

Extending the dynamic range in quantum frequency estimation with sequential weak measurements

Su Direkci^{1,*}, Manuel Endres¹, and Tuvia Gefen²

¹*California Institute of Technology, Pasadena, California 91125, USA*

²*Racah Institute of Physics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel*



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Quantum metrology explores optimal quantum protocols for parameter estimation. In the context of optical atomic clocks, conventional protocols focus on optimal input states and measurements to achieve enhanced sensitivities. However, such protocols are typically limited by phase slip errors inflicted due to the decoherence of the local oscillator. Here, we study schemes to extend the dynamic range and overcome phase slip noise through weak measurements with ancilla qubits. Using coherent spin states, we find optimal weak measurements protocols: we identify optimal measurement strength for any given interrogation time and number of atoms. Then, we combine weak and projective measurements to construct a protocol that asymptotically saturates the noiseless precision limits and outperforms previously proposed methods for phase slip noise suppression.

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In quantum frequency estimation, we seek to estimate an unknown frequency ω , given a dynamic range of $[\omega_0, \omega_0 + \delta\omega]$, with an optimal precision. Maintaining a high precision given an arbitrarily large frequency bandwidth, $\delta\omega$, is one of the fundamental challenges in quantum metrology, and it is typically referred to as the precision-bandwidth trade-off [1–5]. The standard estimation schemes rely on a Ramsey-type experiment: N qubits are initialized to an input state, rotate at the frequency ω for time T , and are subsequently measured. For an infinitesimally small bandwidth, longer evolution times enhance the precision, leading to estimation uncertainty that scales as $1/T$, known as Heisenberg scaling (HS) with respect to time [6–10]. However, a finite bandwidth restricts this improvement: an evolution time that is longer than $\pi/\delta\omega$ leads to phase slip errors that considerably degrade the precision. Consequently, the evolution time in systems such as optical atomic clocks [11–16] is typically constrained by the inverse of the frequency bandwidth [17]. This phenomenon is prominent for the Greenberger-Horne-Zeilinger (GHZ) state, as it provides optimal sensitivity but suffers from a poor dynamic range, limiting its usefulness when $\delta\omega T$ is large [2, 18–20]. It is therefore highly desirable to develop schemes that suppress phase slip errors and achieve nearly optimal precision for a large bandwidth.

Several schemes successfully combine entanglement-enhanced sensing with a relatively large dynamic range, i.e., $\delta\omega \approx \pi/T$ [2, 4, 18, 19, 21, 22]. However, in most practical applications, it is important to extend the dynamic range even further; for example, in atomic clocks, larger

dynamic range would improve the tolerance to laser noise and the clock stability. The fundamental question is, therefore, how to extend the dynamic range beyond this limit. Several methods have been proposed for this purpose, including cascaded protocols [1, 4, 21], quantum deamplification [5], and sequential weak measurements [23, 24]. However, it is unknown whether these schemes saturate the relevant precision bounds, and it is highly desirable to find schemes that are both optimal and easy to implement with state-of-the-art platforms.

We focus in this Letter on the weak measurement approach. While sequential weak measurements were proposed in the context of cavity-based atomic clocks [23, 24], there has been no systematic study of their precision limits in the context of Bayesian frequency estimation. In particular, it is unknown whether such schemes achieve HS or saturate the precision bounds for a large bandwidth, especially for a limited number of atoms (hereafter, HS refers only to the scaling with time). Finally, prior works assumed a large ensemble of atoms measured collectively [23, 24]; devising an optimal weak measurement protocol for circuit-based sensors [25–27] and for a small number of atoms [28] remains an open challenge. We address these questions by designing a tunable sequential weak measurement protocol and analyze its performance for different configurations. We focus on the simplest case where the input state is a coherent spin state (CSS) and propose a protocol that asymptotically saturates the ultimate precision bounds for CSS for an arbitrarily large bandwidth. This protocol also shows HS and outperforms conventional protocols in the finite N regime.

Formulation and Cramér-Rao bounds. We consider a Ramsey-type experiment: N qubits evolve according to the Hamiltonian $\mathcal{H} = \omega \sum_{j=1}^N \sigma_z^j$, where σ_z^j is the Pauli z operator acting on the j th qubit and ω is the unknown frequency to be estimated. We consider the Bayesian setting where ω has a uniform prior distribution $\mathcal{P}_{\delta\omega}(\omega) = \frac{1}{\delta\omega} \chi_{[0, \delta\omega]}(\omega)$ in $[0, \delta\omega]$ (χ is the indicator function) [29]. Note that any uniform prior

*Contact author: sdirekci@caltech.edu

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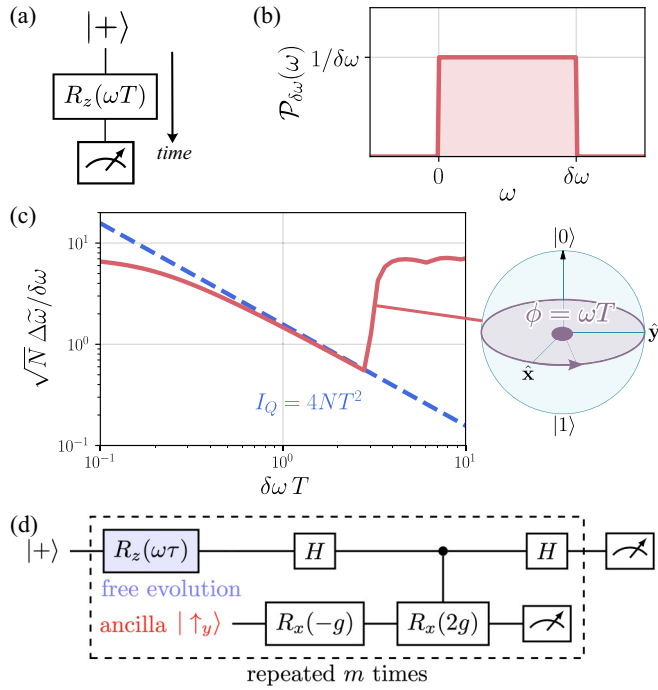


FIG. 1. Dynamic range problem in frequency estimation: (a) Circuit representation of the standard Ramsey experiment, where a strong measurement is applied after a phase accumulation time T . (b) We consider a uniform prior distribution of the unknown frequency $\mathcal{P}_{\delta\omega}(\omega)$ in the interval $[0, \delta\omega]$. (c) The square root of the Bayesian mean squared error $\Delta\tilde{\omega}$, normalized by the prior width $\delta\omega$ and the number of qubits $N = 64$. The solid, red line corresponds to the measurement-optimized BMSE, which suffers from phase slips beyond $\delta\omega T > \pi$, as shown in the Bloch sphere picture. The dashed blue line corresponds to the quantum Fisher information bound of $(\Delta\omega)^{-2} = I_Q = 4NT^2$. (d) Circuit representation of the approach studied here: Ramsey experiment with sequential weak measurements.

distribution in the form of $[\omega_0, \omega_0 + \delta\omega]$ is equivalent to this distribution: we can retrieve the original prior by applying a unitary of $e^{i\omega_0 T \sum_j \sigma_z^j}$ before the measurement. The error in the estimation is quantified by the Bayesian mean squared error (BMSE), which is defined as the mean squared error (MSE) weighted by the prior distribution:

$$(\Delta\tilde{\omega})^2 = \int (\Delta\omega)^2 \mathcal{P}_{\delta\omega}(\omega) d\omega, \quad (1)$$

where $(\Delta\omega)^2 := \sum_x p(x|\omega)(\omega - \omega_{\text{est}}(x))^2$ is the MSE, $(\Delta\tilde{\omega})^2$ is the BMSE, $\{p(x|\omega)\}_x$ are the probabilities of the measurement results, and $\{\omega_{\text{est}}(x)\}$ is the estimator function.

The standard Ramsey experiment is illustrated in Fig. 1(a). The probes are initialized to a CSS, $|+\rangle^{\otimes N} = (|0\rangle + |1\rangle)^{\otimes N}/2^{N/2}$, where $|+\rangle$ is an eigenstate of the Pauli x operator. They are subjected to a free evolution for a duration of T and are finally measured. The BMSE of this protocol, when optimized over all possible measurements, is referred to as the optimal classical interferometer (OCI) [19,30] and is plotted in Fig. 1(c). We observe that for interrogation times where $\delta\omega T > \pi$, the OCI BMSE increases rapidly and converges to the prior width, $\Delta\tilde{\omega} \approx \delta\omega$, signifying that no information is

gained from the estimation. The behavior is caused by phase slips: the unitary transformation describing the free evolution is periodic in ωT . Therefore, the maximum interrogation time using this protocol is limited to $\pi/\delta\omega$, since longer times will give rise to phase slip errors.

To resolve this issue, we propose to perform sequential weak measurements during the free evolution. Weak measurements provide frequent monitoring of the state and allow us to track the oscillations, preventing phase slip errors [37–41]. They therefore can potentially extend the dynamic range beyond π/T .

Our scheme consists of applying a sequence of weak σ_x measurements in time intervals of τ . They are described by the Kraus operators $K_{\pm} = (\cos(g)I \pm \sin(g)\sigma_x)/\sqrt{2}$, where g is a tunable parameter that represents the measurement strength ($g = \pi/4$ corresponds to a projective measurement). They can be realized by weakly entangling each probe to an ancillary qubit and measuring the ancilla (see the Supplemental Material [30]). Each step of the protocol involves the rotation $V_{\omega} = \cos(\omega\tau)I - i\sin(\omega\tau)\sigma_z$, followed by a weak measurement that transforms the state of the sensing qubit ρ as $\rho \rightarrow K_{\pm} \rho K_{\pm}^{\dagger}/\text{Tr}[K_{\pm} \rho K_{\pm}^{\dagger}]$, depending on the measurement outcome. Therefore, the sensing qubits evolve in stochastic quantum trajectories [42]. Given a total interrogation time T and a measurement period τ , this stochastic step is repeated for $m = \lfloor T/\tau \rfloor$ times [43]. Note that, to prevent phase slips, the measurement period must satisfy $\tau < \frac{\pi}{2\delta\omega}$. We study two different protocols, illustrated in Fig. 1(d): the *weak-only* protocol, in which only sequential weak measurements are performed, and the *weak-with-strong* protocol, in which the final weak measurement is replaced by a projective measurement on the sensing qubits.

Let us study the effects of weak measurements and the resulting stochastic dynamics. Initializing the sensing qubit in the state $|+\rangle$, it remains on the σ_x - σ_y plane [44], and is thus fully characterized by the angle θ in this plane. This angle is a random variable that depends on the previous measurement outcomes. It can be calculated iteratively using the following relation: $\theta_{k+1} = -2\omega\tau + \arg(\cos(\theta_k) + x_k \sin(2g) + i \cos(2g) \sin(\theta_k))$, where θ_k is the angle before the k th measurement and $x_k = 0, 1$ corresponds to the outcome of the k th measurement. Notice that in the limit of $g \rightarrow 0$, $\theta_{k+1} = \theta_k - 2\omega\tau$, which corresponds to the free evolution in a standard Ramsey experiment. Each step thus consists of a rotation by an angle of $2\omega\tau$, followed by a small stochastic kick in the phase that degrades the coherent phase accumulation. This effect can be seen more clearly from studying the average dynamics, corresponding to the sequential application of the channel $\Lambda(\rho) = \sum_{i \in \{\pm\}} K_i^{(\omega)} \rho K_i^{(\omega)\dagger}$, where $K_i^{(\omega)} = K_i V_{\omega}$. The sensing qubit undergoes dephasing in this channel with a rate of $\gamma = -\frac{1}{2\tau} \ln(\cos(2g)) \approx g^2/\tau$ in the limit of $g \ll 1$. This dephasing stems from the phase random walk induced by the measurements, a phenomenon that is typically termed as *measurement backaction*. Even though in our protocol we are not limited to the average dynamics but track individual trajectories, we will show that this dephasing still affects the sensitivity.

We are now poised to study the precision limits of this scheme and compare them to the fundamental bounds. The ultimate precision limit is given by the Bayesian Cramér-Rao

bound (or the van Trees inequality) [45–47]

$$(\Delta\tilde{\omega})^2 \geq [\bar{I}_C + I_P]^{-1}, \quad (2)$$

where $\bar{I}_C := \int d\omega \mathcal{P}(\omega) I_C(\omega)$, $I_C = \int dx p(x|\omega) [\partial_\omega \ln p(x|\omega)]^2$ is the classical Fisher information (CFI) and $I_P = \int d\omega \mathcal{P}(\omega) [\partial_\omega \ln \mathcal{P}(\omega)]^2$ denotes the information from the prior distribution. Since we are interested in the large bandwidth limit, i.e., $\delta\omega T \gg 1$, we have that $I_C \gg I_P$, and Eq. (2) reduces to the Cramér-Rao bound (CRB): $(\Delta\tilde{\omega})^2 \gtrsim \bar{I}_C^{-1}$. Furthermore, from the quantum Cramér-Rao bound [48], the CFI is upper bounded by the quantum Fisher information (QFI), i.e., $I_C \leq I_Q$. Then, the QFI is the ultimate limit of precision given an input state, and the benchmark for our parameter estimation. For an N -atom CSS, the QFI about ω is given by $I_Q = 4NT^2$, where N is the number of qubits and T is the interrogation time. This precision limit is referred to as the Heisenberg limit (HL) with respect to the interrogation time T . In contrast, the CFI achieved by repeating a Ramsey protocol for $M = T/\tau$ times is found as $4N\tau^2 \times M = 4NT\tau$, which fails to achieve the HL.

We now turn to quantifying the performance of the proposed protocols. We first compute the CFI of this weak measurement scheme. Even though this is a local quantity, it provides a bound on the BMSE through the Bayesian Cramér-Rao bound [Eq. (2)] and can be calculated analytically. Therefore, we first show that the CFI can approach the QFI. We then demonstrate numerically the saturation of the CFI in the relevant Bayesian setting.

For this purpose, it is useful to define the parameter $\eta = g^2 T/\tau$, which quantifies the amount of backaction on the sensing qubits due to the weak measurements: the backaction decay time was found as $\gamma^{-1} \approx \tau/g^2$; therefore, $\eta = T/\gamma^{-1}$ describes how the total interrogation time compares to the decay time.

Let us start with the weak-only protocol and denote the CFI with this scheme as $\bar{I}_C^w(g, T, \tau)$ [49]. Figure 2 presents $\bar{I}_C^w$ as a function of the interrogation time T , for a fixed measurement strength g and period τ . We observe a trade-off relation in the behavior of $\bar{I}_C^w$: in the weak backaction regime, i.e., for $\eta \ll 1$, $\bar{I}_C^w$ can be approximated as $\bar{I}_C^w \approx \frac{8}{3} N g^2 \frac{T^3}{\tau}$. This T^3 scaling of the CFI is typical for frequency estimation of classical signal [50–55]. Here, even though the scaling with T is more favorable than HS, the absolute performance relative to the HL is poor since $\eta \ll 1$. Conversely, in the strong backaction regime where $\eta \gg 1$, the growth of $\bar{I}_C^w$ is much more restrained, ultimately converging to $\bar{I}_C^w \approx 4NT/\gamma \approx 4N \frac{\tau}{g^2} T$. This linear scaling of the CFI is ubiquitous in sequential measurement schemes [37,56,57] and it is attributed to the state memory loss induced by the sequential measurements; i.e., the phase becomes uncorrelated with its initial value. We can interpret the overall behavior of the CFI as follows: in the weak backaction regime not enough information is extracted from the state, while in the strong backaction regime the decoherence induced by the measurements degrades the CFI and prevents achieving HS with respect to time. The optimal g is thus at the crossover between the two regimes, where the trade-off between extracted information and dissipation is optimal, and HS can be achieved (see Fig. 4 in Ref. [30]).

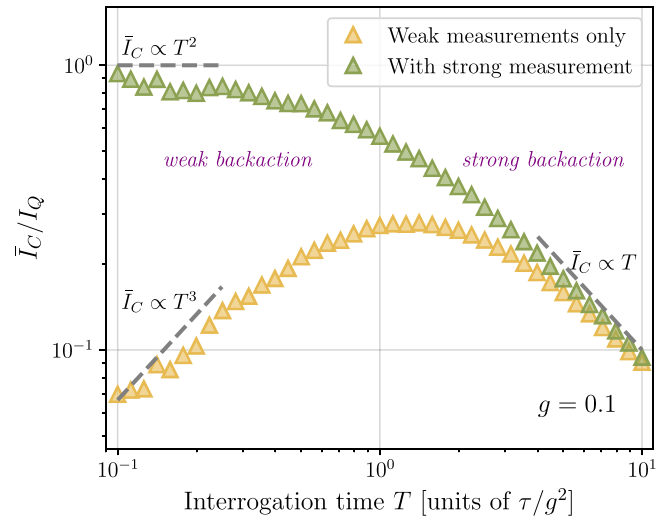


FIG. 2. Average classical Fisher information $\bar{I}_C$, normalized by the QFI, $I_Q = 4NT^2$, for the weak-only protocol (plotted in yellow) and weak-with-strong protocol (plotted in green). We fix the measurement strength as $g = 0.1$ and measurement period as $\tau = 0.1$ s, and vary the interrogation time $T \in [1, 100]$ s. In the weak backaction regime, i.e., $g^2 T/\tau \ll 1$, the CFI for the weak-with-strong protocol saturates I_Q . The CFI for the weak-only protocols is much smaller, yet it increases rapidly as T^3 . Eventually, in the strong backaction regime, $g^2 T/\tau \gg 1$, the two protocols obtain the same CFI of $4NT\tau/g^2$.

The usual experimental scenario is a fixed total time, T , while the weak measurement strength g is tunable. Hence, for a given T , we find that the optimal g corresponds to $g^2 T/\tau \approx \sqrt{3/2}$ [58]. The resulting CFI is then $\bar{I}_C^w \approx 1.11NT^2$. Hence, we can obtain HS with time using this weak-only scheme with an optimal g , where the CFI loses a factor of ≈ 3.60 compared to the QFI limit.

The CFI can be improved if, in addition to the weak measurements, a projective (strong) measurement is applied at the end of the interrogation, which is the weak-with-strong protocol. We denote CFI of this scheme as $\bar{I}_C^{ws}(g, T, \tau)$. In the weak backaction regime ($\eta \ll 1$), $\bar{I}_C^{ws} \approx I_Q = 4NT^2$, where it is maximal for $g = 0$, which saturates the QFI limit (see Fig. 2). However, in the strong backaction regime ($\eta \gg 1$), the final projective measurement does not add any new information and we obtain the same linear CFI scaling $\bar{I}_C^{ws} \approx 4NT\tau/g^2$, as we had for $\bar{I}_C^w$. Hence, weak measurements only reduce I_C^{ws} , but as we will show in the following, they are needed to saturate the CFI given a large dynamic range. A further analysis of the CFI with imperfect weak measurements can be found in Ref. [30].

Saturability of the CRB and the threshold effect. So far we have shown that the CFI of both of the weak measurement protocols, $\bar{I}_C^w, \bar{I}_C^{ws}$, yields HS with appropriate values of g, τ, T . However, these quantities correspond to local estimation of ω , and they are not necessarily saturable in the relevant large bandwidth limit, $\delta\omega T \gg 1$. Hence, we study the minimal η, N for which $\bar{I}_C^w, \bar{I}_C^{ws}$ are saturated, given a uniform prior distribution in $[0, \delta\omega]$. In particular, we focus on whether they can be saturated for small enough η such that HS is achieved. Note that the prior distribution implicitly determines

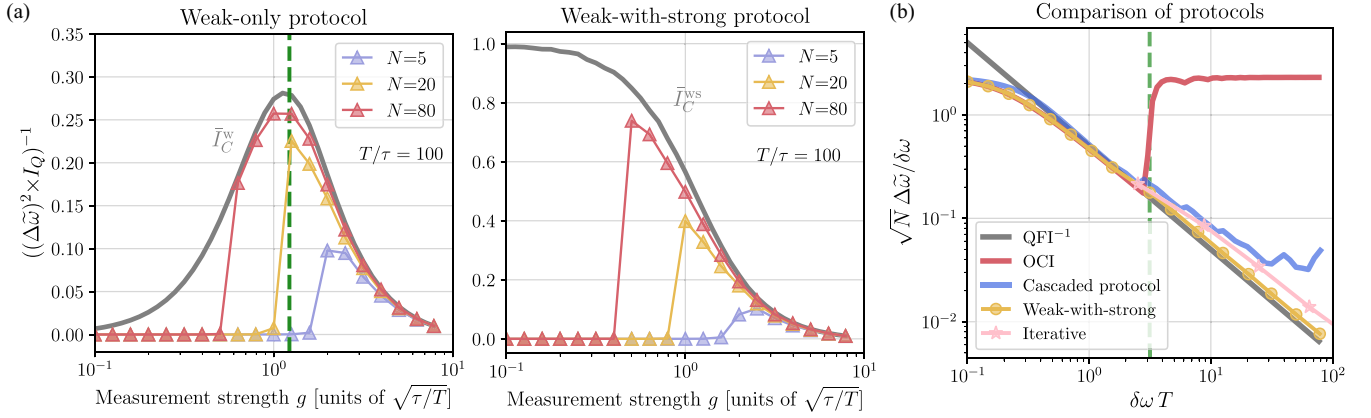


FIG. 3. Performance of the proposed protocol with respect to the interrogation time T and the threshold effect. The frequency ω is sampled from a uniform distribution in $[0, \delta\omega]$, which determines the measurement period as $\tau \lesssim \pi/2\delta\omega$. (a) We fix $\delta\omega = 5\pi$ (i.e., $\tau = 0.1$), $T = 10$, and plot the inverse of BMSE $(\Delta\tilde{\omega})^2$ scaled by the QFI I_Q for the two weak measurement protocols. $(\Delta\tilde{\omega})^2 \times I_Q = 1$ signifies saturating the QFI, and $\bar{I}_C/I_Q$ is plotted with gray, plain lines. We observe a threshold effect such that the attainability of the CFI $\bar{I}_C^w, \bar{I}_C^{ws}$ requires large enough SNR. The green, dashed, vertical line corresponds to $T = \sqrt{3/2}\tau/g^2$, which is a transition point between the weak and strong backaction regimes. Achieving HS requires the threshold to occur before this line (which requires $N > 20$ qubits for this case). (b) We fix $N = 64$, $\delta\omega = \pi$ ($\tau = 0.5$ s) and plot the square root of the BMSE $\Delta\tilde{\omega}$ normalized by the prior width $\delta\omega$, and scaled by the square root of the number of qubits N . The vertical green line corresponds to $\delta\omega T = \pi$: for larger interrogation times, phase slips degrade the sensitivity of standard protocols. We compare the performance of the OCI [19,30], the proposed weak-with-strong protocol, the iterative protocol, and the cascaded protocol [1]. The latter two demonstrate an extended dynamic range; however, the proposed protocol outperforms the cascaded protocol and achieves HS, with an overhead of 1.18 in $\Delta\tilde{\omega}$. The measurement strength g is optimized for each T for the proposed protocol.

the measurement period for the weak measurement protocols, with $\tau \lesssim \pi/2\delta\omega$.

A similar problem was studied extensively in classical sensing, where a signal $s(t_n) = A \cos(2\omega t_n) + v_n$ is sampled at times $\{t_n = n\tau\}_{n=1}^M$ [50,51]. Here, A is the amplitude of the signal and $\{v_n\}_{n=1}^M$ are i.i.d. Gaussian random variables, $v_n \sim N(0, \sigma^2)$. It was shown that this classical frequency estimation problem is plagued by a threshold effect with respect to the signal to noise ratio (SNR), defined as $\frac{A^2}{\sigma^2} \frac{T}{\tau}$. For low SNR, the posterior distribution is very close to the prior distribution, and the MSE is thus very close the prior variance. Above a certain threshold of the SNR, there is a sharp drop in the MSE, and it converges to the CRB. This is a well-known effect in signal estimation, studied in, e.g., Refs. [50–52].

In our case, we also sample a stochastic, time-dependent signal, through the weak measurements. The weak backaction regime in our setting is equivalent to sampling a classical signal due to the negligible backaction. We thus expect a similar threshold behavior, where the SNR in the limit of $g \ll 1$ can be defined as $Ng^2T/\tau = N\eta$. This SNR threshold effect is observed numerically, and it is illustrated in Fig. 3(a) (see also Fig. 6 in Ref. [30]). We plot the BMSE $(\Delta\tilde{\omega})^2$ obtained with the maximum likelihood estimator as a function of T for varying numbers of qubits N [59]. $(\Delta\tilde{\omega})^{-2}$ is compared to the CFI as a benchmark, plotted with gray lines. As the SNR is increased, e.g., by increasing T as in Fig. 3(a), the BMSE starts to rapidly decrease and saturates the CRB.

In order to approach I_Q and achieve HS with time, the CRB needs to be saturated in the regime where $\bar{I}_C^{ws} \approx I_Q$. Therefore, reaching this limit requires (1) weak measurement backaction, $\eta < 1$, and (2) large enough SNR, $N\eta > 1$. As shown in Fig. 3(a), these two conditions are incompatible for

small N : the saturability occurs only in the strong backaction regime. However, there exists a minimal N after which these two conditions are satisfied and HS can be achieved. In particular, we observe that as N increases, the saturability occurs at smaller η .

Let us analytically find the minimal N for which the CRB can be saturated. We aim to compute the parameters of the weak-with-strong protocol for which the BMSE is close to the CFI, i.e., $(\Delta\tilde{\omega})^{-2} = (1 - \epsilon)\bar{I}_C$, $\epsilon \ll 1$. We work in the limit of $\eta \ll 1$ and a large number of probes, $N \gg 1$, for which the problem can be approximated as the sampling of a classical signal. We find that $(\Delta\tilde{\omega})^{-2} = (1 - \epsilon)\bar{I}_C$ is attained for

$$N\eta \approx 2 \ln \left(\frac{\pi^{3/2}}{48\epsilon} \right) + 6 \ln \left(\frac{T}{\tau} \right). \quad (3)$$

For any system parameters g, T, τ , this expression provides the minimal number of probes N for which we are ϵ close to the CFI. Since we can always tune g such that η remains constant, then the minimal N , for any given ϵ , grows as $6 \ln[(2\delta\omega T)/\pi]$. Hence, it grows logarithmically with the number of phase wraps. While this logarithmic dependence is in line with the scaling of N in previous schemes [21], we obtain a smaller overhead.

Finally, we want to explicitly show that this protocol achieves an almost optimal BMSE for long times, i.e., $\delta\omega T \gg 1$, and that it outperforms existing methods. To this end, we plot in Fig. 3(b) the square root of the BMSE $\Delta\tilde{\omega}$, as a function of interrogation time for different sensing schemes for $N = 64$. Our weak-with-strong protocol is compared to the QFI bound, to the OCI, and to two more conventional dynamic range extension schemes: the iterative and the cascaded

protocol [1,4,21,25]. The iterative protocol scales the interrogation time by a constant factor $\nu > 1$ iteratively, whereas the cascaded protocol employs groups of atoms with shorter interrogation times $T/2^i$, $i > 0$, which are used to prevent phase slips [30]. In the Figure, we optimize over ν , which is set to 2.51.

We first observe that our weak-with-strong protocol performs very close to the QFI bound for the entire range of interrogation times, with a maximal overhead of 1.18 in $\Delta\tilde{\omega}$ (compared to 1.75 of the iterative scheme). We thus obtain an almost optimal HS that is immune to phase slip errors and persists in the limit of large $\delta\omega T$. This is in sharp contrast to the OCI that suffers from phase slip errors and becomes inefficient for $\delta\omega T > \pi$. We also observe that our protocol considerably outperforms the cascaded protocol. The intuitive reason behind this metrological gain is that, while the cascaded protocol allows optimization over only a finite set of partitions into blocks, the proposed weak measurement scheme offers greater tunability, as the optimization is over the continuous weak measurement strength g [60].

Conclusions and outlook. In this Letter, we developed and analyzed a weak measurement Ramsey protocol that achieves optimal precision over an extended dynamic range asymptotically. By optimizing over the weak measurement strength, we were able to achieve a nearly optimal sensitivity and HS in the large bandwidth limit, $\delta\omega T \gg 1$, given a large enough number of probes. We identified two

separate conditions for obtaining a high sensitivity and a large bandwidth, and demonstrated that they can be satisfied simultaneously.

Some open questions and directions that can potentially improve the proposed protocol are as follows: first, we can reduce the number of ancillas by coupling only a subset of sensors to ancillas or coupling several sensors to the same ancilla. We can also improve the BMSE in the low SNR limit by adding more ancillas as memory qubits and performing a collective measurement on them [55,61,62]. This may allow achieving HS for smaller N . Finally, we can mitigate the effect of imperfect measurements by using error correction [63–66]. Weak measurements could also be combined with entangled states [24] to achieve HS with both time and the number of qubits, for an extended bandwidth.

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Data availability. The data that support the findings of this article are openly available at Zenodo [67].

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