

Impact of multimode sources on ensemble-based quantum repeaters

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Long-distance entanglement generation at a high rate is a major quantum technological goal yet to be fully realized, with the promise of many interesting applications, such as secure quantum computing on remote servers and quantum cryptography. One possible implementation is using a variant of the scheme proposed by Duan, Lukin, Cirac, and Zoller (DLCZ) [Duan *et al.*, *Nature (London)* **414**, 413 (2001)] by combining atomic-ensemble memories and linear optics with spontaneous parametric down-conversion (SPDC) sources. As we edge closer to the realization of such a technology, the complete details of the underlying components become crucial. In this paper, we consider the impact of the multimode emission from the SPDC source on quantum repeaters based on the DLCZ scheme. We consider two cases, driving the SPDC using short Gaussian pulses and driving the SPDC continuously. For pulsed driving, we find that the use of very narrow laser pulses to drive the SPDC source is crucial to obtain high-fidelity end-to-end entangled states, but this puts demands on the peak intensity. By introducing a maximally allowed laser intensity, we find optimal pulse widths for each swap depth. For continuous driving, we find the temporal acceptance window of clicks relative to the heralding time to be a crucial parameter, and we can similarly optimize the acceptance window for each swap depth. For both cases, we thus identify optimal parameters given experimental limitations and aims. We have thus provided helpful knowledge toward the realization of long-distance entanglement generation using the DLCZ scheme.

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I. INTRODUCTION

A long-distance quantum network, a quantum internet, is one of the central pillars of emerging quantum technologies [1,2]. It would enable a host of exciting quantum applications such as remote blind quantum computing [3,4], quantum key distribution [5,6], and enhanced sensing [7] on a larger scale. Long-distance quantum networks rely on the generation of long-distance entanglement through quantum repeaters [8–10]. Here, the total distance is partitioned into smaller segments, separated by repeater stations containing one or more quantum memories (QMs). Entanglement between the neighboring stations is generated in parallel, and after a successful entanglement generation the entangled states are stored in the QMs. After neighboring pairs have generated entanglement, the entanglement is swapped to cover longer distances. A number of proposed candidates for quantum repeaters have been suggested. A natural choice is to use qubits, e.g., single trapped ions [11], color centers in diamond [12], and quantum dots [13] as the QM. Heralded entanglement protocols can then be used to generate entanglement between distant qubits, and local gate operations can be used to swap the entanglement deterministically.

An alternate proposal was introduced by Duan, Lukin, Cirac, and Zoller (DLCZ) in Ref. [8]. Here, a heralded

entanglement generation protocol creates the entanglement between neighboring QMs based on ensembles of atoms, and a probabilistic Bell-state measurement (BSM) is used to swap the entanglement over longer distances [10]. A crucial advantage of this protocol is the possibility to store many modes in the atomic-ensemble memories, and thus multiplex the entanglement generation rate over the entire network significantly [14]. This has led the DLCZ protocol to become a prime contender for the first real-world implementations of long-distance quantum networks [2]. Since the publication of the DLCZ protocol, a number of different improvements have been published, including alternate detection setups for entanglement generation and swapping [15–17], the use of external sources with the incorporation of multiplexing [18,19], and swapping to other platforms allowing more advanced processing [20–22]. In this work, we focus on the scheme proposed by Simon *et al.* in Ref. [19], where the entanglement is generated by having two spontaneous parametric down-conversion (SPDC) sources, with each source sending one photon into a QM and one photon toward a BSM. A successful detection in the BSM creates entanglement between the photons stored in the two QMs. A key advantage of this protocol is that it is directly compatible with temporal multiplexing. By incorporating multimode memories, the scheme thus allows a significant boost to the quantum communication speed.

To realize the full protocol in practice, it is essential to have a detailed understanding of the possible imperfections and operating condition of the protocol. Previous works have considered various effects such as dark counts [23], details of multiplexing in a repeater scheme [24,25], and distortion of the signal by the QM [14]. Furthermore, crucial work has been done for optimizing the QMs to allow for storage of multiple

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modes [26]. Recent experimental work has also integrated the SPDC sources with QMs [27–32]. However, some issues with the SPDC sources are still outstanding as multiple pairs can be emitted and the multimode nature of the emitted state leads to infidelity [33–35]. In a recent work [35], we examined the multimodality of the SPDC source and how this impacts the applicability as heralded single-photon sources. In this paper, we extend this analysis and examine how the performance of a DLCZ-type quantum repeater is affected by the multimodality of the sources, especially its impact on achievable rates and fidelities. We will furthermore find parameters that optimize the long-distance entanglement generation rate for two different experimental settings: short pulse driving and (quasi)continuous driving.

The paper is structured as follows: In Sec. II, we describe the architecture of the protocol considered, and the operations for generating entanglement, swapping entanglement, and postselection of the output. In Sec. III, we investigate the case when driving the sources with short pulses. In Sec. IV, we investigate the case when driving the sources continuously. In Sec. V, we conclude on the study and discuss further possible investigations.

II. SETUP AND PROTOCOLS

In this section, we present a short summary of the protocol: We will establish two parallel chains, each containing 2^n elementary links of length $L_0 = L/2^n$, where n is the swap depth and L is the total distance. Each repeater node consists of two SPDC sources, two QMs, and a BSM station used for entanglement swapping. Furthermore, we place a BSM station halfway between repeater nodes used for entanglement generation. After the long-distance entanglement has been established in the two chains, we postselect to get one purified entangled pair. We will detail the exact operations for entanglement generation, entanglement swapping, and readout below.

We will in this study examine two main driving fields for operating the sources: driving the SPDC sources using short Gaussian pulses and driving the SPDC sources continuously. The driving fields will give rise to an interaction strength χ with the time dependence of the driving field. In our previous work [35], we studied these two methods and examined the multimodality of the heralded state. In Appendix A, the model of the source is summarized.

When employing Gaussian pulses, which are narrow compared to the cavity decay time $1/\kappa$, to drive the SPDC, the state in the signal beam is heralded upon detection in the idler beam. In this case, the main parameter controlling the multimode nature of the source is the width of the Gaussian pulse σ , and we found that both the single-photon purity and the multiphoton emission probability decrease when increasing σ . On the other hand, when the source is driven continuously, a detection in the idler beam at t_c heralds a photon in the signal beam around this time. We, thus, set an acceptance window T symmetrically around t_c to capture the signal photon. In this case, increasing T increases the probability to have any number of photons, i.e., both the efficiency of extracting single photons and the erroneous probability to collect multiple photons. Finally, if we have captured multiple

photons, the probability that the photons occupy orthogonal modes increases as T increases.

A. Entanglement generation

The entanglement generation process is sketched in Figs. 1(a) and 1(b) for pulsed and continuous driving, respectively. To generate entanglement, two SPDC sources in neighboring nodes are driven. We assume that the source is driven sufficiently weak to only emit up to two photon pairs. The full multimode state is thus given by [35]

$$|\Psi_A\rangle = \left(\sqrt{P_0} + \sqrt{P_1} \sum_{\ell} \sqrt{w_{\ell}} a_{\ell,s}^{\dagger} a_{\ell,i}^{\dagger} + P_1 \sum_{k\ell} \frac{\sqrt{w_i w_j}}{2} a_{\ell,s}^{\dagger} a_{k,s}^{\dagger} a_{\ell,i}^{\dagger} a_{k,i}^{\dagger} \right) |\emptyset\rangle, \quad (1)$$

where A denotes the SPDC source and ℓ denotes a specific mode of the eigendecomposition with weight w_{ℓ} , which can be calculated from the drive $\chi(t)$. $a_{\ell,s}$ ($a_{\ell,i}$) is the annihilation operator of the signal (idler) photon of mode ℓ , and P_1 is the probability of the source emitting one photon pair.

The output from each source is split using a dichroic mirror, such that the light in the signal beam is stored in the QM, while the light in the idler beam is sent over a long distance to the central BSM station. Here, the beams from each node are mixed on a beam splitter, before the beams impinge on single-photon detectors. A detection in any of the two detectors heralds the entanglement between the light in the QM in each node:

$$\rho_{0,AB|t_c} = \frac{\text{Tr}_i[b_{+i}(t_c)|\Psi_A\rangle\langle\Psi_A||\Psi_B\rangle\langle\Psi_B|b_{+i}^{\dagger}(t_c)]}{\langle b_{+i}^{\dagger}(t_c)b_{+i}(t_c)\rangle}, \quad (2)$$

where $\text{Tr}_i[\cdot]$ is the trace over idler photons from both memories, $b_{\pm i}(t_c) = (a_i(t_c) \pm b_i(t_c))/\sqrt{2}$ are the annihilation operators associated with each detector, and we have assumed the detection to be in the “+” mode. The detection is modeled using annihilation operators. While this treatment is strictly speaking incompatible with measurement theory (see, eg., Ref. [36]), it is a good assumption when the state suffers from high losses, which is the case for the long transmission between repeater node and BSM station. Above, we have assumed no dark count events in the detectors. A dark count event would result in the storage of a vacuum state in the QMs, instead of the density matrix given by the equation above. We will discuss the impact of dark counts further in Sec. II C.

For the pulsed case, we will average over all detection times

$$\rho_{0,AB}^{\text{pulsed}} = \frac{\int_{\mathbb{R}} dt_c \langle b_{+i}^{\dagger}(t_c)b_{+i}(t_c)\rangle \rho_{0,AB|t_c}}{\int_{\mathbb{R}} dt \langle b_{+i}^{\dagger}(t)b_{+i}(t)\rangle}. \quad (3)$$

When employing continuous driving, we will introduce an acceptance window chosen symmetrically around t_c and disregard all photons outside the acceptance window, as illustrated

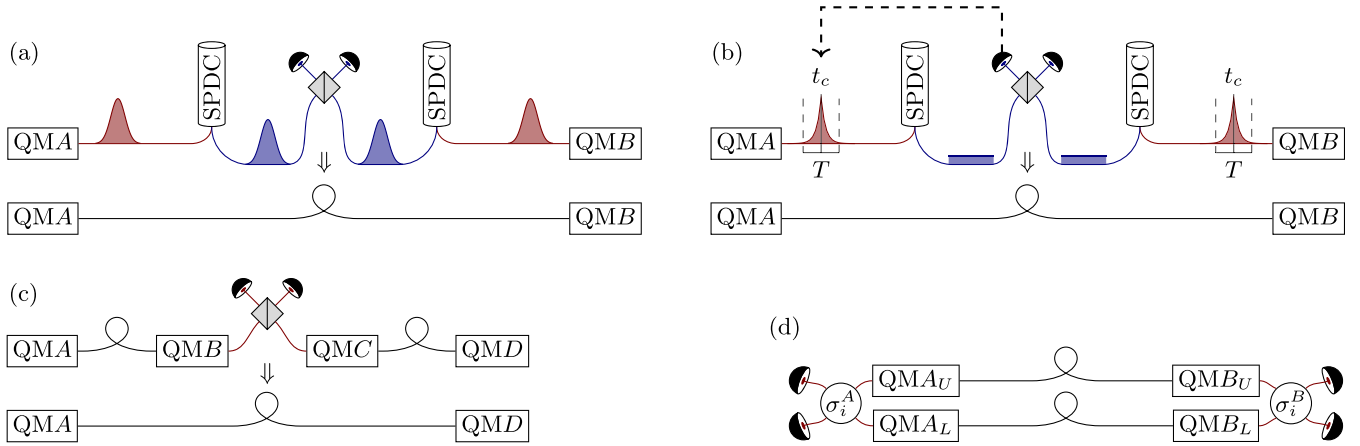


FIG. 1. Schematics of the entanglement generation for pulsed driving [panel (a)] and continuous driving [panel (b)]. The idler photons (blue lines) from both SPDC sources are sent to a central BSM station, whereas the signal photons (red lines) are sent to the respective QM. A detection will herald the entanglement between the light stored in the two QMs. In panel (b), we indicate the postprocessing, where a click has been registered at t_c , and the acceptance window T indicates how much of the light associated with that detection will be included later. (c) Schematic of an entanglement swap. Two neighboring pairs of entangled QMs have been generated. The photons are read out from the central QMs and sent to a BSM based on photon-number-resolving (PNR) detectors. The detection of a single photon swaps the entanglement to the outer QMs. (d) Schematic of the final readout and postselection. Light from the four remaining QMs is read out. At both ends, a Pauli operator acts on the output light before being read by PNR detector.

in Fig. 1(b). We thus have the heralded state

$$\rho_{0,AB|t_c,T}^{\text{cont}} = \text{Tr}_{-T}[\rho_{0,AB|t_c}], \quad (4)$$

where $\text{Tr}_{-T}[\cdot]$ denotes the trace over times outside the interval $[t_c - T/2, t_c + T/2]$.

The probability to successfully generate the elementary link in one attempt, $\mathbf{P}_0$, is given by

$$\mathbf{P}_0 = 2\eta_{\text{LD}} \int_{\mathcal{T}} dt_c \langle b_+^\dagger(t_c) b_+(t_c) \rangle, \quad (5)$$

where $\mathcal{T}$ is the total time looking for photons and $\eta_{\text{LD}} = \eta_d \exp(-L_0/2L_{\text{att}})$ is the efficiency from the node to the heralding station, with η_d the detection efficiency and L_{att} the attenuation length, set to 22 km. For the pulsed driving, we will let $\mathcal{T}$ cover the whole time range; hence, the success probability reduces to

$$\mathbf{P}_0^{\text{pulsed}} = 2\eta_{\text{LD}} P_1 [1 + P_1 (1 + \varpi_1)], \quad (6)$$

where ϖ_1 is the single photon purity, i.e., $\varpi_1 = \sum_i w_i^2$.

For continuous driving, we require that the acceptance window is much shorter than the time spent looking for photons, i.e., $T \ll \mathcal{T}$. Here, the success probability can be calculated directly from the Bogoluibov transformation [35]

$$\mathbf{P}_0^{\text{cont}} = 4\eta_{\text{LD}} \chi^2 \left(1 + 4 \frac{\chi^2}{\kappa^2} \right) \mathcal{T}, \quad (7)$$

where κ is the full width at half maximum (FWHM) linewidth of the cavity around the SPDC source.

Once we obtain a successful heralding detection, the stored photon is entangled between the two memories. We will assume that the source is driven sufficiently weak to only include terms in the density matrix up to linear order in the intensity of the pulse. This allows us to write the density

matrix for any swap depth n using a closed set of coefficients:

$$\begin{aligned} \rho_{n,AB} = & c_{00,n} \rho_{00} + c_{10,n} \rho_{10} + c_{11,n} \rho_{11} \\ & + c_{20,n} \rho_{20} + \rho_{1001,n} + \rho_{2011,n}, \end{aligned} \quad (8)$$

where ρ_{kl} is the normalized diagonal density matrix element with k photons in memory A and l photons in memory B , $c_{kl,n}$ is the coefficient associated with the diagonal density matrix element ρ_{kl} at swap depth n , and $\rho_{ijkl,n}$ is the off-diagonal element in matrix between i photons in memory A and j photons in memory B and k photons in the memory A and l photons in memory B at swap depth n .

B. Entanglement swap

When two neighboring entanglements of the same swap depth, n , have successfully been established, we attempt entanglement swapping to reach swap depth $n + 1$. The entanglement swapping is sketched in Fig. 1(c). The light stored in QMB and QMC is read out with a combined memory efficiency η_m . In reality, the readout process may alter the temporal wave function, as discussed in Ref. [14]. This can be included by modifying the mode functions of the stored light. Since this will likely only have a minor effect on the purity and is highly dependent on the exact memory characteristics, we will for simplicity ignore this effect.

This light is then sent to a BSM, with number-resolving single-photon detectors with efficiency η_d . We can thus introduce a single loss channel, such that the annihilation operators of the stored light undergo the transformation $b_s(t) \rightarrow \sqrt{\eta_m \eta_d} b_d(t) + \sqrt{1 - \eta_m \eta_d} b_\perp(t)$, where b_d is the annihilation operator associated with the detector channel from QMB and $b_\perp$ is the annihilation operator of the loss channel. The light from the quantum memories B and C is mixed evenly on a beam splitter before the detectors, such that $b_{\pm d}(t) = (b_d(t) \pm c_d(t))/\sqrt{2}$. A successful heralding occurs when one

of the two detectors finds exactly one photon, yielding the transformations

$$\rho_{n+1,AD} = \frac{\int dt_s \langle \emptyset_{b_{\pm d}} | b_+(t_s) \text{Tr}_{\perp} [\rho_{n,AB} \otimes \rho_{n,CD}] b_+(t_s) | \emptyset_{b_{\pm d}} \rangle}{\int dt'_s \text{Tr} [\langle b_+(t'_s) \rho_{n,AB} \otimes \rho_{n,CD} b_+^\dagger(t'_s) | \emptyset_{b_{\pm d}} \rangle]} \quad (9)$$

where $|\emptyset_{b_{\pm d}}\rangle$ is the state with no photons in the detector channels. We have chosen the single-click entanglement swapping protocol of the DLCZ scheme, as this is easier to implement and yields the highest rate at low swap depths. Furthermore, this assumption allows us to investigate the influence of the imperfections of the source for the simplest possible scenario.

The transformation of ρ_n to ρ_{n+1} yields a set of recursive equations for the coefficients in Eq. (8). The initial conditions are obtained by combining Eq. (2) with Eq. (3) when using short pulses and Eq. (4) when using continuous driving. In Appendixes B and D, we give the full description of the density matrix for pulsed and continuous driving and express the recursive equations directly.

The probability of a successful entanglement swap is

$$\mathbf{P}_{n+1} = 2\eta_m \eta_d \left[1 - \eta_m \eta_d \left(\frac{c_{10,n}}{2} + c_{11,n} + c_{20,n} \right) + \eta_m^2 \eta_d^2 \frac{c_{20,n}}{2} \right] \times \left[\frac{c_{10,n}}{2} + c_{11,n} + (1 - \eta_m \eta_d) c_{20,n} \right]. \quad (10)$$

C. Final readout and postselection

We sketch the final readout in Fig. 1(d). After creating two entangled pairs between the end nodes, the light from all four remaining QMs is retrieved. We then measure the light at each end in one of the different Pauli bases and postselect on a photon being measured at each end. This postselection suppresses the unwanted vacuum component $[c_{00,n}]$ in Eq. (8) unless mixed with the unlikely two-photon component $[c_{11,n}]$ in Eq. (8). By setting a sufficiently low driving strength in the initial entanglement generation, we can thus obtain a high-fidelity entangled state. Furthermore, since detector dark counts lead to an increase in the population of the vacuum component as discussed above, the postselection has the additional benefit that the infidelity caused by dark counts is suppressed. In the end, these effects will effectively limit the possible driving power, after introducing a target fidelity below [23].

The probability of a successful postselection can be found from the traces of the density matrix

$$\mathbf{P}_{\text{PS}|n} = 2\eta_m^2 \eta_d^2 \left\{ c_{11,n} (1 - \eta_m \eta_d c_{10,n}) + \frac{c_{10,n}}{4} [c_{10,n} + 4(1 - \eta_m \eta_d)(c_{11,n} + c_{20,n})] \right\}, \quad (11)$$

where we have ignored higher terms that do not contribute to the linear order in the driving intensity.

D. Fidelity and rate

The desired state of the readout photons after postselection is $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, in the basis of left side and right side using the notation from above and where $|0\rangle$ ($|1\rangle$) represents a photon from the upper (lower) chain. Assuming

that we get a detection pattern yielding the plus sign in the entanglement generation, we obtain the fidelity of final state

$$F = \frac{(\langle 01| + \langle 10|) \tilde{\rho}_n \otimes \tilde{\rho}_n (|01\rangle + |10\rangle)}{2\mathbf{P}_{\text{PS}|n}} = \frac{1 - \langle \sigma_z^A \sigma_z^B \rangle + \langle \sigma_x^A \sigma_x^B \rangle + \langle \sigma_y^A \sigma_y^B \rangle}{4\mathbf{P}_{\text{PS}|n}}, \quad (12)$$

where the second equality shows how to find the fidelity from measurements of the Pauli operators, $\tilde{\rho}_n$ is ρ_n after readout of all stored photons, and σ_i^A (σ_i^B) is the Pauli- i operator on the left (right) side, defined as

$$\begin{aligned} \sigma_x &= \sum_{\ell} (|1_{\ell}\emptyset\rangle \langle \emptyset 1_{\ell}| + |\emptyset 1_{\ell}\rangle \langle 1_{\ell}\emptyset|), \\ \sigma_y &= -i \sum_{\ell} (|1_{\ell}\emptyset\rangle \langle \emptyset 1_{\ell}| - |\emptyset 1_{\ell}\rangle \langle 1_{\ell}\emptyset|), \\ \sigma_z &= \sum_{\ell} (|1_{\ell}\emptyset\rangle \langle 1_{\ell}\emptyset| - |\emptyset 1_{\ell}\rangle \langle \emptyset 1_{\ell}|), \end{aligned} \quad (13)$$

in the basis where the first (second) index describes photons from the upper (lower) memory. As the coefficients in the density matrix do not depend on the distance directly, the fidelity is independent of the communication distance, if we assume the same swap depth. This is true for any entanglement-based quantum repeater. However, other imperfections, especially dark counts, can introduce distance-dependent infidelity. Furthermore, as the distance is increased, additional swaps are required to combat the exponential decay in communication rate, and this will degrade the fidelity.

The success probabilities for the entanglement generation, entanglement swap, and postselection can be combined into a rate of end-to-end entanglement generation $\tilde{R}$ per. memory without multiplexing [10]:

$$\tilde{R}_n = \frac{c}{L_0} \frac{1}{2^{n+2}} \frac{\mathbf{P}_0 \mathbf{P}_1 \cdots \mathbf{P}_n \mathbf{P}_{\text{PS}}}{(3/2)^{n+1}}, \quad (14)$$

where c is the speed of light in the fiber, and 2^{n+2} is the number of memories, and the $(3/2)^{n+1}$ factor accounts for the establishment of pairs in parallel of entangled memories at all swap depths. This rate is in the absence of multiplexing, and can be used for comparisons between swap depths. However, to achieve higher rates of quantum communication, it will require the inclusion of multiplexing. We include multiplexing in our discussion in Sec. V.

In the limit of vanishing driving power and, e.g., infinitely short pulse for the pulses, the desired state will be obtained with unit fidelity at zero rate. Increasing the driving intensity decreases the fidelity but increases the rate. We can thus set a target fidelity, which will yield the driving power and the total end-to-end entanglement generation time. We will now do this for the two cases separately.

III. PULSED DRIVING

In this section, we investigate the results when driving the SPDC source with a narrow pulse. The SPDC source is embedded in a cavity, and we assume that the pulse will make a single pass through the SPDC. For simplicity, we will assume a Gaussian drive; i.e., the interaction between the driving

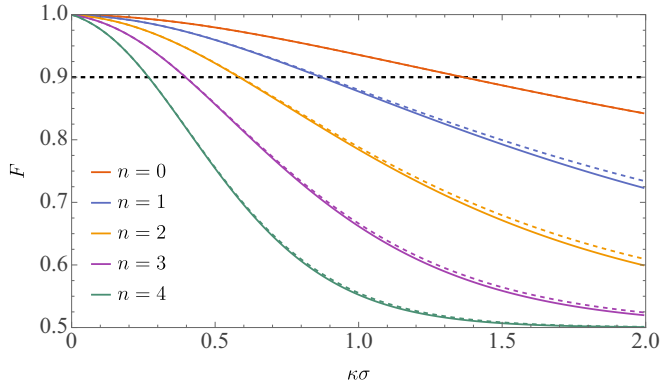


FIG. 2. Fidelity due to the multimode nature. Full lines show the complete model, and dashed lines show the approximated version, from Eq. (15). The horizontal dashed line shows 90% fidelity, which is the target set for later. The lines go in descending order in n when going up the F axis.

pulse and the nonlinear medium χ has the time dependence $\chi(t) = x/(\sqrt{2\pi}\sigma)\exp(-t^2/2\sigma^2)$, where x is a dimensionless characterization of the driving strength. In Appendix B, we give a detailed description of the density matrix in this scenario.

As described above, the multimodality of output state from the SPDC source depends on the width of the Gaussian pulse σ employed. An impure initial state will lead to decreasing coherence after each swap. Conclusively, some infidelity is introduced even to zeroth order in the driving power

$$F = \frac{1 + \sum_{\ell} w_{\ell}^{2^{n+1}}}{2} \approx \frac{1 + \varpi_1^{2^n}}{2}, \quad (15)$$

where w_{ℓ} are the weights of the single-photon mode decomposition and ϖ_1 is the single-photon purity defined as $\sum_{\ell} w_{\ell}^2$. This approximation is equivalent to assuming that the coherence terms are multiplied by a factor of ϖ_1 for each entanglement swap and in the postselection. Using the results from Ref. [35], we can find the eigenmode decomposition of the emitted light using numerical methods.

In Fig. 2, we plot the baseline fidelity, defined in the limit as $P_1 \rightarrow 0$, against different pulse widths σ . In the small σ limit, where the emitted state is pure, the impact to fidelity is negligible. However, as σ increases, the underlying state becomes more impure and thus affects the fidelity of the final state. This is especially damaging for higher swap depth, as more swaps have to be performed. Furthermore, we note that the approximation of Eq. (15) corresponds very closely to the actual values.

A target fidelity will thus effectively set an upper limit on the possible swap depth for a given pulse width, as the fidelity will only further decrease for a nonvanishing P_1 . We can translate the target fidelity into a target value of P_1 given n and σ . In Fig. 3, we plot the obtained value of P_1 using a target fidelity of 90% for the different swap depths and pulse widths. However, a narrow pulse with a large value of P_1 will result in high peak driving intensities, which may represent a challenge for experiments. To investigate this, we set a maximal power for the experiments $I_{\max} = aI_{\text{thres}}$, where I_{thres} is the threshold power at which point the output of the SPDC

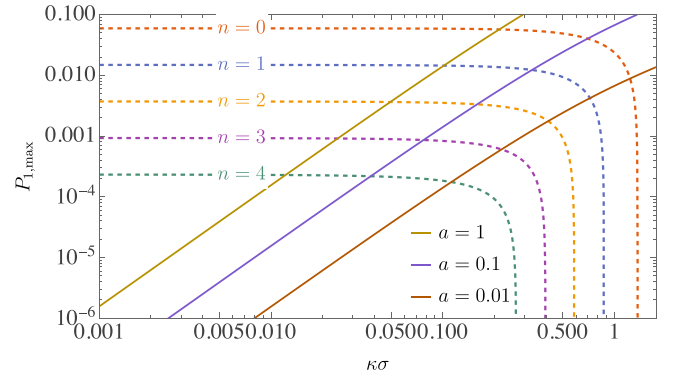


FIG. 3. The full lines show the maximal value P_1 can attain, if the maximal intensity of the laser is $I_{\max} = aI_{\text{thres}}$. The dashed lines show the target P_1 for different swap depths using a target fidelity of 90%. With a source limited by a certain value of a , the ideal pulse width is found by the crossing of the dashed and full lines. We have used $\eta_d = 0.9$ and $\eta_m = 0.8$. The full lines go in ascending order in a when going up the $P_{1,\max}$ axis.

source becomes exponentially enhanced (which corresponds to $\chi = \kappa/2$; see Appendix A for details). We translate the maximal driving intensity into a value of P_1 [to the lowest order in $\chi(t)$] obtaining

$$P_{1,\max} = a \frac{\pi \kappa^2 \sigma^2}{2} e^{\kappa^2 \sigma^2} [1 - \text{erf}(\kappa \sigma)], \quad (16)$$

where the detailed calculation is left for Appendix A 1. Higher values of a allow for more narrow pulses, and thus a higher possible value of P_1 . We plot $P_{1,\max}$ for three different values of a . Since the rate increases with P_1 , we choose the crossing of the lines set by the target fidelity and Eq. (16). We report these pulse widths in Table I for each value of a .

TABLE I. For each combination of swap depth and maximally allowed intensities, using $I_{\max} = aI_{\text{thres}}$, we show the optimal value for the pulse width $\kappa \sigma^*$ and the range of distances above 100 km and up to $n = 4$ where a particular swap depth yields the highest rate. For $a \rightarrow \infty$, the ideal pulse width is $\sigma \rightarrow 0$. We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, and $L_{\text{att}} = 22$ km.

a	0.01		0.1	
	$\kappa \sigma^*$	L (km)	$\kappa \sigma^*$	L (km)
n				
0	1.24	100–105	0.70	100–124
1	0.72	105–252	0.33	124–291
2	0.41	252–627	0.16	291–678
3	0.22	627–1512	0.08	678–1563
4	0.11	1512–	0.04	1563–
a	1		∞	
n	$\kappa \sigma^*$	L (km)	L (km)	
0	0.21	100–133	100–133	
1	0.10	133–298	133–299	
2	0.05	298–683	299–684	
3	0.02	683–1568	684–1568	
4	0.01	1568–	1568–	

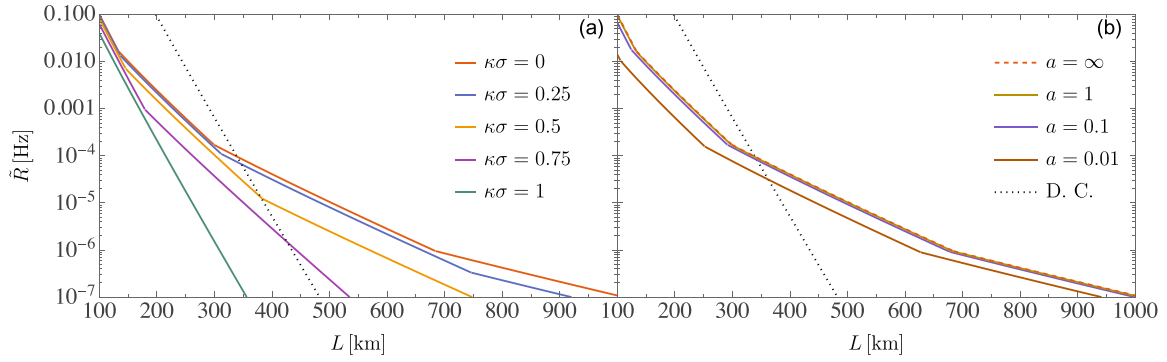


FIG. 4. Unmultiplexed rates per quantum memory for (a) fixed pulse widths σ and (b) fixed maximally allowed laser intensity, $a = I_{\max}/I_{\text{thres}}$. Note that $a = \infty$ in panel (b) corresponds to $\kappa\sigma = 0$ in panel (a). We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, $c = 2 \times 10^5$ km/s, and $L_{\text{att}} = 22$ km. For panel (a) [panel (b)], the lines go in descending (ascending) order in $\kappa\sigma$ (a) when going up the $\tilde{R}$ axis. The black dotted line is a comparison to a repeaterless two-click entanglement protocol with idealized sources (called D. C. for Direct Communication).

We should note that for very short drive pulses the output single photon mode is a decaying exponential in time [35]. When employing cavity-assisted quantum memories, a photon with this mode shape has been shown to be particularly difficult to store [37]. The influence of the mode shape on the memory efficiency has not been included in the present study, as this effect will depend on the exact experimental details.

We can now calculate the rates for different transmission lengths and different swap depths. To compare to direct transmission for a single mode, we also consider a repeaterless two-click entanglement protocol with an unmultiplexed rate of $\eta_{\text{LD}}^2 c/L$. The rate is first maximized over the different swap depths after fixing the pulse width. The result is plotted in Fig. 4(a) for a few values of σ . We see that slower rates are obtained when increasing the pulse width, which is ultimately caused by the increased impurity of the emitted state. Furthermore, we note that for some pulse widths only the lowest swap depths are allowed. In Fig. 4(b), we fix the maximally allowed intensity and optimize the rate for a given length. The optimal pulse width σ^* , i.e., the lowest allowed pulse width for a given swap depth, is chosen according to Fig. 3. We see, unsurprisingly, that a lower allowed intensity yields a lower rate. However, the negative impact of increasing the swap depth is now smaller, as the pulse width can be decreased yielding a higher purity initial state, which decreases the impact on P_1 . Furthermore, we observe that even a modest value of $a = 1$ has essentially no impact on the obtainable rate. The influence of limited driving power is particularly small at long distances since the DLCZ protocol in itself requires very weak driving for long distances. We report in Table I the range of distances where each swap depth yields the highest rate given a .

While the plots and numbers provided are based on a Gaussian pulse drive, we include a brief analysis employing other pulse shapes in Appendix C. We find that the broad trends are similar as for the Gaussian pulses. Narrow pulse shapes lead to approximately single-mode behavior, while broader pulses lead to multiple modes and a faster deterioration of the fidelity of the state. If we fix the peak laser intensity, we find that the allowed values of P_1 to employ are virtually identical across the different pulse shapes. We thus conclude that the shape of the pulses has a very limited effect.

IV. CONTINUOUS DRIVING

In this section, the impacts of driving the SPDC source with a continuous drive are investigated. For the entanglement generation, we will detect a heralding click at time t_c . We will only store photons inside the memory, if they are temporally close to the heralding click t_c . This is set by the length of the acceptance window T . T needs to be chosen sufficiently long to allow the entire pulse to be extracted, but at the same time sufficiently short to avoid additional clicks from pairs generated at different times.

We will write the driving strength as $\chi(t) = x\kappa/2$, where x is dimensionless. The heralded output state is pure to the lowest order in the driving intensity, but becomes multimode to first order. The detailed calculations describing the state at different swap depths are left for Appendix D.

The fidelity and unmultiplexed rate are plotted as functions of the laser intensity in Fig. 5. From the fidelity, we see that a higher swap depth leads to more stringent demands on the laser intensity due to higher impact of mode impurities. As expected, the rate increases with higher intensities. Since we want to obtain both a high rate and fidelity, we thus need to balance these effects.

We find the highest allowed laser intensity to obtain a target fidelity of 90% and plot it in Fig. 6. This laser power is then used to find the unmultiplexed rate per QM. We can factor the rate as $\tilde{R}(n, L, T) = f(n, L)g(n, T)$; hence, it is possible to find the optimal acceptance interval T^* for the different swap depths regardless of the total distance. This is shown in the inset of Fig. 7. Using this optimal acceptance interval for the different swap depths allows us to plot the unmultiplexed rate for the different swap depths over different combined distances, which is shown in Fig. 7. As expected when increasing the distance, introducing entanglement swaps becomes advantageous. We summarize the results in Table II.

V. CONCLUSION AND DISCUSSION

We have investigated the influence of multimode nature of SPDC sources on a DLCZ-type repeater. The impurity of the photon sources is shown to impact the possibly achievable

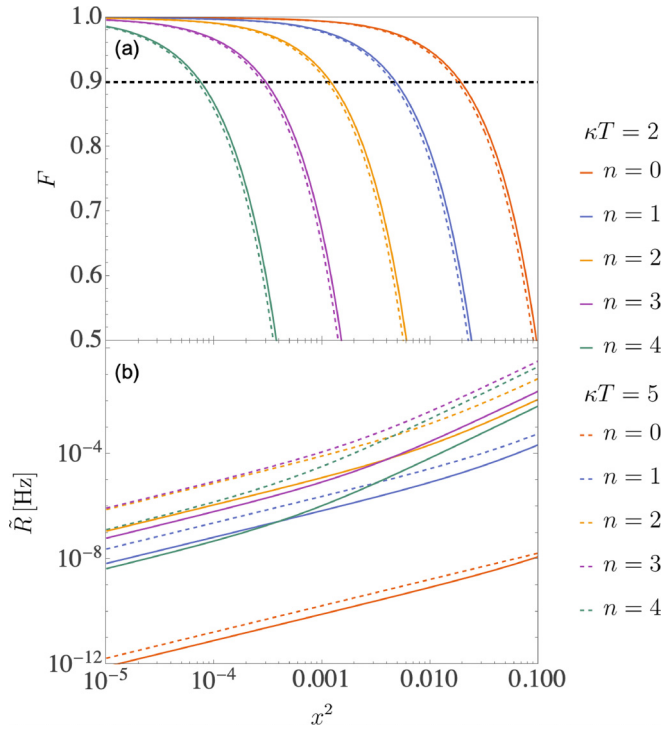


FIG. 5. Fidelity and rate for continuously driven SPDC sources. Full (dashed) lines show the fidelity [panel (a)] and unmultiplexed rate per. quantum memory [panel (b)] for different swap depths at different laser intensities (quantified with $x = 1$ when the laser is at threshold intensity) for $\kappa T = 2$ ($\kappa T = 5$). The dashed black line in panel (a) shows 90% fidelity, which is used as target fidelity. We have used $\eta_d = 0.9$, $\eta_m = 0.8$, $c = 2 \times 10^5$ km/s, $L_{\text{att}} = 22$ km, $\kappa T = 100$, and $L = 500$ km. For panel (a), lines go in descending order in n . For panel (b), the full (dashed) lines go in order 0, 4, 1, 3, and 2 (0, 1, 4, 2, and 3) at $x^2 = 10^{-5}$.

rates in both cases considered in this study. The purity of the output state from the source leads to a deterioration of the fidelity of the final entangled pair the first entanglement swap. This leads to increasingly stringent limitations on the intensity of the driving pulse when adding more swaps.

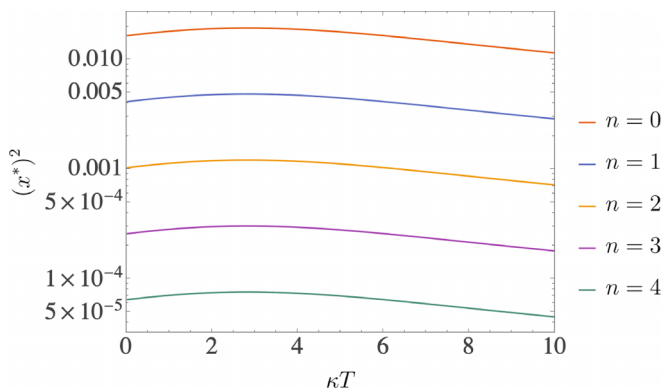


FIG. 6. Optimal laser intensity for obtaining 90% fidelity for continuous driving and different swap depths. We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, $c = 2 \times 10^5$ km/s, and $L_{\text{att}} = 22$ km. The lines go in descending order in n , as one moves up the $(x^*)^2$ axis.

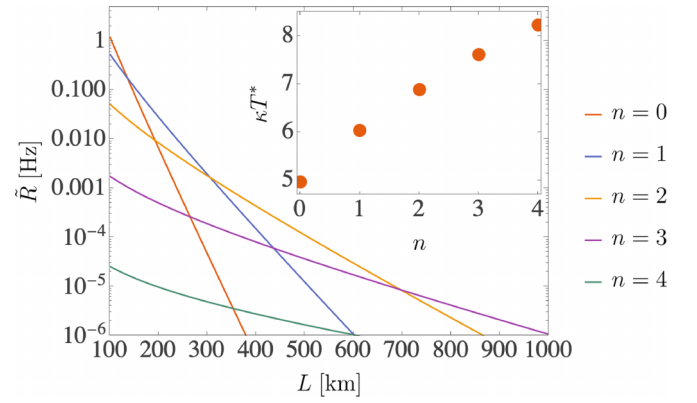


FIG. 7. Rate per mode for continuous driving for the different swap depths using the optimal interval T^* . In the inset, the optimal value for the interval is shown. We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, $c = 2 \times 10^5$ km/s, $L_{\text{att}} = 22$ km, and $\kappa T = 100$. The lines go in descending order in n , as one moves up the $\tilde{R}$ axis at $L = 100$ km.

When driving the source using a short pulse, the purity of the output state decreases when increasing the width of the driving pulse. Hence, when using longer pulses the constraints on the laser power become increasingly demanding when increasing the number of swaps. This ultimately leads to swap depths becoming unable to sustain a high fidelity of entangled pairs between the two ends. As a consequence, using long pulses in a DLCZ-type repeater has a limited advantage over direct transmission. The advantage is much clearer when employing narrow pulses, where high swap depths are possible. We also considered constraining the maximally allowed peak laser power. When increasing the swap depth, where the total laser power has to be decreased to avoid errors from two-photon emissions, we can decrease the pulse width to obtain a more pure state. We find that even for realistic limitations on the laser power, we can operate closely to the ideal single-mode case.

When using continuous driving, we similarly see an impact of the full multimode nature of the SPDC source. To the lowest order in the driving strength, the light from the source is in a pure single-photon state. However, as the laser power increases, the purity of the emitted state decreases due to multiple photons being emitted into other modes. The exact nature of the impurity and the probability of successfully swapping the entanglement depend on the size of the acceptance

TABLE II. The swap depths n with the associated optimal acceptance interval T^* and the range in which this swap depth delivers the highest rate (above 100 km and up to $n = 4$). We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, and $L_{\text{att}} = 22$ km.

n	κT^*	$(x^*)^2$	L (km)
0	5.0	2.0×10^{-2}	100–136
1	6.0	4.5×10^{-3}	136–306
2	6.9	1.0×10^{-3}	306–696
3	7.6	2.3×10^{-4}	696–1588
4	8.2	5.4×10^{-4}	1588 –

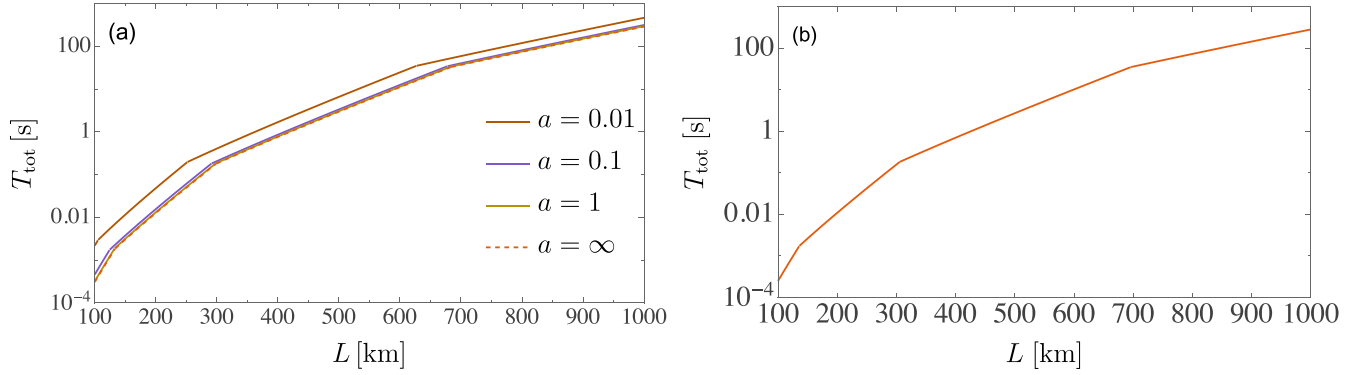


FIG. 8. Time to generate entangled pair when including multiplexing for pulsed [panel (a)] and CW [panel (b)] driving of the SPDC. For both cases, we have set 10 frequency modes and 10 spatial modes for multiplexing, and employ 32 memories. For the pulsed case, we furthermore consider 10 temporal multiplexing modes, while we have set $\kappa\mathcal{T} = 100$ for CW. We stress that one should not compare the numbers directly between these two graphs, as it depends heavily on the specific details of each configuration number used in the two scenarios. We have used a target fidelity of 90%, $\eta_d = 0.9$, $\eta_m = 0.8$, $c = 2 \times 10^5$ km/s, and $L_{\text{att}} = 22$ km. For panel (a), the lines correspond to descending value of a as one goes up the T_{tot} axis.

window. Using smaller acceptance windows limits the probability of getting successful swaps; however, any light stored in the QM will be in approximately pure state. Longer acceptance windows lead to higher chances of successfully swapping the entanglement, however, the state might be more impure due to both multiple photons and these photons occupying multiple modes. Both these effects lead to a decrease in fidelity.

As highlighted in the beginning, the great advantage of atomic ensemble memories is their multiplexing abilities, allowing the rate to be scaled up dramatically. We can thus find the time to generate entanglement with multiplexing through $T_{\text{tot}} = (N_{\text{mm}}N_{\text{mem}}\bar{R})^{-1}$, where N_{mm} is the number of multiplexing modes per. memory and N_{mem} is the number of memories used. We thus treat multiplexing as a set of parallel channels, specifically frequency and spatial for both driving methods, and temporal for pulsed driving. The entanglement is generated in one of these channels at random, but when performing a swap, the light from one of the memories is transferred to the same channel as the light from the other memory. A more detailed analysis of how to treat multiplexing for quantum repeaters is considered in Ref. [24]. The time to generate entanglement including multiplexing is plotted in Fig. 8, where we have used 10 frequency, 10 spatial, and 10 temporal multiplexing modes for the pulsed case, and 10 frequency and 10 spatial multiplexing modes with $\kappa\mathcal{T} = 100$ for the continuous case, and we employ 32 memories in both cases. We see that entanglement generation times around a second are possible at a distance of 500 km to achieve with optimistic near-future efficiencies.

We emphasize that these results are obtained for original DLCZ-repeater scheme. Multiple schemes have been developed since the original proposal, which can achieve higher rates. Concurrently, other sources of entangled photons have been proposed, which could be incorporated into the DLCZ network. In the future, it would be interesting to perform similar investigations of these more advanced protocols and include other sources. We believe that the results will be qualitatively similar, but the details will likely be different.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [38].

APPENDIX A: BRIEF SUMMARY OF THE MODEL OF THE SOURCES

We present a thorough investigation of SPDC sources in Ref [35]. In this Appendix, we briefly summarize how the sources are modeled.

We consider a nonlinear crystal embedded in a cavity with an FWHM linewidth κ . A classical driving field is sent onto the crystal and we consider it to be unaffected by the cavity. This field will interact with the nonlinear crystal according to the Hamiltonian

$$H = \hbar\chi(t)(a_c b_c + a_c^\dagger b_c^\dagger), \quad (\text{A1})$$

where $\chi(t)$ is the interaction strength between the field and the nonlinear crystal with the time dependence of the driving field and a_c (b_c) is the annihilation operator of the signal (idler) cavity mode. The cavity modes described by a_c and b_c leak out into the output signal and idler fields $a(t)$ and $b(t)$. This leads to the input-output relations for the annihilation operators (here shown for the signal field)

$$\dot{a}_c = -\frac{\kappa}{2}a_c + \sqrt{\kappa}a_{\text{in}}(t) - i\chi(t)b_c^\dagger, \quad (\text{A2a})$$

$$a(t) = a_{\text{in}}(t) - \sqrt{\kappa}a_c, \quad (\text{A2b})$$

where $a_{\text{in}}(t)$ is the annihilation operator of the input operator. The differential equations can then be solved for a given driving field, and thus a given $\chi(t)$, since $\chi(t)$ is proportional to the incident field. The input-output relations reveal that if $\chi(t) > \kappa/2$, then the cavity field will increase exponentially. We thus define the threshold intensity as the incident intensity required to reach the threshold and normalize all incident intensities I to the threshold intensity, i.e., $I/I_{\text{thres}} = 4\chi(t)^2/\kappa^2$.

P_1 at threshold calculation

We will in this Appendix use the above parametrization to relate P_1 to the incident intensity relative to the threshold intensity. We recall the driving strength

$$\chi(t) = \frac{x}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}, \quad (\text{A3})$$

where x parametrizes the interaction strength. To find P_1 to the lowest order in x , we have to compute $\int_{\mathbb{R}} dt \langle a^\dagger(t)a(t) \rangle$ to the lowest order in x . With a time-dependent interaction, the probability P_1 is given by

$$P_1 = \int_{\mathbb{R}^2} dt d\tau \chi(t)\chi(\tau) e^{-\kappa|t-\tau|}. \quad (\text{A4})$$

Inserting the Gaussian form of $\chi(t)$ and rewriting to make the inner integral up to t , the integral can be expressed in terms of an integral of an error function

$$P_1 = \frac{x^2}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} dt e^{-\left(\frac{t^2}{2\sigma^2} + \kappa t - \frac{\kappa^2\sigma^2}{2}\right)} \times \left[\text{erf}\left(\frac{t}{\sqrt{2\sigma^2}} - \frac{\kappa\sigma}{\sqrt{2}}\right) + 1 \right]. \quad (\text{A5})$$

This integral can be found in Ref. [39], leading to the result

$$P_1 = x^2 e^{\kappa^2\sigma^2} [1 - \text{erf}(\kappa\sigma)]. \quad (\text{A6})$$

We now define $a = I_{\text{peak}}/I_{\text{thres}} = 2x^2/\pi\sigma^2\kappa^2$; we can solve for x and obtain a value for P_1 :

$$P_{1,\text{max}} = a \frac{\pi\kappa^2\sigma^2}{2} e^{\kappa^2\sigma^2} [1 - \text{erf}(\kappa\sigma)]. \quad (\text{A7})$$

These are the lines plotted in Fig. 3.

APPENDIX B: DENSITY MATRIX FOR THE PULSED CASE

We will in this Appendix consider employing a pulsed driving field and describe the full density matrix, the recursive relations for swapping, and their closed-form solutions. We start with the general expression of the density matrix for the pulsed case:

$$\begin{aligned} \rho_n = & c_{00,n} |\emptyset\rangle\langle\emptyset| |\emptyset\rangle\langle\emptyset| + \frac{c_{10,n}}{2} \sum_{\ell} w_{\ell} (|1_{\ell}\rangle\langle 1_{\ell}| |\emptyset\rangle\langle\emptyset| + |\emptyset\rangle\langle\emptyset| |1_{\ell}\rangle\langle 1_{\ell}|) + c_{11,n} \sum_{\ell,k} w_{\ell} w_k |1_{\ell}\rangle\langle 1_{\ell}| |1_k\rangle\langle 1_k| \\ & + \frac{c_{20,n}}{1+\varpi_1} \sum_{\ell,k \geq \ell} w_{\ell} w_k (|1_{\ell} 1_k\rangle\langle 1_{\ell} 1_k| |\emptyset\rangle\langle\emptyset| + |\emptyset\rangle\langle\emptyset| |1_{\ell} 1_k\rangle\langle 1_{\ell} 1_k|) + \frac{1}{2} \sum_{\ell} [A_{0,n} + A_{1,n} \Lambda_{\ell}] w_{\ell}^2 (|1_{\ell}\rangle\langle 1_{\ell}| |\emptyset\rangle\langle\emptyset| |1_{\ell}\rangle\langle 1_{\ell}| + \text{H.c.}) \\ & + \frac{B_{0,n}}{2} \sum_{\ell} w_{\ell}^2 \sqrt{\frac{2}{1+\varpi_1}} \left[\sqrt{2} w_{\ell} (|1_{\ell}\rangle\langle 1_{\ell}| |\emptyset\rangle\langle\emptyset| |1_{\ell}\rangle\langle 1_{\ell}| + |2_{\ell}\rangle\langle 2_{\ell}| |1_{\ell}\rangle\langle\emptyset| |\emptyset\rangle\langle 1_{\ell}|) \right. \\ & \left. + \sum_{k \neq \ell} w_k (|1_{\ell}\rangle\langle 1_{\ell}| |\emptyset\rangle\langle\emptyset| |1_k\rangle\langle 1_k| |1_{\ell}\rangle\langle 1_{\ell}| + |1_k\rangle\langle 1_k| |1_{\ell}\rangle\langle\emptyset| |\emptyset\rangle\langle 1_{\ell}|) + \text{H.c.} \right], \end{aligned} \quad (\text{B1})$$

where $\Lambda_{\ell} = \sqrt{\frac{2}{1+\varpi_1}} (1 + w_{\ell})$. The initial conditions for the coefficients are

$$c_{00,0} = 0, \quad c_{10,0} = \frac{1 - P_1}{1 + P_1(1 + \varpi_1)}, \quad c_{11,0} = \frac{P_1}{1 + P_1(1 + \varpi_1)}, \quad c_{20,0} = \frac{P_1(1 + \varpi_1)}{1 + P_1(1 + \varpi_1)}, \quad (\text{B2a})$$

$$A_{0,0} = \frac{1 - P_1}{1 + P_1(1 + \varpi_1)}, \quad A_{1,0} = 0, \quad B_{0,0} = \frac{P_1 \sqrt{(1 + \varpi_1)/2}}{1 + P_1(1 + \varpi_1)}. \quad (\text{B2b})$$

We can find the recursive equations for the density matrix after entanglement swapping from Eq. (9). We will, however, ignore the renormalization, i.e., the denominator, as the equations become much easier to combine. Hence, ρ_n is not normalized in the Appendix; instead the trace yields the product $\text{Tr}[\rho_n] = \prod_{i=0}^{n-1} (\mathbf{P}_{n-i}/2)^{2^i}$.

The recursive equations will form two sets of interconnected equations: one involving the diagonal elements, i.e., c_{00} , c_{10} , c_{11} , and c_{20} , and one involving the off-diagonal elements, i.e., A_0 , A_1 , and B_0 .

We start by considering the diagonal elements, and we obtain the recursive equations

$$c_{00,n+1} = \eta_2 \left[\frac{c_{10,n}}{2} + c_{20,n}(1 - \eta_2) \right] \left[c_{00,n} + \frac{c_{10,n}(1 - \eta_2)}{2} + \frac{c_{20,n}(1 - \eta_2)^2}{2} \right], \quad (\text{B3a})$$

$$c_{10,n+1} = \eta_2 \left\{ \left[\frac{c_{10,n}}{2} + c_{20,n}(1 - \eta_2) \right] \left[\frac{c_{10,n}}{2} + c_{11,n}(1 - \eta_2) \right] + c_{11,n} \left[c_{00,n} + \frac{c_{10,n}(1 - \eta_2)}{2} + \frac{c_{20,n}(1 - \eta_2)^2}{2} \right] \right\}, \quad (\text{B3b})$$

$$c_{11,n+1} = \eta_2 \frac{c_{10,n}}{2} c_{11,n}, \quad (\text{B3c})$$

$$c_{20,n+1} = \eta_2 \frac{c_{10,n}}{2} \frac{c_{20,n}}{2}, \quad (\text{B3d})$$

where $\eta_2 = \eta_m \eta_d$. To find closed forms for the recursive equations, we expand the coefficients and their recursive equations into orders of P_1 and solve each separately. Thus, we find closed forms of the equations up to first order in P_1 . As emissions of more than two photon pairs from the source are discarded, terms of the order P_1^2 will not be fully accounted for. Thus, our results will only be valid up to first order in P_1 , regardless of expansion here:

$$c_{00,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} (1 - \eta_2)(2^n - 1) \left\{ 1 - P_1 \left[(1 + \varpi_1)(1 - \eta_2) + 2^n \frac{4 - 3\varpi_1 + 2\eta_2 + 6\varpi_1\eta_2}{3} - 2^{2n+1} \frac{1 - \eta_2}{3} \right] \right\}, \quad (\text{B4a})$$

$$c_{10,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} \{ 1 - P_1 [2(1 + \varpi_1)(1 - \eta_2) + 2^n(2 - \varpi_1 + 2\varpi_1\eta_2) - 2^{2n+1}(1 - \eta_2)] \}, \quad (\text{B4b})$$

$$c_{11,n} = 2^n \left(\frac{\eta_2}{4}\right)^{2^n-1} P_1, \quad (\text{B4c})$$

$$c_{20,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} (1 + \varpi_1) P_1. \quad (\text{B4d})$$

We now do the same for the off-diagonal elements. The recursive equations are

$$A_{0,n+1} = \frac{\eta_2}{4} A_{0,n}^2, \quad (\text{B5a})$$

$$A_{1,n+1} = \frac{\eta_2}{2} A_{0,n} [A_{1,n} + (1 - \eta_2) B_{n,0}], \quad (\text{B5b})$$

$$B_{0,n+1} = \frac{\eta_2}{4} A_{0,n} B_{n,0}. \quad (\text{B5c})$$

These recursive equations can be solved directly. We will only give the answer expanded to first order of P_1 , as higher-order contributions (e.g., three-photon emissions) are not included in the original state:

$$A_{0,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} [1 - 2^n(2 + \varpi_1)P_1], \quad (\text{B6a})$$

$$A_{1,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} (2^n - 1)(1 - \eta_2) \sqrt{\frac{1 + \varpi_1}{2}} P_1, \quad (\text{B6b})$$

$$B_{0,n} = \left(\frac{\eta_2}{4}\right)^{2^n-1} \sqrt{\frac{1 + \varpi_1}{2}} P_1. \quad (\text{B6c})$$

All calculations here can be found in the code base provided in Ref. [38].

APPENDIX C: RESULTS USING ALTERNATIVE PULSE SHAPES

In this Appendix, consider two alternate pulse shapes and how the main results from Sec. III are affected by this change. We will consider an exponentially decaying pulse (i.e., a Lorentzian in the frequency domain) and a hyperbolic secant (Sech), i.e.,

$$\chi_L(t) = \frac{x}{2\gamma} e^{-|t|/\gamma}, \quad \chi_S(t) = \frac{x}{\pi\tau} \text{sech}\left(\frac{t}{\tau}\right). \quad (\text{C1})$$

It is difficult to compare the width of these pulses directly using a single metric, but for simplicity we will here use the standard deviation (SD) of the pulses, which are

$$\sigma_L = \sqrt{2}\gamma, \quad \sigma_S = \frac{\pi}{2}\tau. \quad (\text{C2})$$

We employ the same numerical methods to find the weights of the eigenmodes. This allows us to calculate the baseline fidelity, which is plotted in Fig. 9. We generally see the same trends as for the Gaussian pulse. Narrow pulses lead to approximately single-mode behavior, while broader pulses span more modes. This is especially damaging if we introduce many swaps. Furthermore, similar SD yields better fidelities for the Sech and exponentially decaying pulses. This is an artifact of the long tails of these pulses, when fixing the SD. Had we instead fixed the FWHM, we would have seen the reverse trend, exponentially decaying and Sech pulses having worse baseline fidelities at the same FWHM.

We next consider the interplay if we fix the maximal peak pulse intensity. We thus express P_1 (to lowest order in x) by computing the integral in Eq. (A4), and fixing x , such that the peak intensity is $a\kappa^2/4$, where $a = I_{\text{peak}}/I_{\text{thres}}$. The calculation is straightforward for the exponentially decaying pulse, yielding

$$P_{1,L}(\kappa\sigma_L) = a \frac{\kappa^2\sigma_L^2}{2} \frac{2 + \kappa\sigma_L/\sqrt{2}}{(1 + \kappa\sigma_L/\sqrt{2})^2}. \quad (\text{C3})$$

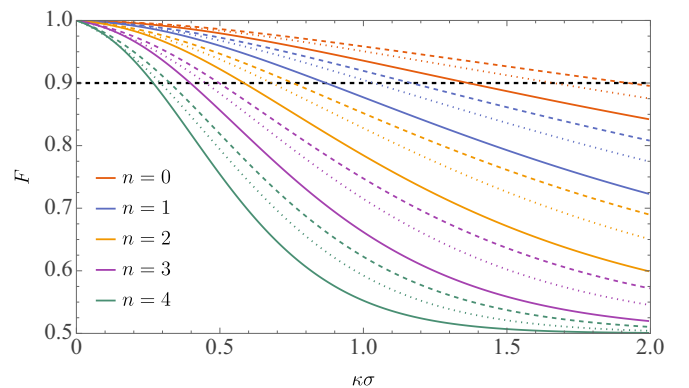


FIG. 9. Baseline fidelity for different swap depths and pulse shapes. Full lines are Gaussian drive, dashed lines are the exponentially decaying, and dotted lines are the Sech pulse.

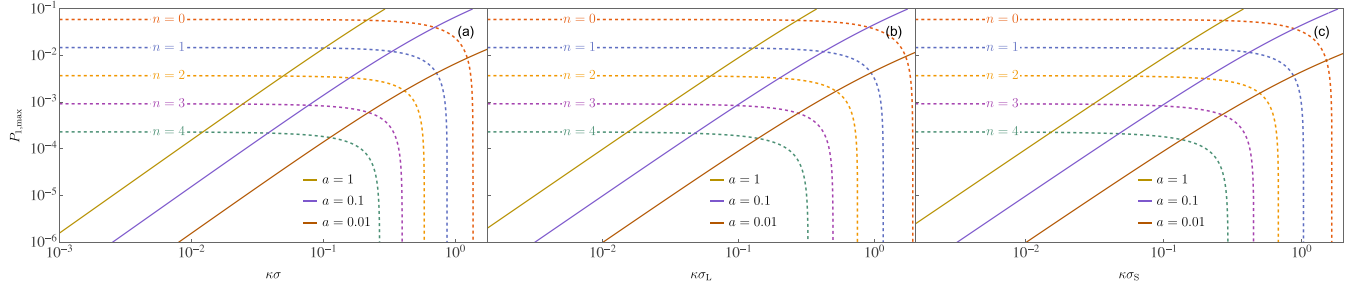


FIG. 10. Full lines show the value of P_1 for different peak intensities, set by $a = I_{\text{peak}}/I_{\text{thres}}$. Dashed lines show P_1 set by a target fidelity of 90%. (a) Gaussian pulse. (b) Exponentially decaying pulse. (c) Hyperbolic secant pulse.

For the Sech pulse, it is slightly more cumbersome. We first Fourier transform the integral to contain a single integral in frequency space. We can then use contour integration to find the integral. Special care has to be taken for $\kappa\tau = (2n + 1)$ for any integer n , as separate poles then coincide. For our interest, this is only relevant when $\kappa\tau = 1$. We find

$$P_{1,s}(\kappa\sigma_s) = a \frac{\kappa^2 \sigma_s^2}{\pi^2} \left[\zeta\left(2, \frac{1 + 2\kappa\sigma_s/\pi}{2}\right) - \zeta\left(2, \frac{1 - 2\kappa\sigma_s/\pi}{2}\right) + \pi^2 \text{sech}^2(\kappa\sigma_s) \right],$$

$$P_{1,s}(1) = a \frac{\kappa^2 \sigma_s^2}{3}, \quad (\text{C4})$$

where $\zeta(s, x) \equiv \sum_{n=0}^{\infty} (n + x)^{-s}$ is the Hurwitz zeta function. P_1 can now be plotted for various SDs set by either the target fidelity for the different swap depths or the maximal peak intensity. This allows us to circumvent the problem of which metric to use for comparison of the width. Instead, we compare the pulse shapes based on the same underlying experimental limitation (maximal peak intensity) and find how well the different pulse shapes perform. This comparison is shown visually in Fig. 10. We note very limited variation between the three different pulse shapes. For the same value of a , the different pulse shapes cross the lines set by the target fidelity at virtually the same value of P_1 . Thus, we can conclude that the actual pulse shape is unimportant, rather the importance lies in the peak intensity.

APPENDIX D: DENSITY MATRIX FOR CONTINUOUS DRIVING

In this Appendix, we describe the density matrix when employing a CW driving laser, and the recursive equations for the coefficients in the density matrix. As noted in the main text, the density matrix depends on the click time t_c and the acceptance window T . We assume that $t_c = 0$, and we will not write the dependence on T for ease of reading. A different detection time will only shift the density matrix to be symmetric around t_c (ignoring for simplicity edge effects where the click time is close to the end of the time interval of the driving).

We will consider the density matrix

$$\begin{aligned} \rho_n = & c_{00,n} |\emptyset\rangle\langle\emptyset| |\emptyset\rangle\langle\emptyset| + \frac{c_{10,n}}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\{t\} \rho_{10}(\{t\}) (|t_1\rangle\langle t_1| |t_2\rangle\langle\emptyset| |\emptyset\rangle\langle\emptyset| + |\emptyset\rangle\langle\emptyset| |t_1\rangle\langle t_1| |t_2\rangle\langle\emptyset|) \\ & + \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\{t\} [A_{0,n} \rho_{A0}(\{t\}) + A_{1,n} \rho_{A1}(\{t\})] (|t_1\rangle\langle t_1| |\emptyset\rangle\langle\emptyset| |\emptyset\rangle\langle\emptyset| |t_2\rangle\langle\emptyset| + \text{H.c.}) \\ & + c_{11,n} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\{t\} \rho_{11}(\{t\}) |t_1\rangle\langle t_1| |t_3\rangle\langle t_3| |t_2\rangle\langle t_4| + \frac{c_{20,n}}{8} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\{t\} \rho_{20|T}(\{t\}) (|t_1\rangle\langle t_1| |t_2\rangle\langle t_3| |t_4\rangle\langle\emptyset| + |\emptyset\rangle\langle\emptyset| |t_1\rangle\langle t_2| |t_3\rangle\langle t_4|) \\ & + \frac{B_{0,n}}{4} \int_{-\frac{T}{2}}^{\frac{T}{2}} d\{t\} \rho_{B0}(\{t\}) (|t_1\rangle\langle t_1| |t_2\rangle\langle t_3| |t_4\rangle\langle\emptyset| |\emptyset\rangle\langle\emptyset| |t_4\rangle\langle\emptyset| + |\emptyset\rangle\langle\emptyset| |t_4\rangle\langle t_1\rangle |t_2\rangle\langle t_3| + \text{H.c.}), \end{aligned} \quad (\text{D1})$$

where $\{t\}$ includes all time coordinates. We take the functions ρ_{ij} to be normalized appropriately. To find the functions and initial values for the coefficients, i.e., the density matrix after entanglement generation, we will follow the methods followed developed in Ref. [35], where the two- and four-time correlation functions after the detection are used to describe the density matrix, up to second order in $x = 2\chi/\kappa$. We write down the useful combinations of the predetection two-time correlation functions up to second order in x , where the idler photon is always taken at $t_c = 0$, i.e., $a_i = a_i(0)$, etc.

$$\frac{\langle a_s^\dagger(t) a_i^\dagger \rangle \langle a_s(t') a_i \rangle}{\langle a_i^\dagger a_i \rangle} = \frac{\kappa}{2} e^{-\frac{\kappa}{2}(|t|+|t'|)} \left\{ 1 + \frac{x^2}{2} \left[\left(1 + \frac{\kappa|t|}{2} \right)^2 + \left(1 + \frac{\kappa|t'|}{2} \right)^2 \right] \right\}, \quad (\text{D2})$$

$$\langle a_s^\dagger(t) a_s(t') \rangle = \frac{\kappa}{2} x^2 e^{-\frac{\kappa}{2}|t-t'|} \left(1 + \frac{\kappa|t-t'|}{2} \right). \quad (\text{D3})$$

We will, as an example, describe how to calculate two of the elements in the density matrix. Consider the element corresponding to two photons in one quantum memory. Only one four-time correlation function contributes up to second order in x :

$$\begin{aligned} \frac{c_{20,0}}{8} [\rho_{n,20}(t_1, t_2, t_3, t_4) + (t_1 \leftrightarrow t_2) + (t_3 \leftrightarrow t_4) + (t_1 \leftrightarrow t_2, t_3 \leftrightarrow t_4)] &= \langle a_s^\dagger(t_4) a_s^\dagger(t_3) a_s(t_2) a_s(t_1) \rangle \\ &= \frac{\langle a_s^\dagger(t_4) a_s^\dagger(t_3) a_s(t_2) a_s(t_1) b_+^\dagger b_+ \rangle}{\langle b_+^\dagger b_+ \rangle} = \frac{\langle a_s^\dagger(t_4) a_s(t_2) \rangle \langle a_s^\dagger(t_3) a_i^\dagger \rangle \langle a_s(t_1) a_i \rangle}{2 \langle a_i^\dagger a_i \rangle} + (t_1 \leftrightarrow t_2) + (t_3 \leftrightarrow t_4) + (t_1 \leftrightarrow t_2, t_3 \leftrightarrow t_4) \\ &\rightarrow \frac{c_{20,0}}{4} \rho_{n,20}(t_1, t_2, t_3, t_4) = \langle a_s^\dagger(t_4) a_s(t_2) \rangle \frac{\langle a_s^\dagger(t_3) a_i^\dagger \rangle \langle a_s(t_1) a_i \rangle}{\langle a_i^\dagger a_i \rangle}, \end{aligned} \quad (\text{D4})$$

where $(t_i \leftrightarrow t_j)$ indicates the first term where t_i and t_j are swapped time coordinates.

To find the single-photon density matrix elements, we have to consider the contribution of two-photon density matrix elements to the two-time correlation functions. In effect, this means that we should subtract appropriate four-time correlation functions from the single-photon correlation function. We will as an example consider the density matrix element with one photon in the one quantum memory:

$$\begin{aligned} \frac{c_{10,0}}{2} \rho_{10}(\{t\}) &= \frac{\langle a_s^\dagger(t_2) a_s(t_1) b_+^\dagger b_+ \rangle}{\langle b_+^\dagger b_+ \rangle} \\ &\quad - \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{ c_{11,0} \rho_{11}(t_1, \tau, t_2, \tau) + \frac{c_{20,0}}{8} [\rho_{20}(t_1, \tau, t_2, \tau) + \rho_{20}(t_1, \tau, \tau, t_2) + \rho_{20}(\tau, t_1, t_2, \tau) + \rho_{20}(\tau, t_1, \tau, t_2)] \right\} \\ &= \frac{\langle a_s^\dagger(t_2) a_i^\dagger \rangle \langle a_s(t_1) a_i \rangle}{\langle a_i^\dagger a_i \rangle} \left(\frac{1}{2} - \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \langle a_s^\dagger(\tau) a_s(\tau) \rangle \right) + \langle a_s^\dagger(t_2) a_s(t_1) \rangle \left(1 - \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \frac{|\langle a_s(\tau) a_i \rangle|^2}{\langle a_i^\dagger a_i \rangle} \right) \\ &\quad - \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \frac{\langle a_s^\dagger(\tau) a_s(t_1) \rangle \langle a_s^\dagger(t_2) a_i^\dagger \rangle \langle a_s(\tau) a_i \rangle + \langle a_s^\dagger(t_2) a_s(\tau) \rangle \langle a_s^\dagger(\tau) a_i^\dagger \rangle \langle a_s(t_1) a_i \rangle}{2 \langle a_i^\dagger a_i \rangle}. \end{aligned} \quad (\text{D5})$$

To write it down properly, we split the coefficient and function into terms attached to a specific order in x , i.e., $c_{10,0} \rho_{10}(\{t\}) = c_{10,0,0} \rho_{10,0}(\{t\}) + c_{10,0,2} \rho_{10,2}(\{t\})$. Following these steps for all the coefficients, we can find the values of the coefficients after the entanglement generation. We will normalize it such that the density matrix functions integrate to 1. Starting with the diagonal elements, i.e., c_{00} , c_{10} , c_{11} , and c_{20} ,

$$c_{00,0,0} = e^{-\frac{\kappa T}{2}}, \quad c_{00,0,2} = -\frac{x^2}{4} e^{-\frac{\kappa T}{2}} \left[(1 - e^{-\frac{\kappa T}{2}})(4 + \kappa T) + 4\kappa T \right], \quad (\text{D6a})$$

$$c_{10,0,0} = 1 - e^{-\frac{\kappa T}{2}}, \quad c_{10,0,2} = -\frac{x^2}{4} \left[2(5 + 2\kappa T) - e^{-\frac{\kappa T}{2}} \left(18 + 13\kappa T + \frac{\kappa^2 T^2}{4} \right) + 2e^{-\kappa T} (4 + \kappa T) \right], \quad (\text{D6b})$$

$$c_{11,0} = \frac{x^2}{2} \kappa T (1 - e^{-\frac{\kappa T}{2}}), \quad c_{20,0} = \frac{x^2}{4} \left[2(5 + \kappa T) - e^{-\frac{\kappa T}{2}} \left(14 + 6\kappa T + \frac{\kappa^2 T^2}{4} \right) + e^{-\kappa T} (4 + \kappa T) \right]. \quad (\text{D6c})$$

As the swapping method is identical for the two cases and since the diagonal matrix coefficients combine in the same way, the recursive equations are the same for the two cases. Hence, we will use the recursive equations for the pulsed case [see Eq. (B3)]. Closed forms for the coefficients can be found using an analytical software like Mathematica, and this is provided in the code base [38]. We can also find the density matrix functions for these elements:

$$\rho_{10,0}(\{t\}) = \frac{\kappa e^{-\frac{\kappa}{2}(|t_1|+|t_2|)}}{2(1 - e^{-\frac{\kappa T}{2}})}, \quad (\text{D7a})$$

$$\begin{aligned} \rho_{10,2}(\{t\}) &= \frac{\kappa e^{-\frac{\kappa(|t_1|+|t_2|)}{2}} \left[4 + 2\kappa T - e^{-\frac{\kappa T}{2}} \left(3 + \frac{\kappa T}{2} \right) + (1 - e^{-\frac{\kappa T}{2}}) \frac{\kappa(|t_1|+|t_2|)}{2} \right] - 2\kappa e^{-\frac{\kappa(T+|t_1|-t_2|)}{2}} (2 + \kappa|t_1 - t_2|)}{2(5 + 2\kappa T) - e^{-\frac{\kappa T}{2}} \left(18 + 13\kappa T + \frac{\kappa^2 T^2}{4} \right) + 2e^{-\kappa T} (4 + \kappa T)} \\ &\quad - \frac{\kappa e^{-\frac{\kappa T}{2}} \left\{ \cosh[\kappa(|t_1| - |t_2|)] \left(3 + \frac{\kappa T}{2} \right) - \frac{e^{\frac{\kappa(|t_1|-|t_2|)}{2}} \kappa|t_1| + e^{\frac{\kappa(|t_2|-|t_1|)}{2}} \kappa|t_2|}{2} \right\}}{2(5 + 2\kappa T) - e^{-\frac{\kappa T}{2}} \left(18 + 13\kappa T + \frac{\kappa^2 T^2}{4} \right) + 2e^{-\kappa T} (4 + \kappa T)}, \end{aligned} \quad (\text{D7b})$$

$$\rho_{11}(\{t\}) = \frac{\kappa \left[e^{-\frac{\kappa}{2}(|t_1|+|t_3|+|t_2-t_4|)} \left(1 + \frac{\kappa|t_2-t_4|}{2} \right) + e^{-\frac{\kappa}{2}(|t_2|+|t_4|+|t_1-t_3|)} \left(1 + \frac{\kappa|t_1-t_3|}{2} \right) \right]}{4T(1 - e^{-\frac{\kappa T}{2}})}, \quad (\text{D7c})$$

$$\rho_{20}(\{t\}) = \frac{4\kappa^2 e^{-\frac{\kappa}{2}(|t_1|+|t_3|+|t_2-t_4|)} \left(1 + \frac{\kappa|t_2-t_4|}{2} \right)}{2(5 + \kappa T) - e^{-\frac{\kappa T}{2}} \left(14 + 6\kappa T + \frac{\kappa^2 T^2}{4} \right) + e^{-\kappa T} (4 + \kappa T)}. \quad (\text{D7d})$$

We now turn to the off-diagonal density matrix elements. The functions are normalized, such that they do not interfere in the swapping process. The coefficients then become

$$A_{0,0} = (1 - e^{-\frac{\kappa T}{2}})(1 - x^2 \kappa T), \quad (\text{D8a})$$

$$A_{1,0} = -\frac{x^2}{4} \left[(10 - 8e^{-\frac{\kappa T}{2}})(1 - e^{-\frac{\kappa T}{2}}) - \kappa T e^{-\frac{\kappa T}{2}} \left(5 + \frac{\kappa T}{4} - 2e^{-\frac{\kappa T}{2}} \right) \right], \quad (\text{D8b})$$

$$B_{0,0} = \frac{x^2}{4} \left[2(5 + \kappa T) - e^{-\frac{\kappa T}{2}} \left(14 + 6\kappa T + \frac{\kappa^2 T^2}{4} \right) + e^{-\kappa T} (4 + \kappa T) \right]. \quad (\text{D8c})$$

As the density matrix for the off-diagonal elements is slightly different in the CW case compared to the pulsed case, the recursive equations for the coefficient are also slightly different:

$$A_{0,n+1} = \eta_2 \frac{A_{0,n}}{4} [A_{0,n} + A_{1,n} + 2(1 - \eta_2)B_{0,n}], \quad (\text{D9a})$$

$$A_{1,n+1} = \eta_2 \frac{A_{0,n}A_{1,n}}{4}, \quad (\text{D9b})$$

$$B_{0,n+1} = \eta_2 \frac{A_{0,n}B_{0,n}}{4}. \quad (\text{D9c})$$

The recursive equations can again be solved using analytical software, and this is again provided in the code base [38]. We also give the off-diagonal matrix functions

$$\rho_{A0}(\{t\}) = \frac{\frac{\kappa}{2} e^{-\frac{\kappa}{2}(|t_1|+|t_2|)}}{1 - e^{-\frac{\kappa T}{2}}}, \quad (\text{D10a})$$

$$\rho_{A1}(\{t\}) = \frac{\kappa e^{-\frac{\kappa(|t_1|+|t_2|)}{2}} \left[4 - \left(3 + \frac{\kappa T}{2} \right) e^{-\frac{\kappa T}{2}} + \left(1 - e^{-\frac{\kappa T}{2}} \frac{\kappa(|t_1|+|t_2|)}{2} \right) \right]}{(10 - 8e^{-\frac{\kappa T}{2}})(1 - e^{-\frac{\kappa T}{2}}) - \kappa T e^{-\frac{\kappa T}{2}} \left(5 + \frac{\kappa T}{4} - 2e^{-\frac{\kappa T}{2}} \right)} - \frac{\kappa e^{-\frac{\kappa T}{2}} \left\{ \cosh[\kappa(|t_1| - |t_2|)] \left(3 + \frac{\kappa T}{2} \right) - \frac{e^{\frac{\kappa(|t_1|-|t_2|)}{2}} \kappa|t_1| + e^{\frac{\kappa(|t_2|-|t_1|)}{2}} \kappa|t_2|}{2} \right\}}{(10 - 8e^{-\frac{\kappa T}{2}})(1 - e^{-\frac{\kappa T}{2}}) - \kappa T e^{-\frac{\kappa T}{2}} \left(5 + \frac{\kappa T}{4} - 2e^{-\frac{\kappa T}{2}} \right)}, \quad (\text{D10b})$$

$$\rho_{B0,0}(\{t\}) = \frac{\kappa^2 e^{-\frac{\kappa}{2}(|t_1|+|t_4|+|t_2-t_3|)} \left(1 + \frac{\kappa|t_2-t_3|}{2} \right)}{(5 + \kappa T) - e^{-\frac{\kappa T}{2}} \left(7 + 3\kappa T + \frac{\kappa^2 T^2}{8} \right) + e^{-\kappa T} \frac{4+\kappa T}{2}}. \quad (\text{D10c})$$

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