

Entanglement thresholds for sequential nonlocality sharing in star networks

Ming-Xiao Li^{1,2,*}, Yuqi Li^{3,*†}, Rui-Bin Xu^{1,2,*}, Mo-Ran Zhu^{1,2}, Haitao Ma^{3,‡},
Chang-Yue Zhang^{4,§} and Zhu-Jun Zheng^{1,2,||}

¹*School of Mathematics, South China University of Technology, Guangzhou 510640, People's Republic of China*

²*Laboratory of Quantum Science and Engineering, South China University of Technology, Guangzhou 510641, People's Republic of China*

³*College of Mathematics Science, Harbin Engineering University, Harbin, Heilongjiang 150001, People's Republic of China*

⁴*Department of Mathematics, College of Information Science and Technology, Jinan University, Guangzhou 510632, People's Republic of China*



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Quantum networks promise scalable distributed quantum information processing. Realizing this potential depends critically on the efficient use of nonclassical resources. Sequential sharing of quantum nonlocality provides a resource-efficient route for recycling entanglement across multiple observers. In network settings, however, such recycling is typically constrained by a trade-off between the number of active branches and the achievable sharing depth. Here, we investigate this problem in star networks using probabilistic projective measurements. We establish an explicit entanglement threshold above which nonlocality can be shared across all branches up to any target depth, and below which we determine the optimal trade-off between the number of active branches and the achievable number of sequential rounds. We further analyze the protocol's robustness to noise and its experimental observability, clarifying its practical feasibility. These results establish quantitative design principles and entanglement requirements for scalable nonlocality recycling in quantum networks.

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I. INTRODUCTION

Quantum networks are expected to serve as the physical platform for secure communication [1–7], distributed quantum computation [8–13], and network-enhanced sensing [14–16]. Considerable progress has been achieved in recent years [17–22], including long-distance quantum key distribution [23–26], metropolitan-scale entanglement distribution [27–29], and the development of quantum repeater architectures. Yet the transition from proof-of-principle demonstrations to scalable network operation remains limited by a basic experimental bottleneck: High-quality quantum correlations are costly to prepare, fragile under transmission and measurement, and difficult to distribute across many users simultaneously. For this reason, an important practical question is not only how to generate nonlocal correlations [30–44] but also how to reuse them efficiently once they have been established.

Quantum nonlocality sharing provides a direct pathway to recycling quantum resources [45–62]. In this setting, multiple

observers can sequentially extract correlations from a single entangled state, while preserving its ability to demonstrate nonlocality for subsequent use. The central issue in these protocols is balancing information extraction against state disturbance. Traditionally, this balance has been achieved through unsharp measurements. More recently, probabilistic projective measurement (PPM) has emerged as a powerful alternative, offering a more experimentally feasible approach. Recent successful experimental demonstrations of sequential sharing in tripartite systems have opened new technological pathways for the distributed utilization of quantum nonlocality [63]. However, extending this capability to full quantum networks introduces qualitatively new challenges. In a network, early measurements affect not only later observers on the same branch but also the nonlocal resources available across different branches [64,65].

Ideally, one would like to realize sequential sharing across all branches of a network for arbitrarily many rounds, corresponding to a full-depth, full-breadth regime. In practice, however, this ideal scenario is generally constrained by the finite available network entanglement and the difficulty of tailoring measurement settings. Under such constraints, full-network sharing is governed by an intrinsic depth-breadth trade-off: Extending the sharing depth along certain branches typically requires sacrificing the nonlocality on others, and vice versa. Therefore, two fundamental questions arise for full-network sequential sharing under limited resources: whether there exists a minimum entanglement requirement for sustaining full-network sequential sharing? When this regime is unattainable, what trade-off fundamentally limits the simultaneously sharable network depth and breadth?

*These authors contributed equally to this work.

†Contact author: liyuyi@hrbeu.edu.cn

‡Contact author: hmamath@hrbeu.edu.cn

§Contact author: cyzhang@jnu.edu.cn

||Contact author: zhengzj@scut.edu.cn

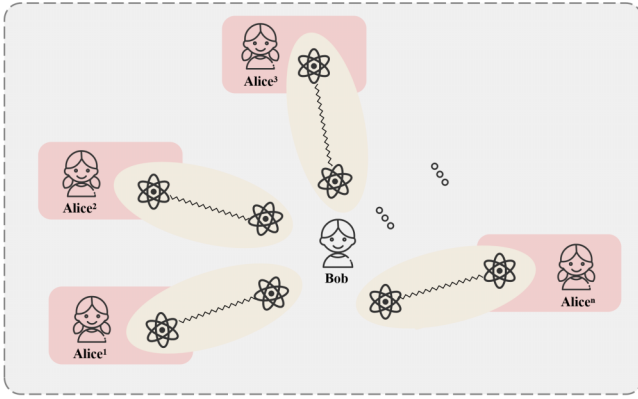


FIG. 1. The n -star network. The central party Bob shares a bipartite entangled state with each of the n peripheral Alice¹, Alice², ..., Alice ^{n} via independent sources.

In this work, we identify an entanglement threshold that supports ideal full-network nonlocality sharing by introducing a sharing protocol assisted with PPM. For entanglement below this threshold, we characterize the achievable numbers of sharable network branches and sequential sharing rounds, and derive an explicit trade-off between sharing depth and network breadth. From a practical perspective, we further present two experimentally relevant theoretical analyses. First, we extend the sharing scenario to noisy settings and characterize the effect of noise on the admissible measurement strength, thereby providing guidance for parameter selection under realistic decoherence. Second, we compare the detectability of the proposed PPM protocol with that of weak-measurement-based protocols, showing improved detectability for the same number of sharing rounds. Our results clarify the achievable regimes of nonlocality recycling and establish a framework for recycling nonlocal correlations in quantum networks.

II. NETWORK NONLOCALITY

Beyond Bell nonlocality in bipartite scenarios, recent research has increasingly focused on understanding quantum correlations in more complex networked structures. Among these, star topology networks have proven to be a particularly powerful framework, facilitating efficient management of quantum correlations, optimized resource distribution, and scalable quantum communication [35,38].

An n -star network, as depicted in Fig. 1, comprises a central node, Bob, connected to n peripheral nodes, Alice ^{i} ($i \in \{1, \dots, n\}$), where Bob shares a bipartite entangled state $\rho_{A^i B}$ with each Alice ^{i} via independent sources. Each party performs local measurements: Alice ^{i} and Bob choose inputs x_i and y from the set $\{0, 1\}$, yielding outcomes $a_i, b \in \{+1, -1\}$, respectively. The resulting joint probability distribution is given by

$$P(a_1, \dots, a_n, b | x_1, \dots, x_n, y) = \text{Tr} \left[(A_{x_1}^1 \otimes \dots \otimes A_{x_n}^n \otimes B_y) (\rho_{A^1 B^1} \otimes \dots \otimes \rho_{A^n B^n}) \right], \quad (1)$$

where $A_{x_i}^i$ and B_y denote the observables of the respective parties. The network is said to be n local if the probability distribution admits a hidden-variable model. In this case, it

takes the following form:

$$P(a_1 \dots a_n b | x_1 \dots x_n y) = \int \left(\prod_{i=1}^n d\lambda_i \mu_i(\lambda_i) P(a_i | x_i, \lambda_i) \right) P(b | y, \lambda_1, \dots, \lambda_n), \quad (2)$$

where $\mu_i(\lambda_i)$ is the distribution of the hidden variable λ_i .

To derive testable constraints from this model, it is convenient to introduce the correlation functions:

$$\begin{aligned} \langle A_{x_1}^1 \dots A_{x_n}^n B_y \rangle &= \sum_{a_1, \dots, a_n, b} (-1)^{b + \sum_i a_i} P(a_1 \dots a_n b | x_1 \dots x_n y). \end{aligned} \quad (3)$$

For n -local probability distributions, these correlations are bounded by the following inequality [35]:

$$S_n = \sqrt[n]{|I_n|} + \sqrt[n]{|J_n|} \leq 2, \quad (4)$$

where

$$\begin{aligned} I_n &= \sum_{x_1, \dots, x_n} \langle A_{x_1}^1 \dots A_{x_n}^n B_0 \rangle, \\ J_n &= \sum_{x_1, \dots, x_n} (-1)^{\sum_i x_i} \langle A_{x_1}^1 \dots A_{x_n}^n B_1 \rangle. \end{aligned} \quad (5)$$

A violation of this classical bound certifies that the n -star network exhibits nonlocal correlations.

To characterize the entanglement of quantum systems, one useful measure is the concurrence [66]. For an arbitrary two-qubit pure state, the concurrence is given by

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr}(\rho_A^2))}. \quad (6)$$

Here, we use C to denote the concurrence of the quantum state $|\psi\rangle_{AB} = \cos \theta |00\rangle + \sin \theta |11\rangle$, which is given by $\sin 2\theta$.

III. LIMITATIONS OF NETWORK NONLOCALITY SHARING

Recycling quantum resources through nonlocality sharing requires controlling the disturbance induced by sequential measurements. As information is extracted sequentially, entanglement degrades and may eventually destroy correlations across the network branches. We therefore begin by asking under what conditions the central node can share nonlocal correlations with all n peripheral nodes simultaneously up to an arbitrary depth k .

Answering this question reveals whether the trade-off between sharing depth and network breadth is fundamental and, if not, what entanglement resource is required to sustain multiround operations across the network. In this section, we establish the quantitative relationship among the target depth k , the available entanglement resource C , and the maximum number of branches m that can simultaneously sustain nonlocal correlations.

PPM protocol in the star network. To establish the existence of a valid sharing strategy, we consider a PPM-based sequential protocol to realize nonlocality sharing along m branches of the network. Each observer Alice ^{i,j} on an active branch is equipped with a biased classical coin, which

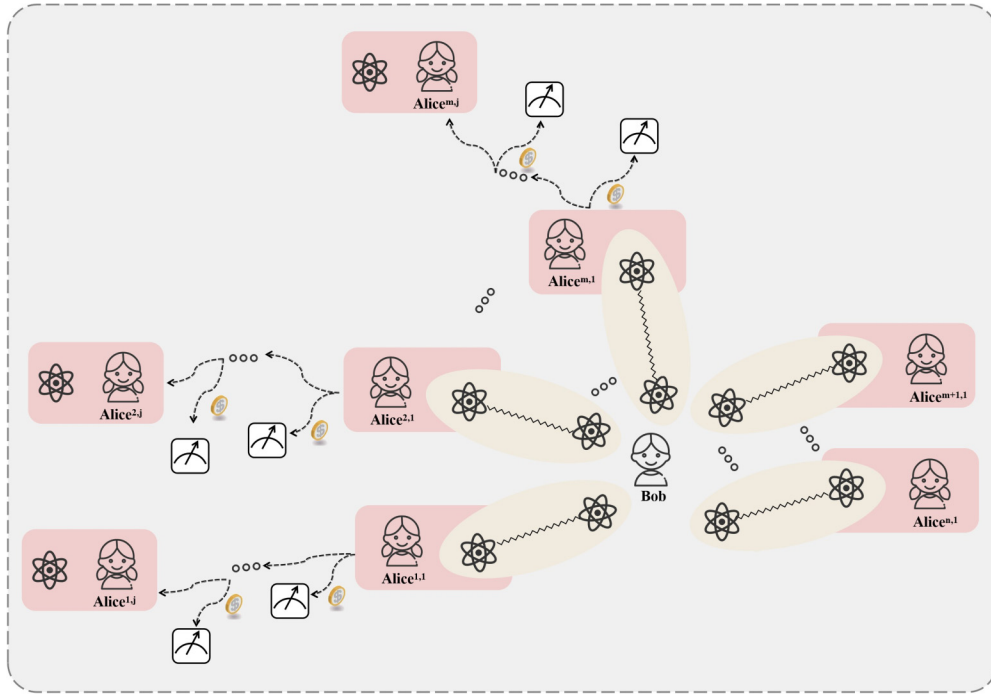


FIG. 2. Sequential nonlocality-sharing protocol. The central node Bob shares an entangled state with each peripheral $Alice^{i,1}$ ($i = 1, \dots, n$). On m branches, sequential observers ($Alice^{i,j}$) perform the PPM strategy: If the input is $x_{i1} = 0$, $Alice^{i,j}$ measures σ_x . If the input is $x_{i1} = 1$, she flips a biased classical coin; on “heads” she measures σ_z , while on “tails” she leaves the state unchanged. Each observer then passes the postmeasurement state to the next observer in the same branch, repeating the PPM up to the final layer. The remaining $n - m$ branches are preserved until the final measurement to detect network nonlocality.

yields “heads” with probability $\alpha_{ij} \in (0, 1)$ and “tails” with probability $1 - \alpha_{ij}$. For a first-generation node $Alice^{i,1}$, the measurement process is determined by a classical input bit $x_{i1} \in \{0, 1\}$:

(1) if $x_{i1} = 0$, $Alice^{i,1}$ performs the projective measurement $A_0^{i,1} = \sigma_x$;

(2) if $x_{i1} = 1$, $Alice^{i,1}$ flips her coin: Upon obtaining “heads,” she performs the projective measurement $A_1^{i,1} = \sigma_z$, whereas upon obtaining “tails,” she leaves the subsystem unmeasured.

The postmeasurement state is then passed to the next observer on the same branch. The same procedure is then repeated sequentially: Each observer $Alice^{i,j}$ receives the incoming state, performs the operation specified by x_{ij} and α_{ij} , and forwards the resulting state to $Alice^{i,j+1}$. This process continues up to the $(k - 1)$ th generation on each active branch.

Consequently, after $k - 1$ sequential rounds on the m active branches, Bob remains correlated with the k th-generation observers on those branches, while on the remaining $n - m$ branches he is still correlated with the first-generation observers, as illustrated in Fig. 2. To test the resulting correlations, all Alice nodes that remain correlated with Bob perform a final PPM round, whereas Bob chooses his measurement according to the input $y \in \{0, 1\}$, corresponding to the observables

$$\begin{aligned} B_0 &= (\sin \delta \sigma_z + \cos \delta \sigma_x)^{\otimes n}, \\ B_1 &= (-\sin \delta \sigma_z + \cos \delta \sigma_x)^{\otimes n}. \end{aligned} \quad (7)$$

Therefore, achieving network nonlocality sharing reduces to finding, for a given initial entanglement resource, a compatible choice of the bias sequence α_{ij} and Bob’s measurement parameter δ . The parameters α_{ij} determine the measurement strength of the PPM at each round and hence govern the trade-off between information extraction and disturbance of the residual state. Bob’s measurement setting also plays a crucial role in the protocol performance, and its optimization depends on the initial entangled resource. In what follows, we impose a uniform bias along the m active branches at the j th sharing round, namely, $\alpha_{1j} = \dots = \alpha_{mj} = \alpha_j$.

To quantify the correlations produced at round j , we consider the corresponding network nonlocality expression $S_n^{m,j}$.

Lemma 1. When nonlocality is shared across m branches, the network nonlocality expression in the j th round can be written as

$$S_n^{m,j} = 2 Q_j^{m/n} (\cos \delta + \sin \delta)^{(n-m)/n}, \quad (8)$$

where

$$Q_j = \cos \delta \sin 2\theta P_j + \sin \delta \left[\frac{\alpha_j}{2^{j-1}} + \cos 2\theta (1 - \alpha_j) \right], \quad (9)$$

with $P_1 = 1$ and $P_j = \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2}$.

The details of the derivation are provided in Appendix B. In particular, for the full-branch case $m = n$, the above expression reduces to $S_n^{n,j} = 2Q_j$. Hence, full-network nonlocality sharing up to round k is guaranteed once $Q_j > 1$ holds for every $j = 1, \dots, k$. We now establish a sufficient condition on the initial entanglement to satisfy this requirement.

Theorem 1. For any given integer k , define $C(k) = 2^{1-k}\sqrt{4^{k-1} - 1}$. If each source distributes a pure entangled state with concurrence $C \in (C(k), 1]$, then there exists a PPM sequential sharing protocol with a bias sequence $\{\alpha_j\}_{j=1}^k$ and a state-dependent measurement parameter δ for Bob such that, for every round $j \in \{1, \dots, k\}$,

$$S_n^{n,j} = \sqrt[n]{|I_n^{n,j}|} + \sqrt[n]{|J_n^{n,j}|} > 2. \quad (10)$$

Hence, network nonlocality sharing along all n branches is achievable up to the k th round.

Proof. We prove the theorem by explicitly constructing a feasible bias sequence $\{\alpha_j\}_{j=1}^k$ together with a suitable choice of Bob's measurement parameter δ . Let each source distribute the pure state $|\psi\rangle_{A^{1,1}B^1} = \cos\theta|00\rangle + \sin\theta|11\rangle$, whose concurrence is $C = \sin 2\theta$. By Lemma 1, successful k -round nonlocality sharing across n branches is equivalent to the condition

$$\alpha_j > \frac{2^{j-1}(1 - \cos\delta \sin 2\theta P_j - \sin\delta \cos 2\theta)}{\sin\delta(1 - 2^{j-1} \cos 2\theta)} \quad (11)$$

for $1 \leq j \leq k$.

We distinguish two cases based on the source entanglement.

Case 1: $0 < \theta < \pi/4$ (nonmaximally entangled sources). Choose Bob's measurement parameter as

$$\delta = \frac{\pi}{2} - 2\theta.$$

Then, Eq. (11) reduces to

$$\alpha_j > \frac{2^{j-1} \cos^2 \delta (1 - P_j)}{\sin \delta (1 - 2^{j-1} \sin \delta)}. \quad (12)$$

Choose $\alpha_1 \in (0, 1)$ sufficiently small so as to satisfy the first-round violation condition. Then, for $2 \leq j \leq k$, we recursively define

$$\alpha_j = (1 + \epsilon) \frac{2^{j-1} \cos^2 \delta (1 - P_j)}{\sin \delta (1 - 2^{j-1} \sin \delta)}, \quad (13)$$

where $\epsilon > 0$ is arbitrarily small, $P_1 = 1$, and $P_j = \prod_{l=1}^{j-1} \frac{2 - \alpha_l}{2}$. By construction, this sequence strictly satisfies the lower bound in Eq. (12).

Case 2: $\theta = \pi/4$ (maximally entangled sources). This is the limiting case of the choice $\delta = \pi/2 - 2\theta$, which formally gives $\delta = 0$. Since Eq. (11) involves $\sin \delta$ in the denominator, we treat this case by taking $\delta > 0$ arbitrarily small and considering the limit $\delta \rightarrow 0^+$. Using $\cos 2\theta = 0$ and $\sin 2\theta = 1$, Eq. (11) becomes

$$\alpha_j > \frac{2^{j-1}(1 - P_j \cos \delta)}{\sin \delta}. \quad (14)$$

In particular, for the first round, $\alpha_1 > (1 - \cos \delta)/\sin \delta$. Since the right-hand side tends to 0 as $\delta \rightarrow 0^+$, any fixed $\alpha_1 > 0$ satisfies this condition for sufficiently small δ . For $2 \leq j \leq k$, we define

$$\alpha_j = (1 + \epsilon) \frac{2^{j-1}(1 - P_j \cos \delta)}{\sin \delta}. \quad (15)$$

Again, this choice strictly satisfies Eq. (14).

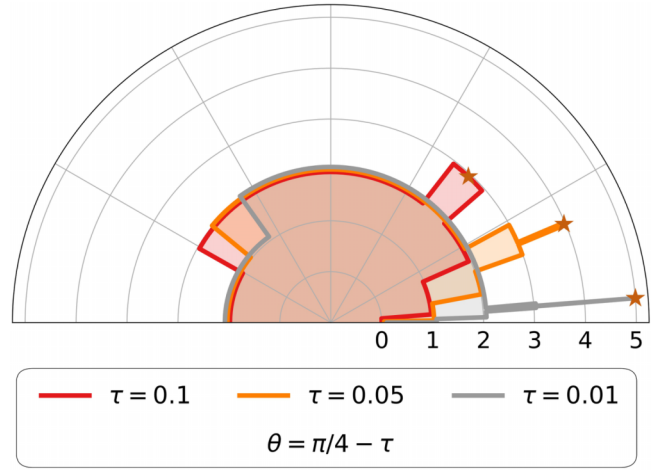


FIG. 3. Achievable sharing rounds. The figure employs a semi-circular coordinate system where the angular position directly represents Bob's measurement parameter δ , ranging from $\delta = 0$ at the right end point to $\delta = \pi/4$ at the left end point. The contours are labeled from 1 to 5, indicating the value of sharing rounds. We map the nonlocality-sharing capability for quantum states with different initial parameter θ . The pentagram markers in the figure denote parameter configurations that satisfy the relation $\delta = \pi/2 - 2\theta$, which is constructed in the proof of Theorem 1. Adherence to this relation guarantees, within our protocol, the achievement of the target sequential sharing round k for a given entanglement resource $C > C(k)$.

As detailed in Appendix A, if the concurrence satisfies

$$C > C(k) = 2^{1-k}\sqrt{4^{k-1} - 1},$$

the above constructed sequence is strictly increasing and, as $\alpha_1 \rightarrow 0$, remains bounded within $(0, 1)$, ensuring it constitutes a valid probability sequence. Therefore, in both cases, there exist a state-dependent measurement parameter δ and a valid bias sequence $\{\alpha_j\}_{j=1}^k$ such that, for every $C \in (C(k), 1]$, network nonlocality sharing along all n branches is achievable up to the k th round. ■

Theorem 1 establishes a sufficient entanglement condition for k -round nonlocality sharing, together with the corresponding measurement settings α_j and δ . This provides a concrete basis for extending multipartite experimental schemes to network nonlocality-sharing scenarios. In Fig. 3, we present the sharing performance for initial states with concurrence $C = \sin 2\theta$. The numerical results corroborate the analytical predictions: For fixed θ , the optimal choice is $\delta = \pi/2 - 2\theta$, and k -round nonlocality sharing is guaranteed whenever $C \in (C(k), 1]$.

The scenario changes distinctively at the threshold $C = C(k)$. At this critical point, the sufficient condition of Theorem 1 no longer guarantees full n -branch sharing. Substituting $C = 2^{1-k}\sqrt{4^{k-1} - 1}$ into the nonlocality inequality in Eq. (10) for the k th round yields

$$S_n^{n,k} = 2(\cos^2 \delta P_k + \sin^2 \delta) < 2, \quad (16)$$

which implies that the sharing fails. This motivates the question of how much of one dimension must be sacrificed to

preserve the other. The following theorem establishes an achievable trade-off at $C = C(k)$.

Theorem 2. For any given integer k , suppose each source distributes an entangled pure state with concurrence $C = C(k)$. Then, there exists a protocol for which the achievable sharing branches m and sharing rounds j satisfy the relation $m + j = n + k - 1$.

Proof. We consider the same star-network scenario as in Theorem 1, with Bob's measurement parameter set to $\delta = \frac{\pi}{2} - 2\theta$.

We first consider the full-branch case $m = n$. Since the fixed resource $C(k)$ is strictly greater than the threshold $C(k-1)$ required for $k-1$ rounds, Theorem 1 guarantees the existence of a valid probability sequence $\{\alpha_1, \dots, \alpha_{k-1}\}$ that satisfies

$$S_n^{n,j} > 2, \quad j = 1, \dots, k-1.$$

Hence, full-network nonlocality sharing is achievable up to round $k-1$.

We next turn to the case $m = n-1$. By Lemma 1, the condition $S_n^{n-1,j} > 2$ is equivalent to

$$Q_j > (\cos \delta + \sin \delta)^{-1/(n-1)}.$$

Since $(\cos \delta + \sin \delta)^{-1/(n-1)} < 1$ for $\delta \in (0, \pi/2)$, this requirement is strictly weaker than the full-branch condition $Q_j > 1$. Consequently, the same choice of bias sequence that realizes $S_n^{n,j} > 2$ for $j = 1, \dots, k-1$ also ensures

$$S_n^{n-1,j} > 2, \quad j = 1, \dots, k-1.$$

It remains to examine the k th round. At the threshold $C = C(k)$, the relation $\sin 2\theta = 2^{1-k} \sqrt{4^{k-1} - 1}$ together with $\delta = \frac{\pi}{2} - 2\theta$ implies

$$\sin \delta = 2^{k-1} \sin^2 \delta.$$

Under this relation, the corresponding quantity in Lemma 1 simplifies to

$$Q_k = \cos^2 \delta P_k + \sin^2 \delta.$$

By choosing α_1 sufficiently small, the recursively defined bias sequence yields $P_k \rightarrow 1$. Hence,

$$Q_k \rightarrow \cos^2 \delta + \sin^2 \delta = 1.$$

Substituting this into Lemma 1, we find

$$\lim_{\alpha_1 \rightarrow 0} S_n^{n-1,k} = 2(\cos \delta + \sin \delta)^{1/n} > 2, \quad \delta \in (0, \pi/2).$$

Therefore, by continuity, there exists a valid choice of sufficiently small $\alpha_1 > 0$ such that

$$S_n^{n-1,k} > 2.$$

This shows that nonlocality sharing across $n-1$ branches is achievable up to the k th round.

Combining the two parts, we conclude that at this threshold, one can trade one branch for one additional round: Full sharing across all n branches remains achievable up to round $k-1$, while reducing the breadth to $n-1$ branches enables sharing up to round k , which satisfy

$$m + j = n + k - 1.$$

This establishes the claimed trade-off. ■

IV. NETWORK NONLOCALITY SHARING UNDER NOISE

Until now, our analysis has assumed ideal pure entangled states, a common but idealized setting in quantum information processing. In realistic settings, however, environmental coupling and imperfections in state preparation inevitably introduce noise. For a given initial entanglement resource, the protocol is specified by a compatible choice of the PPM bias parameters α_{ij} and Bob's measurement parameter δ . The central question in the noisy setting is therefore how decoherence modifies this compatibility. To address this question, we derive the nonlocality-violation conditions in the presence of noise and use them to characterize how noise constrains the admissible measurement strengths, modifies the extractable information at each round, and ultimately limits the achievable sequential sharing depth.

In what follows, we focus on how noise modifies the admissible PPM strengths, while keeping Bob's measurement parameter fixed at the noiseless benchmark value $\delta = \pi/2 - 2\theta$. In the noiseless case, this choice maximizes the achievable sharing depth (see Fig. 3). We first consider the case where each initial bipartite entangled source is subjected to independent depolarizing noise. The resulting mixed state shared between Bob and the i th Alice becomes

$$\rho_{A^{i,1}B^i} = (1-p)|\psi\rangle\langle\psi| + p\frac{\mathbb{1}}{4}, \quad i = 1, \dots, n, \quad (17)$$

with $|\psi\rangle = \cos \theta|00\rangle + \sin \theta|11\rangle$. Assuming uniform noise strength p across all branches, the violation condition [Eq. (C5)] establishes a direct functional relationship between the required measurement probabilities and the noise parameter. Specifically, for the protocol to sustain sequential sharing up to the k th round, the probabilities $\{\alpha_j\}$ can be constructed as

$$\alpha_j = (1+\epsilon) \frac{2^{j-1}((1-p)^{-1} - \sin^2 2\theta P_j - \cos^2 2\theta)}{\cos 2\theta - 2^{j-1} \cos^2 2\theta}. \quad (18)$$

As a second noise model, we consider amplitude damping. In this setting, the entangled states initially shared between Bob and the Alices are described by

$$\rho_{A^{i,1}B^i} = \sum_j (I \otimes K_{ij}) |\psi\rangle\langle\psi| (I \otimes K_{ij})^\dagger, \quad (19)$$

where the Kraus operators are

$$K_1 = \begin{bmatrix} 0 & \sqrt{p_i} \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p_i} \end{bmatrix}. \quad (20)$$

Again assuming identical damping strength p across all branches, the violation condition in Eq. (C11) directly relates the damping parameter to the admissible PPM strengths. For $\epsilon > 0$, network nonlocality can be shared up to the k th round provided that, for all $1 \leq j \leq k$, the parameters $\{\alpha_j\}$ are chosen as

$$\alpha_j = (1+\epsilon) \frac{2^{j-1}(1 - \sqrt{1-p} \sin^2 2\theta P_j - \cos^2 2\theta)}{(1 - 2p \sin^2 \theta) \cos 2\theta - 2^{j-1} \cos^2 2\theta}. \quad (21)$$

For consistency with the noiseless optimization discussed above, we set $\epsilon = 10^{-10}$ in the following numerical simulations. Figure 4 shows how the achievable number of sharing rounds is affected by the noise strength. In contrast to the noiseless case, we find that although pure states with higher

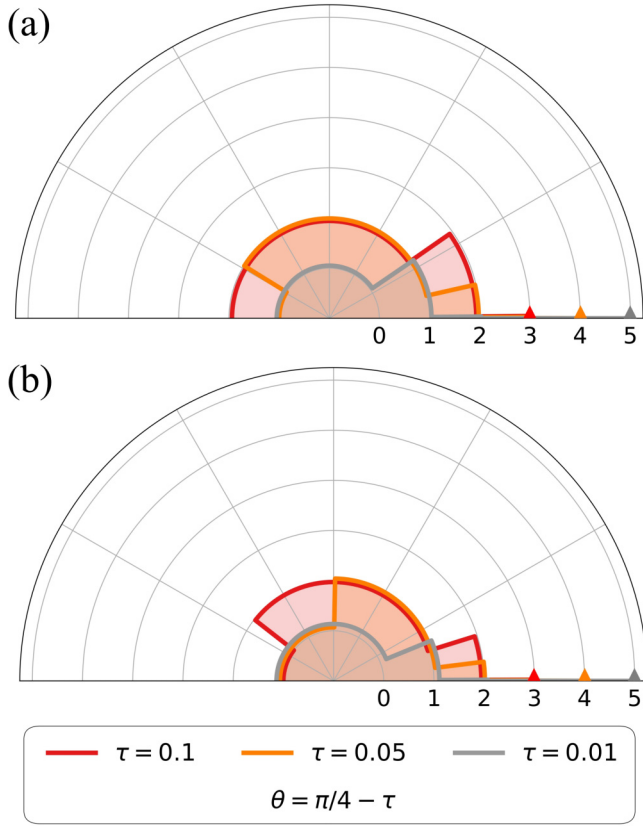


FIG. 4. Sharing capability under noises. The angular coordinate represents the noise strength, with zero noise at the right end point and increasing toward the left. Contour lines labeled from 1 to 5 indicate the maximum number of achievable sharing rounds k for different initial states. Panel (a) covers depolarizing noise, with the noise strength ranging from 0 to 0.1. Panel (b) corresponds to amplitude damping noise, with the damping parameter ranging from 0 to 0.3. The mapping illustrates how the noise level influences the sharing robustness under varying initial entanglement.

initial entanglement yield a larger sharing depth in the absence of noise, they are also more susceptible to decoherence. Consequently, once noise is present, the admissible PPM strengths and the extractable information per round are more strongly restricted for highly entangled states, so that beyond a certain noise level they can be outperformed by states with lower initial entanglement.

V. COMPARISON WITH UNSHARP MEASUREMENT

Beyond noise robustness, another important practical aspect of any sequential sharing protocol is the detectability of the generated nonlocal correlations. Even if many rounds of sharing are theoretically possible, the protocol is of limited experimental value when the resulting nonlocality violations are too weak to be reliably observed. Stronger violations relax the required measurement precision and improve tolerance to imperfections. Weak measurements constitute a common approach to sequentially recycling quantum correlations, as the reduced interaction strength helps limit state disturbance for later observers [45–48]. We therefore assess the

detectability of the PPM protocol by taking the weak-measurement protocol as a benchmark.

We consider the same star-network setting introduced in the previous section where Bob serves as the central node and simultaneously distributes to each Alice the entangled state $|\psi\rangle_{A^{i,1}B^i} = \cos\theta|00\rangle + \sin\theta|11\rangle$. Each node Alice ^{i,j} chooses an input $x_{ij} \in \{0, 1\}$ and performs either the projective measurement $A_0^{i,j} = \sigma_z$ or the unsharp measurement $A_1^{i,j} = \gamma_{ij}\sigma_x$, with γ_{ij} quantifying the degree of disturbance. All Alices within the same generation employ the same unsharpness parameter $\gamma_{1j} = \dots = \gamma_{nj} = \gamma_j$. Bob chooses $y \in \{0, 1\}$ and performs the projective measurements $B_0 = (\sin\omega\sigma_x + \cos\omega\sigma_z)^{\otimes n}$ and $B_1 = (-\sin\omega\sigma_x + \cos\omega\sigma_z)^{\otimes n}$.

Building on the criterion introduced in Ref. [48], the function $S_n^{n,j}$ can be expressed in terms of the two largest singular values λ_1 and λ_2 ($\lambda_1 > \lambda_2$) of the state's correlation matrix [67]:

$$S_n^{n,j} = 2^{2-j} \left(\gamma_j \sqrt{\lambda_2} \sin\omega + \sqrt{\lambda_1} \cos\omega \prod_{l=1}^{j-1} (1 + \sqrt{1 - \gamma_l^2}) \right). \quad (22)$$

For the quantum state $|\psi\rangle_{A^{i,1}B}$, we have $\lambda_1 = 1$ and $\lambda_2 = \sin 2\theta$. Substituting these values into Eq. (22) yields the following condition for violation:

$$\gamma_j > \frac{2^{j-1} - \cos\omega \prod_{l=1}^{j-1} (1 + \sqrt{1 - \gamma_l^2})}{\sqrt{\sin 2\theta} \sin\omega}. \quad (23)$$

To ensure a rigorous comparison, we establish the violation thresholds for both schemes using a unified auxiliary parameter $\epsilon > 0$. The measurement parameters for the unsharp protocol are set as

$$\gamma_j = (1 + \epsilon) \frac{2^{j-1} - \cos\omega \prod_{l=1}^{j-1} (1 + \sqrt{1 - \gamma_l^2})}{\sqrt{\sin 2\theta} \sin\omega}, \quad (24)$$

while the corresponding probabilities for the PPM protocol follow from Eq. (11):

$$\alpha_j = (1 + \epsilon) \frac{2^{j-1} (1 - \sin^2 2\theta P_j - \cos^2 2\theta)}{\cos 2\theta - 2^{j-1} \cos^2 2\theta}. \quad (25)$$

With these analytical forms established, we compare the detectability of the two protocols by evaluating the magnitude of violation per round at the same maximal sharing depth. For consistency, we fix the initial state parameter to $\theta = \pi/4 - 0.01$ and the auxiliary constant to $\epsilon = 10^{-2}$ in both scenarios. Specific measurement parameters are set to $\alpha_1 = 10^{-10}$ for PPM protocol, while for the weak-measurement protocol, we adopt the setting $\omega = (\pi/4) \times 10^{-7}$ from Ref. [48]. As illustrated in Fig. 5, although both approaches successfully achieve a sharing depth of $k = 5$, the PPM protocol delivers a consistently stronger violation in each round. This stronger signal enhances experimental visibility and therefore provides a clear practical advantage for verifying network nonlocality.

We emphasize that the present comparison is limited to existing unsharp-measurement-based network-sharing protocols. Whether other weak-measurement variants can offer improved detectability and operational advantages remains an

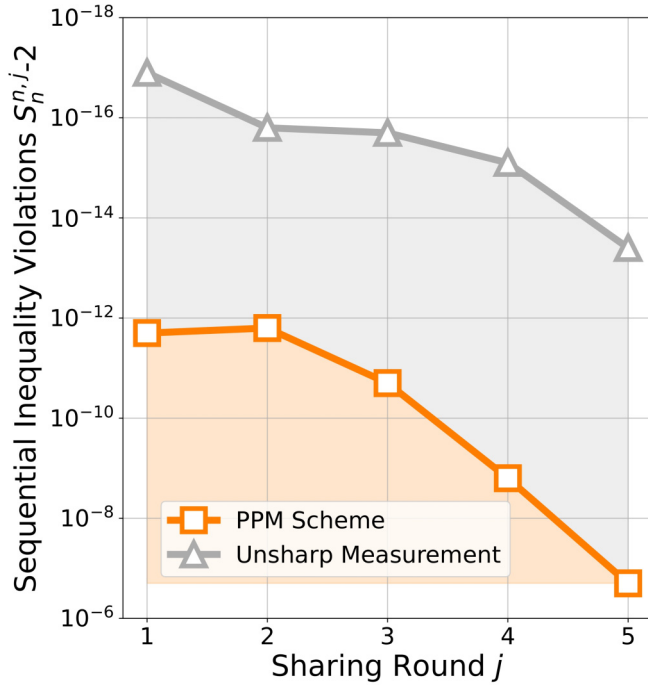


FIG. 5. Comparison with unsharp measurements. The orange line corresponds to the PPM protocol, while the gray line represents the unsharp measurement protocol. For the same sharing depth, the PPM protocol yields consistently stronger nonlocality violations across all rounds, indicating superior experimental detectability.

interesting direction for future work. For example, postselected weak measurement may provide additional advantages, as recent studies suggest its potential for enhancing measurement precision and amplifying weak signals [68,69].

VI. CONCLUSION

In this work, we studied sequential nonlocality sharing in star quantum networks using a PPM protocol. For a target sharing depth k , we identified the corresponding entanglement threshold $C(k)$ by explicitly constructing a feasible bias sequence that realizes the protocol. Above this threshold, nonlocality can be shared across all branches up to the desired depth, showing that the usual trade-off between sharing depth and network breadth is not fundamental when sufficient entanglement is available. At the threshold, in contrast, the two objectives can no longer be achieved simultaneously: Retaining all n branches limits the protocol to round $k - 1$, whereas reaching round k requires reducing the number of active branches to $n - 1$. These results clarify how the available entanglement resource constrains sequential nonlocality sharing in star networks.

We then extended our analysis to realistic settings by incorporating depolarizing and amplitude-damping noise. We have provided a complete protocol that supports k -round sharing under realistic noise. Our numerical results show that, once realistic imperfections are taken into account, the relation between entanglement and sequential sharing performance becomes more subtle: Although more highly entangled states support greater sharing depth in the ideal case, they can become more fragile under noise and may be outperformed by less entangled states at finite noise strength. We further compared PPM with existing unsharp-measurement-based protocols and found that, for the same maximal sharing depth, PPM yields stronger nonlocality violations at each round, potentially improving experimental detectability.

Several theoretical and experimental directions follow from these findings. On the theoretical side, an important next step is to determine how these results extend beyond star networks, where network topology may introduce new scaling laws and resource trade-offs. Introducing the framework of quantum resource theory into nonlocality sharing could further clarify the limits of quantum correlation reuse. More broadly, it will be interesting to explore whether other quantum resources, such as coherence, quantum discord, and magic states, can assist sequential nonlocality sharing in distributed settings. On the experimental side, our results provide useful guidance for future implementations, although extending current tripartite sharing demonstrations to network architectures will require improved control over photon loss, accumulated imperfections, and the realization of biased measurements.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CONSTRUCTION OF SHARING PROTOCOL

In this Appendix, we present a detailed and rigorous construction of the probability sequences $\{\alpha_j\}_j$ employed in the PPM protocol. These sequences are a central ingredient of our scheme, as they control the trade-off between information extraction and disturbance in each measurement round, and hence determine whether network nonlocality can be shared sequentially over multiple rounds.

Lemma A1. For any integer $k \geq 2$, and pure entangled two-qubit states, characterized by the concurrence, $2^{1-k}\sqrt{4^{k-1}-1} < C = \sin 2\theta < 1$, there exists an increasing sequence $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ satisfying

$$\alpha_1 > 0 \quad \text{and} \quad 0 < \frac{2^{j-1} \cos^2 \delta (1 - P_j)}{\sin \delta (1 - 2^{j-1} \sin \delta)} < \alpha_j < 1, \quad \forall 2 \leq j \leq k,$$

where $\delta = \frac{\pi}{2} - 2\theta$ and $P_j = \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2}$.

The proof of this lemma can be found in Ref. [58].

Lemma A2. For any integer $k \geq 2$, there exists a parameter δ and an increasing sequence $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ satisfying

$$\alpha_1 > 0 \quad \text{and} \quad 0 < \frac{2^{j-1} (1 - \cos \delta \cdot P_j)}{\sin \delta} < \alpha_j < 1, \quad \forall 2 \leq j \leq k, \quad (\text{A1})$$

where again $P_j = \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2}$.

Proof. We explicitly construct such a sequence. Let $\epsilon > 0$ be an arbitrarily small constant and define

$$\alpha_1 > 0, \quad \alpha_j = (1 + \epsilon) \frac{2^{j-1} (1 - \cos \delta \cdot P_j)}{\sin \delta}, \quad 2 \leq j \leq k. \quad (\text{A2})$$

We first show that the sequence can be made arbitrarily small. Each α_j can be expressed as a polynomial in α_1 , whose constant term is $\frac{2^{j-1}(1+\epsilon)(1-\cos \delta)}{\sin \delta}$. This term vanishes in the limit $\delta \rightarrow 0$, thus guaranteeing that

$$\lim_{\alpha_1 \rightarrow 0^+} \alpha_j = 0 \quad \text{for all } j. \quad (\text{A3})$$

Next, we establish the monotonicity of the sequence. Choosing α_1 such that

$$\frac{1 - \cos \delta}{\sin \delta} < \alpha_1 < \frac{2(1 - \cos \delta)}{\sin \delta},$$

we obtain for $j = 2$,

$$\frac{\alpha_2}{\alpha_1} = (1 + \epsilon) \left(\frac{\cos \delta}{\sin \delta} + \frac{2(1 - \cos \delta)}{\alpha_1 \sin \delta} \right) > (1 + \epsilon) \left(1 + \frac{\cos \delta}{\sin \delta} \right) > 1.$$

For the general case $j \geq 3$, we compute

$$\frac{\alpha_j}{\alpha_{j-1}} = 2 \times \frac{1 - \cos \delta, P_j}{1 - \cos \delta, P_{j-1}} = 2 \times \frac{1 - \cos \delta, P_{j-1} \left(1 - \frac{\alpha_{j-1}}{2}\right)}{1 - \cos \delta, P_{j-1}} = 2 + \frac{\cos \delta, P_{j-1} \alpha_{j-1}}{1 - \cos \delta, P_{j-1}} > 2 + \frac{2^{j-1} P_{j-1}}{\sin \delta} > 1. \quad (\text{A4})$$

Finally, by taking the joint limit $\alpha_1, \delta \rightarrow 0^+$, we ensure that all α_j lie in the interval $(0, 1)$, while the strict inequalities $\alpha_j > \alpha_{j-1}$ are preserved. This completes the proof. ■

APPENDIX B: DERIVATION OF THE NETWORK NONLOCALITY INEQUALITY UNDER THE PPM PROTOCOL

In this Appendix, we present a complete derivation of the network nonlocality inequality associated with the PPM protocol in the n -star network. We explicitly compute the average postmeasurement state after each sharing round and evaluate the resulting correlators between the central node and the peripheral observers. This Appendix provides the full technical justification for the expression of $S_n^{m,j}$ used in the main text.

Lemma B1. Let $\rho_j (j = 2, 3, \dots)$ denote the average state after all observers Alice^{1,j-1}, Alice^{2,j-1}, ..., Alice^{m,j-1} have performed their probabilistic projective measurements and passed their respective postmeasurement states to the subsequent observers Alice^{1,j}, Alice^{2,j}, ..., Alice^{m,j}, under the assumptions of uniform measurement bias $\alpha_{1j} = \dots = \alpha_{mj} = \alpha_j$, ρ_j has the form

$$\rho_j = \sum_{p+q=0}^m \sum_{\{\beta_1, \dots, \beta_p\} \subset \{1, \dots, m\}} \sum_{\{\gamma_1, \dots, \gamma_q\} \subset \{1, \dots, m\} \setminus \{\beta_1, \dots, \beta_p\}} \frac{\alpha_{j-1}^q (3 - \alpha_{j-1})^{m-p-q}}{2^{2m}} \mathcal{A}_0^{\beta_1 \dots \beta_p, j-1} \mathcal{A}_1^{\gamma_1 \dots \gamma_q, j-1} \rho_{j-1} \mathcal{A}_0^{\beta_1 \dots \beta_p, j-1} \mathcal{A}_1^{\gamma_1 \dots \gamma_q, j-1}, \quad (\text{B1})$$

where $\mathcal{A}_0^{\beta_1 \dots \beta_p, j-1} = A_0^{\beta_1, j-1} \otimes \dots \otimes A_0^{\beta_p, j-1}$ and $\mathcal{A}_1^{\gamma_1 \dots \gamma_q, j-1} = A_1^{\gamma_1, j-1} \otimes \dots \otimes A_1^{\gamma_q, j-1}$.

Proof. For an n -star network state $\rho = \bigotimes_{i=1}^n \rho_{A^i B^i}$, where Alice¹, ..., Alice^m perform projective measurements $A_{\pm|x_i}^i = \frac{1}{2}(\mathbb{1} \pm A_{x_i}^i)$, $i = 1 \dots m$, a direct induction argument shows that averaging over all outcomes yields

$$\tilde{\rho} = \sum_{r=0}^m \sum_{\substack{\{i_1, \dots, i_r\} \\ \subset \{1, \dots, m\}}} \mathcal{A}_{x_{i_1} \dots x_{i_r}}^{i_1 \dots i_r} \rho \mathcal{A}_{x_{i_1} \dots x_{i_r}}^{i_1 \dots i_r}, \quad (\text{B2})$$

where $\mathcal{A}_{x_{i_1} \dots x_{i_r}}^{i_1 \dots i_r} = A_{x_{i_1}}^{i_1} \otimes A_{x_{i_2}}^{i_2} \otimes \dots \otimes A_{x_{i_r}}^{i_r}$. This is proved via mathematical induction: For $k = 1$, the statement holds because

$$\tilde{\rho} = \frac{\mathbb{1} + A_{x_1}^1}{2} \rho \frac{\mathbb{1} + A_{x_1}^1}{2} + \frac{\mathbb{1} - A_{x_1}^1}{2} \rho \frac{\mathbb{1} - A_{x_1}^1}{2} = \frac{1}{2} (\rho + A_{x_1}^1 \rho A_{x_1}^1). \quad (\text{B3})$$

Assume that the statement holds for $k = m - 1$. Then, for $k = m$, we have

$$\begin{aligned} \tilde{\rho} &= \frac{1}{2^{m-1}} \left[\left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) A_{x_1}^1 \dots A_{x_{m-1}}^{m-1} \rho A_{x_1}^1 \dots A_{x_{m-1}}^{m-1} \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) + \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) A_{x_1}^1 \dots A_{x_{m-1}}^{m-1} \rho A_{x_1}^1 \dots A_{x_{m-1}}^{m-1} \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) \right. \\ &\quad + \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) \sum_{\substack{\{i_1, \dots, i_{m-2}\} \\ \subset \{1, \dots, m-1\}}} A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-2}}}^{i_{m-2}} \rho A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-2}}}^{i_{m-2}} \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) \\ &\quad + \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) \sum_{\substack{\{i_1, \dots, i_{m-2}\} \\ \subset \{1, \dots, m-1\}}} A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-2}}}^{i_{m-2}} \rho A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-2}}}^{i_{m-2}} \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) + \dots + \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) \sum_{i_1=1}^m A_{x_{i_1}}^{i_1} \rho A_{x_{i_1}}^{i_1} \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) \\ &\quad \left. + \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) \sum_{i_1=1}^m A_{x_{i_1}}^{i_1} \rho A_{x_{i_1}}^{i_1} \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) + \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) \rho \left(\frac{\mathbb{1} + A_{x_m}^m}{2} \right) + \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) \rho \left(\frac{\mathbb{1} - A_{x_m}^m}{2} \right) \right] \\ &= \frac{1}{2^m} \left[A_{x_1}^1 A_{x_2}^2 \dots A_{x_m}^m \rho A_{x_1}^1 A_{x_2}^2 \dots A_{x_m}^m + \sum_{\substack{\{i_1, \dots, i_{m-1}\} \\ \subset \{1, \dots, m\}}} A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-1}}}^{i_{m-1}} \rho A_{x_{i_1}}^{i_1} \dots A_{x_{i_{m-1}}}^{i_{m-1}} + \dots + \sum_{i_1=1}^m A_{x_{i_1}}^{i_1} \rho A_{x_{i_1}}^{i_1} + \rho \right]. \end{aligned}$$

Thus, by induction, Eq. (B2) is proved. Consequently, under PPM protocol the state ρ_j shared by Alice^{1,j}, Alice^{2,j}, ..., Alice^{m,j}, ..., Alice^{m+1,1}, ..., Alice^{n,1} and Bob has the form

$$\begin{aligned} \rho_j &= \frac{1}{2^m} \sum_{p+q=0}^m \sum_{\{\beta_1, \dots, \beta_p\} \subset \{1, \dots, m\}} \sum_{\{\gamma_1, \dots, \gamma_q\} \subset \{1, \dots, m\} \setminus \{\beta_1, \dots, \beta_p\}} \sum_{x_{\beta_s} \in \{0, 2\}} \sum_{x_{\gamma_t} \in \{1, 2\}} \frac{\alpha_{j-1}^q (1 - \alpha_{j-1})^{m-p-q}}{2^{p+q}} \\ &\quad \times A_{x_{\beta_1}}^{\beta_1, j-1} \dots A_{x_{\beta_p}}^{\beta_p, j-1} A_{x_{\gamma_1}}^{\gamma_1, j-1} \dots A_{x_{\gamma_q}}^{\gamma_q, j-1} \rho_{j-1} A_{x_{\beta_1}}^{\beta_1, j-1} \dots A_{x_{\beta_p}}^{\beta_p, j-1} A_{x_{\gamma_1}}^{\gamma_1, j-1} \dots A_{x_{\gamma_q}}^{\gamma_q, j-1}, \end{aligned} \quad (\text{B4})$$

where $A_2^{l_s, j} = A_2^{g_t, j} = \mathbb{1}$. Based on our PPM protocol, we can assume that among the m systems, p systems perform measurement A_0 , q systems perform measurement A_1 , and the remaining systems undergo no measurement. By exhaustively considering all possible measurement configurations, we obtain the first three summation symbols together with the probability $\alpha_{j-1}^q (1 - \alpha_{j-1})^{m-p-q}$. Furthermore, according to the structure of the postmeasurement quantum state established in Eq. (B2), the remaining two summation symbols naturally arise, accompanied by the normalization factor $\frac{1}{2^{p+q}}$. Noting that each term in ρ_j admits the form

$$A_0^{\beta_1, j-1} \dots A_0^{\beta_p, j-1} A_1^{\gamma_1, j-1} \dots A_1^{\gamma_q, j-1} \rho_{j-1} A_0^{\beta_1, j-1} \dots A_0^{\beta_p, j-1} A_1^{\gamma_1, j-1} \dots A_1^{\gamma_q, j-1},$$

we can determine the corresponding coefficient for each such term as

$$\begin{aligned} &\sum_{u+v=0}^{m-p-q} \binom{m-p-q}{u} \binom{m-p-q-u}{v} \frac{\alpha_{j-1}^{q+v} (1 - \alpha_{j-1})^{m-(p+u)-(q+v)}}{2^{(p+u)+(q+v)}} \\ &= \left(\frac{1}{2} \right)^p \left(\frac{\alpha_{j-1}}{2} \right)^q \left(1 - \frac{\alpha_{j-1}}{2} + \frac{1}{2} \right)^{m-p-q} = \frac{\alpha_{j-1}^q (3 - \alpha_{j-1})^{m-p-q}}{2^m}. \end{aligned} \quad (\text{B5})$$

Consequently, we recover the expression of ρ_j given in Eq. (B1). ■

Lemma B2. Under our PPM protocol, the network composed of Alice^{1,j}, ..., Alice^{m,j}, Alice^{m+1,1}, ..., Alice^{n,1}, and Bob is nonlocal if

$$S_n^{m,j} = \sqrt[n]{I_n^{m,j}} + \sqrt[n]{J_n^{m,j}} = 2 \left(\cos \delta \sin 2\theta \prod_{l=1}^{j-1} \frac{2 - \alpha_l}{2} + \sin \delta \frac{\alpha_j}{2^{j-1}} + \sin \delta \cos 2\theta (1 - \alpha_j) \right)^{\frac{m}{n}} (\cos \delta + \sin \delta)^{\frac{n-m}{n}} > 2. \quad (\text{B6})$$

Proof. Under the PPM protocol, the quantum network comprised by Alice^{1,j}, ..., Alice^{m,j}, Alice^{m+1,1}, ..., Alice^{n,1}, and Bob exhibits network nonlocality whenever

$$S_n^{m,j} = \sqrt[n]{|I_n^{m,j}|} + \sqrt[n]{|J_n^{m,j}|} > 2, \quad (\text{B7})$$

where the correlators $I_n^{m,j}$ and $J_n^{m,j}$ are defined as

$$I_n^{m,j} = \sum_{x_{m+1}, \dots, x_n \in \{0,1\}} \sum_{h+f=0}^m \sum_{\{\xi_1, \dots, \xi_h\} \subset \{1, \dots, m\}} \sum_{\{\zeta_1, \dots, \zeta_f\} \subset \{1, \dots, m\} \setminus \{\xi_1, \dots, \xi_h\}} \times \alpha_j^f (1 - \alpha_j)^{m-h-f} \text{Tr}[(\mathcal{A}_0^{\xi_1 \dots \xi_h, j} \mathcal{A}_1^{\zeta_1 \dots \zeta_f, j} \mathcal{A}_{x_{m+1} \dots x_n}^{m+1 \dots n, j} B_0) \rho_j]. \quad (\text{B8})$$

$$J_n^{m,j} = \sum_{x_{m+1}, \dots, x_n \in \{0,1\}} \sum_{h+f=0}^m \sum_{\{\xi_1, \dots, \xi_h\} \subset \{1, \dots, m\}} \sum_{\{\zeta_1, \dots, \zeta_f\} \subset \{1, \dots, m\} \setminus \{\xi_1, \dots, \xi_h\}} \times (-1)^{m-h+\sum_{i=m+1}^n x_i} \alpha_j^f (1 - \alpha_j)^{m-h-f} \text{Tr}[(\mathcal{A}_0^{\xi_1 \dots \xi_h, j} \mathcal{A}_1^{\zeta_1 \dots \zeta_f, j} \mathcal{A}_{x_{m+1} \dots x_n}^{m+1 \dots n, j} B_1) \rho_j]. \quad (\text{B9})$$

Equations (B8) and (B9) perform a summation over all possible configurations of the m systems, where each system undergoes one of three options: an A_0 measurement, an A_1 measurement, or no measurement. A weight of α_j is assigned to the cases where an A_1 measurement is performed, and a weight of $1 - \alpha_j$ is assigned to the cases with no measurement.

We now proceed to analyze each term in Eqs. (B8) and (B9). To simplify notation, we introduce

$$M_{j,\pm}^{h,f} = \sum_{x_{m+1}, \dots, x_n \in \{0,1\}} (\pm 1)^{\sum_{i=m+1}^n x_i} \mathcal{A}_0^{\xi_1 \dots \xi_h, j} \mathcal{A}_1^{\zeta_1 \dots \zeta_f, j} \mathcal{A}_{x_{m+1} \dots x_n}^{m+1 \dots n, j}.$$

In the case where, among the m systems, h systems undergo the A_0 measurement and f systems undergo the A_1 measurement, we have

$$\begin{aligned} \text{Tr}[(M_{j,+}^{h,f} B_0) \rho_j] &= \sum_{p+q=0}^m \sum_{\{\beta_1, \dots, \beta_p\} \subset \{1, \dots, m\}} \sum_{\{\gamma_1, \dots, \gamma_q\} \subset \{1, \dots, m\} \setminus \{\beta_1, \dots, \beta_p\}} \frac{\alpha_{j-1}^q (3 - \alpha_{j-1})^{m-p-q}}{2^m} \\ &\quad \times \text{Tr}[(M_j^{h,f} B_0) \mathcal{A}_0^{p,j-1} \mathcal{A}_1^{q,j-1} \rho_{j-1} \mathcal{A}_0^{p,j-1} \mathcal{A}_1^{q,j-1}] \\ &= \sum_{p+q=0}^m \sum_{\substack{w+u=\max\{0, p-(m-h-f)\} \\ w \leq h, u \leq f}}^{\min\{p, h+f\}} \sum_{\substack{v+z=\max\{0, q-(m-h-f-(p-w-u))\} \\ v \leq h-w, z \leq f-u}}^{\min\{q, h+f-w-u\}} \\ &\quad \times (-1)^{u+v} \binom{h}{w} \binom{f}{u} \binom{h-w}{v} \binom{f-u}{z} \binom{m-h-f}{p-w-u} \\ &\quad \times \binom{m-h-f-(p-w-u)}{q-u-z} \frac{\alpha_{j-1}^q (3 - \alpha_{j-1})^{m-p-q}}{2^{2m}} \text{Tr}[(M_j^{h,f} B_0) \rho_{j-1}]. \end{aligned} \quad (\text{B10})$$

Using the identities $A_0^2 = A_1^2 = \mathbb{1}$ and $A_0 A_1 = -A_1 A_0$, together with the cyclicity of the trace, the above expression can be reorganized by grouping the m systems into three subsets (see Fig. 6):

- (1) S_1 : the h systems measured with A_0 ,
- (2) S_2 : the f systems measured with A_1 ,
- (3) S_3 : the remaining $m - h - f$ systems.

Among the p measurements of A_0 acting on ρ_{j-1} , suppose w are in S_1 , u are in S_2 , and the remaining $p - w - u$ are in S_3 . Because each of the u measurements of A_0 in S_2 anticommutes with the A_1 measurements in that same group, a factor $(-1)^u$ arises. Summing over all such assignments yields the combinatorial factor

$$(-1)^u \binom{h}{w} \binom{f}{u} \binom{m-h-f}{p-w-u}.$$

Proceeding analogously for the q measurements of A_1 produces an additional factor

$$(-1)^v \binom{h-w}{v} \binom{f-u}{z} \binom{m-h-f-(p-w-u)}{q-v-z},$$

where the sign factor $(-1)^v$ originates from anticommutations with measurements in the respective subgroups.

Carrying out the summations explicitly and applying the binomial theorem, we find

$$\sum_{q,v,z} (-1)^v \binom{h-w}{v} \binom{f-u}{z} \binom{m-h-f-(p-w-u)}{q-v-z} \frac{\alpha_{j-1}^q (3 - \alpha_{j-1})^{m-p-q}}{2^{2m}} = \frac{(3 - 2\alpha_{j-1})^{h-w} 3^{m-h-w}}{2^{2m}}. \quad (\text{B11})$$

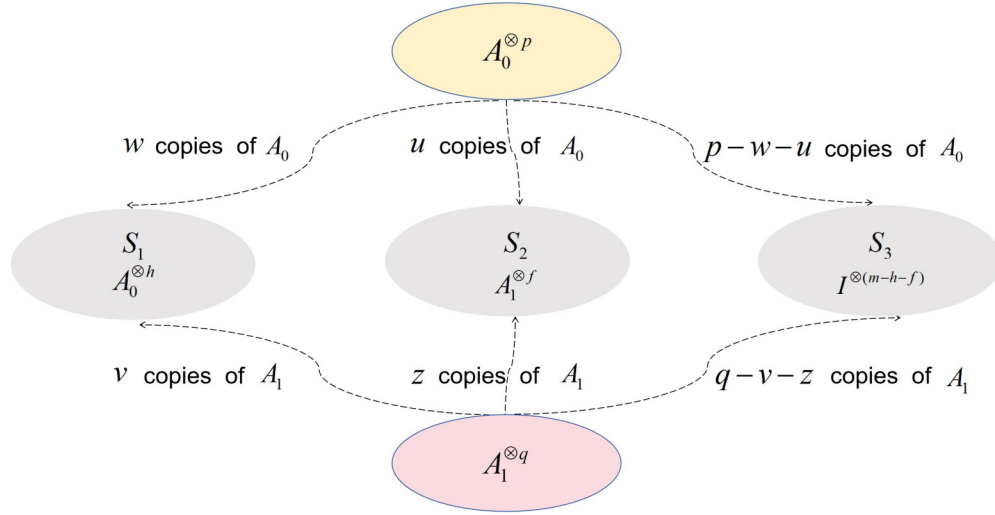


FIG. 6. Schematic explanation of the derivation of Eq. (B10). The n measurement systems are divided into three groups, as shown in the central ellipses: h systems measured with A_0 , f with A_1 , and the remaining $m - h - f$ with no measurement. The top and bottom ellipses represent the p measurements of A_0 and q measurements of A_1 performed on the state ρ_{j-1} , respectively. The arrows indicate how the measurement choices at step j are correlated with the system positions from step $j - 1$.

Substituting this result back into Eq. (B10) and summing over p, w, u finally gives

$$\text{Tr}[(M_{j,+}^{h,f} B_0) \rho_j] = \frac{(2 - \alpha_j)^h}{2^{h+f}} \text{Tr}[(M_{j,+}^{h,f} B_0) \rho_{j-1}]. \quad (\text{B12})$$

Iterating this relation back to the initial state ρ_1 , we obtain

$$\text{Tr}[(M_{j,+}^{h,f} B_0) \rho_j] = \prod_{l=1}^{j-1} \frac{(2 - \alpha_l)^h}{2^{h+f}} (\cos \delta \sin 2\theta)^h (\sin \delta)^f (\sin \delta \cos 2\theta)^{m-h-f} (\cos \delta + \sin \delta)^{n-m} \quad (\text{B13})$$

and similarly,

$$\text{Tr}[(M_{j,-}^{h,f} B_1) \rho_j] = (-1)^{m-h} \prod_{l=1}^{j-1} \frac{(2 - \alpha_l)^h}{2^{h+f}} (\cos \delta \sin 2\theta)^h (\sin \delta)^f (\sin \delta \cos 2\theta)^{m-h-f} (\cos \delta + \sin \delta)^{n-m}. \quad (\text{B14})$$

Substituting Eqs. (B13) and (B14) into Eqs. (B8) and (B9), respectively, we finally obtain

$$\begin{aligned} I_n^{m,j} &= J_n^{m,j} = \sum_{h+f=0}^m \binom{m}{h} \binom{m-h}{f} \alpha_j^f (1 - \alpha_j)^{m-h-f} \\ &\quad \times \prod_{l=1}^{j-1} \frac{(2 - \alpha_l)^h}{2^{h+f}} (\cos \delta \sin 2\theta)^h (\sin \delta)^f (\sin \delta \cos 2\theta)^{m-h-f} (\cos \delta + \sin \delta)^{n-m} \\ &= \left(\cos \delta \sin 2\theta \prod_{l=1}^{j-1} \frac{2 - \alpha_l}{2} + \sin \delta \frac{\alpha_j}{2^{j-1}} + \sin \delta \cos 2\theta (1 - \alpha_j) \right)^m (\cos \delta + \sin \delta)^{n-m}. \end{aligned} \quad (\text{B15})$$

Therefore, the network is nonlocal whenever

$$S_n^{m,j} = 2 \left(\cos \delta \sin 2\theta \prod_{l=1}^{j-1} \frac{2 - \alpha_l}{2} + \sin \delta \frac{\alpha_j}{2^{j-1}} + \sin \delta \cos 2\theta (1 - \alpha_j) \right)^{\frac{m}{n}} (\cos \delta + \sin \delta)^{\frac{n-m}{n}} > 2. \quad (\text{B16})$$

■

APPENDIX C: NETWORK NONLOCALITY SHARING UNDER NOISES

In this Appendix, we extend the analysis of the PPM protocol to realistic scenarios in which the initially distributed entangled states are affected by environmental noise. We focus on two paradigmatic noise models—depolarizing noise and amplitude

damping noise—and investigate how they modify the conditions under which network nonlocality can be sequentially shared across all branches of the network.

We first consider the case in which each initial bipartite state is independently subjected to a depolarizing channel, yielding

$$\rho_{A^i,1B^i} = (1-p)|\psi\rangle\langle\psi| + p\frac{\mathbb{I}}{4}, \quad i = 1, \dots, n, \quad (\text{C1})$$

where $|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ and $p \in [0, 1]$ denotes the noise strength. Throughout this Appendix, we assume that the noise parameter p is identical for all shared states. Using Eqs. (B13) and (B14), the expectation values entering the correlation functions can be written as

$$\begin{aligned} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_j\} &= \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_1\} \\ &= (1-p)^n \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\cos\delta \sin 2\theta)^h (\sin\delta)^f (\sin\delta \cos 2\theta)^{n-h-f} \end{aligned} \quad (\text{C2})$$

and

$$\begin{aligned} \text{Tr}\{(M_{j,-}^{h,f} B_0)\rho_j\} &= \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_1\} \\ &= (1-p)^n (-1)^{n-h} \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\cos\delta \sin 2\theta)^h (\sin\delta)^f (\sin\delta \cos 2\theta)^{n-h-f}. \end{aligned} \quad (\text{C3})$$

Consequently, the correlators take the form

$$\begin{aligned} I_n^{n,j} = J_n^{n,j} &= \sum_{h+f=0}^n \binom{n}{h} \binom{n-h}{f} \alpha_j^f (1-\alpha_j)^{n-h-f} \\ &\times \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\cos\delta \sin 2\theta)^h (\sin\delta)^f (\sin\delta \cos 2\theta)^{n-h-f} (1-p)^n \\ &= \left(\cos\delta \sin 2\theta \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2} + \sin\delta \frac{\alpha_j}{2^{j-1}} + \sin\delta \cos 2\theta (1-\alpha_j) \right)^n (1-p)^n. \end{aligned} \quad (\text{C4})$$

In particular, for the choice $\delta + 2\theta = \pi/2$, network nonlocality can be shared up to the k th round under depolarizing noise if, for all $1 \leq j \leq k$,

$$\alpha_j > \frac{2^{j-1}((1-p)^{-1} - \sin^2 2\theta \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2} - \cos^2 2\theta)}{\cos 2\theta (1 - 2^{j-1} \cos 2\theta)}. \quad (\text{C5})$$

We now turn to the amplitude damping noise model. Each initial state undergoes an amplitude damping channel, resulting in

$$\rho_{A^i,1B^i} = \sum_j (I \otimes K_j) |\psi\rangle\langle\psi| (I \otimes K_j)^\dagger = \begin{bmatrix} \cos^2\theta & 0 & 0 & \cos\theta \sin\theta \sqrt{1-p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p \sin^2\theta & 0 \\ \cos\theta \sin\theta \sqrt{1-p} & 0 & 0 & (1-p) \sin^2\theta \end{bmatrix}, \quad i = 1, \dots, n, \quad (\text{C6})$$

where

$$K_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}. \quad (\text{C7})$$

Assuming again identical noise parameters for all shared states, Eqs. (B13) and (B14) yield

$$\begin{aligned} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_j\} &= \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_1\} \\ &= \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\sqrt{1-p} \cos\delta \sin 2\theta)^h ((1-2p \sin^2\theta) \sin\delta)^f (\sin\delta \cos 2\theta)^{n-h-f} \end{aligned} \quad (\text{C8})$$

and

$$\begin{aligned} \text{Tr}\{(M_{j,-}^{h,f} B_0)\rho_j\} &= \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} \text{Tr}\{(M_{j,+}^{h,f} B_0)\rho_1\} \\ &= (-1)^{n-h} \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\sqrt{1-p} \cos \delta \sin 2\theta)^h ((1-2p \sin^2 \theta) \sin \delta)^f (\sin \delta \cos 2\theta)^{n-h-f}. \end{aligned} \quad (\text{C9})$$

Accordingly, the correlations become

$$\begin{aligned} I_n^{n,j} &= J_n^{n,j} = \sum_{h+f=0}^n \binom{n}{h} \binom{n-h}{f} \alpha_j^f (1-\alpha_j)^{n-h-f} \\ &\quad \times \prod_{l=1}^{j-1} \frac{(2-\alpha_l)^h}{2^{h+f}} (\sqrt{1-p} \cos \delta \sin 2\theta)^h ((1-2p \sin^2 \theta) \sin \delta)^f (\sin \delta \cos 2\theta)^{n-h-f} \\ &= \left(\sqrt{1-p} \cos \delta \sin 2\theta \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2} + (1-2p \sin^2 \theta) \sin \delta \frac{\alpha_j}{2^{j-1}} + \sin \delta \cos 2\theta (1-\alpha_j) \right)^n. \end{aligned} \quad (\text{C10})$$

For $\delta + 2\theta = \pi/2$, network nonlocality can therefore be shared up to the k th round under amplitude damping noise if, for all $1 \leq j \leq k$,

$$\alpha_j > \frac{2^{j-1} (1 - \sqrt{1-p} \sin^2 2\theta \prod_{l=1}^{j-1} \frac{2-\alpha_l}{2} - \cos^2 2\theta)}{(1-2p \sin^2 \theta) \cos 2\theta - 2^{j-1} \cos^2 2\theta}. \quad (\text{C11})$$

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