

# Emergent symmetries and phases in quantum spin chains coupled to a Kuramoto model

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Floquet theory is a widely used framework to describe the dynamics of periodically driven quantum systems. The usual scenario to describe such systems is to consider the effect of an external control with a definite period in time that can act either locally or globally on the system of interest. However, apart from periodicity, such drives typically lack classical correlations or additional structure. In this work, we consider drives with intrinsic dynamics that undergo self-organization, leading to periodic steady states with emergent symmetries. To substantiate our results, we consider two examples of one-dimensional quantum spin chains coupled to a classical Kuramoto model. First, we investigate a Kuramoto model with all-to-all coupling driving a one-dimensional quantum Ising chain into a time-periodic steady state with an emergent translational symmetry. Next, we consider a Kuramoto model in a zigzag lattice driving an XX spin chain. The dynamics of traveling waves in the Kuramoto model trimerizes the lattice, effectively inducing topological behavior that can be exploited to perform topological pumping. Our results can be experimentally implemented in digital and analog near-term quantum devices.

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## I. INTRODUCTION

Self-organization is one of the most intriguing phenomena in nature [1]. Individual units can form complex patterns and structures by following simple dynamical rules [2]. These exist due to the delicate balance between nonlinear behavior, energy gain, and dissipation [3]. Synchronization provides a paradigmatic example of self-organization in complex dynamical systems [4–6], extending even into the quantum regime [7–13].

The Kuramoto model of coupled phase oscillators [14] is a cornerstone of synchronization theory. It exhibits a rich variety of dynamical patterns including traveling waves [15], chimera states [16,17], and spiral and scroll waves [18,19], among others. Applications range from electrical networks [20] to neuronal dynamics [21] and chemical oscillators [22], to mention only a few.

Driven quantum many-body systems also display rich and diverse behavior [23]. For example, Floquet systems have been studied in early works on nuclear magnetic resonance [24,25]. There high-frequency expansions were used to derive effective Hamiltonians describing the dynamics at stroboscopic times [26–28]. In the context of many-body systems, the convergence of the Magnus expansion [26] is intimately related to heating phenomena [29–31]. High-frequency expansions can also be used to realize phases of matter with no equilibrium counterparts [32–34].

Floquet systems host thermal [35,36], many-body localized [37], ergodic-localized junctions [38,39], and time crystalline phases [40–42], as well as many-body quantum scars [43,44]. More recently, the role of quasiperiodic drives in thermalization and complex dynamics has been explored [45,46]. A common feature of these approaches, however, is that the drive is externally imposed and lacks internal dynamics or structure.

This raises a natural question: Which kind of behaviors can emerge in a many-body system subject to a drive with its own dynamics?

In this work, we address this question by considering a quantum many-body system subject to a drive whose dynamics are governed by the Kuramoto model [14]. Specifically, we couple a one-dimensional quantum spin chain [47] to the Kuramoto model [14], solving the classical oscillator equations for a given network topology and using the resulting dynamics as a local drive for the spin chain. Initially, the quantum system has no symmetry, as the oscillators are initialized with random phases. However, as the Kuramoto dynamics evolve and patterns form, we investigate how these structures influence the quantum dynamics.

We focus on two different scenarios. First, we couple a quantum Ising chain in a transverse field [48] to a Kuramoto model with all-to-all coupling. The latter exhibits a transition to synchronized motion after a transient period. This effectively drives the system from a structureless initial state with no symmetries before the transient time to a system with translational invariance in space and time, which are emergent symmetries during the dynamics. We also consider a Kuramoto model in a zigzag lattice with unidirectional coupling acting as a drive for a quantum XX model [49]. After a transient time, the classical drive forms a periodic traveling wave pattern that breaks time-reversal symmetry of

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the quantum model. This induces an emergent topological behavior that can be exploited to perform topological pumping of edge excitations in the chain.

The field of driven quantum systems beyond Floquet and quasiperiodic drives is widely unexplored. A recent work has predicted that quasiperiodic drives induce complete Hilbert-space ergodicity [46]. Our main contribution is to show that the dynamics of a self-organized drive induces rich behavior on our quantum spin chain and is responsible for emergent symmetries. In future works, it would be interesting to investigate quantum signatures of pattern formation in nonintegrable models and to explore the phases that appear in this scenario.

The paper is structured as follows. In Sec. II, we discuss a semiclassical approach to an XY spin chain coupled to a network of quantum Stuart-Landau oscillators. In the semiclassical limit of the oscillators, we decouple the quantum fluctuations from the spin chain by ignoring the back action. This allows us to investigate the effect of the mean field on the XY spin chain by adiabatically eliminating the amplitude dynamics of the oscillator network. Section III discusses the model we investigate in this work, which is a spin chain coupled to a Kuramoto model. Sections IV and V discuss in detail the physical implications of the dynamics of Kuramoto model and the role of emergent symmetries. Finally, we make concluding remarks and an outlook in Sec. VI.

## II. XY SPIN CHAIN COUPLED TO A NETWORK OF QUANTUM STUART-LANDAU OSCILLATORS: A SEMICLASSICAL APPROACH

Similarly to Ref. [9], we consider a quantum network consisting of a ring of  $N$  coupled Stuart-Landau oscillators. Such a network can be described by the master equation for the density matrix  $\rho(t)$  [12]:

$$\dot{\rho} = -\frac{i}{\hbar}[\hat{H}_{\text{XYSL}}, \rho] + 2 \sum_{l=1}^N [\kappa_1 \mathcal{D}(\hat{A}_l^\dagger) + \kappa_2 \mathcal{D}(\hat{A}_l^2)] \rho, \quad (1)$$

where  $\hat{A}_l^\dagger$  and  $\hat{A}_l$  are creation and annihilation operators of bosonic particles and  $\mathcal{D}(\hat{O})\rho = \hat{O}\rho\hat{O}^\dagger - \frac{1}{2}(\hat{O}^\dagger\hat{O}\rho + \rho\hat{O}^\dagger\hat{O})$  describes dissipative processes with rates  $\kappa_1, \kappa_2 > 0$ .

The coherent part of the Hamiltonian describes the coupling between the quantum oscillators and the Ising chain and is given by

$$\begin{aligned} \hat{H}_{\text{XYSL}} = & -\hbar \sum_{i=1}^N (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y) - \hbar \sum_{i=1}^N \omega_i \hat{A}_i^\dagger \hat{A}_i \\ & + \frac{\hbar G}{2} \sum_{i=1}^N (\hat{A}_i + \hat{A}_i^\dagger) \sigma_i^z \\ & + i \sum_{i,j=1}^N K_{i,j} (e^{i\beta_j} \hat{A}_i^\dagger \hat{A}_j - e^{-i\beta_j} \hat{A}_i \hat{A}_j^\dagger), \end{aligned} \quad (2)$$

where  $\sigma_j^x, \sigma_j^y$ , and  $\sigma_j^z$  are Pauli operators acting on the  $j$ th qubit.

Next, we will formalize a semiclassical expansion of the bosonic modes  $\hat{A}_l(t) = \alpha_l(t) + \hat{a}_l$ , where  $\alpha(t) = [\alpha_1(t), \dots, \alpha_N(t)]$  is the semiclassical trajectory and  $\hat{a} =$

$(\hat{a}_1, \dots, \hat{a}_N)$  describes the quantum fluctuations. We will work in the semiclassical regime where the magnitude of  $\alpha_i(t)$  is much larger than the quantum fluctuations described by  $\hat{a}_i$ .

Let us consider the bosonic displacement operator  $\hat{D}[\alpha(t)] = \exp[\alpha(t) \cdot \hat{a}^\dagger - \alpha^*(t) \cdot \hat{a}]$ , defining the time-dependent coherent states  $|\alpha_i(t)\rangle = \hat{D}[\alpha(t)]|0_i\rangle$ , where  $|0_i\rangle$  is the vacuum state of the  $i$ th oscillator.

Following the same method as in Ref. [9], we will consider an expansion of the density operator  $\rho_\alpha(t) = \hat{D}^\dagger[\alpha(t)]\rho(t)\hat{D}[\alpha(t)]$  in a comoving frame defined by the semiclassical trajectory  $\alpha(t)$ . This allows us to obtain a master equation for the density operator

$$\dot{\rho}_\alpha \approx -\frac{i}{\hbar}[\hat{H}_\alpha, \rho_\alpha] + 2 \sum_{i=1}^N [\kappa_1 \mathcal{D}(\hat{a}_i^\dagger) + 4\kappa_2 |\alpha_i|^2 \mathcal{D}(\hat{a}_i)]. \quad (3)$$

In addition, the coherent dynamics of the fluctuations are governed by a Hamiltonian driven by the mean field, as follows:

$$\begin{aligned} \frac{H_\alpha}{\hbar} = & \frac{G}{2} \sum_{i=1}^N (\hat{a}_i + \hat{a}_i^\dagger + \alpha_i + \alpha_i^*) \sigma_i^z \\ & - \sum_{i=1}^N (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y) \\ & - \sum_{i=1}^N \omega_i \hat{a}_i^\dagger \hat{a}_i + i \sum_{i=1}^N \kappa_2 (\alpha_i^*)^2 \hat{a}_i^2 \\ & + i \sum_{i,j=1}^N K_{i,j} e^{i\beta_j} \hat{a}_i^\dagger \hat{a}_j + \text{H.c.} \end{aligned} \quad (4)$$

In turn, the mean field defining the comoving frame satisfies the Stuart-Landau equation

$$\dot{\alpha}_i(t) = i\omega_i \alpha_i(t) + \alpha_i(t)(\kappa_1 - 2\kappa_2 |\alpha_i(t)|^2) + \sum_{i,j=1}^N K_{i,j} e^{i\beta_j} \alpha_j(t). \quad (5)$$

In Eq. (6), the coupling between the spin chain and the oscillators is via the quantum fluctuations that are smaller in magnitude in comparison to the mean field  $|\hat{a}_i| \ll |\alpha_i|$ . For this reason, we can neglect the coupling between the spin chain and the bosonic fluctuations to obtain an effective theory for the spin chain:

$$\hat{H}(t) = \frac{\hbar G}{2} \sum_{i=1}^N (\alpha_i + \alpha_i^*) \sigma_i^z - \hbar \sum_{i=1}^N (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y). \quad (6)$$

Here, the motion of the classical oscillator drives the quantum dynamics of the spin chain. Further, we can use the polar decomposition  $\alpha_i(t) = r_i(t)e^{i\theta_i(t)}$  in terms of amplitude  $r_i(t)$  and phase  $\theta_i(t)$ . The equations of motion for these variables are

$$\begin{aligned} \dot{r}_i = & r_i(\kappa_1 - 2\kappa_2 r_i^2) + r_i \sum_{i,j=1}^N K_{i,j} r_j \cos(\theta_j - \theta_i + \beta_j), \\ \dot{\theta}_i = & \omega_i + \sum_{i,j=1}^N K_{i,j} \frac{r_j}{r_i} \sin(\theta_j - \theta_i + \beta_j). \end{aligned} \quad (7)$$

In the absence of coupling, i.e.,  $K_{i,j}$ , each Stuart-Landau oscillator exhibits a limit cycle with radius  $R = \kappa_1/2\kappa_2$ . At the technical level, our semiclassical approach is only valid in the limit  $\kappa_1 \gg \kappa_2$  when  $|\hat{a}_i| \ll R$ . When the two-photon loss dominates the dynamics, the size of the limit cycle is small and quantum fluctuations become dominant.

Next, if we assume that the coupling is weak in comparison to the one-photon loss rate  $|K_{i,j}| \ll \kappa_1$ , one can adiabatically eliminate the amplitude dynamics [50]. In this situation, the relevant dynamics involve the phases of the oscillators and are governed by the Kuramoto model [14].

Motivated by the previous discussion, in the next section we will investigate how phase dynamics can be used to drive a quantum model.

### III. XY SPIN CHAIN IN A TRANSVERSE FIELD COUPLED TO A CLASSICAL KURAMOTO MODEL

In the previous section, we performed a semiclassical approximation on a network of quantum Stuart-Landau oscillator. We show that by diabatically eliminating the amplitude dynamics in Eq. (7), we can effectively obtain an approximate solution  $\alpha_i(t) \approx R e^{i\theta_i}$  of Eq. (5). By replacing this in Eq. (6), we obtain a time-dependent XY quantum spin chain in a transverse field [47,51,52]:

$$\hat{H}(t) = \hbar \sum_{i=1}^N g_i(t) \sigma_i^z - \hbar \sum_{i=1}^N (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y). \quad (8)$$

The local time-dependent modulation of the transverse field is given by  $g_i(t) = \tilde{G} \cos[\theta_i(t)]$  with amplitude  $\tilde{G} = GR$ , while  $J_x$  and  $J_y$  are the anisotropic interaction strengths in  $x$  and  $y$  directions, respectively.

From now on, we will focus on the effect of the phase dynamics on the quantum spin chain. Thus, the time-dependent phases  $\{\theta_i(t)\}$  are obtained by solving the classical equation for the Kuramoto model of coupled phase oscillators [14]

$$\dot{\theta}_i(t) = \omega_i + \sum_{j=1}^N K_{i,j} \sin[\theta_j(t) - \theta_i(t) + \beta_j], \quad (9)$$

where  $\omega_i$  are local frequencies drawn from a normal distribution, while  $\beta_j$  is a phase delay. The initial phases  $\theta_i(0) \in [-\pi, \pi]$  are chosen randomly from a uniform distribution. The Kuramoto model is a paradigmatic model in nonlinear dynamics that shows the transition to synchronized behavior. In our work, its solution acts as a classical drive for the quantum spin chain, as we illustrate in Fig. 1.

It is instructive to represent our spin chain in terms of fermions. To do this, we use the Jordan Wigner transformation [53]

$$\hat{f}_j = \frac{1}{2} \left( \prod_{l=1}^{j-1} \sigma_l^z \right) (\sigma_j^x + i \sigma_j^y), \quad (10)$$

where  $\sigma_l^z = 1 - 2\hat{f}_l^\dagger \hat{f}_l$ , and  $\hat{f}_j^\dagger$  and  $\hat{f}_j$  are the fermionic creation and annihilation operators satisfying the anticommutation relations  $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{i,j}$  and  $\{\hat{f}_i, \hat{f}_j\} = \{\hat{f}_i^\dagger, \hat{f}_j^\dagger\} = 0$ .

After applying the Jordan Wigner transformation to the Ising model in Eq. (8), we obtain the fermionic quadratic

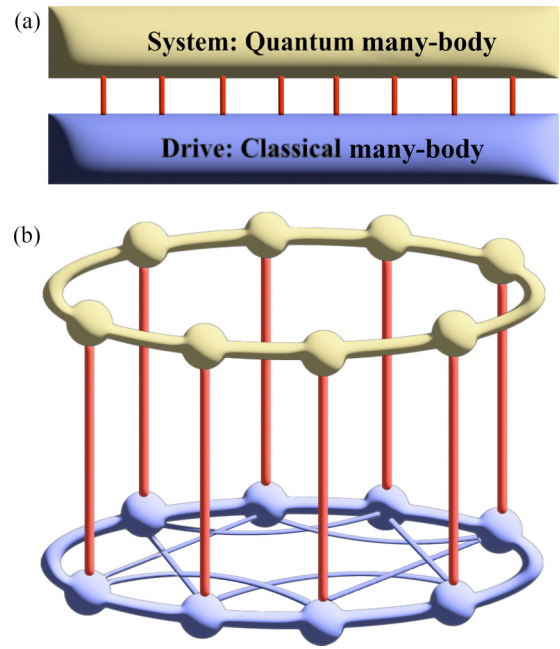


FIG. 1. (a) A quantum many-body system dynamics driven by a classical many-body system. The classical drive acts locally on the quantum many-body system. Panel (b) illustrates a many-body lattice system with short range interactions coupled to a network with arbitrary connectivity.

Hamiltonian [48]

$$\begin{aligned} \hat{H}(t) = & -\hbar \sum_{j=1}^N g_j(t) (1 - 2\hat{f}_j^\dagger \hat{f}_j) \\ & - \hbar \sum_{j=1}^{N-1} (J_x + J_y) (\hat{f}_j^\dagger \hat{f}_{j+1} + \hat{f}_{j+1}^\dagger \hat{f}_j) \\ & - \hbar \sum_{j=1}^{N-1} (J_x - J_y) (\hat{f}_j^\dagger \hat{f}_{j+1}^\dagger + \hat{f}_{j+1} \hat{f}_j), \end{aligned} \quad (11)$$

which can also be written as

$$\begin{aligned} \hat{H}(t) = & i\hbar \sum_{j=1}^N g_j(t) \hat{a}_{2j-1} \hat{a}_{2j} \\ & + i\hbar \sum_{j=1}^{N-1} (J_x \hat{a}_{2j} \hat{a}_{2j+1} - J_y \hat{a}_{2j-1} \hat{a}_{2j+2}) \end{aligned} \quad (12)$$

in terms of Majorana fermions  $\hat{a}_{2j-1} = \hat{f}_j^\dagger + \hat{f}_j$  and  $\hat{a}_{2j} = i(\hat{f}_j^\dagger - \hat{f}_j)$  [54,55] satisfying the anticommutation relations  $\{\hat{a}_m, \hat{a}_n\} = \delta_{m,n}$  given that  $\hat{a}_n = \hat{a}_n^\dagger$ .

The continuous limit of our model is intimately related to the Dirac equation with a mass that depends on space and time (see Appendix A). This type of situation appears when Dirac fermions are coupled to gravitational fields as we discuss in Appendix B. In our work, we will focus on the discrete model and we will not discuss the physics of the continuous limit.

To investigate quantum fingerprints of self-organization, we numerically solve the Heisenberg equations of motion

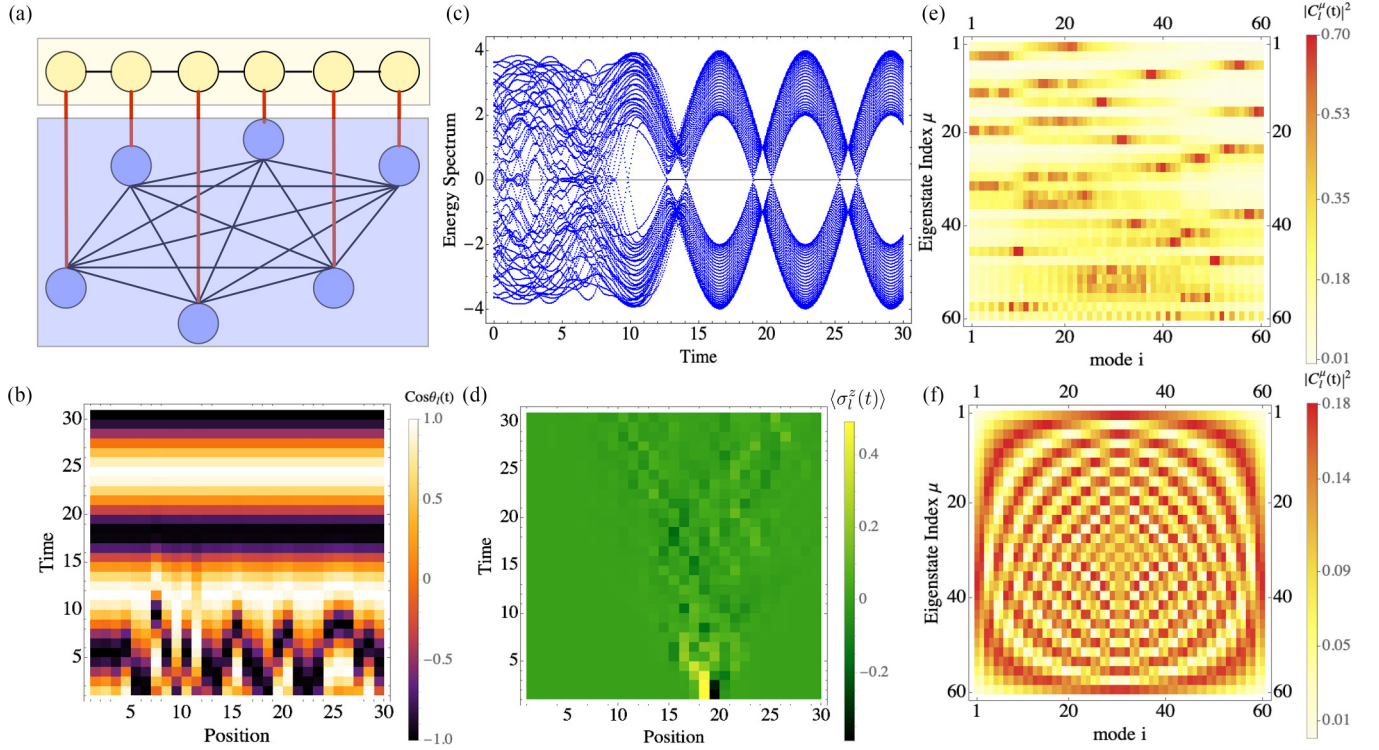


FIG. 2. Panel (a) illustrates a one-dimensional quantum Ising chain with  $N = 30$  sites that is coupled to a network of  $N = 30$  phase oscillators with all-to-all coupling described by the Kuramoto model. Panel (b) shows the time evolution of the transverse local fields  $g_i(t) = \tilde{G} \cos[\theta_i(t)]$  and the emergence of synchronized motion. Panel (c) depicts the instantaneous spectrum and the formation of two energy bands periodically oscillating in time. Panel (d) shows the time evolution of a localized energy eigenstate of the Hamiltonian  $\hat{H}(0)$ . Panels (e) and (f) depict two snapshots of the eigenstates before ( $Jt = 5$ ) and after ( $Jt = 23$ ) synchronization. We have chosen parameters  $\tilde{G} = 3J$ ,  $\tilde{K} = 0.5J$ , and  $\omega = 0.5J$ .

for the fermionic operators in several situations. This enables us to explore how symmetries emerge in our quantum system as a consequence of self-organization in the Kuramoto model.

#### IV. EMERGENT SYMMETRIES IN THE ISING MODEL

In this section, we will consider a concrete example of a one-dimensional quantum Ising model in a transverse field [48,51], which is obtained from Eq. (8) by setting  $J_y = 0$ . For simplicity, we consider a spin chain with open boundary conditions as depicted in Fig. 2(a). We also consider a Kuramoto model of phase oscillators  $\dot{\theta}_i(t) = \omega + \tilde{K} \sum_{j=1}^N \sin[\theta_j(t) - \theta_i(t)]$  in a network with all-to-all coupling  $K_{i,j} = \tilde{K}$ , uniform frequencies  $\omega_i = \omega$ , and with no phase delay  $\beta_j = 0$  [see Fig. 2(a)].

As a first step to investigate the dynamics, we numerically solve the equations of motion for the Kuramoto model. As we choose random initial conditions, the local transverse field  $g_i(t) = \tilde{G} \cos[\theta_i(t)]$  breaks the translational invariance of the quantum Ising chain as we show in Fig. 2(b). In this figure, one can see that the system synchronizes after a transient time and the phases  $\{\theta_i(t)\}$  for each  $i$  become identical and oscillate in time with a period  $T$ .

Next, we will discuss how the onset of synchronization influences the quantum dynamics of the spin chain. To study the dynamics, it is convenient to work with the Hamiltonian (12) in terms of Majorana fermions [54,55] by setting  $J_y = 0$ .

From this, we can derive the Heisenberg equations of motion

$$\begin{aligned} i \frac{d\hat{a}_{2i-1}}{dt} &= i g_i(t) \hat{a}_{2i} - i J_x \hat{a}_{2(i-1)}, \\ i \frac{d\hat{a}_{2i}}{dt} &= -i g_i(t) \hat{a}_{2i-1} + i J_x \hat{a}_{2(i+1)-1}. \end{aligned} \quad (13)$$

##### A. Quantum dynamics and emergent symmetries

Before synchronization, the Kuramoto model has random phases and no spatial or temporal translational invariance. However, when the phase oscillators synchronize, there are emergent translational symmetries in space and time. Effectively, the system becomes a Floquet system with time-periodic Hamiltonian  $\hat{H}(t+T) = \hat{H}(t)$  and the quasimomentum  $k$  becomes a good quantum number because the system is translationally invariant with  $\theta_i(t) = \theta_j(t) = \Omega t + \psi_0$  for all  $i \neq j$ . In this regime, the system is described by the Hamiltonian in the quasimomentum space [48]:

$$\begin{aligned} \hat{H}(t) &= -2 \sum_k [g(t) + J \cos k] (\hat{F}_k^\dagger \hat{F}_k - \hat{F}_{-k}^\dagger \hat{F}_{-k}) \\ &\quad + 2J_x \sum_k \sin k (\hat{F}_k^\dagger \hat{F}_{-k}^\dagger + \hat{F}_{-k} \hat{F}_k), \end{aligned} \quad (14)$$

where  $g(t) = \tilde{G} \cos(\Omega t + \psi_0)$  and  $\hat{F}_k$  are fermionic operators. In this example, discrete translational invariance in space and time are emergent symmetries in our quantum model.

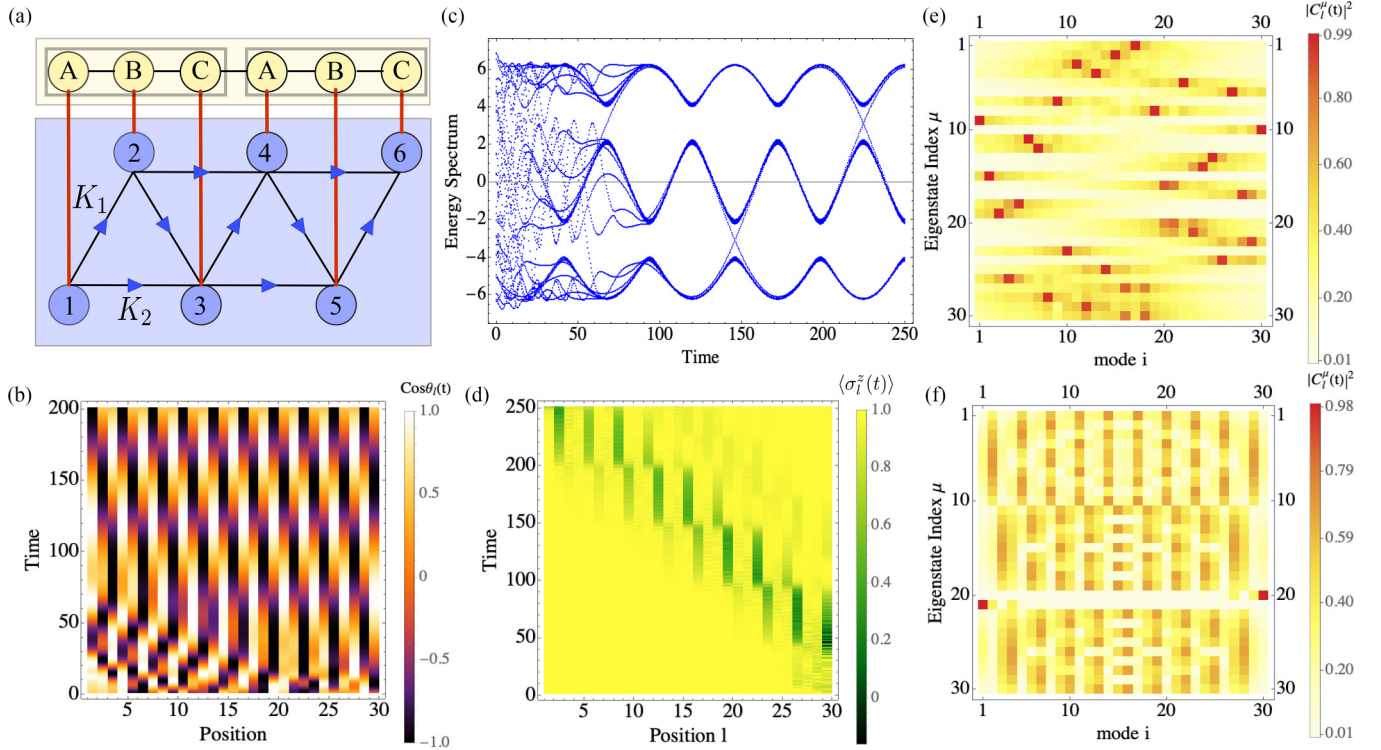


FIG. 3. Panel (a) illustrates a one-dimensional quantum Ising chain with  $N = 30$  sites that is coupled to a Kuramoto model in a zigzag lattice with unidirectional coupling. Panel (b) shows the time evolution of the transverse local fields  $g_i(t) = \tilde{G} \cos[\theta_i(t)]$  and the emergence of traveling waves after the system synchronizes. Panel (c) depicts the instantaneous spectrum and the emergence of three energy bands periodically oscillating in time. Panel (d) shows the time evolution of a localized energy eigenstate of the Hamiltonian  $\hat{H}(0)$  and how it is pumped along the lattice. Panels (e) and (f) depict two snapshots of the eigenstates before ( $\omega t = 50$ ) and after ( $\omega t = 150$ ) synchronization. We have chosen parameters  $\tilde{G} = 3J$ ,  $K_1 = 0.2J$ ,  $K_2 = 0.1J$ , and  $\omega = 0.04J$ .

Let us analyze the instantaneous spectrum shown in Fig. 2(c). In this figure, one clearly see that the spectrum does not have much structure before synchronization takes place. When the system synchronizes, there are two well-defined energy bands that exhibit periodic oscillations. This can be attributed to the existence of translational invariance in our system in space and time. Figure 2(d) shows the dynamics of the expectation value  $\langle \sigma_i^z(t) \rangle$ . The initial state was chosen to be a localized energy eigenstate of the Hamiltonian  $\hat{H}(0)$ . As expected, before classical synchronization takes place, the state remains localized until synchronization occurs, after which it spreads ballistically along the chain. The dynamics of the Kuramoto model also influences the behavior of the eigenstates before and after synchronization takes place as we depict in Figs. 2(e) and 2(f), respectively. There one can see that when the phase oscillators are synchronized, there is a structure of the eigenstates that shows oscillations in space associated with the quasimomentum.

## V. EMERGENT SYMMETRY BREAKING AND TOPOLOGICAL PUMPING IN THE XX MODEL

So far, we have discussed how the dynamics of a classical Kuramoto model is responsible for the emergent symmetries of our quantum spin chain. Next, we will consider a different situation where the classical phase oscillators restore spatial symmetries while breaking time-reversal symmetry in an XX

model [ $J_x = J_y = J$  in Eq. (8)] with open boundary conditions [see Fig. 3(a)]. Here, we consider a Kuramoto model  $\dot{\theta}_i(t) = \omega + \sum_{r=1}^2 K_r \sum_{j=1}^{N-1} \sin[\theta_{i+r}(t) - \theta_i(t) - 2r\pi/3]$  with unidirectional coupling and phase delay  $\beta_r = -2r\pi/3$ . As we depict in Fig. 3, the oscillators are arranged in a zigzag geometry with couplings  $K_1$  and  $K_2$ .

We numerically solve the equations of motion for the phase oscillators with random initial conditions. Similarly to the Ising model, at transient times before synchronization appears, the phases are random and the dynamics breaks the translational invariance of the quantum model in space. After synchronization, the Kuramoto dynamics exhibits periodic traveling waves [56] with a very low frequency in time as we show in Fig. 3(b). This effectively generates a pattern of phases in space that moves with a preferred direction that breaks time reversal in the quantum model.

As the XX model preserves the total number of fermions, to explore its dynamics we need to solve the Heisenberg equations of motion for the fermionic operators

$$i \frac{d\hat{f}_i}{dt} = -2g_i(t)\hat{f}_i - 2J(\hat{f}_{i+1} + \hat{f}_{i-1}). \quad (15)$$

The solution of the Kuramoto model acts as a local transverse field for the XX chain. Thus, by solving the equation of motion we can explore quantum signatures of the traveling waves. Figure 3(c) shows the instantaneous spectrum. At short times, it does not exhibit any structure as the phase oscillators

are unsynchronized. When synchronization takes place, the traveling wave pattern effectively trimerizes the spin chain [57] because the transverse field reads  $g_i(t) = \tilde{G} \cos[\Omega t + 2\pi i/3 + \phi_0]$ . For this reason, we see three well-defined bands associated with the traveling wave. This behavior resembles a situation encountered in the theory of the quantum Hall effect and it is intimately related to topological pumping [58,59]. The traveling wave acts as an effective adiabatic drive breaking time-reversal symmetry, which allows us to adiabatically pump charges in the system as we show in Fig. 3(d). Before synchronization, the eigenstates are mostly localized due to the random transverse fields. However, when the system is synchronized, the eigenstates exhibit a spatial structure due to the trimerization of the lattice.

One of the key points of this example is that while the Kuramoto drive is responsible for the emergence of a well-defined trimer structure, it also breaks the time-reversal symmetry giving rise to a topological transport process.

## VI. CONCLUSIONS

In summary, we have shown quantum signatures of a self-organization by considering a Kuramoto model coupled to a spin chain. In particular, we have demonstrated how symmetries can emerge due to pattern formation in our drive. Of course, we are not restricted to one-dimensional spin chains. In fact, our work is also intimately related to quantum simulation of electronic structure in quantum computers driven by the motion of nuclei within the framework of the Born-Oppenheimer approximation [60]. In this context, one needs to solve the equation of motion for the classical nuclei, and the solution modulates the one- and two-particle integrals of the electronic structure Hamiltonian as we discuss in Appendix C.

In future works, it would be interesting to study two- or three-dimensional lattices driven by Kuramoto or coupled Stuart-Landau oscillators to generate more complex dynamics. Recent works have shown that even a quasiperiodic drive can induce complex dynamics leading to complete Hilbert-space ergodicity [46]. Our approach can open an avenue to study complex dynamics in many-body systems. For example, it would be interesting to explore which phases are generated when a system is subject to a self-organized drive.

Our work can be experimentally realized in analog quantum simulators such as one-dimensional arrays of superconducting qubits [39]. By using a classical computer, one can obtain the solution  $\{\theta_i(t)\}$  of the Kuramoto model. The Z control lines of the superconducting chip allow to modulate the frequencies of the qubits in time to achieve the desired time dependence  $g_i(t) = \tilde{G} \cos[\theta_i(t)]$ . The local magnetization  $\langle \sigma_i^z(t) \rangle$  can be measured by using dispersive readout of the local qubits [61].

## ACKNOWLEDGMENT

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

## APPENDIX A: CONTINUOUS LIMIT OF THE MODEL: DIRAC EQUATION WITH TIME-DEPENDENT MASS

The first step to systematically define the continuous limit is to consider a lattice constant  $b$  defining the spatial coordinate  $x_j = jb$  associated with the  $j$ th lattice site [62]. We also define fermionic field operators depending on this coordinate as  $\hat{\chi}_1(x_j) = \hat{a}_{2i-1}/\sqrt{b}$  and  $\hat{\chi}_2(x_j) = \hat{a}_{2i}/\sqrt{b}$ . This will ensure the right anticommutation relations of the field operators in the continuous limit. Next, let us evaluate the approximate form of the field operators under spatial translations [51], as follows:

$$\hat{\chi}_1(x_j \pm b) = \hat{a}_{2(j\pm1)-1}/\sqrt{b} \approx \hat{\chi}_1(x_j) \pm b\partial_x \hat{\chi}_1(x_j),$$

$$\hat{\chi}_2(x_j \pm b) = \hat{a}_{2(j\pm1)}/\sqrt{b} \approx \hat{\chi}_2(x_j) \pm b\partial_x \hat{\chi}_2(x_j). \quad (\text{A1})$$

After inserting these relations in Hamiltonian equation (12), we can derive the Hamiltonian in the continuous limit:

$$\hat{H}(t) = i\hbar \int \hat{\chi}_2(x) [-\Delta(x, t) + c\partial_x] \hat{\chi}_1(x) + E_0(t), \quad (\text{A2})$$

where  $\Delta(x, t) = g(x, t) - J_x - J_y$ ,  $c = (J_x - J_y)b$ , and  $E_0(t)$  is a time-dependent energy. Correspondingly, we can also write the continuous limit of the equations of motion, as follows:

$$i\partial_t \hat{\chi}_1(x) = i[\Delta(x, t) + c\partial_x] \hat{\chi}_2(x),$$

$$i\partial_t \hat{\chi}_2(x) = -i[\Delta(x, t) - c\partial_x] \hat{\chi}_1(x). \quad (\text{A3})$$

By considering the Dirac matrices  $\gamma^0 = -\sigma^y$  and  $\gamma^1 = i\sigma^z$ , one can write the equations of motion as a Dirac equation  $i[\gamma^\mu \partial_\mu - m(\mathbf{x})] \hat{\chi}(\mathbf{x})$  in  $1+1$  space time with coordinates  $\mathbf{x} = (X_0, X_1)$  (see Ref. [62]). Here,  $X_0 = ct$  and  $X_1 = x$  denote the temporal and spatial coordinates, respectively. We also have defined the mass term  $m(\mathbf{x}) = \Delta(x, t)/c$  and the Dirac spinor field  $\hat{\chi}^\dagger(\mathbf{x}) = [\hat{\chi}_1(\mathbf{x}), \hat{\chi}_2(\mathbf{x})]$ .

In the effective field theory, the mass term is also modulated by the dynamics of the Kuramoto model in Eq. (9) in the continuous limit:

$$\partial_t \theta(x, t) = \omega(x) + \int K(x-y) \sin[\theta(x, t) - \theta(y, t)]. \quad (\text{A4})$$

Thus, as  $m(\mathbf{x}) = \Delta(x, t)/c = \tilde{G} \cos[\theta(x, t)]/c$ , the dynamics of the Kuramoto model modulates the mass of the Dirac equation in space and time.

## APPENDIX B: DIRAC FERMION COUPLED TO A GRAVITATIONAL FIELD

Another example is the relativistic dynamics of electrons coupled to a semiclassical gravitational field. In this case, the time dependence of the external field enters as a time-dependent mass of the Dirac fermion, governed by the Hamiltonian [63]

$$\hat{H}(t) = \int dx \hat{\psi} [-i\gamma^\mu \partial_{\mu} + m_{\text{eff}}(t)] \hat{\psi}(x), \quad (\text{B1})$$

where the effective time-dependent mass depends on the conformal factor of the metric as in Ref. [63]:

$$ds^2 = \Omega(x, t)(dt^2 - dx^2). \quad (\text{B2})$$

A similar Dirac Hamiltonian can be also obtained for the Friedmann-Robertson-Walker metric with a scale factor that can be obtained by solving the Einstein's equations.

The common characteristic of these two examples is that a classical field satisfying equations of motion can drive the dynamics of a quantum field. It is important to note that the previous two examples are rather general nonrelativistic and relativistic field theories so they are low-energy descriptions of a plethora of phenomena in diverse systems.

### APPENDIX C: ELECTRONIC STRUCTURE DRIVEN BY THE NUCLEAR MOTION

To illustrate the generality of our approach, let us start by considering a generic example of quantum fields in second quantization that are driven by classical fields. One of them is inspired by the problem of electronic structure in quantum chemistry under the Born-Oppenheimer approximation [60].

The electronic structure problem can be written in terms of a nonrelativistic field theory governed by the Hamiltonian [64]

$$\begin{aligned} \hat{H}(t) = & \int dx \, \hat{\psi}^\dagger(x) \left\{ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}[x, \mathbf{R}(t)] \right\} \hat{\psi}(x) \\ & + \iint dx \, dx' \, \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') U(x, x') \hat{\psi}(x') \hat{\psi}(x), \quad (\text{C1}) \end{aligned}$$

where  $\hat{\psi}(x)$  denotes a fermionic quantum field satisfying the anticommutation relation  $\{\hat{\psi}(x), \hat{\psi}^\dagger(x')\} = \delta(x - x')$ . Here, the external potential is the interaction energy between the electrons and the nuclei,

$$V_{\text{ext}}[x, \mathbf{R}(t)] = - \sum_{\nu=1}^M \frac{Z_\nu}{|x - \mathbf{R}(t)|}, \quad (\text{C2})$$

that is modulated by the positions  $\mathbf{R}(t) = [\mathbf{R}_1(t), \dots, \mathbf{R}_M(t)]$  of the  $M$  nuclei in the molecule that follow a classical trajectory that is a solution of a classical equation of motion

$$m_i \frac{d^2}{dt^2} \mathbf{R}_i(t) = -\nabla_{\mathbf{R}_i} U[\mathbf{R}(t)], \quad (\text{C3})$$

where  $U[\mathbf{R}(t)]$  is the potential energy. This is extremely important in the theoretical investigation of scattering and chemical reactions. Similar models can be considered by problems of interacting bosons in external potentials.

One can decompose the fermionic field operator in terms of molecular or atomic orbitals as  $\hat{\psi}(x) = \sum_i \phi_i[x, \mathbf{R}(t)] \hat{a}_i$ , and by replacing this in Eq. (C1), we obtain the Hamiltonian

$$\hat{H}[\mathbf{R}(t)] = \sum_{i,j} h_{i,j}[\mathbf{R}(t)] \hat{a}_i^\dagger \hat{a}_j + \sum_{i,j} V_{i,j,k,l}[\mathbf{R}(t)] \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l. \quad (\text{C4})$$

Depending on the chosen basis, the atomic or molecular orbitals may depend on the nuclear coordinates, and for this reason, the nuclear motion induces a time dependence in the one- and two-particle integrals  $h_{i,j}[\mathbf{R}(t)]$  and  $V_{i,j,k,l}[\mathbf{R}(t)]$ , respectively.

To simulate the electronic structure problem in a quantum computer, it is convenient to organize the fermionic operators  $\hat{a}_j$  in a one-dimensional array and to use Eq. (10) to map the fermionic operators to qubits. By using this, we obtain the spin Hamiltonian

$$\hat{H}_{\text{P}}(\mathbf{R}) = \sum_J B_J[\mathbf{R}(t)] \hat{P}_J, \quad (\text{C5})$$

where we define the Pauli words  $\hat{P}_J = \bigotimes_j^{2K} \sigma_j^J$ , which are elements of the Pauli group. The operator  $\sigma_j^J$  is one of the Pauli operators  $\sigma_j^x, \sigma_j^y, \sigma_j^z$  and the identity  $\hat{1}$  acting on the  $j$ th qubit. Note that again there is parametric dependence of the coefficients  $B_J[\mathbf{R}(t)]$  on the nuclear trajectories  $\mathbf{R}(t)$ . The nuclei act as an external drive for the spin system.

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