

Interplay of resources for universal continuous-variable quantum computing

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Quantum resource theories identify the features of quantum computers that provide their computational advantage over classical systems. We investigate the resources driving the complexity of classical simulation in the standard model of continuous-variable quantum computing, and their interplay enabling computational universality. Specifically, we uncover a new property in continuous-variable circuits, analogous to coherence in discrete-variable systems, termed *symplectic coherence*, defined as the ability of Gaussian gates to mix position and momentum operators. Using quadrature propagation across multiple computational paths, we develop an efficient classical simulation algorithm for continuous-variable computations with symplectic coherence constant in the size of the circuit. This establishes symplectic coherence as a necessary resource for universality in continuous-variable quantum computing, alongside non-Gaussianity and entanglement. Via the Gottesman-Kitaev-Preskill encoding, we show that the interplay of these three continuous-variable quantum resources mirrors the discrete-variable relationship between coherence, magic, and entanglement.

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I. INTRODUCTION

Universal quantum computers may offer significant advantages over their classical counterparts [1–5]. Understanding the origin of these advantages is crucial for advancing quantum technologies. This requires identifying scenarios where classical simulation of quantum computations—replicating quantum behavior with classical algorithms—becomes inefficient [6–10]. Quantum resource theory provides a powerful framework for addressing this challenge [11–16], revealing in the context of quantum computing how the absence or limited presence of specific quantum features, known as quantum resources, enables efficient classical simulation, and thus shedding light on the requirements for quantum computational advantages.

For discrete-variable quantum computations (DVQCs), such as those based on qubits, quantum computational advantage can be understood as relying on the interplay of three resources: magic, entanglement, and coherence [11,17–20]. The absence of any one of these resources in a quantum computation enables efficient classical simulation, e.g., using methods based on the Gottesman-Knill theorem [21,22] or matrix product states [23].

Continuous-variable quantum computations (CVQCs) [24–26] operate instead on infinite-dimensional Hilbert spaces, where operators such as position and momentum

have a continuous spectrum. These computations are implemented in photonic and superconducting systems, platforms that offer enhanced noise robustness [27] and enable unprecedented levels of entanglement [28]. Notably, these platforms have achieved the experimental milestone of quantum error correction beyond the breakeven point using Gottesman-Kitaev-Preskill (GKP) encoding [29,30].

The GKP encoding maps DV quantum states and operations from a finite-dimensional Hilbert space to a (closure of a) subspace of the infinite-dimensional Hilbert space through the use of logical CV quantum states and operations. As such, it bridges the two computational paradigms and enables established DV simulation techniques to be applied to GKP-encoded CV quantum computations [31,32].

For general CV quantum computations beyond GKP-encoded ones, the subclass of Gaussian computations is known to be efficiently simulatable on classical computers [33], underpinning non-Gaussianity as a key quantum resource alongside entanglement [34]. Various measures have been developed to quantify non-Gaussianity [13,33,35–39], and its role in the complexity of classical simulation has been extensively studied [39–47]. However, not a single quantum resource by itself is sufficient for achieving universal CV quantum computations: In particular, some classes of CV quantum computations with high non-Gaussianity can still be efficiently simulated classically [31,32]. How the interplay of multiple quantum resources prevents efficient classical simulation and enables computational universality in the CV setting remains underexplored (see Fig. 1).

In this work, we bridge this gap by identifying three physical resources whose interplay determines the complexity of CV quantum computations: non-Gaussianity, entanglement, and a new resource that we call *symplectic coherence*. Informally, this latter resource is responsible for mixing position and momentum quadratures in CV quantum computations.

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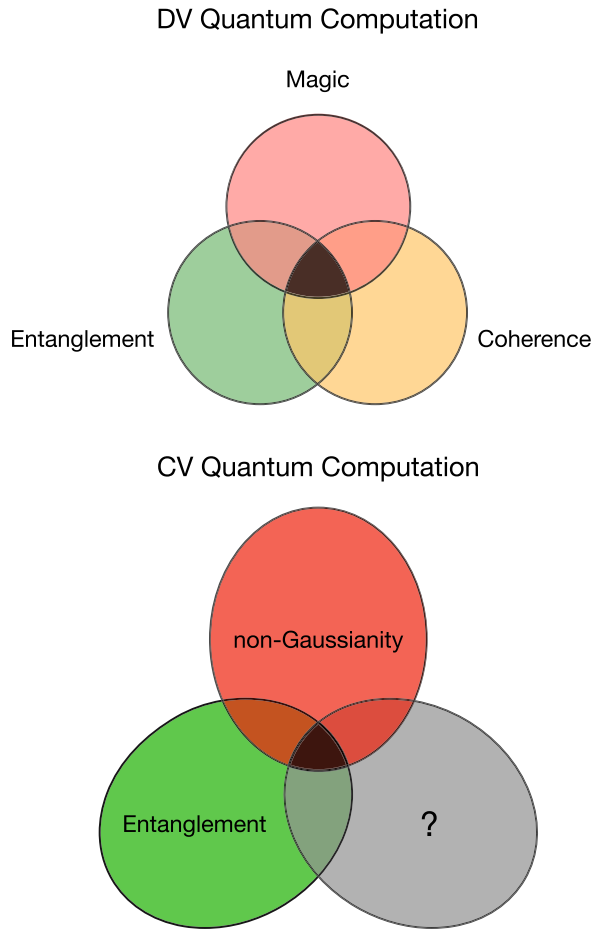


FIG. 1. Interplay of resources that determine the complexity of classical simulation of quantum computations, with the black Reuleaux triangle in the middle indicating the regime of universal quantum computation. While universality in DV quantum computations can be seen as resulting from the interplay of magic, entanglement, and coherence, whether such an interplay of resources can be seen in a universal CV quantum computation has remained an open question.

We show that symplectic coherence is indeed a quantum computational resource, by introducing a classical algorithm for simulating CV quantum computations, which is efficient for circuits with symplectic coherence constant in the size of the circuit. This algorithm is based on evolving CV quantum operators over simultaneous computational paths (Fig. 4), analogous to recent simulation techniques based on evolution over Pauli paths introduced in the DV setting [48–52], but largely unexplored in the CV regime. As a consequence, the absence of any one of the three resources—non-Gaussianity, entanglement, or symplectic coherence—leads to efficient classical simulation, while their combined presence enables universal quantum computation.

Through the GKP encoding, we show that this CV resource interplay can be interpreted as a generalization of the interplay of quantum computational resources observed in DV systems: CV entanglement naturally generalizes DV entanglement, CV non-Gaussianity plays the role of DV magic, as the recent literature suggests [53–55], and CV symplectic coherence

corresponds to DV coherence. This result relies on identifying the resources brought by each quantum gate in universal gate sets in DV computations, GKP-encoded CV computations, and general CV computations.

II. PRELIMINARIES

We refer the reader to Appendix A for preliminaries on Gaussian and non-Gaussian quantum information as well as on the standard model of CVQC introduced by Lloyd and Braunstein [56]. There, we show that the universal gate set for CVQC is given by

$$\{e^{i\gamma\hat{q}_1^3}, \hat{O}, \hat{R}, \hat{D}\}, \quad (1)$$

where $e^{i\gamma\hat{q}_1^3}$ is the cubic phase gate, $\hat{O}$ are the orthogonal gates, $\hat{R}$ are the phase shifters, and $\hat{D}$ is the displacement gate. Whereas $\hat{O}$, $\hat{R}$, and $\hat{D}$ are Gaussian unitary gates, $e^{i\gamma\hat{q}_1^3}$ is a non-Gaussian unitary gate. More information on these gates is also available in Appendix A.

We now define the central concept of this work: *symplectic coherence*.

III. SYMPLECTIC COHERENCE OF GAUSSIAN GATES

The evolution of the quantum state of a physical system is governed by a unitary operator that characterizes the system’s dynamics. One way to describe this unitary evolution is by specifying its action on the position and momentum quadratures. Gaussian unitary operators transform each quadrature as a symplectic and affine combination of the others, redistributing the information encoded in the state. In contrast, non-Gaussian unitary operators, such as the cubic phase gate in Eq. (A2), alter the nature of the information by introducing nonlinearities.

Focusing on Gaussian unitary operators, recall that their action on the quadrature operators is described by a symplectic matrix S and a displacement vector d . Hereafter, a Gaussian unitary operator is said to exhibit *symplectic coherence* if its associated symplectic matrix, expressed in the basis $\Gamma = [\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m]$, is not in a block-diagonal form consisting of two $m \times m$ matrices, i.e., if it mixes the position and momentum quadratures. The term “symplectic coherence” draws an informal analogy to quantum coherence; however, they represent vastly different physical concepts, as clarified in Appendix B. So while a beam splitter with no nonzero off-block-diagonal terms lacks symplectic coherence, a Fourier gate with nonzero off-block-diagonal terms exhibits it. We define Gaussian unitaries with no nonzero off-block-diagonal terms in their associated symplectic matrix as “block-diagonal symplectic” Gaussian unitaries. This terminology aligns with the block-diagonal Gaussian states introduced in Refs. [57,58], which are Gaussian states without direct correlations between position and momentum quadratures. The block-diagonal Gaussian states in Refs. [57,58] motivate the exploration of symplectic coherence as a resource in quantum states, defined as the presence of off-diagonal blocks in their covariance matrix. Hereafter, however, we focus on symplectic coherence as a property of Gaussian unitary gates. Moreover, inspired by the definition of T count in DV computations [59], we define the symplectic

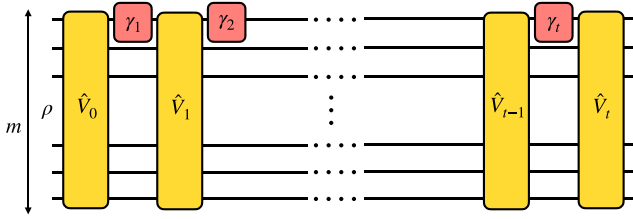


FIG. 2. Universal circuit for CV quantum computations. $\gamma_1, \dots, \gamma_t$ are cubic phase gates, whereas $\hat{V}_0, \dots, \hat{V}_t$ are m -mode passive linear unitary gates, up to a displacement.

coherence of a CV quantum circuit (i.e., a list of CV quantum gates) as follows:

Definition 1 (*Symplectic coherence of a CV quantum circuit*). The symplectic coherence of a CV quantum circuit is defined as the number of Gaussian unitary gates with nonzero off-block-diagonal matrices in their symplectic matrix.

To understand how this property arises in the context of universal CVQC, we recall the universal gate set for CVQC considered in this work: $\{e^{i\gamma\hat{q}_1^3}, \hat{O}, \hat{R}, \hat{D}\}$. Note that while this gate set may seem restrictive, any unitary gate generated by polynomial Hamiltonian can be generated using Gaussian unitary plus cubic phase gates [60]. Since any physical gate can be effectively described by a unitary gate generated by a polynomial Hamiltonian [61], Eq. (1) represents a universal gate set for any physical CV computation.

In this gate set, the only Gaussian gates that may introduce symplectic coherence are phase-shift gates. This raises the question of how the presence of phase-shift gates inducing symplectic coherence may affect our ability to classically simulate CVQC. The next section addresses this question and demonstrates that the absence of symplectic coherence in a CVQC enables its efficient classical simulation, establishing symplectic coherence as a new computational resource.

IV. SYMPLECTIC COHERENCE AS A QUANTUM COMPUTATIONAL RESOURCE

In this section, we consider the problem of computing the expectation value of a Hamiltonian that is polynomial in position and momentum quadratures (a prominent example being the total particle number operator), at the output of a universal CV quantum circuit: Given $m, t, d \geq 0$ and the quadrature moments of an m -mode input state ρ (by which we mean analytical expressions for moments of arbitrary degree of its position and momentum quadratures; see Appendix F for details) evolving over an m -mode unitary circuit

$$\hat{O}_\gamma = \hat{V}_t e^{i\gamma_t \hat{q}_1^3} \hat{V}_{t-1} \dots \hat{V}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{V}_0, \quad (2)$$

where $\gamma = (\gamma_1, \dots, \gamma_t)$ and where $\hat{V}_0, \dots, \hat{V}_t$ are passive linear unitary gates up to a multimode displacement (see Fig. 2), what is the cost of classically computing the expectation value of an operator $H(\hat{q}, \hat{p}) := H(\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m)$, polynomial of degree d in position and momentum quadratures for the output quantum state $\hat{O}_\gamma \rho \hat{O}_\gamma^\dagger$ of the circuit? Note that $\hat{O}_\gamma$ describes a universal circuit for CVQC since displaced passive linear unitary gates (involving orthogonal, phase-shift, and displacement gates) together with cubic phase gates form a universal gate set for CVQC according to Eq. (1).

This problem is generally hard to solve classically: For instance, evolving the position quadrature $\hat{q}_1$ through t cubic phase gates with cubicity γ interleaved with Fourier gates produces terms with coefficients of the form $\hat{q}_1^{2^t}$, where repeated squaring after each layer leads to doubly exponential growth [62]. Similarly, quadrature evolution in circuits with passive linear unitary gates and cubic phase gates typically results in doubly exponentially large numbers, making classical simulation inefficient.

However, without symplectic coherence, i.e., if $\hat{V}_0, \dots, \hat{V}_t$ are displaced orthogonal gates denoted by $\hat{O}_0, \dots, \hat{O}_t$, we show that the degree and number of terms remains tractable, and that the circuit can be classically simulated efficiently. This is summarized by the following result:

Theorem 1. Given the quadrature moments of an m -mode input state ρ , and an m -mode unitary $\hat{O}_\gamma$ given by

$$\hat{O}_\gamma = \hat{O}_t e^{i\gamma_t \hat{q}_1^3} \hat{O}_{t-1} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0, \quad (3)$$

where $\hat{O}_0, \dots, \hat{O}_t$ are displaced orthogonal gates, the expectation value of an operator $H(\hat{q}, \hat{p})$, polynomial of degree d in the quadrature operators, given by

$$\text{Tr}[\hat{O}_\gamma \rho \hat{O}_\gamma^\dagger H(\hat{q}, \hat{p})] = \text{Tr}[\rho \hat{O}_\gamma^\dagger H(\hat{q}, \hat{p}) \hat{O}_\gamma], \quad (4)$$

can be computed in time $\mathcal{O}(m^{3d} + t^2 m^6)$.

The proof of Theorem 1 is given in Appendix D and relies on a new simulation technique, which we call the quadrature backpropagation technique. The intuition behind this technique has been explained in Appendix C, and is reminiscent of a classical simulation technique for DV quantum computations based on Pauli backpropagation [48–52], which to the best of our knowledge has not been applied in the CV setting until our work.

Theorem 1 establishes symplectic coherence as a computational resource, as its absence enables efficient classical simulation of quantum expectation values of constant-degree polynomial Hamiltonians. Physically, this result can be understood as follows: A cubic phase gate introduces non-Gaussianity by adding a quadratic term to the momentum quadrature ($\hat{p} \rightarrow \hat{p} + 6\gamma\hat{q}^2$). If the displaced passive linear unitary gates do not mix $\hat{q}$ and $\hat{p}$, the nonlinearity remains confined to a quadratic term in the position quadrature, remaining unaffected by subsequent cubic phase gates. In other words, the displaced passive linear unitary gates do not amplify the non-Gaussianity. However, when passive linear unitary gates do mix $\hat{q}$ and $\hat{p}$, inducing symplectic coherence, the nonlinearity grows with successive layers, making the evolution intractable.

Note that the quantum circuits for which Theorem 1 provides an efficient classical simulation algorithm can have arbitrary high entanglement and non-Gaussianity, induced by block-diagonal symplectic gates and cubic phase gates, respectively. As a result, these include circuits that are hard to simulate classically using existing methods that only take advantage of limited non-Gaussianity [39–47].

We have shown how CV quantum computations without symplectic coherence may be simulated classically. We now further explore whether circuits with low but nonzero symplectic coherence can also be efficiently simulated, similar to

CV circuits with low non-Gaussianity [40,42,44,63] or low entanglement [64]. We address this in the following theorem:

Theorem 2. Given the quadrature moments of an m -mode input state ρ , and an m -mode unitary $\hat{U}_t$ given by

$$\hat{U}_t = \hat{O}_{\gamma_c} \hat{R}_1(\theta_c) \hat{O}_{\gamma_{c-1}} \cdots \hat{R}_1(\theta_1) \hat{O}_{\gamma_0}, \quad (5)$$

evolving the input state, where $\hat{O}_{\gamma_i}$ describes $t+1$ displaced orthogonal gates interleaved with t cubic phase gates as in Theorem 1, and $\hat{R}_1(\theta_i)$ are single-mode phase-shift gates in the first mode $\forall i \in \{0, 1, \dots, c\}$, the expectation value of an operator $H(\hat{q}, \hat{p})$, polynomial of degree d in the quadrature operators, given by

$$\text{Tr}[\hat{U}_t \rho \hat{U}_t^\dagger H(\hat{q}, \hat{p})] = \text{Tr}[\rho \hat{U}_t^\dagger H(\hat{q}, \hat{p}) \hat{U}_t], \quad (6)$$

can be computed in time $\mathcal{O}(m^{d^2 c+1} + (c+1)t^2 m^7)$.

Note that phase-shift gates in Eq. (5) act on the first mode without loss of generality, as the phase-shift gates on other modes can be shifted to the first one by an orthogonal (SWAP) gate, which can then be absorbed into the corresponding block of displaced orthogonal unitary gates. The proof of Theorem 2 is detailed in Appendix E and also relies on the quadrature backpropagation technique, explained in Appendix C.

In particular, when $d = \mathcal{O}(1)$, $t = \mathcal{O}(\text{poly } m)$, and $c = \mathcal{O}(1)$, the circuit remains efficiently simulable. This demonstrates that CV quantum circuits with low symplectic coherence, i.e., with a constant number of phase-shift gates, can also be simulated efficiently classically, thereby strengthening the role of symplectic coherence as a computational resource in CVQC. Note that we have focused on quantifying symplectic coherence based on the number of phase-shift gates generating symplectic coherence, but we expect that similar results may be obtained based on other quantifiers, for instance, using the angles of these phase-shift gates.

V. CV VERSUS DV RESOURCE INTERPLAY

Based on the gate set $\{e^{i\gamma\hat{q}_i^3}, \hat{O}, \hat{R}, \hat{D}\}$ in Eq. (1), we have found that the computational complexity of universal CV quantum circuits arises from the interplay of three computational resources: non-Gaussianity (introduced by non-Gaussian gates such as the cubic phase gate), entanglement (generated by block-diagonal symplectic gates such as orthogonal gates), and symplectic coherence (induced by phase-shift gates, such as the Fourier gate). Note that, while we considered orthogonal gates as the one inducing entanglement in the universal gate set, any block-diagonal symplectic Gaussian unitary gate can be taken as entanglement-inducing gate without changing the complexity of the simulation algorithms obtained in the previous section. While displacement gates are also necessary for universal CV quantum computations, they are not expected to impact the complexity of simulating CVQC as their action is analogous to applying a “classical” force that shifts the position and momentum quadratures of a quantum harmonic oscillator, without introducing fundamentally quantum effects.

It is natural to ask how this interplay of CV resources relates to that observed in DV quantum computations. In the latter case, computational universality stems from the interaction of magic (induced by gates such as T), entanglement (induced by gates such as CNOT), and coherence (induced by

gates such as Hadamard). The GKP encoding, which maps DV states and operations to a CV subspace, provides a useful way of comparing the DV and CV resource interplays. Within the GKP code space, the emergence of computational complexity can be understood directly using the DV resource-theoretic framework, but extending this interplay beyond GKP-encoded CVQC is less straightforward. Indeed, while the connections between CV non-Gaussianity and DV magic [53–55], and between mode entanglement and logical qubit entanglement [65], are well understood within the GKP encoding [29], a CV computational resource analogous to DV coherence is missing. As we shall explain in what follows, symplectic coherence fills this gap.

To better understand the relationship between the DV and CV resource interplays, we consider each DV resource, specify the corresponding DV gate inducing it within the gate set $\{T, \text{CNOT}, H\}$, obtain its GKP-encoded counterpart, and identify the CV family of gates this GKP-encoded gate belongs to. Finally, we give the CV resource tied to that family of gates. The goal is to determine which CV computational resource is uniquely provided by each gate in the universal GKP-encoded gate set. For magic, the associated DV gate is the T gate, which maps to the cubic phase gate in the GKP-encoded model. The cubic phase gate, in turn, introduces non-Gaussianity in CV computations. Next, entanglement in DV systems, introduced by the CNOT gate, corresponds to the two-mode sum gate in the GKP encoding. The SUM gates mix position quadratures with position quadratures of two modes (and similarly, momentum quadratures with momentum quadratures of two modes) [29] and hence belong to the class of block-diagonal symplectic Gaussian unitary gates and induce entanglement in CV computations. Finally, coherence in DV systems, induced by the Hadamard gate, corresponds to the Fourier transform (F) gate in the GKP encoding. Since the Fourier gate belongs to the class of phase-shift gates, it induces symplectic coherence in CV computations.

Note that Theorems 1 and 2 may be applied to the case of input physical GKP states, as efficient analytical formulas for their quadrature moments can be derived, as we detail in Appendix F.

In conclusion, the interplay of magic, entanglement, and coherence in GKP-encoded DV quantum computations maps to that of non-Gaussianity, entanglement, and symplectic coherence in CV computations, identifying symplectic coherence as a CV analog of coherence in the context of universal quantum computations.

VI. CONCLUSION

We introduced symplectic coherence, the property of a unitary that mixes position and momentum quadratures. After identifying the role of phase-shift gates in inducing it, we showed its role as a resource for universal CVQC. In particular, we derived an efficient algorithm for simulating CV circuits with symplectic coherence constant in the size of the circuit, i.e., a constant number of phase-shift gates, using a CV path simulation method based on quadrature backpropagation. Finally, through the GKP encoding, we linked CVQC and DVQC, showing how symplectic coherence completes the resource framework for universal CVQC: CV entanglement,

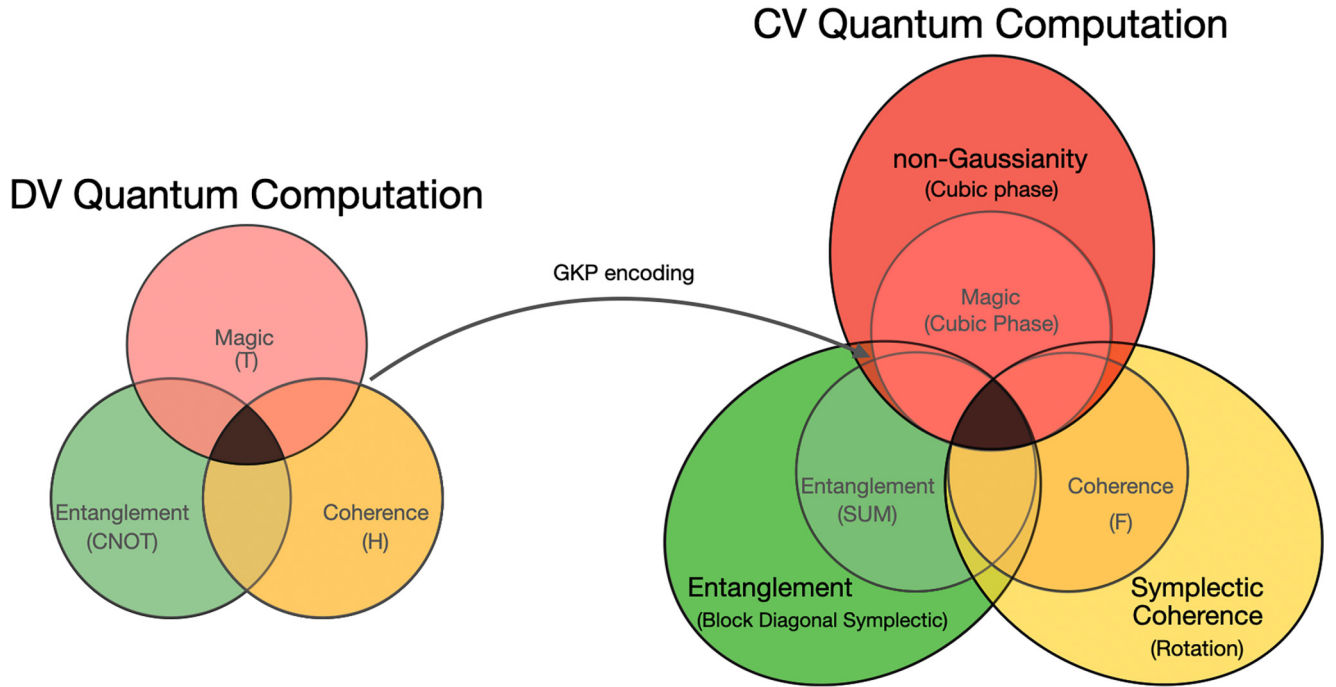


FIG. 3. The interplay of CV non-Gaussianity, entanglement, and symplectic coherence generalizes the resource interplay of DV magic, entanglement, and coherence in GKP-encoded quantum computations. Between parentheses are quantum gates inducing each computational resources. The circles on the right-hand side denote the resource interplay in the GKP-encoded space within general CV quantum computations. While T, CNOT, H, cubic phase, SUM, and F gates are standard quantum gates in the literature, “block-diagonal symplectic” refers to Gaussian unitary gates whose symplectic matrices have a block-diagonal form, i.e., they map position quadratures to position quadratures and momentum quadratures to momentum quadratures, and “rotation” refers to gates that may mix position and momentum quadratures within a single mode.

non-Gaussianity, and symplectic coherence directly parallel DV entanglement, magic, and coherence (Fig. 3).

Our work opens several research directions. One is to investigate the measurement of symplectic coherence in experiments implementing CV quantum circuits. Here, a naive approach would be to perform process tomography of each Gaussian gate in the circuit to determine whether it mixes position and momentum quadratures. A more efficient method could be to directly learn the symplectic matrix associated with each Gaussian gate [66]. Another interesting research direction is to determine whether the absence of symplectic coherence enables efficient classical sampling and strong simulation, beyond expectation value simulation. In the DV regime, circuits with at most logarithmically many coherence-inducing gates, such as Hadamard gates (or Fourier gates in GKP-encoded computations), are known to be efficiently simulatable [16,67–69]; we proved the CV analog for a constant number of such gates, while the logarithmic case remains open—likely requiring improved algorithms. The path simulation formalism introduced here, successful in analyzing noisy DV circuits [48,50–52], could also be extended to noisy CVQC. Finally, another direction is to formalize symplectic coherence for states and operations within a resource-theoretic framework and study its operational role beyond quantum computing.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PRELIMINARIES

We refer the reader to Ref. [70] for quantum information theory material, and to Refs. [24–26] for background on CV quantum information, which is the branch of quantum information dealing with observables taking a continuous spectrum of values.

A mode refers to the degree of freedom associated with a specific quantum field in a CV system, such as single spatial or frequency mode of light, and is the equivalent of qubit in the CV regime. In this paper, $m \in \mathbb{N}^*$ denotes the number of modes in the system, and $|0\rangle$ is the vacuum state. $\hat{a}$ and $\hat{a}^\dagger$ refer to the single-mode annihilation and creation operators, respectively, satisfying $[\hat{a}, \hat{a}^\dagger] = \mathbb{I}$. $\hat{a}$ and $\hat{a}^\dagger$ are related to the position and momentum quadrature operators as $\hat{q} = \hat{a} + \hat{a}^\dagger$, $\hat{p} = -i(\hat{a} - \hat{a}^\dagger)$, with the convention $\hbar = 2$. Furthermore,

$\hat{q}$ and $\hat{p}$ satisfy the commutation relation $[\hat{q}, \hat{p}] = 2i\mathbb{I}$. The particle number operator is given as $\hat{n} = \hat{a}^\dagger \hat{a}$.

Product of unitary operations generated by Hamiltonians that are quadratic in the quadrature operators of the modes are called Gaussian unitary operations, and states produced by applying a Gaussian unitary operation to the vacuum state are Gaussian states. The action of an m -mode Gaussian unitary operation $\hat{G}$ on the vector of quadratures $\mathbf{\Gamma} = [\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m]$ is given by

$$\hat{G}^\dagger \mathbf{\Gamma} \hat{G} = S\mathbf{\Gamma} + \mathbf{d}, \quad (\text{A1})$$

where S is a $2m \times 2m$ symplectic matrix transforming quadratures under $\hat{G}$, and $\mathbf{d} \in \mathbb{R}^{2m}$ is a displacement vector. Passive linear operations, induced by beam splitters and phase shifters in quantum optics, are Gaussian transformations that preserve the total particle number. Among those, orthogonal gates $\hat{O}$ are passive linear unitary operations whose associated symplectic matrix written in the quadrature basis $\mathbf{\Gamma} = [\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m]$ can be put in a block-diagonal form with two $m \times m$ orthogonal matrices. In particular, they map position quadratures to position quadratures and momentum quadratures to momentum quadratures. On the other hand, phase-shift gates $\hat{R}(\theta) = e^{i\theta\hat{n}}$, with $\theta \in [0, 2\pi)$, are passive linear unitary operators that can mix position and momentum quadratures within a mode. Single-mode displacement operators are $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$, with $\alpha \in \mathbb{C}$, whereas the shearing gate is given by $e^{is\hat{q}^2}$, with $s \in \mathbb{R}$. The Bloch-Messiah decomposition [25] states that any multimode Gaussian unitary gate can be decomposed into a product of orthogonal gates, phase-shift gates, displacement gates, and shearing gates.

Since Gaussian gates acting on Gaussian states can be efficiently simulated classically [33], non-Gaussian gates are necessary for enabling quantum advantages. One prominent example of a non-Gaussian gate is the cubic phase gate $e^{i\gamma\hat{q}^3}$; its action on the quadratures is given by [71]

$$e^{-i\gamma\hat{q}^3} \hat{q} e^{i\gamma\hat{q}^3} = \hat{q}, \quad e^{-i\gamma\hat{q}^3} \hat{p} e^{i\gamma\hat{q}^3} = \hat{p} + 6\gamma\hat{q}^2. \quad (\text{A2})$$

A universal gate set comprises gates capable of performing any computation within a given quantum computational model. In the seminal model of CVQC introduced by Lloyd and Braunstein [56], the universal gate set consists of Gaussian unitary gates and a single non-Gaussian gate acting on an input vacuum state. Within this framework, universality is defined as the ability to implement any unitary that can be generated by the specific family of polynomial Hamiltonians, i.e., polynomial functions of the position and momentum quadrature operators. While this definition may seem restrictive at first sight, recent work demonstrates that any physical unitary operation can be approximated to arbitrary precision by a unitary evolution generated by such a polynomial Hamiltonian [61]. Since a combination of Gaussian unitary gates and a cubic phase gate can approximate any unitary characterized by a polynomial Hamiltonian with arbitrary precision [56, 72], they indeed constitute a universal gate set. By the Bloch-Messiah decomposition of Gaussian unitary gates, a universal gate set for CVQC can therefore be given as [65] $\{e^{i\gamma\hat{q}_1^3}, \hat{O}, \hat{R}, \hat{D}, e^{is\hat{q}_1^2}\}$, where $\hat{O}$, $\hat{R}$, and $\hat{D}$ refer to an orthogonal gate, a phase-shift gate, and a displacement gate, respectively. Note that cubic phase gates and shearing gates are applied on the first mode

without loss of generality, up to an orthogonal (SWAP) gate. Further, following Ref. [72], each of the shearing gates can be decomposed as displacement gates and cubic phase gates. This reduces the universal gate set for CVQC to

$$\{e^{i\gamma\hat{q}_1^3}, \hat{O}, \hat{R}, \hat{D}\}. \quad (\text{A3})$$

Note that GKP-encoded computations naturally fit within this CV quantum computing model, as the gate set for this encoding is generated by polynomial Hamiltonians, and the required input GKP states can be approximated up to arbitrary precision using unitary gates generated by polynomial Hamiltonians [61]. There is much to be said about the GKP encoding, but in this work we mainly use it as a tool to bridge the DV and CV computational paradigms. The interested reader can refer to Refs. [29, 73] for more details on the GKP encoding. In particular, under this encoding, the T, CNOT, and H gates are mapped to the cubic phase gate ($e^{i\gamma\hat{q}^3}$), the SUM gate ($e^{i\frac{\hat{q}_1\hat{p}_2}{2}}$) [65], and the Fourier transform F gate ($e^{i\frac{\pi}{2}\hat{n}}$), respectively.

APPENDIX B: COHERENCE AND SYMPLECTIC COHERENCE

The term *symplectic coherence* denotes the ability of Gaussian gates to mix position and momentum quadratures, or equivalently the presence of nonzero off-block-diagonal elements in the symplectic matrix of a Gaussian unitary. The terminology is inspired by an analogy with quantum coherence, which is defined by the presence of nonzero off-diagonal elements in the density matrix of a quantum state in a fixed basis. We clarify below the conceptual distinction between the two notions and the role of symplectic coherence in CV computation.

Quantum coherence refers to the ability of a quantum state to exist in a superposition of distinguishable states and underlies macroscopic quantum effects in lasers [74] and superfluids [75]. From an information-theoretic perspective, coherence has been extensively studied as a resource [19]. A general resource-theoretic framework was established for finite-dimensional systems in Ref. [18], with operational interpretations for state conversion later identified [76]. Extensions to infinite-dimensional and CV settings were subsequently developed under physical constraints such as finite energy or Gaussianity [77–80] and its properties were studied in infinite-dimensional systems [81].

Only recently has coherence been identified as a computational resource: In discrete-variable systems, *path coherence* was shown to enable quantum advantage through its interplay with magic and entanglement [16]. An analogous resource in CV computation has been lacking. In this work, we show that this role is played by symplectic coherence (Fig. 3). Importantly, despite the similarity in terminology, the two concepts are fundamentally different. Quantum coherence is a *state-level* resource describing superposition, whereas symplectic coherence is a *gate-level* resource describing the mixing of noncommuting quadratures by Gaussian unitaries. Thus, although they play analogous roles in DV and CV computation when considering specific, universal gate sets, coherence and symplectic coherence correspond to distinct physical notions.

We note that the quantification of symplectic coherence at the level of quantum states and its operational significance are explored further in Ref. [82], beyond the scope of this work.

APPENDIX C: QUADRATURE BACKPROPAGATION SIMULATION TECHNIQUE INTUITION

Here, we illustrate the quadrature backpropagation technique by giving an intuition on how it helps in proving Theorems 1 and 2. We leave the formal proofs for later in the Appendixes.

Proof sketch of Theorem 1. Theorem 1 relies on backpropagating single quadrature operators through the circuit $\hat{O}_\gamma$, which are then combined to get an efficient description of $\hat{O}_\gamma^\dagger H(\hat{q}, \hat{p}) \hat{O}_\gamma$, i.e., the backpropagated version of $H(\hat{q}, \hat{p})$. This backpropagation of single quadratures is made efficient based on the quadrature backpropagation simulation technique, where instead of evolving a single quadrature over the full unitary circuit in Eq. (3), we simultaneously evolve it over t simpler circuits (computational paths), each involving the displaced orthogonal unitary gates $\hat{O}_0, \dots, \hat{O}_t$, together with a single cubic phase gate, and we sum the results. In this sum over paths, the cubicity of the single cubic phase gate is the same across all paths, and the contribution of each path reflects the fraction of the total cubicity brought by the original cubic phase gates in the circuit (see Fig. 4). This is formalized in the proof of Theorem 1 in Appendix D.

Proof sketch of Theorem 2. As for Theorem 1, Theorem 2 proceeds by backpropagating $2m$ single quadrature operators through the unitary circuits, and multiplying the quadratures to obtain the backpropagated version of $H(\hat{q}, \hat{p})$ under the unitary. However, during the backpropagation, the degree of a single-mode quadrature doubles at every layer due to the phase-shift gate mixing position and momentum quadratures, leading to a degree of 2^{c+1} after $c+1$ layers. At each layer, we sum over all computational paths to express the evolution as a single polynomial, using Theorem 1. This results in a total of $(c+1) \times t$ computational paths, allowing each final backpropagated quadrature operator to be obtained in time $\mathcal{O}(m^{3 \times 2^c} + (c+1)t^2 m^7)$. Finally, multiplying the backpropagated quadratures introduces an extra time complexity resulting in the one given in the theorem. The proof is formalized in Appendix E.

APPENDIX D: PROOF OF THEOREM 1

We restate Theorem 1 first.

Theorem D1. Given the quadrature moments of an m -mode input state ρ and an m -mode unitary $\hat{O}_\gamma$ given by

$$\hat{O}_\gamma = \hat{O}_t e^{i\gamma_t \hat{q}_1^3} \hat{O}_{t-1} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0, \quad (\text{D1})$$

evolving the input state, where $\hat{O}_1, \dots, \hat{O}_{t-1}$ are displaced orthogonal gates, the expectation value of an operator $H(\hat{q}, \hat{p})$, polynomial of degree d in the quadrature operators, given by

$$\text{Tr}[\hat{O}_\gamma \rho \hat{O}_\gamma^\dagger H(\hat{q}, \hat{p})] = \text{Tr}[\rho \hat{O}_\gamma^\dagger H(\hat{q}, \hat{p}) \hat{O}_\gamma], \quad (\text{D2})$$

can be computed in time $\mathcal{O}(m^{3d} + t^2 m^6)$.

Proof. We first give the theorem for the quadrature backpropagation technique described in the main text.

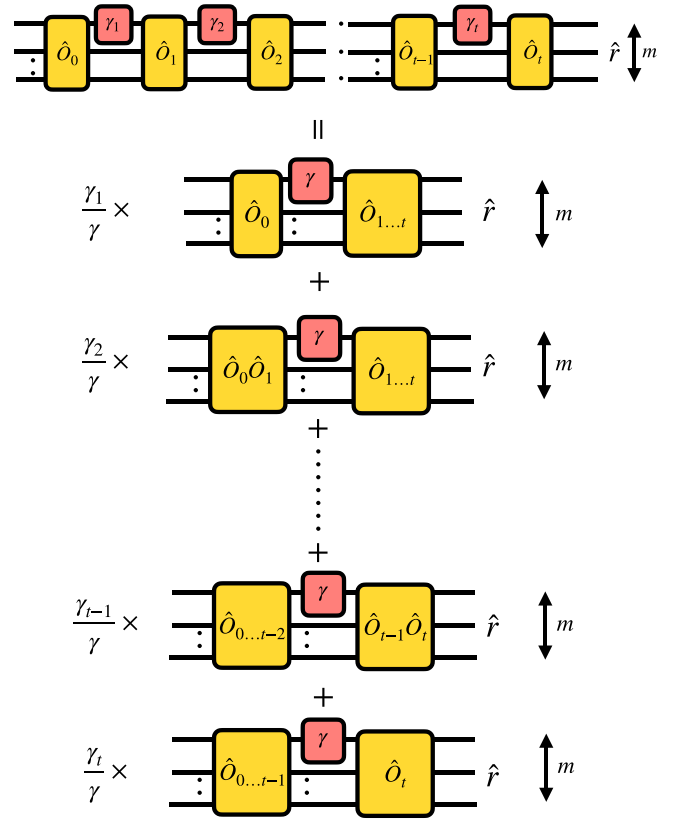


FIG. 4. Quadrature backpropagation in the absence of symplectic coherence, where $\hat{r} \in \{\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m\}$. The evolution (through backpropagation) of a quadrature over a circuit with a displaced orthogonal unitary between cubic phase gates can be seen as simultaneous evolution over t circuits with a single cubic phase gate and displaced passive linear unitary. Here, $\gamma = \sum_{i=1}^t \gamma_i$ and $\hat{O}_{i \dots j} = \hat{O}_i \dots \hat{O}_j$, for $j < i$ and $j, i \in \{0, \dots, t\}$.

Theorem D2 (Quadrature backpropagation technique). With $\hat{r} \in \{\hat{q}_1, \hat{p}_1, \dots, \hat{q}_m, \hat{p}_m\}$ and $\hat{O}_0, \dots, \hat{O}_t$ being m -mode displaced orthogonal unitary gates, the evolution of $\hat{r}$ over $\hat{O}_t e^{i\gamma_t \hat{q}_1^3} \hat{O}_{t-1} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0$ can be seen as the weighted sum of its evolution over t unitary gates with the displaced orthogonal unitary gates and a single (modified) cubic phase gate (Fig. 4). That is,

$$\begin{aligned} & \hat{O}_0^\dagger e^{-i\gamma_1 \hat{q}_1^3} \hat{O}_1^\dagger \dots \hat{O}_{t-1}^\dagger e^{-i\gamma_t \hat{q}_1^3} \hat{O}_t^\dagger \hat{r} \hat{O}_t e^{i\gamma_t \hat{q}_1^3} \hat{O}_{t-1} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0 \\ &= \sum_{i=1}^t \frac{\gamma_i}{\gamma} (\hat{O}_{i-1 \dots 0}^\dagger e^{-i\gamma_i \hat{q}_1^3} \hat{O}_{i \dots i}^\dagger \hat{r} \hat{O}_{i \dots i} e^{i\gamma_i \hat{q}_1^3} \hat{O}_{i-1 \dots 0}), \end{aligned} \quad (\text{D3})$$

where we have defined

$$\hat{O}_{j \dots i} := \hat{O}_j \dots \hat{O}_i, \quad (\text{D4})$$

for $i, j \in \{1, \dots, t\}$ and $j < i$, and $\gamma := \sum_{i=1}^t \gamma_i$.

Proof. We will prove that this relation holds for position and momentum quadratures separately.

For $\hat{q} \in \{\hat{q}_1, \dots, \hat{q}_m\}$,

$$\begin{aligned} & (\hat{O}_0^\dagger e^{-i\gamma_1 \hat{q}_1^3} \hat{O}_1^\dagger \dots e^{-i\gamma_{t-1} \hat{q}_1^3} \hat{O}_{t-1}^\dagger e^{-i\gamma_t \hat{q}_1^3} \hat{O}_t^\dagger) \hat{q} (\hat{O}_t e^{i\gamma_t \hat{q}_1^3} \\ & \times \hat{O}_{t-1} e^{i\gamma_{t-1} \hat{q}_1^3} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0) = \hat{O}_{t \dots 0}^\dagger \hat{q} \hat{O}_{t \dots 0}, \end{aligned} \quad (\text{D5})$$

since the position quadratures are unaffected by the cubic phase gates and the orthogonal gates mixes the position quadratures with position quadratures. Writing

$$\hat{O}_{t\dots 0}^\dagger \hat{q} \hat{O}_{t\dots 0} = \frac{1}{\gamma} \sum_{i=1}^t \gamma_i \hat{O}_{t\dots i}^\dagger \hat{q} \hat{O}_{t\dots i}, \quad (\text{D6})$$

the theorem is proved for $\hat{q}$.

Let S_i be the $2m \times 2m$ symplectic matrix that characterizes the transformation of quadratures under $\hat{O}_i$. The entry $(S_i)_{q_i, p_j}$ represents the coefficient of $\hat{p}_j$ in the transformation of $\hat{q}_i$ under $\hat{O}_i$, with similar interpretations for other quadratures. Then, given $\hat{p} \in \{\hat{p}_1, \dots, \hat{p}_m\}$, its evolution follows as

$$\begin{aligned} & (\hat{O}_0^\dagger e^{-i\gamma_1 \hat{q}_1^3} \hat{O}_1^\dagger \dots e^{-i\gamma_{t-1} \hat{q}_{t-1}^3} \hat{O}_{t-1}^\dagger e^{-i\gamma_t \hat{q}_t^3} \hat{O}_t^\dagger) \hat{p} (\hat{O}_t e^{i\gamma_t \hat{q}_t^3} \hat{O}_{t-1} e^{i\gamma_{t-1} \hat{q}_{t-1}^3} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0) \\ &= (\hat{O}_0^\dagger e^{-i\gamma_1 \hat{q}_1^3} \hat{O}_1^\dagger \dots e^{-i\gamma_{t-1} \hat{q}_{t-1}^3} \hat{O}_{t-1}^\dagger) (\hat{O}_t^\dagger \hat{p} \hat{O}_t + 6\gamma_t (S_t)_{p, p_1} \hat{q}_1^2) (\hat{O}_{t-1} e^{i\gamma_{t-1} \hat{q}_{t-1}^3} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0) \\ &= (\hat{O}_0^\dagger e^{-i\gamma_1 \hat{q}_1^3} \hat{O}_1^\dagger \dots \hat{O}_{t-2}^\dagger) (\hat{O}_{t-1}^\dagger \hat{p} \hat{O}_{t-1} + 6\gamma_{t-1} (S_{t-1})_{p, p_1} \hat{q}_1^2) (\hat{O}_{t-2} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0) + 6\gamma_t (S_t)_{p, p_1} \hat{O}_{t-1\dots 0}^\dagger \hat{q}_1^2 \hat{O}_{t-1\dots 0} \\ &\quad \vdots \\ &= \hat{O}_{t\dots 0}^\dagger \hat{p} \hat{O}_{t\dots 0} + \sum_{i=1}^t 6\gamma_i (S_i \dots S_t)_{p, p_1} \hat{O}_{i-1\dots 0}^\dagger \hat{q}_1^2 \hat{O}_{i-1\dots 0}. \end{aligned} \quad (\text{D7})$$

Since the cubic phase gate affects only the $\hat{p}_1$ quadrature, transforming it as $\hat{p}_1 \rightarrow \hat{p}_1 + 6\gamma \hat{q}_1^2$, while $\hat{q}_1^2$ remains unchanged by subsequent cubic phase gates. Therefore, the resulting evolution remains quadratic in the position quadratures. Writing

$$\hat{O}_{t\dots 0}^\dagger \hat{p} \hat{O}_{t\dots 0} = \frac{1}{\gamma} \sum_{i=1}^t \gamma_i \hat{O}_{t\dots i}^\dagger \hat{p} \hat{O}_{t\dots i}, \quad (\text{D8})$$

and grouping $\frac{\gamma_i}{\gamma} \hat{O}_{t\dots i}^\dagger \hat{p} \hat{O}_{t\dots i}$ with $6\gamma_i (S_i \dots S_t)_{p, p_1} \hat{O}_{i-1\dots 0}^\dagger \hat{q}_1^2 \hat{O}_{i-1\dots 0}$ for all $i \in \{1, \dots, t\}$, we get the statement of the theorem. ■

The quadrature backpropagation technique allows us to evolve the single-degree quadratures in $H(\hat{q}, \hat{p})$ as degree 2 polynomials. This is formalized in the following lemma:

Lemma D 1. Given an m -mode unitary

$$\hat{O}_\gamma = \hat{O}_t e^{i\gamma_t \hat{q}_t^3} \hat{O}_{t-1} \dots \hat{O}_1 e^{i\gamma_1 \hat{q}_1^3} \hat{O}_0, \quad (\text{D9})$$

such that $\hat{O}_0, \dots, \hat{O}_t$ are m -mode displaced orthogonal unitary gates, the evolution of a quadrature $\hat{r} \in \{\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m\}$ over $\hat{O}_\gamma$ is described by a quadratic degree polynomial $f_r^{(2)}(\hat{q}, \hat{p}, \hat{O}_\gamma)$, which can be computed in time $\mathcal{O}(t^2 m^5)$.

Proof. Lemma D 1 follows from the fact that a quadrature evolution according to Theorem D 2 simplifies to a quadrature evolution in t circuits with $(t+1)$ orthogonal unitary gates and a single cubic phase gate. The evolution of the quadrature along each of these simplified circuits is described as a degree 2 polynomial with $\mathcal{O}(m^2)$ terms. Given the description of $\hat{O}_0, \dots, \hat{O}_t$, the description of $\hat{O}_i \times \dots \times \hat{O}_j$ can be calculated in time $\mathcal{O}((j-i)m^3)$. Since each of the t simplified circuits have $t+1$ orthogonal unitary gates, this implies that the degree 2 polynomial along each of the paths can be given in time $\mathcal{O}((t+1)m^3)$. Further, since we have to sum the coefficient of each of the $\mathcal{O}(m^2)$ terms in t such polynomials, this gives the time complexity of computing $f_r^{(2)}(\hat{q}, \hat{p}, \hat{O}_\gamma)$ to

be of the order $\mathcal{O}(t(t+1)m^5) \sim \mathcal{O}(t^2 m^5)$ upto lower-degree polynomial factors in t . ■

With Lemma D 1, Theorem D 1 can be easily seen by noting that with the quadrature backpropagation picture, the single-degree quadratures involved in the description of $H(\hat{q}, \hat{p})$ can be evolved independently as quadratic degree polynomials in time $\mathcal{O}(t^2 m^6)$, which corresponds to the time calculating the evolution of each $\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m$, as all of them may be involved in the description of $H(\mathbf{q}, \mathbf{p})$. Then, we need to multiply $\mathcal{O}(m^{2d})$ terms to get the evolution of terms of degree d in $H(\mathbf{q}, \mathbf{p})$, and similarly for terms of degree $0, 1, 2, \dots, d-1$. The general form of $H(\hat{q}, \hat{p})$ is

$$\sum_{\alpha_1 + \alpha_2 + \dots + \alpha_m + \beta_1 + \dots + \beta_m \leq d} \hat{q}_1^{\alpha_1} \dots \hat{q}_m^{\alpha_m} \hat{p}_1^{\beta_1} \dots \hat{p}_m^{\beta_m}, \quad (\text{D10})$$

with $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{N}$. Note that any other ordering of the operators can be put in this form by the application of the appropriate commutation relation. This gives the total number of terms as

$$\sum_{n=0}^d \binom{2m+n-1}{n} = \binom{2m+d}{d} = \mathcal{O}(m^d), \quad (\text{D11})$$

where the right-hand side follows from the hockey-stick identity. Therefore, we have at most $\mathcal{O}(m^d)$ terms and each one of them can be described in time at most $\mathcal{O}(m^{2d})$. This gives the total time to describe $\hat{O}_\gamma^\dagger H(\hat{q}, \hat{p}) \hat{O}_\gamma$ to be $\mathcal{O}(m^{3d} + t^2 m^6)$. And since we have the efficient description of the input state ρ , this also gives the time complexity of computing its expectation value. ■

APPENDIX E: PROOF OF THEOREM 2

We restate Theorem 2.

Theorem E 1. Given the quadrature moments of an m -mode input state ρ and an m -mode unitary $\hat{U}_t$ given by

$$\hat{U}_t = \hat{O}_{\gamma_t} \hat{R}_1(\theta_c) \hat{O}_{\gamma_{t-1}} \dots \hat{R}_1(\theta_1) \hat{O}_{\gamma_0}, \quad (\text{E1})$$

evolving the input state, where $\hat{O}_{y_i}$ describes $t + 1$ displaced orthogonal gates interleaved with t cubic phase gates, and $\hat{R}_1(\theta_i)$ are single-mode rotation gates in the first mode $\forall i \in \{0, 1, \dots, c\}$, the expectation value of an operator $H(\hat{q}, \hat{p})$, polynomial of degree d in the quadrature operators, given by

$$\text{Tr}[\hat{U}_t \rho \hat{U}_t^\dagger H(\hat{q}, \hat{p})] = \text{Tr}[\rho \hat{U}_t^\dagger H(\hat{q}, \hat{p}) \hat{U}_t], \quad (\text{E2})$$

can be computed in time $\mathcal{O}(m^{d2^{c+1}} + (c+1)t^2m^7)$.

Proof. The application of Lemma D 1 and Theorem 1 for the evolution of a single-degree quadrature over each of $\hat{O}_{y_i}$ gives the following lemma:

Lemma E 1. Given an m -mode unitary

$$\hat{U}_t = \hat{O}_{y_c} \hat{R}_1(\theta_c) \hat{O}_{y_{c-1}} \cdots \hat{R}_1(\theta_1) \hat{O}_{y_0}, \quad (\text{E3})$$

such that $\hat{O}_{y_i}$ describes $t + 1$ displaced orthogonal unitary gates interleaved with t cubic phase gates, and $\hat{R}_1(\theta_i)$ are single-mode rotation gates in the first mode $\forall i \in \{0, 1, \dots, c\}$, the evolution of a quadrature $\hat{r} \in \{\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m\}$ over $\hat{U}_t$ is described by a polynomial of degree 2^{c+1} and number of terms $\mathcal{O}(m^{2^{c+1}})$, whose description can be given in time $\mathcal{O}(m^{3 \times 2^c} + (c+1)t^2m^6)$.

Proof. This lemma follows from the iterative structure of the evolution. Starting with quadrature operator $\hat{r} \in \{\hat{q}_1, \dots, \hat{q}_m, \hat{p}_1, \dots, \hat{p}_m\}$, the application of $\hat{O}_{y_c}$ using the quadrature backpropagation technique (Lemma D 1) transforms it into a polynomial of degree at most 2 with $\mathcal{O}(m^2)$ terms in the quadratures where the quadratic terms involve only the position quadratures. Using Theorem 1, the description of this polynomial can be computed in time $\mathcal{O}(t^2m^6 + m^3)$. Applying the rotation $\hat{R}_1(\theta_c)$ to this polynomial produces another polynomial of degree 2 with $\mathcal{O}((m+1)^2)$ terms. This transformation is computationally efficient since it involves a single rotation gate, and crucially, it introduces a quadratic term of the form $\hat{p}_1^2$ by transforming the quadrature $\hat{q}_1^2$ as $(\cos(\theta_c)\hat{q}_1 + \sin(\theta_c)\hat{p}_1)^2$.

Therefore, applying $\hat{O}_{y_{c-1}}$ to this polynomial results in an evolution where the previously introduced $\hat{p}_1^2$ transforms as $(\hat{p}_1 + 6\gamma\hat{q}_1^2)^2$, leading to a polynomial of degree 4, with quadratic terms involving only the position quadratures, as the subsequent displaced orthogonal unitaries acting on $\hat{q}_1^4$ do not mix position quadratures with momentum quadratures. Using Theorem 1, the description of this polynomial can be obtained in time $\mathcal{O}(t^2m^6 + m^6)$. The application of $\hat{R}_1(\theta_{c-1})$ on the quartic term $\hat{q}_1^4$ then introduces a term of the form $\hat{p}_1^4$, and repeating the process with the next layer $\hat{O}_{y_{c-2}}$ results in a polynomial of degree 8. This pattern continues, with each additional layer doubling the polynomial degree. Therefore, by induction, the degree of the evolved single-degree quadrature $\hat{r}$ transforms as

$$1 \xrightarrow{\hat{O}_{y_c}} 2 \xrightarrow{\hat{R}_1(\theta_c)\hat{O}_{y_{c-1}}} 2^2 \xrightarrow{\hat{R}_1(\theta_{c-1})\hat{O}_{y_{c-2}}} 2^3 \cdots \xrightarrow{\hat{R}_1(\theta_1)\hat{O}_{y_0}} 2^{c+1}, \quad (\text{E4})$$

whereas the number of terms in the polynomial corresponding to the evolved form of $\hat{r}$ transform as

$$1 \xrightarrow{\hat{O}_{y_c}} \mathcal{O}(m^2) \xrightarrow{\hat{R}_1(\theta_c)\hat{O}_{y_{c-1}}} \mathcal{O}(m^{2^2}) \xrightarrow{\hat{R}_1(\theta_{c-1})\hat{O}_{y_{c-2}}} \mathcal{O}(m^{2^3}) \cdots \cdots \xrightarrow{\hat{R}_1(\theta_1)\hat{O}_{y_0}} \mathcal{O}(m^{2^{c+1}}). \quad (\text{E5})$$

Therefore, after $c + 1$ layers, the single-degree quadrature $\hat{r}$ evolves to a polynomial of degree 2^{c+1} in the quadratures, with the number of terms of the order $\mathcal{O}(m^{2^{c+1}})$.

Applying Theorem 1 at each step, the total time required to describe the evolved quadrature after $(c + 1)$ layers is given by $\mathcal{O}((c+1)t^2m^6 + \sum_{i=0}^c m^{3 \times 2^i})$, which accounts for the cumulative time needed to describe the evolved polynomial after each of the $c + 1$ layers. This expression simplifies to $\mathcal{O}(m^{3 \times 2^c} + (c+1)t^2m^6)$, as the term $m^{3 \times 2^c}$ dominates due to the double-exponential growth in $m^{3 \times 2^i}$. ■

The complexity of computing the expectation value of the $H(\hat{q}, \hat{p})$ is determined by first evolving all $2m$ single-degree quadratures into polynomials of degree 2^{c+1} , each containing $\mathcal{O}(m^{2^{c+1}})$ terms. According to Lemma E 2, this requires $\mathcal{O}(m(m^{3 \times 2^c} + (c+1)t^2m^6)) = \mathcal{O}(m^{3 \times 2^c+1} + (c+1)t^2m^7)$ time. Next, describing an evolved quadrature of degree d involves multiplying $\mathcal{O}(m^{d2^{c+1}})$ terms, and $H(\hat{q}, \hat{p})$ consists of at most $\mathcal{O}(m^d)$ terms of degree at most d . Combining these contributions, given the efficient description of ρ , the overall complexity of computing $\text{Tr}[\rho \hat{U}_t^\dagger H(\hat{q}, \hat{p}) \hat{U}_t]$ is given by $\mathcal{O}(m^{d+d2^{c+1}} + m^{3 \times 2^c} + (c+1)t^2m^7)$, which simplifies to $\mathcal{O}(m^{d2^{c+1}} + (c+1)t^2m^7)$, up to lower-degree polynomial factors in m . ■

APPENDIX F: EFFICIENT COMPUTATION OF SINGLE-MODE QUADRATURE MOMENTS

In Theorems 1 and 2, we provide classical simulation algorithms for computing expectation values at the output of continuous-variable quantum circuits. The running time of these algorithms is essentially independent of the input state, as long as its quadrature moments can be computed efficiently. To that end, we derive explicit formulas for the quadrature moments of several single-mode states, i.e., for the expectation value of $\hat{q}^k \hat{p}^l$ for arbitrary $k, l \in \mathbb{N}$. These include the following:

- (1) Gaussian states [Eq. (F1)].
- (2) Finite superpositions of Fock states [Eq. (F13)].
- (3) Cat states [Eq. (F21)].
- (4) GKP states [Eq. (F24)].

1. Gaussian states

For single-mode Gaussian states, we have the following formula for moments of quadratures.

Lemma F 1. Given a single-mode Gaussian state ρ_G with covariance matrix

$$V = \begin{bmatrix} V_q & V_{qp} \\ V_{qp} & V_p \end{bmatrix}$$

and displacement vector $\mathbf{d} = [r_q, r_p]$, we have

$$\begin{aligned} \text{Tr}[\rho_G \hat{q}^k \hat{p}^l] &= \sum_{u=0}^k \sum_{v=0}^l \binom{k}{u} \binom{l}{v} \delta_{(u+v) \pmod{2}, 0} r_q^{k-u} r_p^{l-v} \\ &\times \sum_{s=0}^{u/2} \sum_{t=0}^{v/2} \delta_{u-2s, v-2t} \binom{u}{2s} (2s-1)!! \binom{v}{2t} \\ &\times (2t-1)!! (v-2t)! \end{aligned}$$

$$\times V_q^s V_p^t (V_{qp} + i)^{(u+v)/2-(s+t)}. \quad (\text{F1})$$

Proof. Define

$$\Delta\hat{q} = \hat{q} - r_q, \Delta\hat{p} = \hat{p} - r_p, \quad (\text{F2})$$

such that

$$\begin{aligned} \text{Tr}[\rho_G \Delta\hat{q}^2] &= V_q, \\ \text{Tr}[\rho_G \Delta\hat{p}^2] &= V_p, \\ \text{Tr}[\rho_G \Delta\hat{q} \Delta\hat{p}] &= V_{qp} + i. \end{aligned} \quad (\text{F3})$$

Therefore,

$$\begin{aligned} \text{Tr}[\rho_G \hat{q}^k \hat{p}^l] &= \text{Tr}[\rho_G (r_q + \Delta\hat{q})^k (r_p + \Delta\hat{p})^l] \\ &= \sum_{u=0}^k \sum_{v=0}^l \binom{k}{u} \binom{l}{v} r_q^{k-u} r_p^{l-v} \text{Tr}[\rho_G \Delta\hat{q}^u \Delta\hat{p}^v]. \end{aligned} \quad (\text{F4})$$

Now, $\Delta\hat{q}$ and $\Delta\hat{p}$ are position and momentum quadratures of a Gaussian state with zero first moments. To see this, note that $\Delta\hat{q}$ (and similarly $\Delta\hat{p}$) can equivalently be seen as $\hat{D}(r_q)\hat{q}\hat{D}(-r_q)$. Applying this displacement operator on ρ_G makes its displacement vector equal to zero. Therefore, we can apply Wick's theorem [83], which states that $\text{Tr}[\rho_G \Delta\hat{q}^u \Delta\hat{p}^v]$ can be computed by grouping $\Delta\hat{q}^u \Delta\hat{p}^v$ into pairs of two (called pair contractions), calculating the expectation value of each pair over ρ , and multiplying these expectation values. Finally, we sum over possible pairings.

For each such contraction, we get s pairs of $\Delta\hat{q}$ (terms of the form $\text{Tr}[\rho \Delta\hat{q}^2]$), t pairs of $\Delta\hat{p}$ (terms of the form $\text{Tr}[\rho \Delta\hat{p}^2]$), and $(u+v)/2 - (s+t)$ pairs of $\hat{q}$ and $\hat{p}$ (terms of the form $\text{Tr}[\rho \Delta\hat{q} \Delta\hat{p}]$). Now given s and t , there are $\binom{u}{2s-1}!!$ ways to obtain s pairs of $\Delta\hat{q}$, as can be seen from the following argument: First, we need to pick out $2s$ q 's out of u that will be paired with each other; this can be done in $\binom{u}{2s}$ ways. The $2s$ q 's can be paired with each other in $(2s-1)!!$ ways (picking the first $\hat{q}$, we have to select one out of $2s-1$ $\hat{q}$'s to pair it with, then picking the second $\hat{q}$, we have to select one out of $2s-3$ $\hat{q}$'s to pair it with, and so on). A similar argument for t pairs of $\hat{p}$'s shows that there are $\binom{v}{2t}(2t-1)!!$ ways one can get t pairs of $\Delta\hat{p}$. Finally, the remaining $u-2s$ $\hat{q}$'s should be paired with the remaining $v-2t$ $\hat{p}$'s. We require $u-2s = v-2t$ for there to be a $\hat{p}$ for every $\hat{q}$. There are $(v-2t)! = (u-2s)!$ ways of obtaining $(u+v)/2 - (s+t)$ pairs of $\Delta\hat{q}$ and $\Delta\hat{p}$, as can be seen as follows: For the first $\Delta\hat{q}$, there are $\binom{v-2t}{1} = (v-2t)$ ways to pick a $\Delta\hat{p}$ to pair it with; for the second $\Delta\hat{q}$, there are $v-2t-1$ choices; and so on.

Note that $u+v$ needs to be even for the trace to give a nonzero value, otherwise there will be one $\Delta\hat{q}$ or $\Delta\hat{p}$ that will be unpaired and make the term zero, since the first moments are zero.

Combining these arguments, we obtain

$$\begin{aligned} \text{Tr}[\rho_G \Delta\hat{q}^u \Delta\hat{p}^v] &= \sum_{s=0}^{u/2} \sum_{t=0}^{v/2} \delta_{u-2s, v-2t} \binom{u}{2s} (2s-1)!! \binom{v}{2t} (2t-1)!! (v-2t)! \\ &\quad \times \text{Tr}[\rho_G \Delta\hat{q}^2]^s \text{Tr}[\rho_G \Delta\hat{p}^2]^t \text{Tr}[\rho_G \Delta\hat{q} \Delta\hat{p}]^{(u+v)/2-(s+t)}, \text{ if } (u+v) \pmod{2} = 0 \\ &= 0, \text{ otherwise.} \end{aligned} \quad (\text{F5})$$

Using Eq. (F3), we finally get

$$\begin{aligned} \text{Tr}[\rho_G \hat{q}^k \hat{p}^l] &= \sum_{u=0}^k \sum_{v=0}^l \binom{k}{u} \binom{l}{v} \delta_{(u+v) \pmod{2}, 0} r_q^{k-u} r_p^{l-v} \sum_{s=0}^{u/2} \sum_{t=0}^{v/2} \delta_{u-2s, v-2t} \binom{u}{2s} (2s-1)!! \binom{v}{2t} (2t-1)!! (v-2t)! \\ &\quad \times V_q^s V_p^t (V_{qp} + i)^{(u+v)/2-(s+t)}. \end{aligned} \quad (\text{F6})$$

■

2. Superposition of Fock states

To derive the formula for moments of quadratures of superpositions of Fock states, we give the following lemma:

Lemma F2. For Fock states $|s\rangle$ and $|t\rangle$,

$$\begin{aligned} \langle s | \hat{q}^k \hat{p}^l | t \rangle &:= f_{st}^{kl} = (-i)^l \sum_{\substack{j=0 \\ (k-j) \pmod{2}=0}}^k \sum_{\substack{j=0 \\ (l-j) \pmod{2}=0}}^l \sum_{r=0}^i \sum_{m=0}^j \frac{(-1)^{\frac{k-j}{2}+m-r}}{2^{\frac{k-i+l-j}{2}}} \frac{k!!}{i! j! \left(\frac{k-i}{2}\right)! \left(\frac{l-j}{2}\right)!} \binom{i}{r} \binom{j}{m} \\ &\quad \times \sqrt{(t+j-2m+i-r) \cdots (t+j-2m+i-2r+1)} \\ &\quad \times \sqrt{(t+j-2m+i-r) \cdots (t+j-2m+1)} \\ &\quad \times \sqrt{(t+j-m) \cdots (t+j-2m+1)} \\ &\quad \times \sqrt{(t+j-m) \cdots (t+1)} \delta_{s, t+j-2m+i-2r}. \end{aligned} \quad (\text{F7})$$

Proof. Since $\hat{a}$ and $\hat{a}^\dagger$ are noncommutative operators such that $[[a, a^\dagger], a] = [[a, a^\dagger], a^\dagger] = 0$ since $[\hat{a}, \hat{a}^\dagger] = \mathbb{I}$, we have [84]

$$(\hat{a} + \hat{a}^\dagger)^k = \sum_{\substack{i=0 \\ (k-i) \bmod 2=0}}^k \sum_{r=0}^i \frac{(-1)^{(k-i)/2}}{2^{(k-i)/2}} \frac{k!}{i! \left(\frac{k-i}{2}\right)!} \binom{i}{r} \hat{a}^r (\hat{a}^\dagger)^{i-r}. \quad (\text{F8})$$

Similarly,

$$(\hat{a} - \hat{a}^\dagger)^l = \sum_{\substack{j=0 \\ (l-j) \bmod 2=0}}^l \sum_{m=0}^j \frac{1}{2^{(l-j)/2}} \frac{l!}{j! \left(\frac{l-j}{2}\right)!} \binom{j}{m} \hat{a}^m (-\hat{a}^\dagger)^{j-m}. \quad (\text{F9})$$

Therefore,

$$\begin{aligned} \langle s | \hat{q}^k \hat{p}^l | t \rangle &= (-i)^l \langle s | (\hat{a} + \hat{a}^\dagger)^k (\hat{a} - \hat{a}^\dagger)^l | q \rangle = (-i)^l \sum_{\substack{i=0 \\ (k-i) \bmod 2=0}}^k \sum_{\substack{j=0 \\ (l-j) \bmod 2=0}}^l \sum_{r=0}^i \sum_{m=0}^j \frac{(-1)^{\frac{k-i}{2}+j-m}}{2^{\frac{k-i}{2}}} \frac{k!l!}{i!j! \left(\frac{k-i}{2}\right)! \left(\frac{l-j}{2}\right)!} \binom{i}{r} \binom{j}{m} \\ &\times \langle s | \hat{a}^r (\hat{a}^\dagger)^{i-r} \hat{a}^m (\hat{a}^\dagger)^{j-m} | t \rangle. \end{aligned} \quad (\text{F10})$$

Using

$$(\hat{a}^\dagger)^r |t\rangle = \sqrt{(t+r) \cdots (t+1)} |t+r\rangle, \quad \hat{a}^r |t\rangle = \sqrt{t \cdots (t-r+1)} |t-r\rangle, \quad \langle s | t \rangle = \delta_{s,t}, \quad (\text{F11})$$

we obtain the formula for the moment of the quadrature as given in the lemma [Eq. (F7)]. ■

As a direct consequence, we have the following lemma for finite superpositions of Fock states:

Lemma F3. Given a superposition of Fock states

$$|\psi\rangle_{\text{Fock}} = \sum_{t=0}^d c_t |t\rangle, \quad (\text{F12})$$

we have

$$\langle \psi | \hat{q}^k \hat{p}^l | \psi \rangle_{\text{Fock}} = \sum_{s,t=0}^d c_t c_s^* f_{st}^{kl}, \quad (\text{F13})$$

where f_{st}^{kl} is given by Eq. (F7).

3. Cat states and GKP states

To find the expectation value of $\hat{q}^k \hat{p}^l$ for cat states and physical GKP states, first we give the following lemma:

Lemma F4. For coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$,

$$\langle \alpha_2 | \hat{q}^k \hat{p}^l | \alpha_1 \rangle = i^{k+l} \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_2^* \alpha_1\right) H_k\left(-i \frac{\alpha_2^* + \alpha_1}{2}\right) H_l\left(\frac{\alpha_1 - \alpha_2^*}{2}\right), \quad (\text{F14})$$

where $H_n(x)$ is the n th Hermite polynomial.

Proof. First we define the operator

$$\hat{O} = e^{s\hat{q}+t\hat{p}} \quad (\text{F15})$$

such that

$$\hat{q}^k \hat{p}^l = \frac{\partial^l \partial^k}{\partial t^l \partial s^k} \hat{O} \Big|_{s,t=0} \quad (\text{F16})$$

and

$$\begin{aligned} \langle \alpha_2 | \hat{O} | \alpha_1 \rangle &= \exp(s^2 + t^2) \langle \alpha_2 | \exp((s+it)\hat{a}^\dagger) \exp((s-it)\hat{a}) | \alpha_1 \rangle \\ &= \exp(s^2 + t^2) \times \exp((s+it)\alpha_2^* + (s-it)\alpha_1) \times \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_2^* \alpha_1\right) \\ &= \exp(s^2 + s(\alpha_2^* + \alpha_1)) \times \exp(t^2 + t(i\alpha_2^* - i\alpha_1)) \times \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_2^* \alpha_1\right) \\ &= \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_2^* \alpha_1\right) \sum_{n=0}^{\infty} H_n\left(\frac{-i(\alpha_2^* + \alpha_1)}{2}\right) \frac{(is)^n}{n!} \sum_{m=0}^{\infty} H_m\left(\frac{(\alpha_2^* - \alpha_1)}{2}\right) \frac{(it)^m}{m!}, \end{aligned} \quad (\text{F17})$$

where in the last line we have used the generating function for the Hermite polynomials

$$e^{2tz-t^2} = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!}. \quad (\text{F18})$$

Therefore, we get

$$\begin{aligned} \langle \alpha_2 | \hat{q}^k \hat{p}^l | \alpha_1 \rangle &= \langle \alpha_2 | \frac{\partial^l \partial^k}{\partial t^l \partial s^k} \hat{O}|_{s,t=0} | \alpha_1 \rangle = \frac{\partial^l \partial^k}{\partial t^l \partial s^k} \langle \alpha_2 | \hat{O} | \alpha_1 \rangle |_{s,t=0} \\ &= i^{k+l} \exp\left(-\frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2 + \alpha_2^* \alpha_1\right) H_k\left(-i\frac{\alpha_2^* + \alpha_1}{2}\right) H_l\left(\frac{\alpha_2^* - \alpha_1}{2}\right). \end{aligned} \quad (\text{F19})$$

■

Now, we are ready to give formulas for moments of quadratures for cat and GKP states.

Lemma F5. For the cat state

$$|\psi\rangle_{\text{cat}\pm} = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{\mathcal{N}_{\pm}}}, \quad (\text{F20})$$

with $\mathcal{N}_{\pm} = 2(1 \pm e^{-2|\alpha|^2})$, the expectation value of $\hat{q}^k \hat{p}^l$ is given by

$$\begin{aligned} \langle \psi | \hat{q}^k \hat{p}^l | \psi \rangle_{\text{cat}} &= \frac{i^{k+l}}{\mathcal{N}_{\pm}} (H_k(-i \operatorname{Re}(\alpha)) H_l(-i \operatorname{Im}(\alpha)) + H_k(i \operatorname{Re}(\alpha)) H_l(i \operatorname{Im}(\alpha))) \\ &\quad \pm e^{-2|\alpha|^2} H_k(-i \operatorname{Im}(\alpha)) H_l(-\operatorname{Re}(\alpha)) \pm e^{-2|\alpha|^2} H_k(i \operatorname{Im}(\alpha)) H_l(\operatorname{Re}(\alpha)), \end{aligned} \quad (\text{F21})$$

where $\operatorname{Re}(\alpha)$ and $\operatorname{Im}(\alpha)$ are real and imaginary parts of α , respectively.

This follows by writing

$$\langle \psi | \hat{q}^k \hat{p}^l | \psi \rangle_{\text{cat}} = \frac{1}{\mathcal{N}_{\pm}} (\langle \alpha | \hat{q}^k \hat{p}^l | \alpha \rangle + \langle -\alpha | \hat{q}^k \hat{p}^l | -\alpha \rangle \pm \langle -\alpha | \hat{q}^k \hat{p}^l | \alpha \rangle \pm \langle \alpha | \hat{q}^k \hat{p}^l | -\alpha \rangle) \quad (\text{F22})$$

and applying Lemma F4.

For physical GKP states as defined in Eq. (7.7b) of Ref. [73] (note that expectation value of the momentum operator is ill-defined for the ideal GKP states, as can be inferred from the fact that their position wavefunctions is a Dirac comb), we have the following lemma:

Lemma F6. We recall the expression of a physical GKP state [Eq. (7.7b) of Ref. [73]]:

$$|\mu^{\Delta}\rangle = \frac{1}{\sqrt{\mathcal{N}_{\Delta,\mu}}} \sum_{\vec{n} \in \mathbb{Z}^2} e^{-\frac{\pi}{2} \Delta^2 [(2n_1 + \mu)^2 + n_2^2] - \pi(2n_1 + \mu)n_2} |\sqrt{\pi/2}(2n_1 + \mu + in_2)\rangle, \quad (\text{F23})$$

where $|\sqrt{\pi/2}(2n_1 + \mu + in_2)\rangle$ is a coherent state of amplitude $\sqrt{\pi/2}(2n_1 + \mu + in_2)$, $\mu \in \{0, 1\}$, and $\mathcal{N}_{\Delta,\mu}$ is a normalization constant that can be expressed in terms of Jacobi theta functions [Eq. (2) of Ref. [85]]. The exact expression is given in Proposition 3 of Ref. [85] (compared to the notations therein, we picked $\sigma_x^2, \sigma_p^2 = \Delta$, $\Gamma = \sqrt{\pi}$, and $\mu = j$). We have

$$\begin{aligned} \langle \mu^{\Delta} | \hat{q}^k \hat{p}^l | \mu^{\Delta} \rangle &= \frac{i^{k+l}}{\mathcal{N}_{\Delta,\mu}} \sum_{\vec{n} \in \mathbb{Z}^2} \sum_{\vec{m} \in \mathbb{Z}^2} e^{-\frac{\pi}{2} \Delta^2 [(2n_1 + \mu)^2 + n_2^2] - \pi(2n_1 + \mu)n_2} e^{-\frac{\pi}{2} \Delta^2 [(2m_1 + \mu)^2 + m_2^2] - \pi(2m_1 + \mu)m_2} \\ &\quad \times \exp\left(-\frac{1}{2}((2n_1 + \mu)^2 + (2m_1 + \mu)^2 + n_2^2 + m_2^2)\right) H_k\left(-i\frac{2n_1 + 2m_1 + 2\mu + i(n_2 - m_2)}{2}\right) \\ &\quad \times H_l\left(\frac{2m_1 - 2n_1 - i(m_2 + n_2)}{2}\right). \end{aligned} \quad (\text{F24})$$

This is again a direct consequence of Lemma F4.

These last expressions involve infinite sums, but since the term inside the sums involves a rapidly decaying Gaussian function multiplied by a polynomial, we can cut off the sum at some value of $\vec{m}, \vec{n}$ depending on the parameters Δ, k, l and the precision required. To estimate the necessary cutoff, we show below that for all $|x| \geq \max(2k/\Delta^2, 2/(\Delta^2 |\ln(\epsilon)|))$, we have $|x|^k e^{-\Delta^2 x^2} \leq \epsilon$. We expect to see similar scalings for the necessary cutoffs for $\vec{m}, \vec{n}$, with x being replaced by the entries of $\vec{m}$ and $\vec{n}$.

We now prove the claim: Writing

$$|x|^k e^{-\Delta^2 x^2} \leq \epsilon, \quad (\text{F25})$$

and taking the logarithm on both sides and assuming $\epsilon < 1$, we obtain

$$\frac{\Delta^2}{k} x^2 - \ln |x| \geq \frac{1}{k} |\ln(\epsilon)|. \quad (\text{F26})$$

Since $\ln |x| \leq |x| \leq \Delta^2/(2k)x^2$ for $|x| \geq 2k/\Delta^2$, we have that for $|x| \geq 2k/\Delta^2$:

$$\frac{\Delta^2}{k}x^2 - \ln |x| \geq \frac{\Delta^2}{2k}x^2 \geq \frac{|\ln \epsilon|}{k} \quad (\text{F27})$$

for $|x| \geq 2/(\Delta |\ln(\epsilon)|)$. Therefore, for $|x| \geq \max(2k/\Delta^2, 2/(\Delta^2 |\ln(\epsilon)|))$, we have $|x|^k e^{-\Delta^2 x^2} \leq \epsilon$.

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