{x+y}{2})} \langle \xi(y \mp u \mp \eta) | \otimes \langle \xi(x) |. \quad (\text{C4})$$

Due to the band structure of the matrices (12), for a proof of Eq. (15), it is enough to check the following properties [below we adopt simplified notations $L(m) \rightarrow L_m$, $X(m) \rightarrow X_m$, etc.]:

$$\begin{aligned} U(L_m \otimes L'_{m+1}) &= (L'_m \otimes L_{m+1})U, \\ U(L_m \otimes X_{m+1} + X'_m \otimes L'_{m-1}) &= (L'_m \otimes X'_{m+1} + X_m \otimes L_{m-1})U, \\ U(X'_m \otimes X_{m-1}) &= (X_m \otimes X'_{m-1})U, \end{aligned} \quad (\text{C5})$$

which hold for arbitrary m , and the same set for the conjugated operators in Eq. (12).

Define the following operators:

$$\mathcal{L}_m = |\psi(m\eta + b)\rangle\langle\xi(-u - m\eta - b)|, \quad \mathcal{L}'_m = |\psi(u + m\eta + b)\rangle\langle\xi(-m\eta - b)|, \quad (\text{C6})$$

$$\mathcal{X}_m = |\psi(m\eta - u + a)\rangle\langle\xi(-m\eta - a)|, \quad \mathcal{X}'_m = |\psi(m\eta + a)\rangle\langle\xi(u - m\eta - a)|. \quad (\text{C7})$$

By using Eqs. (C1)–(C4) repeatedly, we can get the following identities:

$$\tilde{U}(\mathcal{L}_m \otimes \mathcal{L}'_{m+1}) = (\mathcal{L}'_m \otimes \mathcal{L}_{m+1})\tilde{U}, \quad (\text{C8})$$

$$\tilde{U}(\mathcal{X}'_m \otimes \mathcal{X}_{m-1}) = (\mathcal{X}_m \otimes \mathcal{X}'_{m-1})\tilde{U}, \quad (\text{C9})$$

$$\begin{aligned} & \tilde{U}\left(\tilde{\theta}_2\left(\frac{a+b}{2} + m\eta - \eta\right)\mathcal{L}_m \otimes \mathcal{X}_{m+1} + \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta + \eta\right)\mathcal{X}'_m \otimes \mathcal{L}'_{m-1}\right) \\ &= \left(\tilde{\theta}_2\left(\frac{a+b}{2} + m\eta - \eta\right)\mathcal{L}'_m \otimes \mathcal{X}'_{m+1} + \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta + \eta\right)\mathcal{X}_m \otimes \mathcal{L}_{m-1}\right)\tilde{U}. \end{aligned} \quad (\text{C10})$$

We suppose

$$L_m = \frac{\mathcal{L}_m}{\alpha_m}, \quad L'_m = \frac{\mathcal{L}'_m}{\alpha'_m}, \quad X_m = \frac{\mathcal{X}_m}{\beta_m}, \quad X'_m = \frac{\mathcal{X}'_m}{\beta'_m}. \quad (\text{C11})$$

From Eq. (C5), the following constraints need to be satisfied:

$$\alpha_m \alpha'_{m+1} = \alpha'_m \alpha_{m+1}, \quad \beta_m \beta'_{m-1} = \beta'_m \beta_{m-1}, \quad (\text{C12})$$

$$\begin{aligned} \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta - \eta\right)\alpha_m \beta_{m+1} &= \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta + \eta\right)\beta'_m \alpha'_{m-1} \\ &= \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta - \eta\right)\alpha'_m \beta'_{m+1} = \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta + \eta\right)\beta_m \alpha_{m-1}. \end{aligned} \quad (\text{C13})$$

A simple solution is

$$\alpha_m = \alpha'_m = \beta_m = \beta'_m = \tilde{\theta}_2\left(\frac{a+b}{2} + m\eta\right). \quad (\text{C14})$$

Reinstalling the original notations [$L_m \rightarrow L(m)$, $X_m \rightarrow X(m)$, etc.] and letting $b = a$, we obtain Eq. (15).

APPENDIX D: RESOLVING EQ. (19): FINDING THE LEFT AUXILIARY VECTOR

We first introduce some identities regarding theta functions

$$\theta_1(u)\theta_1(v) + \theta_4(u)\theta_4(v) = \tilde{\theta}_4\left(\frac{u+v}{2}\right)\tilde{\theta}_3\left(\frac{u-v}{2}\right), \quad (\text{D1})$$

$$\theta_1(u)\theta_4(v) - \theta_1(v)\theta_4(u) = \tilde{\theta}_2\left(\frac{u+v}{2}\right)\tilde{\theta}_1\left(\frac{u-v}{2}\right), \quad (\text{D2})$$

$$\begin{aligned} & \tilde{\theta}_1(x)\tilde{\theta}_2(y)\tilde{\theta}_3(z)\tilde{\theta}_4(w) - \tilde{\theta}_1\left(\frac{x+y+z-w}{2}\right)\tilde{\theta}_2\left(\frac{x+y-z+w}{2}\right)\tilde{\theta}_3\left(\frac{x-y+z+w}{2}\right)\tilde{\theta}_4\left(\frac{y+z+w-x}{2}\right) \\ & - \tilde{\theta}_1\left(\frac{w+x-y-z}{2}\right)\tilde{\theta}_2\left(\frac{w-x+y-z}{2}\right)\tilde{\theta}_3\left(\frac{w-x-y+z}{2}\right)\tilde{\theta}_4\left(\frac{w+x+y+z}{2}\right) = 0. \end{aligned} \quad (\text{D3})$$

Other useful identities derived from Eqs. (C1) and (C2) are

$$U|\psi(\beta)\rangle \otimes |\psi(u + \beta + \eta)\rangle = \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)}|\psi(u + \beta)\rangle \otimes |\psi(\beta + \eta)\rangle, \quad (\text{D4})$$

$$U|\psi(\beta)\rangle \otimes |\psi(\beta - \eta + u)\rangle = \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta + u)}{\tilde{\theta}_2(\beta)}|\psi(\beta - u)\rangle \otimes |\psi(\beta - \eta)\rangle + \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\beta - \eta)}{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\beta)}|\psi(\beta + u)\rangle \otimes |\psi(\beta + \eta)\rangle, \quad (\text{D5})$$

$$U|\psi(\beta)\rangle \otimes |\psi(\eta - u + \beta)\rangle = \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)}|\psi(\beta + u)\rangle \otimes |\psi(\beta + \eta)\rangle + \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\beta + \eta)}{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\beta)}|\psi(\beta - u)\rangle \otimes |\psi(\beta - \eta)\rangle, \quad (\text{D6})$$

where κ is defined as $\kappa = \left[\frac{\tilde{\theta}_1(\eta+u)\tilde{\theta}_1(\eta-u)}{\tilde{\theta}_1(\eta)\tilde{\theta}_1(\eta)}\right]^{\frac{1}{2}}$.

We postulate the solution of Eq. (19) in the form

$$\langle L| = \sum_{j,j'=0,1} l_{j,j'} \langle j|_a \langle j'|_b, \quad (D7)$$

for appropriately chosen parameter a in all equations in Eq. (14).

First, without losing generality, we postulate the left reset channel to target the qubit state

$$\rho_L = \rho_0 = \frac{|\psi(\beta)\rangle\langle\psi(\beta)|}{\langle\psi(\beta)|\psi(\beta)\rangle}, \quad \rho_n = \frac{|\psi(\beta + n\eta)\rangle\langle\psi(\beta + n\eta)|}{\langle\psi(\beta + n\eta)|\psi(\beta + n\eta)\rangle}, \quad \rho'_n = \frac{|\psi(u + \beta + n\eta)\rangle\langle\psi(u + \beta + n\eta)|}{\langle\psi(u + \beta + n\eta)|\psi(u + \beta + n\eta)\rangle}, \quad (D8)$$

where $\beta = \alpha_L = w_0 + iw_1$ is an arbitrary complex number.

Postulating the constants $a = \beta$ in Eq. (14), we write Eq. (19) in components

$$\sum_{j,j'=0,1} l_{j,j'} (L_1^+)_{j,k} (T_1^+)_{j',k'} = \sum_{j,j'=0,1} l_{j,j'} \mathcal{K}_L[(L_1^-)_{j,k} (T_1^-)_{j',k'}], \quad k, k' = 0, 1. \quad (D9)$$

Before analyzing Eq. (D9), we denote $L(m) \equiv L_m$, $L'(m) \equiv L'_m$, $X(m) \equiv X_m$, $X'(m) \equiv X'_m$, $L(m)^\dagger \equiv T_m$, $L'(m)^\dagger \equiv T'_m$, $X(m)^\dagger \equiv Y_m$, $X'(m)^\dagger \equiv Y'_m$, and their definitions are as follows:

$$\begin{aligned} L_m &= \frac{1}{C_m} |\psi(m\eta + \beta)\rangle \langle \xi(-u - m\eta - \beta)|, \\ L'_m &= \frac{1}{C_m} |\psi(u + m\eta + \beta)\rangle \langle \xi(-m\eta - \beta)|, \\ X_m &= \frac{1}{C_m} |\psi(m\eta - u + \beta)\rangle \langle \xi(-m\eta - \beta)|, \\ X'_m &= \frac{1}{C_m} |\psi(m\eta + \beta)\rangle \langle \xi(u - m\eta - \beta)|, \end{aligned} \quad (D10)$$

$$C_m = \tilde{\theta}_2(\beta + m\eta), \quad (D11)$$

and

$$\begin{aligned} T_m &= \frac{1}{C_m^*} |\xi(-u - m\eta - \beta)\rangle \langle \psi(m\eta + \beta)|, \\ T'_m &= \frac{1}{C_m^*} |\xi(-m\eta - \beta)\rangle \langle \psi(u + m\eta + \beta)|, \\ Y_m &= \frac{1}{C_m^*} |\xi(-m\eta - \beta)\rangle \langle \psi(m\eta - u + \beta)|, \\ Y'_m &= \frac{1}{C_m^*} |\xi(u - m\eta - \beta)\rangle \langle \psi(m\eta + \beta)|. \end{aligned} \quad (D12)$$

Expression of $l_{0,0}$. Taking $(k, k') = (0, 0)$ in Eq. (D9) yields

$$l_{0,0} L_1 T_1 = l_{0,0} \mathcal{K}_L[L'_1 T'_1]. \quad (D13)$$

The definition of $\mathcal{K}_L$ is

$$\mathcal{K}_L(A) = \text{tr}_1[U(\rho_L \otimes A)U^\dagger]. \quad (D14)$$

By substituting $A = L'_1 T'_1$, we obtain

$$\begin{aligned} \mathcal{K}_L[L'_1 T'_1] &= \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \text{tr}_1[U|\psi(\beta)\rangle \otimes \psi(\beta + u + \eta) \langle \psi(\beta) | \otimes \langle \psi(\beta + u + \eta) | U^\dagger] \\ &\stackrel{(D4)}{=} \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \langle \psi(\beta + u) | \psi(\beta + u) \rangle \langle \psi(\eta + \beta) | \langle \psi(\eta + \beta) |. \end{aligned} \quad (D15)$$

In both case (A) and case (B), we find

$$\frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle \langle \psi(\beta + u) | \psi(\beta + u) \rangle}{\langle \xi(-u - \eta - \beta) | \xi(-u - \eta - \beta) \rangle \langle \psi(\beta) | \psi(\beta) \rangle} \stackrel{(D1)}{=} 1, \quad (D16)$$

which directly leads to Eq. (D13). Since Eq. (D13) holds for arbitrary $l_{0,0}$, we set $l_{0,0} = 1$.

Expression of $l_{0,1}$ and $l_{1,0}$. For the element $(k, k') = (0, 1)$ of Eq. (D9), we get

$$L_1 Y_1' + l_{0,1} L_1 T_{-1} = \mathcal{K}_L[L_1' Y_1] + l_{0,1} \mathcal{K}_L[L_1' T_{-1}']. \quad (\text{D17})$$

Using Eqs. (D4)–(D6) repeatedly, we have

$$\begin{aligned} \mathcal{K}_L[L_1' Y_1] &= \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) | \\ &+ \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) |, \end{aligned} \quad (\text{D18})$$

$$\begin{aligned} \mathcal{K}_L[L_1' T_{-1}'] &= \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_1 C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) | \\ &+ \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_1 \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) |. \end{aligned} \quad (\text{D19})$$

Comparing the two sides of Eq. (D17) gives the following identities:

$$\begin{aligned} &\frac{\langle \xi(-u - \eta - \beta) | \xi(u - \eta - \beta) \rangle}{C_1^* \langle \psi(\beta + u) | \psi(\beta + u) \rangle} l_{0,0} - \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} l_{0,0} \\ &- \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} l_{0,1} = 0, \end{aligned} \quad (\text{D20})$$

$$\begin{aligned} &\frac{\langle \xi(-u - \eta - \beta) | \xi(-u + \eta - \beta) \rangle}{C_{-1}^* \langle \psi(\beta - u) | \psi(\beta + u) \rangle} l_{0,1} - \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} l_{0,0} \\ &- \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u + \eta)}{\tilde{\theta}_1(\eta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} l_{0,1} = 0. \end{aligned} \quad (\text{D21})$$

From Eq. (D20), we can derive the expression of $l_{0,1}$, as follows:

$$\begin{aligned} \text{case (A): } l_{0,1} &\stackrel{(\text{D1})}{=} \frac{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta^* - u) \tilde{\theta}_3(iw_1 + \eta) \tilde{\theta}_4(w_0)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta^* - \eta) \tilde{\theta}_3(iw_1) \tilde{\theta}_4(w_0 + \eta)} - \frac{\tilde{\theta}_1(\eta - u) \tilde{\theta}_2(\beta^*) \tilde{\theta}_3(iw_1 + \eta + u) \tilde{\theta}_4(w_0) \tilde{\theta}_4(w_0)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta^* - \eta) \tilde{\theta}_3(iw_1) \tilde{\theta}_4(w_0 + \eta) \tilde{\theta}_4(w_0 + u)} \\ &\stackrel{(\text{D3})}{=} \frac{\tilde{\theta}_4(w_0) \tilde{\theta}_3(u + iw_1)}{\tilde{\theta}_4(u + w_0) \tilde{\theta}_3(iw_1)}, \end{aligned} \quad (\text{D22})$$

$$\begin{aligned} \text{case (B): } l_{0,1} &\stackrel{(\text{D1})}{=} \frac{\tilde{\theta}_1(\eta) \tilde{\theta}_2(u + \beta^*) \tilde{\theta}_3(iw_1) \tilde{\theta}_4(w_0 + \eta)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta^* + \eta) \tilde{\theta}_3(iw_1 + \eta) \tilde{\theta}_4(w_0)} - \frac{\tilde{\theta}_1(\eta - u) \tilde{\theta}_2(\beta^*) \tilde{\theta}_3(iw_1) \tilde{\theta}_3(iw_1) \tilde{\theta}_4(u + \eta + w_0)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta^* + \eta) \tilde{\theta}_3(iw_1 + \eta) \tilde{\theta}_3(u + iw_1) \tilde{\theta}_4(w_0)} \\ &\stackrel{(\text{D3})}{=} \frac{\tilde{\theta}_4(u + w_0) \tilde{\theta}_3(iw_1)}{\tilde{\theta}_4(w_0) \tilde{\theta}_3(u + iw_1)}. \end{aligned} \quad (\text{D23})$$

One can verify that $l_{0,1}$ in Eq. (D22) or (D23) also satisfies Eq. (D21).

Analogously, taking $(k, k') = (1, 0)$ in Eq. (D9) can give $l_{1,0}$. After some tedious calculations, we finally obtain

$$l_{1,0} = l_{0,1}^*. \quad (\text{D24})$$

Expression of $l_{1,1}$. Setting $(k, k') = (1, 0)$ in Eq. (D9) yields the following equation:

$$l_{0,0} X_1' Y_1' + l_{1,0} L_{-1} Y_1' + l_{0,1} X_1' T_{-1} + l_{1,1} L_{-1} T_{-1} = l_{0,0} \mathcal{K}_L[X_1 Y_1] + l_{1,0} \mathcal{K}_L[L_{-1}' Y_1] + l_{0,1} \mathcal{K}_L[X_1 T_{-1}'] + l_{1,1} \mathcal{K}_L[L_{-1}' T_{-1}']. \quad (\text{D25})$$

With the help of Eqs. (D4)–(D6), we arrive at the following equations:

$$\begin{aligned} \mathcal{K}_L[X_1 Y_1] &= \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) | \\ &+ \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) | \\ &+ \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta + \eta) | \\ &+ \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta - \eta) |, \end{aligned} \quad (\text{D26})$$

$$\begin{aligned}
\mathcal{K}_L[L'_1 Y_1] = & \frac{\langle \xi(\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_{-1} C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta + u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta + \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_{-1} \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta + u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta - \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(-\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) |, \quad (D27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_L[X_1 T'_{-1}] = & \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_1 C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) | \\
& + \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_1 \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) | \\
& + \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta - \eta) | \\
& + \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta + \eta) |, \quad (D28)
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_L[L'_1 T'_{-1}] = & \frac{\langle \xi(\eta - \beta) | \xi(\eta - \beta) \rangle}{C_{-1} C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta + u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta - \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(\eta - \beta) \rangle}{C_{-1} \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta + u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta - u) \rangle | \psi(\beta - \eta) \rangle \langle \psi(\beta + \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(\eta - \beta) \rangle}{C_{-1}^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* + u^*)}{\tilde{\theta}_2(\beta^*)} \langle \psi(\beta - u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta - \eta) | \\
& + \frac{\langle \xi(\eta - \beta) | \xi(\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} \langle \psi(\beta + u) | \psi(\beta + u) \rangle | \psi(\beta + \eta) \rangle \langle \psi(\beta + \eta) |. \quad (D29)
\end{aligned}$$

We see that both sides of Eq. (D25) are linear combinations of $|\psi(\beta + m_1 \eta)\rangle \langle \psi(\beta + m_2 \eta)|$, $m_1, m_2 = \pm 1$. Comparing the coefficients of $|\psi(\beta + \eta)\rangle \langle \psi(\beta + \eta)|$ gives

$$\begin{aligned}
& \frac{\langle \xi(u - \eta - \beta) | \xi(u - \eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta + u) | \psi(\beta + u) \rangle} l_{0,0} \\
& = l_{0,0} \frac{\langle \xi(-\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1 C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} + l_{1,0} \frac{\langle \xi(\eta - \beta) | \xi(-\eta - \beta) \rangle}{C_1^* \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_2(\beta^* - u^*)}{\tilde{\theta}_2(\beta^*)} \\
& + l_{0,1} \frac{\langle \xi(-\eta - \beta) | \xi(\eta - \beta) \rangle}{C_1 \langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_2(\beta - u)}{\tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)} + l_{1,1} \frac{\langle \xi(\eta - \beta) | \xi(\eta - \beta) \rangle}{\langle \psi(\beta) | \psi(\beta) \rangle} \frac{1}{\kappa} \frac{\tilde{\theta}_1(u)}{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta)} \frac{1}{\kappa^*} \frac{\tilde{\theta}_1(u^*)}{\tilde{\theta}_1(\eta^*) \tilde{\theta}_2(\beta^*)}. \quad (D30)
\end{aligned}$$

By substituting the expression of $l_{0,0}$, $l_{0,1}$, and $l_{1,0}$ into Eq. (D30), we can get

$$\begin{aligned}
\text{case (A): } l_{1,1} & \stackrel{(D1)}{=} - \frac{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta - u) \tilde{\theta}_3(iw_1 + u) \tilde{\theta}_4(w_0 + \eta)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta + \eta) \tilde{\theta}_3(iw_1 - \eta) \tilde{\theta}_4(w_0 + u)} + \frac{\tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta^* - u) \tilde{\theta}_3(u - iw_1) \tilde{\theta}_4(\eta - w_0)}{\tilde{\theta}_1(u) \tilde{\theta}_2(\beta^* - \eta) \tilde{\theta}_3(iw_1 - \eta) \tilde{\theta}_4(w_0 + u)} \\
& - \frac{\tilde{\theta}_1(\eta - u) \tilde{\theta}_1(\eta + u) \tilde{\theta}_2(\beta) \tilde{\theta}_2(\beta^*) \tilde{\theta}_3(iw_1 + \eta) \tilde{\theta}_4(u - w_0)}{\tilde{\theta}_1(u) \tilde{\theta}_1(u) \tilde{\theta}_2(\beta + \eta) \tilde{\theta}_2(\beta^* - \eta) \tilde{\theta}_3(iw_1 - \eta) \tilde{\theta}_4(w_0 + u)} \\
& + \frac{\tilde{\theta}_1(\eta) \tilde{\theta}_1(\eta) \tilde{\theta}_2(\beta - u) \tilde{\theta}_2(\beta^* - u) \tilde{\theta}_3(iw_1 + \eta)}{\tilde{\theta}_1(u) \tilde{\theta}_1(u) \tilde{\theta}_2(\beta + \eta) \tilde{\theta}_2(\beta^* - \eta) \tilde{\theta}_3(iw_1 - \eta)} \stackrel{(D3)}{=} \frac{\tilde{\theta}_4(u - w_0)}{\tilde{\theta}_4(u + w_0)}, \quad (D31)
\end{aligned}$$

$$\begin{aligned}
\text{case (B): } l_{1,1} &\stackrel{\text{(D1)}}{=} \frac{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\beta^*+u)\tilde{\theta}_3(\eta-iw_1)\tilde{\theta}_4(u-w_0)}{\tilde{\theta}_1(u)\tilde{\theta}_2(\beta^*+\eta)\tilde{\theta}_3(u+iw_1)\tilde{\theta}_4(w_0-\eta)} + \frac{\tilde{\theta}_1(\eta)\tilde{\theta}_1(\eta)\tilde{\theta}_2(\beta-u)\tilde{\theta}_2(\beta^*+u)\tilde{\theta}_4(w_0+\eta)}{\tilde{\theta}_1(u)\tilde{\theta}_1(u)\tilde{\theta}_2(\beta+\eta)\tilde{\theta}_2(\beta^*+\eta)\tilde{\theta}_4(w_0-\eta)} \\
&\quad - \frac{\tilde{\theta}_1(\eta-u)\tilde{\theta}_1(\eta+u)\tilde{\theta}_2(\beta)\tilde{\theta}_2(\beta^*)\tilde{\theta}_3(u-iw_1)\tilde{\theta}_4(w_0+\eta)}{\tilde{\theta}_1(u)\tilde{\theta}_1(u)\tilde{\theta}_2(\beta+\eta)\tilde{\theta}_2(\beta^*+\eta)\tilde{\theta}_3(u+iw_1)\tilde{\theta}_4(w_0-\eta)} \\
&\quad - \frac{\tilde{\theta}_1(\eta)\tilde{\theta}_2(\beta-u)\tilde{\theta}_3(\eta+iw_1)\tilde{\theta}_4(w_0+u)}{\tilde{\theta}_1(u)\tilde{\theta}_2(\beta+\eta)\tilde{\theta}_3(u+iw_1)\tilde{\theta}_4(w_0-\eta)} \stackrel{\text{(D3)}}{=} \frac{\tilde{\theta}_3(u-iw_1)}{\tilde{\theta}_3(u+iw_1)}. \tag{D32}
\end{aligned}$$

Although Eq. (D25) is overdetermined, comparing the coefficients of $|\psi(\beta-\eta)\rangle\langle\psi(\beta+\eta)|$, $|\psi(\beta+\eta)\rangle\langle\psi(\beta-\eta)|$, and $|\psi(\beta-\eta)\rangle\langle\psi(\beta-\eta)|$ in Eq. (D25) yields the same expression for $l_{1,1}$, namely, Eq. (D31) or (D32).

APPENDIX E: RESOLVING EQ. (20): FINDING THE RIGHT AUXILIARY VECTOR $|\mathbf{R}\rangle$

Let us introduce the following identities derived from Eq. (C2):

$$\begin{aligned}
U|\psi(k\eta+\beta)\rangle \otimes |\psi(\delta)\rangle &= \frac{\tilde{\theta}_2\left(\frac{\delta+\eta+k\eta+\beta+u}{2}\right)}{\kappa \tilde{\theta}_2\left(\frac{\delta+\eta+k\eta+\beta-u}{2}\right)} |\psi(\beta+k\eta-u)\rangle \otimes |\psi(\delta-u)\rangle \\
&\quad + \frac{\tilde{\theta}_1(u)\tilde{\theta}_2\left(\frac{\delta-\eta+k\eta+\beta-u}{2}\right)}{\kappa \tilde{\theta}_1(\eta)\tilde{\theta}_2\left(\frac{\delta+\eta+k\eta+\beta-u}{2}\right)} |\psi(\delta+\eta)\rangle \otimes |\psi(k\eta+\beta+\eta)\rangle, \tag{E1}
\end{aligned}$$

$$\begin{aligned}
U|\psi(k\eta+\beta)\rangle \otimes |\psi(\delta)\rangle &= \frac{\tilde{\theta}_2\left(\frac{\delta-\eta+k\eta+\beta-u}{2}\right)}{\kappa \tilde{\theta}_2\left(\frac{\delta-\eta+k\eta+\beta+u}{2}\right)} |\psi(\beta+k\eta+u)\rangle \otimes |\psi(\delta+u)\rangle \\
&\quad + \frac{\tilde{\theta}_1(u)\tilde{\theta}_2\left(\frac{\delta+\eta+k\eta+\beta+u}{2}\right)}{\kappa \tilde{\theta}_1(\eta)\tilde{\theta}_2\left(\frac{\delta-\eta+k\eta+\beta+u}{2}\right)} |\psi(\delta-\eta)\rangle \otimes |\psi(k\eta+\beta-\eta)\rangle. \tag{E2}
\end{aligned}$$

One can write Eq. (20) in components. The element (k, k') of Eq. (20) is

$$\sum_{j,j'=0}^{\infty} r_{j,j'}(L_N^-)_{k,j}(T_N^-)_{k',j'} = \sum_{j,j'=0}^{\infty} r_{j,j'}\mathcal{K}_R[(L_N^+)_{k,j}(T_N^+)_{k',j'}]. \tag{E3}$$

The left-hand side of Eq. (E3) can be rewritten as

$$\begin{aligned}
\text{LHS} &= r_{k,k'}(L_N^-)_{k,k}(T_N^-)_{k',k'} + r_{k,k'+1}(L_N^-)_{k,k}(T_N^-)_{k',k'+1} + r_{k+1,k'}(L_N^-)_{k,k+1}(T_N^-)_{k',k'} + r_{k+1,k'+1}(L_N^-)_{k,k+1}(T_N^-)_{k',k'+1} \\
&= r_{k,k'}L'_{N-2k}T'_{N-2k'} + r_{k,k'+1}L'_{N-2k}Y_{N-2k'} + r_{k+1,k'}X_{N-2k}T'_{N-2k'} + r_{k+1,k'+1}X_{N-2k}Y_{N-2k'} \\
&= \frac{1}{C_k^*C_{k'}^*} \langle \xi(-\bar{k}\eta-\beta) | \xi(-\bar{k}'\eta-\beta) \rangle [r_{k,k'}|\psi(u+\bar{k}\eta+\beta)\rangle\langle\psi(u+\bar{k}'\eta+\beta)| \\
&\quad + r_{k,k'+1}|\psi(u+\bar{k}\eta+\beta)\rangle\langle\psi(\bar{k}'\eta-u+\beta)| + r_{k+1,k'}|\psi(\bar{k}\eta-u+\beta)\rangle\langle\psi(u+\bar{k}'\eta+\beta)| \\
&\quad + r_{k+1,k'+1}|\psi(\bar{k}\eta-u+\beta)\rangle\langle\psi(\bar{k}'\eta-u+\beta)|], \tag{E4}
\end{aligned}$$

where $\bar{k} = N-2k$, $\bar{k}' = N-2k'$, and $\beta = \alpha_L$. The right-hand side of Eq. (E3) reads

$$\begin{aligned}
\text{RHS} &= r_{k,k'}\mathcal{K}_R[(L_N^+)_{k,k}(T_N^+)_{k',k'}] + r_{k,k'+1}\mathcal{K}_R[(L_N^+)_{k,k}(T_N^+)_{k',k'+1}] \\
&\quad + r_{k+1,k'}\mathcal{K}_R[(L_N^+)_{k,k+1}(T_N^+)_{k',k'}] + r_{k+1,k'+1}\mathcal{K}_R[(L_N^+)_{k,k+1}(T_N^+)_{k',k'+1}] \\
&= r_{k,k'}\mathcal{K}_R[L_{N-2k}T_{N-2k'}] + r_{k,k'+1}\mathcal{K}_R[L_{N-2k}Y'_{N-2k'}] + r_{k+1,k'}\mathcal{K}_R[X'_{N-2k}T_{N-2k'}] + r_{k+1,k'+1}\mathcal{K}_R[X'_{N-2k}Y'_{N-2k'}] \\
&= \frac{1}{C_k^*C_{k'}^*} [r_{k,k'}\langle \xi(-u-\bar{k}\eta-\beta) | \xi(-u-\bar{k}'\eta-\beta) \rangle + r_{k,k'+1}\langle \xi(-u-\bar{k}\eta-\beta) | \xi(u-\bar{k}'\eta-\beta) \rangle \\
&\quad + r_{k+1,k'}\langle \xi(u-\bar{k}\eta-\beta) | \xi(-u-\bar{k}'\eta-\beta) \rangle + r_{k+1,k'+1}\langle \xi(u-\bar{k}\eta-\beta) | \xi(u-\bar{k}'\eta-\beta) \rangle] \\
&\quad \times \mathcal{K}_R[|\psi(\bar{k}\eta+\beta)\rangle\langle\psi(\bar{k}'\eta+\beta)|]. \tag{E5}
\end{aligned}$$

In the following, we will demonstrate that our ansatz in the main text [Eqs. (24)–(29) and (31)] satisfies Eq. (E3) for arbitrary k and k' . For convenience, we will omit the overall factor $\frac{r_{k,k'}}{C_k^*C_{k'}^*}$ in the proof.

Case (B). To be consistent with the notation in the main text, we first rewrote $\alpha_{L,R}$ as

$$\alpha_L = \beta = \Gamma - N\eta - \eta, \quad \alpha_R = \delta = u + \Gamma + 2\nu. \tag{E6}$$

Multiplying Eq. (E4) from the left by $\langle \xi(\bar{k}\eta + \beta + u) |$ and from the right by $|\xi(\bar{k}'\eta + \beta + u)\rangle$ and using Eqs. (D1) and (D2), the left-hand side becomes

$$\begin{aligned} & \langle \xi(\bar{k}\eta + \beta + u) | \text{LHS} | \xi(\bar{k}'\eta + \beta + u) \rangle \\ &= \frac{r_{k,k'}^{k+1,k'+1}}{r_{k,k'}} \langle \xi(-\bar{k}\eta - \beta) | \xi(-\bar{k}'\eta - \beta) \rangle \langle \xi(\bar{k}\eta + \beta + u) | \psi(\bar{k}\eta - u + \beta) \rangle \langle \psi(\bar{k}'\eta - u + \beta) | \xi(\bar{k}'\eta + \beta + u) \rangle \\ &= - \frac{\tilde{\theta}_1^2(u) \tilde{\theta}_1(k\eta + v) \tilde{\theta}_1(k'\eta + v^*) \tilde{\theta}_2(\Gamma - (2k+1)\eta) \tilde{\theta}_2(\Gamma^* - (2k'+1)\eta) \tilde{\theta}_2(\Gamma - k\eta + v + u) \tilde{\theta}_2(\Gamma^* - k'\eta + v^* - u)}{\tilde{\theta}_1((k+1)\eta + v + u) \tilde{\theta}_1((k'+1)\eta + v^* - u) \tilde{\theta}_2(\Gamma - (k+1)\eta + v) \tilde{\theta}_2(\Gamma^* - (k'+1)\eta + v^*)} \\ & \quad \times \frac{\tilde{\theta}_3(i\Gamma_1 - (k - k')\eta) \tilde{\theta}_4(\Gamma_0 - (k + k' + 1)\eta) \tilde{\theta}_4((k + k' + 2)\eta - \Gamma_0)}{\tilde{\theta}_4((k + k')\eta - \Gamma_0)}. \end{aligned} \quad (\text{E7})$$

The c term in Eq. (E5), appearing in the first and second lines of the last step, can be simplified as

$$\begin{aligned} & \langle \xi(-u - \bar{k}\eta - \beta) | \xi(-u - \bar{k}'\eta - \beta) \rangle + \frac{r_{k,k'}^{k,k'+1}}{r_{k,k'}} \langle \xi(-u - \bar{k}\eta - \beta) | \xi(u - \bar{k}'\eta - \beta) \rangle \\ & + \frac{r_{k,k'}^{k+1,k'}}{r_{k,k'}} \langle \xi(u - \bar{k}\eta - \beta) | \xi(-u - \bar{k}'\eta - \beta) \rangle + \frac{r_{k,k'}^{k+1,k'+1}}{r_{k,k'}} \langle \xi(u - \bar{k}\eta - \beta) | \xi(u - \bar{k}'\eta - \beta) \rangle \\ & \stackrel{(\text{D1})}{=} \tilde{\theta}_4(\Gamma_0 - (k + k' + 1)\eta) \tilde{\theta}_3(u + i\Gamma_1 - (k - k')\eta) + \frac{r_{k,k'}^{k,k'+1}}{r_{k,k'}} \tilde{\theta}_4(u + \Gamma_0 - (k + k' + 1)\eta) \tilde{\theta}_3(i\Gamma_1 - (k - k')\eta) \\ & + \frac{r_{k,k'}^{k+1,k'}}{r_{k,k'}} \tilde{\theta}_4(u - \Gamma_0 + (k + k' + 1)\eta) \tilde{\theta}_3(i\Gamma_1 - (k - k')\eta) + \frac{r_{k,k'}^{k+1,k'+1}}{r_{k,k'}} \tilde{\theta}_4(\Gamma_0 - (k + k' + 1)\eta) \tilde{\theta}_3(u - i\Gamma_1 + (k - k')\eta) \\ & \stackrel{(\text{D3})}{=} \frac{\tilde{\theta}_1(\eta + u) \tilde{\theta}_1(\eta - u) \tilde{\theta}_2(\Gamma - (2k+1)\eta) \tilde{\theta}_2(\Gamma^* - (2k'+1)\eta)}{\tilde{\theta}_1((k+1)\eta + v + u) \tilde{\theta}_1((k'+1)\eta + v^* - u) \tilde{\theta}_2(\Gamma - (k+1)\eta + v) \tilde{\theta}_2(\Gamma^* - (k'+1)\eta + v^*)} \\ & \quad \times \frac{\tilde{\theta}_3(i\Gamma_1 + v - v^* + u) \tilde{\theta}_4(\Gamma_0 - (k + k' + 1)\eta) \tilde{\theta}_4(\Gamma_0 + v + v^*)}{\tilde{\theta}_4((k + k')\eta - \Gamma_0)}. \end{aligned} \quad (\text{E8})$$

Since $\langle \xi(x) | \psi(x) \rangle = 0$, multiplying the third line of the last step in Eq. (E5) on the left by $\langle \xi(\bar{k}\eta + \beta + u) |$ and on the right by $|\xi(\bar{k}'\eta + \beta + u)\rangle$, and applying Eqs. (D1), (D2), and (E2), we obtain

$$\begin{aligned} & \langle \xi(\bar{k}\eta + \beta + u) | \mathcal{K}_R[|\psi(\bar{k}\eta + \beta)\rangle \langle \psi(\bar{k}'\eta + \beta)|] | \xi(\bar{k}'\eta + \beta + u) \rangle \\ &= \frac{\tilde{\theta}_1(u) \tilde{\theta}_2(\frac{\delta + \eta + \bar{k}\eta + \beta + u}{2})}{\kappa \tilde{\theta}_1(\eta) \tilde{\theta}_2(\frac{\delta - \eta + \bar{k}\eta + \beta + u}{2})} \left[\frac{\tilde{\theta}_1(u) \tilde{\theta}_2(\frac{\delta + \eta + \bar{k}'\eta + \beta + u}{2})}{\kappa \tilde{\theta}_1(\eta) \tilde{\theta}_2(\frac{\delta - \eta + \bar{k}'\eta + \beta + u}{2})} \right]^* \frac{\langle \psi(\bar{k}'\eta + \beta - \eta) | \psi(\bar{k}\eta + \beta - \eta) \rangle}{\langle \psi(\delta) | \psi(\delta) \rangle} \\ & \quad \times \langle \xi(\bar{k}\eta + \beta + u) | \psi(\delta - \eta) \rangle \langle \psi(\delta - \eta) | \xi(\bar{k}'\eta + \beta + u) \rangle \\ &= - \frac{\tilde{\theta}_1^2(u) \tilde{\theta}_1(v + k\eta) \tilde{\theta}_1(v^* + k'\eta) \tilde{\theta}_2(\Gamma - k\eta + v + u) \tilde{\theta}_2(\Gamma^* - k'\eta + v^* - u)}{\tilde{\theta}_1(\eta - u) \tilde{\theta}_1(\eta + u)} \\ & \quad \times \frac{\tilde{\theta}_3(i\Gamma_1 - (k - k')\eta) \tilde{\theta}_4(\Gamma_0 - (k + k' + 2)\eta)}{\tilde{\theta}_3(i\Gamma_1 + v - v^* + u) \tilde{\theta}_4(\Gamma_0 + v + v^*)}. \end{aligned} \quad (\text{E9})$$

From Eqs. (E7)–(E9), we can easily conclude that

$$\langle \xi(\bar{k}\eta + \beta + u) | \text{LHS} | \xi(\bar{k}'\eta + \beta + u) \rangle = \langle \xi(\bar{k}\eta + \beta + u) | \text{RHS} | \xi(\bar{k}'\eta + \beta + u) \rangle.$$

Furthermore, by repeating the above procedure, one can obtain

$$\langle \xi(\bar{k}\eta + \beta + \epsilon u) | \text{LHS} | \xi(\bar{k}'\eta + \beta + \epsilon' u) \rangle = \langle \xi(\bar{k}\eta + \beta + \epsilon u) | \text{RHS} | \xi(\bar{k}'\eta + \beta + \epsilon' u) \rangle, \quad \epsilon, \epsilon' = \pm 1. \quad (\text{E10})$$

Since both sides of Eq. (E3) are 2×2 matrices, the four equations presented in Eq. (E10) consequently lead to Eq. (E3).

Case (A). Now, we consider the case (A). Multiplying Eq. (E4) from the left by $\langle \xi(\bar{k}\eta + \beta + u) |$ and from the right by $|\xi(\bar{k}'\eta + \beta + u)\rangle$ and using Eqs. (D1) and (D2), the left-hand side reads

$$\begin{aligned} & \langle \xi(\bar{k}\eta + \beta + u) | \text{LHS} | \xi(\bar{k}'\eta + \beta + u) \rangle \\ &= \frac{r_{k,k'}^{k+1,k'+1}}{r_{k,k'}} \langle \xi(-\bar{k}\eta - \beta) | \xi(-\bar{k}'\eta - \beta) \rangle \langle \xi(\bar{k}\eta + \beta + u) | \psi(\bar{k}\eta - u + \beta) \rangle \langle \psi(\bar{k}'\eta - u + \beta) | \xi(\bar{k}'\eta + \beta + u) \rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{\tilde{\theta}_1^2(u)\tilde{\theta}_1(v+k\eta)\tilde{\theta}_1(v^*-k'\eta)\tilde{\theta}_2(\Gamma-(2k+1)\eta)\tilde{\theta}_2(\Gamma^*+(2k'+1)\eta)\tilde{\theta}_2(\Gamma-k\eta+v+u)\tilde{\theta}_2(\Gamma^*+k'\eta+v^*+u)}{\tilde{\theta}_1(u+(k+1)\eta+v)\tilde{\theta}_1(u-(k'+1)\eta+v^*)\tilde{\theta}_2(\Gamma-(k+1)\eta+v)\tilde{\theta}_2(\Gamma^*+(k'+1)\eta+v^*)} \\
&\times \frac{\tilde{\theta}_3(i\Gamma_1-(k+k'+1)\eta)\tilde{\theta}_3(i\Gamma_1-(k+k'+2)\eta)\tilde{\theta}_4(\Gamma_0-(k-k')\eta)}{\tilde{\theta}_3(i\Gamma_1-(k+k')\eta)}. \tag{E11}
\end{aligned}$$

The c term in Eq. (E5), appearing in the first and second lines of the last step, can be simplified as

$$\begin{aligned}
&\langle \xi(-u-\bar{k}\eta-\beta)|\xi(-u-\bar{k}'\eta-\beta)\rangle + \frac{r_{k,k'+1}}{r_{k,k'}}\langle \xi(-u-\bar{k}\eta-\beta)|\xi(u-\bar{k}'\eta-\beta)\rangle \\
&+ \frac{r_{k+1,k'}}{r_{k,k'}}\langle \xi(u-\bar{k}\eta-\beta)|\xi(-u-\bar{k}'\eta-\beta)\rangle + \frac{r_{k+1,k'+1}}{r_{k,k'}}\langle \xi(u-\bar{k}\eta-\beta)|\xi(u-\bar{k}'\eta-\beta)\rangle \\
&\stackrel{(D1)}{=} \tilde{\theta}_3(i\Gamma_1-(k+k'+1)\eta)\tilde{\theta}_4(u+\Gamma_0-(k-k')\eta) + \frac{r_{k,k'+1}}{r_{k,k'}}\tilde{\theta}_3(u+i\Gamma_1-(k+k'+1)\eta)\tilde{\theta}_4(\Gamma_0-(k-k')\eta) \\
&+ \frac{r_{k+1,k'}}{r_{k,k'}}\tilde{\theta}_3(u-i\Gamma_1+(k+k'+1)\eta)\tilde{\theta}_4(\Gamma_0-(k-k')\eta) + \frac{r_{k+1,k'+1}}{r_{k,k'}}\tilde{\theta}_3(i\Gamma_1-(k+k'+1)\eta)\tilde{\theta}_4(u-\Gamma_0+(k-k')\eta) \\
&\stackrel{(D3)}{=} \frac{\tilde{\theta}_1(u-\eta)\tilde{\theta}_1(u+\eta)\tilde{\theta}_2(\Gamma-(2k+1)\eta)\tilde{\theta}_2(\Gamma^*+(2k'+1)\eta)}{\tilde{\theta}_1(u+(k+1)\eta+v)\tilde{\theta}_1(u-(k'+1)\eta+v^*)\tilde{\theta}_2(\Gamma-(k+1)\eta+v)\tilde{\theta}_2(\Gamma^*+(k'+1)\eta+v^*)} \\
&\times \frac{\tilde{\theta}_3(i\Gamma_1+v-v^*)\tilde{\theta}_3(i\Gamma_1-(k+k'+1)\eta)\tilde{\theta}_4(\Gamma_0+v+v^*+u)}{\tilde{\theta}_3(i\Gamma_1-(k+k')\eta)}. \tag{E12}
\end{aligned}$$

Multiplying the third line in the last step of Eq. (E5) on the left by $\langle \xi(\bar{k}\eta+\beta+u)|$ and on the right by $|\xi(\bar{k}'\eta+\beta+u)\rangle$, and applying Eqs. (D1), (D2), and (E2), we arrive at

$$\begin{aligned}
&\langle \xi(\bar{k}\eta+\beta+u)|\mathcal{K}_R[|\psi(\bar{k}\eta+\beta)\rangle\langle\psi(\bar{k}'\eta+\beta)|]|\xi(\bar{k}'\eta+\beta+u)\rangle \\
&= \frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\frac{\delta+\eta+\bar{k}\eta+\beta+u}{2})}{\kappa\tilde{\theta}_1(\eta)\tilde{\theta}_2(\frac{\delta-\eta+\bar{k}\eta+\beta+u}{2})} \left[\frac{\tilde{\theta}_1(u)\tilde{\theta}_2(\frac{\delta+\eta+\bar{k}'\eta+\beta+u}{2})}{\kappa\tilde{\theta}_1(\eta)\tilde{\theta}_2(\frac{\delta-\eta+\bar{k}'\eta+\beta+u}{2})} \right]^* \frac{\langle\psi(\bar{k}'\eta+\beta-\eta)|\psi(\bar{k}\eta+\beta-\eta)\rangle}{\langle\psi(\delta)|\psi(\delta)\rangle} \\
&\times \langle \xi(\bar{k}\eta+\beta+u)|\psi(\delta-\eta)\rangle\langle\psi(\delta-\eta)|\xi(\bar{k}'\eta+\beta+u)\rangle \\
&= \frac{\tilde{\theta}_1^2(u)\tilde{\theta}_1(v+k\eta)\tilde{\theta}_1(v^*-k'\eta)\tilde{\theta}_2(\Gamma-k\eta+v+u)\tilde{\theta}_2(\Gamma^*+k'\eta+v^*+u)}{\tilde{\theta}_1(u-\eta)\tilde{\theta}_1(u+\eta)} \\
&\times \frac{\tilde{\theta}_3(i\Gamma_1-(k+k'+2)\eta)\tilde{\theta}_4(\Gamma_0-(k-k')\eta)}{\tilde{\theta}_3(i\Gamma_1+v-v^*)\tilde{\theta}_4(\Gamma_0+v+v^*+u)}. \tag{E13}
\end{aligned}$$

Comparing Eq. (E11) with Eqs. (E12) and (E13), we find

$$\langle \xi(\bar{k}\eta+\beta+u)|\text{LHS}|\xi(\bar{k}'\eta+\beta+u)\rangle = \langle \xi(\bar{k}\eta+\beta+u)|\text{RHS}|\xi(\bar{k}'\eta+\beta+u)\rangle.$$

Analogously, by repeating the above procedure, one can verify

$$\langle \xi(\bar{k}\eta+\beta+\epsilon u)|\text{LHS}|\xi(\bar{k}'\eta+\beta+\epsilon' u)\rangle = \langle \xi(\bar{k}\eta+\beta+\epsilon u)|\text{RHS}|\xi(\bar{k}'\eta+\beta+\epsilon' u)\rangle, \quad \epsilon, \epsilon' = \pm 1.$$

In this way, the correctness of Eq. (E3) in case (A) is also proved.

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