

Quantum geometry of bosonic Bogoliubov quasiparticles

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Bosonic Bogoliubov-de Gennes (BBdG) Hamiltonians describe the excitations of weakly interacting Bose condensates as well as photonic systems under parametric driving. Their topological features have been studied mainly by utilizing a generalized symplectic version of the Berry curvature and related Chern numbers. However, a full characterization of geometrical features in BBdG systems is still lacking. Here, we propose a symplectic quantum geometric tensor (SQGT), whose imaginary part leads to the previously studied symplectic Berry curvature, while the real part gives rise to a symplectic quantum metric, providing a natural distance measure in the space of bosonic Bogoliubov modes. The SQGT is directly related to observable properties of BBdG systems. We show how to measure all components of the SQGT by extracting excitation rates in response to periodic modulations of the system's parameters. Moreover, we connect the symplectic Berry curvature to a generalized symplectic anomalous velocity term for Bogoliubov-Bloch wave packets. We test our results for a bosonic Bogoliubov-Haldane model.

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Introduction. Bosonic Bogoliubov-de Gennes (BBdG) Hamiltonians naturally arise in systems of weakly interacting bosons [1–4], or in photonic systems subjected to parametric driving, generating squeezing of light [5–10]. They also describe magnonic [11–13], phononic [14], and mechanical systems [15]. Different from their fermionic (BCS) counterparts, BBdG systems are diagonalized in a paraunitary or symplectic manner [2,16]. Their topological features [17,18] have first been studied in the context of magnonic crystals [11,12] by utilizing a generalized symplectic Berry curvature and Chern number for Bogoliubov bands. A generalization of the conventional bulk-boundary correspondence [19,20] to the BBdG case can even lead to chiral edge states with properties distinct from their particle-number-conserving counterparts [6,21–24]. Recent works, building on the tenfold Altland-Zirnbauer (AZ) classification of topological insulators [25–29], corroborate the notions of the symplectic Berry curvature and Chern number [30,31] and provide a topological classification of quadratic BBdG systems [32–37].

Although topological features of BBdG systems have recently received some attention [7,23,24,36,38–58], the geometry of BBdG systems is still lacking a complete characterization. In particle-number-conserving systems, the Berry curvature is related to the imaginary part of the so-called quantum geometric tensor (QGT) [59,60], and the real part of the QGT, known as quantum metric, provides a natural distance measure in state space [60,61]. Here, we generalize

these concepts to the non-number-conserving BBdG case and propose a symplectic quantum geometric tensor (SQGT) [62]. We show that the imaginary part of the SQGT is related to the symplectic Berry curvature investigated previously and that its real part, which we dub symplectic quantum metric, defines a natural distance measure in the space of bosonic Bogoliubov modes. Previous proposals for measurable signatures of the topology of BBdG systems have only focused on global properties, such as probing bosonic chiral edge modes for nonzero symplectic Chern numbers [6,24]. Here, inspired by Refs. [63,64], we propose, for the first time, to measure the full local SQGT from excitation rates to other Bogoliubov modes in response to weak periodic driving [Fig. 1(a)]. As a further probe, we also show that, analogous to the number-conserving case [65–67], the symplectic Berry curvature naturally arises as a transverse velocity term when a Bogoliubov Bloch wave packet is subjected to an external force [Fig. 1(b)].

Framework. Consider a BBdG Hamiltonian

$$\hat{H} = \sum_{i,j=1}^N K_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} (G_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger + G_{ji}^* \hat{a}_i \hat{a}_j) \\ = \frac{1}{2} \hat{\Psi}^\dagger \mathcal{M} \hat{\Psi} - \frac{1}{2} \text{Tr}\{K\}, \quad \mathcal{M} = \begin{pmatrix} K & G \\ G^* & K^* \end{pmatrix}, \quad (1)$$

with $K_{ij} = K_{ji}^* \in \mathbb{C}$ (Hermitian), $G_{ij} = G_{ji} \in \mathbb{C}$ (symmetric) forming the Hermitian $2N \times 2N$ matrix $\mathcal{M}$ [68], and Nambu spinor $\hat{\Psi} = (\hat{a}_1 \dots \hat{a}_N \hat{a}_1^\dagger \dots \hat{a}_N^\dagger)^T$ of bosonic annihilation (creation) operators $\hat{a}_i^{(\dagger)}$ [2,16,69]. The Hamiltonian can be diagonalized [2,16] (see also Supplemental Material [70]),

$$\hat{H} = \frac{1}{2} \hat{\Phi}^\dagger W^\dagger \mathcal{M} W \hat{\Phi} - \frac{1}{2} \text{Tr}\{K\} = \sum_{n=1}^N \omega_n \hat{b}_n^\dagger \hat{b}_n + C, \quad (2)$$

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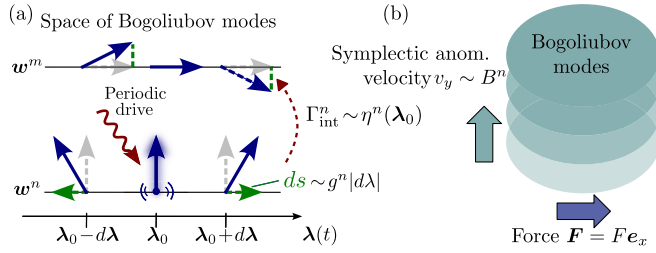


FIG. 1. (a) Illustration of the symplectic quantum metric providing a distance measure ds between infinitesimally close Bogoliubov modes $\mathbf{w}(\lambda_0)$ and $\mathbf{w}(\lambda_0 + d\lambda)$ [cf. Eq. (11)]. The integrated excitation rate Γ_{int}^n of a Bogoliubov system in response to (weak) periodic perturbations is governed by the symplectic quantum geometry [cf. Eqs. (14) and (16)]. (b) Illustration of the transverse velocity a Bogoliubov Bloch wave packet experiences in response to an external force, which is proportional to the symplectic Berry curvature [cf. Eq. (18)].

by a Bogoliubov transformation $\hat{\Psi} = W\hat{\Phi}$, where $\hat{\Phi} = (\hat{b}_1 \dots \hat{b}_N \hat{b}_1^\dagger \dots \hat{b}_N^\dagger)^T$ and

$$W = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}, \quad \text{with} \quad \begin{cases} W^\dagger \tau_z W = W \tau_z W^\dagger = \tau_z, \\ W^\dagger \mathcal{M} W = \Omega. \end{cases} \quad (3)$$

Here, $C \equiv \sum_n \omega_n/2 - \text{Tr}\{K\}/2$; U and V are complex $N \times N$ matrices, so that $\hat{a}_i = \sum_n [U_{in} \hat{b}_n + (V^*)_{in} \hat{b}_n^\dagger]$. We only consider thermodynamically stable systems where $\mathcal{M}$ is positive definite such that $\Omega = \text{diag}(\omega_1, \dots, \omega_N, \omega_1, \dots, \omega_N)$ consists of strictly positive excitation energies $\omega_n > 0$ [30,69]. The transformation W is paraunitary, as indicated by the relation involving the diagonal matrix τ_z , where $\tau_i = \sigma_i \otimes \mathbb{1}_N$ with Pauli matrices σ_i , $i \in \{x, y, z\}$, and the $N \times N$ identity matrix $\mathbb{1}_N$. This ensures that the operators $\hat{b}_n^{(\dagger)}$ fulfill bosonic commutation relations. Representing the BdG Hamiltonian (1) via position-momentum quadratures $(\hat{q}_j, \hat{p}_j) = ((\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}, i(\hat{a}_j^\dagger - \hat{a}_j)/\sqrt{2})$ leads to symplectic transformation matrices $\tilde{W} \in \text{Sp}(2N, \mathbb{R})$ [30,71], thence the commonly used label “symplectic” [6,23] (see also Supplemental Material [70]). The Bogoliubov ground state $|\psi_0\rangle$ is the quasiparticle vacuum, $\hat{b}_n|\psi_0\rangle = 0 \forall n$. It is generally a multimode squeezed state with a fluctuating nonzero number of $\hat{a}_i$ bosons [6,47,72]. Here, we will focus on the geometrical properties of the quasiparticle excitations on top of $|\psi_0\rangle$ [7].

Symplectic quantum geometric tensor. The Bogoliubov modes are associated with the vectors $\mathbf{w}_n$ given by the columns of $W = (\mathbf{w}^1, \dots, \mathbf{w}^{2N})$. For $n \leq N$, they describe quasiparticles, $\hat{b}_n^\dagger = \hat{\Psi}^\dagger \tau_z \mathbf{w}_n$, and for $N > n$ the quasiholes, $\hat{b}_n = -\hat{\Psi}^\dagger \tau_z \mathbf{w}^{N+n}$ [21,47,70]. They play a role akin to single-particle wave functions [7] and solve the eigenvalue equation [70]

$$D\mathbf{w}^n = \tilde{\Omega}_n \mathbf{w}^n, \quad \text{with } D \equiv \tau_z \mathcal{M} \text{ and } \tilde{\Omega}_n = (\tau_z \Omega)_{nn}, \quad (4)$$

where the dynamical matrix D is pseudo-Hermitian, $D = \tau_z D^\dagger \tau_z$ [16,30,31,73], and obeys particle-hole symmetry (PHS) $D = -\tau_x D^* \tau_x$ [74], so that the excitation energies $\tilde{\Omega}_n = s_n \omega_n$ are positive ($s_n = +1$) for quasiparticles ($n \leq N$) and negative ($s_n = -1$) for quasiholes ($n > N$), with $\omega_{n+N} = -\omega_n$. The Bogoliubov modes $\mathbf{w}^n$ obey the generalized τ_z inner product structure [73] $(\mathbf{w}^m)^\dagger \tau_z \mathbf{w}^n = (\tau_z)_{mn} = \delta_{mn} s_n$ [70].

Thus, the projector P^n onto the n th Bogoliubov mode and its orthogonal complement Q^n are naturally defined as [11]

$$P^n = W \Gamma^n W^{-1} = W \Gamma^n \tau_z W^\dagger \tau_z, \quad Q^n = \mathbb{1}_{2N} - P^n, \quad (5)$$

where $(\Gamma^n)_{lm} = \delta_{nl} \delta_{lm}$. Then $P^n P^m = \delta_{nm} P^n$ and $\sum_{n=1}^{2N} P^n = \mathbb{1}_{2N}$ [11]. Both P^n and Q^n are pseudo-Hermitian, $\tau_z X^n \tau_z = (X^n)^\dagger$ for $X^n \in \{P^n, Q^n\}$. The projectors onto the particle and hole parts are not independent but also related via the PHS, $P^n = \tau_x (P^{n+N})^* \tau_x$ [70].

Let the Hamiltonian (1) now depend on a set of dimensionless parameters $\lambda = (\lambda^1, \dots, \lambda^r)$, $\hat{H} = \hat{H}(\lambda)$ and have, for simplicity, a nondegenerate spectrum $\tilde{\Omega}_n \neq \tilde{\Omega}_m, \forall \lambda$ (the generalization to the degenerate case is straightforward). Then, motivated by the definition of the conventional QGT [59,61], we define the SQGT as

$$\eta_{\mu\nu}^n = \text{Tr}\{\partial_\mu P^n Q^n \partial_\nu P^n\}, \quad (6)$$

where $\partial_\mu \equiv \partial/\partial \lambda^\mu$. The SQGT is $U(1)$ gauge-invariant, since the projectors P_n and Q_n do not depend on the phase of the modes $\mathbf{w}^n(\lambda)$. The pseudo-Hermiticity of P^n and Q^n (5) implies that the SQGT is Hermitian, $(\eta_{\mu\nu}^n)^* = \eta_{\nu\mu}^n$. By employing completeness, $\sum_n P^n = \mathbb{1}_{2N}$, and orthogonality, $(\mathbf{w}^m)^\dagger \tau_z \mathbf{w}^n = \delta_{mn} s_n$, it can be expressed as [70]

$$\eta_{\mu\nu}^n = s_n (\partial_\mu \mathbf{w}^n)^\dagger \tau_z Q^n \partial_\nu \mathbf{w}^n. \quad (7)$$

In the Supplemental Material [70], we also derive the spectral representation

$$\eta_{\mu\nu}^n = s_n \sum_{m \neq n} s_m \frac{(\mathbf{w}^n)^\dagger (\partial_\mu \mathcal{M}) \mathbf{w}^m (\mathbf{w}^m)^\dagger (\partial_\nu \mathcal{M}) \mathbf{w}^n}{(\tilde{\Omega}_n - \tilde{\Omega}_m)^2}. \quad (8)$$

A first indication that the so-defined SQGT (6) is meaningful is that its imaginary part is proportional to the previously defined symplectic Berry curvature $B_{\mu\nu}^n = i s_n [(\partial_\mu \mathbf{w}^n)^\dagger \tau_z (\partial_\nu \mathbf{w}^n) - (\partial_\nu \mathbf{w}^n)^\dagger \tau_z (\partial_\mu \mathbf{w}^n)]$ [11,12,24],

$$B_{\mu\nu}^n = -2\text{Im}[\eta_{\mu\nu}^n] = -2\text{Im}[\text{Tr}\{\partial_\mu P^n Q^n \partial_\nu P^n\}], \quad (9)$$

which is real-valued and antisymmetric, $B_{\mu\nu}^n = -B_{\nu\mu}^n$. From this quantity, a symplectic (first) Chern number $C^n \equiv \frac{1}{2\pi} \int_S B^n$ was defined for closed two-dimensional parameter spaces S [11,12] and found to describe a generalized bulk-boundary correspondence for BBdG systems [6,21–24]. We show in the Supplemental Material [70] that the symplectic Berry curvature admits a local conservation law, i.e., $\sum_n B_{\mu\nu}^n = 0$. Different from the conservation law for particle-number conserving systems, this implies that the sum of the Berry curvature over the particle subspace can be nonzero, and may only be compensated by the hole contributions. This situation typically occurs in driven photonic BBdG systems, where the Hamiltonian (1) is dynamically stable [75]. As a consequence, the sum of the Chern numbers over the particle subspace can be nonzero, which is distinct from the conventional case [46]. We, moreover, show below that the symplectic Berry curvature describes the anomalous (transverse) velocity induced by an external force in Bogoliubov Bloch bands.

We now demonstrate that also the real part of the SQGT,

$$g_{\mu\nu}^n = \text{Re}[\eta_{\mu\nu}^n] = \text{Re}[\text{Tr}\{\partial_\mu P^n Q^n \partial_\nu P^n\}], \quad (10)$$

carries physical meaning. We dub it symplectic quantum metric, since it directly generalizes the conventional quantum metric to the BBdG case. It is symmetric, $g_{\mu\nu}^n = g_{\nu\mu}^n$. Note that the PHS relates the particle and hole parts of the SQGT to each other via $\eta_{\mu\nu}^n = (\eta_{\mu\nu}^{n+N})^*$ [70], implying that the symplectic quantum metric (10) is the same for particles and holes, $g_{\mu\nu}^n = g_{\mu\nu}^{n+N}$, whereas the symplectic Berry curvature (9) switches sign, $B_{\mu\nu}^n = -B_{\mu\nu}^{n+N}$.

Unlike the symplectic Berry curvature, which is only nonzero for systems with broken time-reversal or inversion symmetry [65], the symplectic quantum metric can already be nonzero without these constraints.

The conventional quantum metric (Fubini-Study metric) [59–61, 76–78] is given by the real part of the conventional QGT and defines a natural distance measure in the (projective) Hilbert space. It is used to characterize quantum phase transitions [79–85], excitation rates, and quantum fluctuations [60, 64, 86] (properties that have been exploited to measure the quantum metric in various experimental setups [87–94]).

The symplectic quantum metric $g_{\mu\nu}$ naturally describes the distance ds^2 between two infinitesimally close Bogoliubov modes $\mathbf{w}(\lambda)$ and $\mathbf{w}(\lambda + d\lambda)$,

$$ds^2 = 1 - |\mathbf{w}^\dagger(\lambda)\tau_z\mathbf{w}(\lambda + d\lambda)|^2 = g_{\mu\nu}d\lambda^\mu d\lambda^\nu, \quad (11)$$

with summation over repeated indices. Here, we first defined ds^2 via the fidelity $|\mathbf{w}^\dagger(\lambda)\tau_z\mathbf{w}(\lambda + d\lambda)|^2$ between both modes corresponding to the inner-product structure described above. The second identity follows from expanding $\mathbf{w}(\lambda + d\lambda)$ to second order in $d\lambda$, giving $ds^2 = \{(\mathbf{w}^\dagger\tau_z\mathbf{w})[(\partial_\mu\mathbf{w})^\dagger\tau_z(\partial_\nu\mathbf{w})] - (\partial_\mu\mathbf{w})^\dagger\tau_z\mathbf{w}\mathbf{w}^\dagger\tau_z(\partial_\nu\mathbf{w})\}d\lambda^\mu d\lambda^\nu = \eta_{\mu\nu}d\lambda^\mu d\lambda^\nu = \frac{1}{2}(\eta_{\mu\nu} + \eta_{\nu\mu})d\lambda^\mu d\lambda^\nu = g_{\mu\nu}d\lambda^\mu d\lambda^\nu$ [70].

Relation to linear response and probing. We will now show that the SQGT determines the transition rates of Bogoliubov excitations between different modes in linear response to weak external forcing. This provides another physical interpretation of this quantity and opens the door for measuring its components similarly to the protocols that have been used to extract the quantum geometry of free fermions [63, 64, 86, 95].

Assuming a quasiparticle in a superposition of (unperturbed) Bogoliubov modes, $\psi(t) = \sum_{n=1}^N \psi_n(t)\mathbf{w}^n(\lambda_0) \in \mathbb{C}^{2N}$, its dynamics is described by the effective time-dependent Schrödinger equation $-i\partial_t\psi(t) = D(t)\psi(t)$ [47] (see Supplemental Material [70]), with $D(t)$ being the dynamical matrix (4), which becomes time dependent in response to external forcing (external forces generally also induce terms in the Hamiltonian that do not conserve the number of Bogoliubov quasiparticles, which are not captured by $D(t)$, which will be discussed below).

We assume driving in the form of a sinusoidal modulation $\lambda^\nu = \lambda_0^\nu + 2(A_\nu/\omega)\cos(\omega t - \varphi_\nu)$ of the system parameters around a mean value λ_0 . For weak driving ($A_\nu/\omega \ll 1$), the resulting time dependence of the dynamical matrix $D(t) = D[\lambda(t)]$ is captured by the linear approximation $D[\lambda(t)] \simeq D(\lambda_0) + \sum_\nu \partial_\nu D(\lambda_0)2(A_\nu/\omega)\cos(\omega t - \varphi_\nu)$. For the sake of simplicity inspired by Ref. [64], we focus on a two-parameter drive with relative phase φ , for which

$$D[\lambda(t)] = D(\lambda_0) + \partial_1 D(\lambda_0)2(A_1/\omega)\cos(\omega t) \pm \partial_2 D(\lambda_0)2(A_2/\omega)\cos(\omega t - \varphi). \quad (12)$$

The driving amplitudes A_ν shall be switched on suddenly at time $t = 0$, and we assume, for simplicity, that the system is prepared in mode n , $\psi(0) = \mathbf{w}^n(\lambda_0)$. Applying time-dependent perturbation theory in first order (linear response theory), we obtain the probabilities

$$p_{mn}^\pm(\omega, t) = |(\mathbf{w}^m)^\dagger(A_1\partial_1\mathcal{M} \pm A_2e^{i\varphi}\partial_2\mathcal{M})\mathbf{w}^n|^2 \times (2\pi t/\omega^2)\delta(\tilde{\Omega}_{mn} - \omega) \quad (13)$$

for finding the system in state $\mathbf{w}^m$ at time t , where $\tilde{\Omega}_{mn} \equiv \tilde{\Omega}_m - \tilde{\Omega}_n$. Note that all quantities are evaluated for $\lambda = \lambda_0$, whenever the argument (λ) is dropped, i.e., $\mathbf{w}^m \equiv \mathbf{w}^m(\lambda_0)$ and $\tilde{\Omega}_m \equiv \tilde{\Omega}_m(\lambda_0)$. The total excitation rate $\Gamma(\omega)$ into all other Bogoliubov modes $m \neq n$ is then given by $\Gamma^{n,\pm}(\omega) = t^{-1} \sum_{m \neq n} s_m p_{mn}^\pm(\omega, t)$. For an initial quasiparticle state, $n \leq N$, the sum can be restricted to $m \leq N$, since there is no quasiparticle-to-quasihole transition for a single-quasiparticle initial state [70].

To probe the SQGT, we consider the total excitation rate integrated over all (relevant) probing frequencies ω [63, 64], $\Gamma_{\text{int}}^n = \int d\omega \Gamma^{n,\pm}(\omega)$. For the modulation of a single parameter ($A_1 \equiv A, A_2 = 0$), we find

$$\Gamma_{\text{int}}^n = 2\pi A^2 \sum_{m \neq n} \frac{|(\mathbf{w}^m)^\dagger(\partial_1\mathcal{M})\mathbf{w}^n|^2}{s_m(\tilde{\Omega}_m - \tilde{\Omega}_n)^2} = 2\pi A^2 g_{11}^n. \quad (14)$$

For the second identity, we have used the spectral representation (10) of the SQGT, whose diagonal elements correspond to the symmetric part given by the symplectic quantum metric $\eta_{\nu\nu}^n = g_{\nu\nu}^n$. Thus, we can probe these diagonal elements by measuring the frequency integrated response to a modulation of λ^ν .

Next, we now consider the driving of both parameters λ^1 and λ^2 with two different phase lags φ and equal amplitudes $A_1 = A_2 = A$, to obtain the frequency-integrated transition rates

$$\Gamma_{\text{int}}^{n,\pm} = 2\pi A^2 \begin{cases} g_{11}^n \pm 2g_{12}^n + g_{22}^n, & \varphi = 0, \\ g_{11}^n \mp B_{12}^n + g_{22}^n, & \varphi = \pi/2, \end{cases} \quad (15)$$

the differences of which,

$$\Delta\Gamma_{\text{int}}^n \equiv \Gamma_{\text{int}}^{n,+} - \Gamma_{\text{int}}^{n,-} = \begin{cases} 8\pi A^2 g_{12}^n, & \varphi = 0, \\ -4\pi A^2 B_{12}^n, & \varphi = \pi/2, \end{cases} \quad (16)$$

are directly proportional to the real and imaginary parts of the off-diagonal element η_{12}^n of the SQGT, given by g_{12}^n and $-B_{12}^n/2$. In this way, all off-diagonal components $\eta_{\mu\nu}^n = g_{\mu\nu}^n - iB_{\mu\nu}^n/2$ can be extracted, allowing for a complete reconstruction of the SQGT from quasiparticle scattering rates.

We can generalize the above results to initial states of multiple quasiparticles. A generic perturbation $V(t) = \sum_{ij}\{A_{ij}(t)\hat{a}_i^\dagger\hat{a}_j + \frac{1}{2}[B_{ij}(t)\hat{a}_i^\dagger\hat{a}_j^\dagger + \text{H.c.}]\} + \sum_i\{C_i(t)\hat{a}_i^\dagger + \text{H.c.}\}$, takes the form $V(t) = \sum_{nm}\{A'_{nm}(t)\hat{b}_n^\dagger\hat{b}_m + \frac{1}{2}[B'_{nm}(t)\hat{b}_n^\dagger\hat{b}_m^\dagger + \text{H.c.}]\} + \sum_n\{C'_n(t)\hat{b}_n^\dagger + \text{H.c.}\} + D'(t)$ with respect to the unperturbed Bogoliubov operators $\hat{b}_n^{(\dagger)}$. Here, terms $\propto A'$ correspond to scattering between quasiparticle (or hole) states, whereas terms $\propto B'$ are associated with the scattering of a quasiparticle into a quasihole or vice versa. The terms $\propto C'$, which correspond to the creation of a single quasiparticle (or hole), can result from the scattering of an $\hat{a}_i$

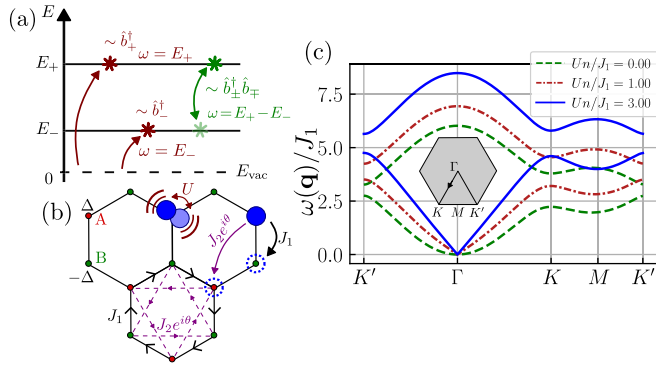


FIG. 2. (a) Sketch of the measurement scheme: selecting transitions by the frequency of the drive, first the Bogoliubov mode n is populated (red), and then a transition to another mode m is induced (green). (b) Illustration of the Bogoliubov-Haldane model. (c) Bogoliubov energy spectrum along high-symmetry path in the first Brillouin zone (inset) for $J_2 = 0.1J_1$, $\theta = \pi/2$, $\Delta = 3\sqrt{3}J_2/2$, and different Un/J_1 .

boson out of (or into) a Bose condensate, which is described by a macroscopic wave function that is not in equilibrium with respect to the perturbation. All these processes are associated with corresponding bosonic enhancement factors. To reconstruct the full SQGT, it is sufficient to consider the scattering of single quasiparticles. For discrete energies ω_n , this can be achieved by using that the different processes can be addressed selectively by appropriate driving frequencies [96]. As illustrated in Fig. 2(a), starting from the ground state, one can first exploit the B' or C' terms to populate the state n and then modulate the system resonantly with transitions $n \rightarrow m$ of A' type to extract information about the SQGT.

Application to Bogoliubov-Haldane model. As a concrete example, we study weakly interacting bosons in a two-dimensional hexagonal tight-binding lattice with two sublattice states, $s = A, B$, per unit cell. The single-particle terms correspond to the Haldane model [97], as it was recently realized with ultracold atoms in optical lattices [95,98,99]. They are characterized by real (complex) tunneling matrix elements $-J_1$ ($-J_2 e^{i\theta}$) between nearest (next-nearest) neighbors and on-site potentials Δ ($-\Delta$) on A (B) sites [Fig. 2(b)]. Due to the lattice symmetry, quasimomentum $\mathbf{q} = (q_x, q_y)$ is conserved by these processes, which is treated as a parameter in the following. By treating a weak repulsive on-site interaction U at a filling of n bosons per site in a self-consistent Bogoliubov approximation with quantum fluctuations on top of a Gross-Pitaevskii mean-field condensate, we obtain a Bogoliubov-Haldane model [70], $\hat{H}^{(B)} = \sum_{\mathbf{q} \neq 0} \sum_{s,s'} \hat{a}_{\mathbf{q}s}^\dagger [\mathcal{H}(\mathbf{q}) + \mathcal{H}_1]_{ss'} \hat{a}_{\mathbf{q}s'} + (\hat{a}_{\mathbf{q}s}^\dagger [\mathcal{H}_2]_{ss'} \hat{a}_{-\mathbf{q}s'}^\dagger + \text{H.c.})$. Here, $\mathcal{H}(\mathbf{q}) + \mathcal{H}_1$ contains the single-particle tight-binding Hamiltonian including a mean-field interaction contribution, and $\mathcal{H}_2$ stems exclusively from the interactions which in experiments with ultracold bosons are tunable via Feshbach resonances [100]. The Bogoliubov spectrum is shown in Fig. 2(c).

We test our proposal for extracting the symplectic quantum geometric tensor from excitation rates to other Bogoliubov modes by simulating the time evolution for initially

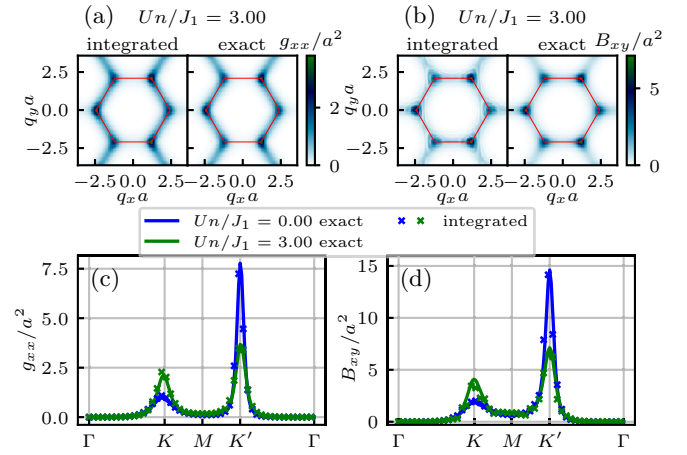


FIG. 3. Symplectic quantum metric component $g_{xx}(\mathbf{q})$ and Berry curvature $B_{xy}(\mathbf{q})$ for the upper particle band of the Bogoliubov-Haldane model [70] obtained via integrated excitation rates (14)–(16) and via exact diagonalization (6). (a) Symplectic quantum metric component $g_{xx}(\mathbf{q})$ for $Un/J_1 = 3$ shown in 2D quasimomentum space from numerical integration of excitation rates (left) and exact diagonalization (right). (b) Same as in panel (a) but for the symplectic Berry curvature $B_{xy}(\mathbf{q})$. (c) Exact (solid line) and integrated (crosses) results for the symplectic quantum metric component $g_{xx}(\mathbf{q})$ along a high-symmetry path [cf. inset in Fig. 2(c)] for two interactions strengths $Un/J_1 = 0$ (noninteracting Haldane model) and $Un/J_1 = 3$. (d) Same as in panel (c) but for the symplectic Berry curvature $B_{xy}(\mathbf{q})$. Simulation parameters: $A/J_1 = 0.01$, $J_1 t/\hbar = 10$ (final integration time), $\hbar\omega/J_1 \in [0.1(0.2), 5.5]$ for $Un/J_1 = 3$ (0) with a frequency spacing of $\delta\omega = 0.05$. Other parameters are given in Fig. 2.

populated Bogoliubov modes corresponding to the upper particle band. The parameter modulation (12) is induced by periodically shaking the Bogoliubov-Haldane model along the lattice directions x and y . In the reference frame comoving with the lattice, the corresponding inertial forces directly induce modulations of the parameters q_x and q_y , respectively [70]. In Figs. 3(a) and 3(b), we present the exact symplectic quantum metric $g_{xx}^+(\mathbf{q})$ and Berry curvature $B_{xy}^+(\mathbf{q})$ of the upper quasiparticle for $Un/J_1 = 3$ together with the numerically integrated excitation rates obtained from the time evolution [70]. In Figs. 3(c) and 3(d), we compare the exact and integrated values for the same quantities along a high-symmetry path in quasimomentum space for $Un/J_1 = 0$ (corresponding to the bare number-conserving Haldane model) and $Un/J_1 = 3$. We find very good agreement between the exact and integrated symplectic quantum geometric quantities, validating our scheme for extracting the components of the SQGT. Results for $g_{xx}^+(\mathbf{q})$ and $g_{xy}^+(\mathbf{q})$ are presented in the Supplemental Material [70]. Experimentally, the Bogoliubov excitation fraction can be measured by relating the presence of Bogoliubov quasiparticles to the quasimomentum distribution of the original particles, which can be observed using time-of-flight imaging techniques [70,101,102].

Symplectic anomalous velocity. We show that, like in the number-conserving case [65–67,103], a Bogoliubov lattice system acquires an anomalous transverse velocity proportional to the symplectic Berry curvature and transverse to

an applied homogeneous force $\mathbf{F}$. Consider a quasiparticle prepared in the Bogoliubov-Bloch mode $\mathbf{w}^n(\mathbf{q}_0)$. In response to the force, the quasimomentum becomes $\mathbf{q}(t) = \mathbf{q}_0 + \mathbf{F}t$, inducing a modulation of the dynamical matrix $D[\mathbf{q}(t)]$. In first-order adiabatic perturbation theory with respect to $\mathbf{F}$, the time-evolved state is given by [70]

$$\mathbf{w}^n(\mathbf{q}) + i \sum_{m \neq n} \mathbf{w}^m(\mathbf{q}) \frac{(\mathbf{w}^m(\mathbf{q}))^\dagger (\nabla_{\mathbf{q}} \mathcal{M}) \mathbf{w}^n(\mathbf{q}) \cdot \mathbf{F}}{s_m [\tilde{\Omega}_n(\mathbf{q}) - \tilde{\Omega}_m(\mathbf{q})]^2}. \quad (17)$$

The expectation value of the Bogoliubov velocity operator $\mathbf{v}(\mathbf{q}) = \nabla_{\mathbf{q}} D(\mathbf{q})$ then reads

$$\langle \mathbf{v}(\mathbf{q}) \rangle = \nabla_{\mathbf{q}} \tilde{\Omega}_n(\mathbf{q}) + \mathbf{B}_n(\mathbf{q}) \times \mathbf{F}, \quad (18)$$

with $\mathbf{B}_n(\mathbf{q}) \equiv \epsilon_{\mu\nu\kappa} B_{\mu\nu}^n(\mathbf{q}) \mathbf{e}_\kappa$. The first term corresponds to the slope of the occupied Bogoliubov energy band, and the second term provides the symplectic anomalous transverse velocity. This result is complementary to that of Ref. [103], where the dynamics of Bogoliubov-Bloch wave packets has also been studied using a time-dependent variational principle, but without the connection to the symplectic Berry curvature. The effect of the symplectic anomalous velocity can be verified in semiclassical dynamics of Bogoliubov-Bloch wave packets or by measuring the modified thermal depletion of a BEC due to the presence of the symplectic Berry curvature [103].

Conclusion. We have introduced the manifestly gauge-invariant SQGT for BBdG systems, whose imaginary part is proportional to the symplectic Berry curvature. The real part of the SQGT defines a symplectic quantum metric, which is shown to provide a natural distance measure in the space of Bogoliubov modes. All components of the symplectic quantum geometric tensor can be measured via tracking Bogoliubov excitation rates and the symplectic Berry curvature describes an anomalous velocity in Bogoliubov Bloch systems. State-of-the-art experiments with bosonic ultracold atoms [104–107] are well suited to test our proposals. For future work, it will be interesting to connect recent topological notions for mixed quantum states [108–110] to thermal BBdG systems and to see whether Euler invariants [111–113] may be generalized to the BBdG context.

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Data Availability The data that support the findings of this article are openly available at [143].

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