

On-demand positioning of topological states via non-Hermitian defects

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(Received 13 October 2025; accepted 2 April 2026; published 24 April 2026)

We show that a single non-Hermitian defect embedded in an otherwise Hermitian topological lattice can anomalously relocate an edge state into the bulk without altering the topology. Using one- and two-dimensional models, including the Su-Schrieffer-Heeger chain and honeycomb lattices, we demonstrate that tuning an asymmetric coupling or a local parity-time-symmetric perturbation shifts the edge mode to the defect site, where it becomes strongly localized. In two dimensions, the defect further acts as a momentum-bandpass filter, selectively relocating a narrow subset of edge modes from the continuum of boundary states. This relocation is distinct from the non-Hermitian skin effect, which drives collective bulk accumulation, and from trivial defect-induced modes, which lack topological origin. The effect is robust against disorder and system size, enabling programmable, on-demand positioning of topological states for applications in photonic routing, switching, and robust sensing.

DOI: [10.1103/PhysRevResearch.8.023074](#)

I. INTRODUCTION

Topological lattices host boundary states protected by bulk invariants, which makes them robust against moderate disorder and fabrication imperfections [1–11]. In one and two dimensions, such as Su-Schrieffer-Heeger (SSH) chains and Honeycomb lattice, these modes are exponentially localized at boundaries and are usually considered immovable: As long as the bulk topology remains unchanged, the edge state should remain pinned to the system edge.

In parallel, non-Hermitian (NH) physics has revealed phenomena—including exceptional points (EPs), the non-Hermitian skin effect (NHSE), and defect-induced modes—that challenge conventional bulk-boundary correspondence [8,12–21]. Several works have studied NH defects in trivial and topological lattices [18,22–29], showing how gain/loss or asymmetric couplings generate localized states or shift phase boundaries. However, these scenarios either redistribute entire spectra, as in the NHSE, or create new defect-bound modes. They do not address whether a single NH defect far from the boundary can directly control the position of a *preexisting* topological edge state.

Unlike prior work where non-Hermiticity globally modifies spectra or induces defect-bound resonances, we show that a single local defect can nonlocally relocate a preexisting topological edge mode—revealing an unexpected interplay between topology and exceptional-point physics, where non-Hermiticity acts as a programmable “topological handle” controlling mode anchoring. For example, as illustrated in

Figs. 1(a) and 1(b), adding an alternating balanced gain-loss profile to the right edge of a bearded-zigzag honeycomb lattice delocalizes the topological edge mode from its conventional position on the left and relocates it on the right edge on the opposite side. Similarly, in an SSH chain with a single non-Hermitian defect [Figs. 1(c) and 1(d)], the zero mode relocates to the defect site instead of remaining on the left edge as conventionally expected. To clarify the origin of this surprising behavior, we first employ the SSH chain as a minimal model. This choice provides the simplest one-dimensional topological lattice with a single, well-isolated edge mode under the chosen termination, allowing the defect-induced relocation to be identified unambiguously without interference from additional boundary states. Using this model, we show that relocation arises from a wave-matching condition and occurs around an EP, when non-Hermiticity—introduced via asymmetric coupling or a local parity-time (*PT*) dimer—matches the stronger lattice coupling. Beyond the EP, the topological state is restored to its conventional location at the edge. Unlike the NHSE or defect-induced modes, the relocated state preserves both its topological origin and exponential localization.

To demonstrate the generality of this mechanism beyond one dimension, we further extend it to two-dimensional honeycomb lattices. In this case, introducing alternating gain and loss at the boundary where no topological edge state is expected in the Hermitian limit causes the edge state to migrate across the lattice and localize at that boundary. Importantly, both this relocation and the subsequent restoration of the edge state to its conventional boundary beyond the EP underscore that our findings are not reducible to a simple geometric truncation (“cut”) or trivial perturbation of the topological lattice.

Although our analysis is presented in SSH language, the same equations arise from the coupled-mode theory of optical ring resonators restricted to a whispering-gallery mode, where couplings and local gain/loss can be engineered with high precision. These results demonstrate that non-Hermitian

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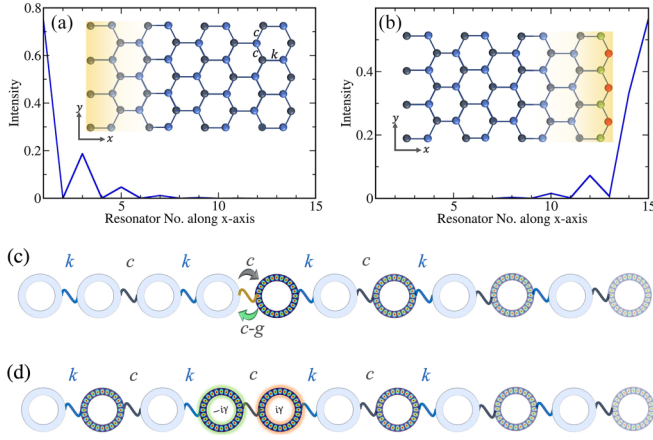


FIG. 1. (a), (b) Relocation of a topological edge state in a two-dimensional honeycomb lattice. Inset of (a): Schematic of the bearded-zigzag geometry, with the expected edge state highlighted in yellow at the left boundary. Main panel of (a): Edge-state profile at $q = 0$ for $c = k = 0.5$. (b) When an alternating gain-loss profile with $\gamma = 2c \cos(q) = 1$, (here we made $q = 0$), is applied to the rightmost sites, the edge state relocates to the right boundary. This momentum-selective relocation acts as a momentum-filter condition selecting a single q mode. (c), (d) SSH lattice with a single non-Hermitian defect, equivalently realized by coupled optical ring resonators supporting a single whispering-gallery mode, with couplings $k < c$. (c) Asymmetric intradimer coupling, where forward and backward hoppings differ; for $g \approx c$, the edge state relocates from the left boundary and localizes at the defect bond. (d) Local parity-time-symmetric defect, implemented by balanced gain and loss $\pm\gamma$; for $\gamma \approx c$, the edge state relocates and appears at the PT defect. In both cases, the bulk topology remains unchanged.

defect engineering provides an experimentally feasible route for programmable control of topological states in representative one- and two-dimensional lattice models.

II. MODEL

Our proposal can be realized in diverse physical settings [6–8, 11, 30–37]. We consider the canonical SSH lattice, characterized by intracell and intercell couplings c and k ($k < c$), which supports a zero-energy topological state localized at the left edge. In the optical ring resonator arrays restricted to a single whispering-gallery mode, k and c correspond to the effective coupling strengths determined by the relative spacing between adjacent resonators, while non-Hermitian perturbations can be implemented through engineered asymmetric couplings or site-selective gain and loss. Figure 1 illustrates both the abstract SSH lattice and its direct realization in coupled microring resonators.

The eigenvalue equations governing the modal amplitude ψ_n at site n are

$$\begin{aligned} E\psi_1 &= k\psi_2 \quad (\text{a}), \\ E\psi_n &= k\psi_{n-1} + c\psi_{n+1}, \quad n \in \text{even} \quad (\text{b}), \\ E\psi_n &= c\psi_{n-1} + k\psi_{n+1}, \quad n \in \text{odd}, \quad n \neq 1, \mathcal{N} \quad (\text{c}), \\ E\psi_{\mathcal{N}} &= c\psi_{\mathcal{N}-1} \quad (\text{d}), \end{aligned} \quad (1)$$

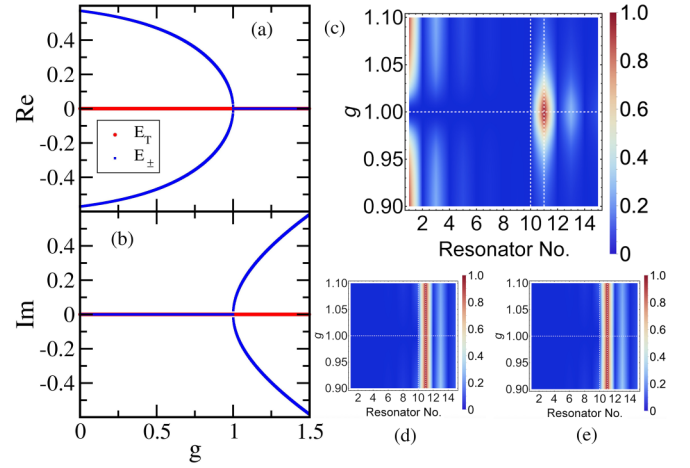


FIG. 2. (a), (b) Real and imaginary parts of the eigenvalues of the topological state E_T (red) and the defect-induced bound states $E_{\pm}$ (blue) as functions of the asymmetric coupling g ($c = 1, k = 0.5$). The defect modes coalesce at a second-order exceptional point near $g = c$. (c) Spatial profile of the topological zero mode as a function of g . The asymmetric coupling is applied between resonators 10 and 11 ($m = 5$), highlighted by the vertical white dashed line. Around $g = c$, the maximum intensity of the zero mode relocates to the site ψ_{2m+1} , while away from this point it remains localized near the left edge as expected—even in the presence of non-Hermiticity. (d), (e) Spatial profiles of the defect-bound states associated with $E_{\pm}$ as functions of g . We emphasize that near $g \approx c$, the topological mode becomes strongly localized at the defect bond and spatially overlaps with the defect-bound subspace, while remaining a distinct eigenstate.

where $\mathcal{N}$ is an odd number identifying the total number of sites in the lattice. It is known that in this configuration, the topological zero mode is exponentially localized at the left boundary.

A. Asymmetric-coupling defect

We introduce a single asymmetric-coupling defect on bond $(2m, 2m+1)$, as shown in Fig. 1(c). Here, m is an integer chosen once from the range $1 \leq m \leq \frac{\mathcal{N}-1}{2}$ and denotes the position of the defect dimer in the lattice. In all other dimers, the SSH equations remain those of Eq. (1), while at the defective bond, the eigenvalue equations become

$$\begin{aligned} E\psi_{2m} &= k\psi_{2m-1} + (c - g)\psi_{2m+1}, \\ E\psi_{2m+1} &= c\psi_{2m} + k\psi_{2m+2}, \end{aligned} \quad (2)$$

where $g > 0$ quantifies the coupling asymmetry. For $g = 0$, the model reverts to the Hermitian SSH chain.

1. Defect-bound modes and exceptional point

Any $g \neq 0$ pulls two defect-induced bound states $E_{\pm}$ out of the bands and into the gap. As shown in Figs. 2(a) and 2(b), these nontopological localized modes have real eigenvalues $E_{\pm} \neq 0$ in the exact phase, which for a sufficiently large lattice occurs for $g < c$. At a specific value of g (in our case $g \simeq c$), the two defect modes coalesce at a second-order exceptional point: Their eigenvalues become degenerate at zero and their eigenvectors coalesce. For $g > c$, they enter the broken phase

with $\text{Re } E_{\pm} = 0$ and $\text{Im } E_{\pm} \neq 0$ [25]. In contrast, the original topological zero mode $E_T = 0$ remains real for all g , indicating that the bulk topology is unchanged [Figs. 2(a) and 2(b)].

2. Relocation of the topological zero mode

To isolate the effect of a single defect on the topological state, we place it away from the left edge. In our numerical simulations shown in Figs. 2(c)–2(e), the defect is introduced between sites 10 and 11, where the conventional left-edge zero mode (in the absence of non-Hermiticity) has negligible weight. Figures 2(c), 2(d), and 2(e), respectively, show the spatial profiles of the zero mode and the two defect-induced bound states as a function of g . For $g < c$, the zero mode and bound states occupy their expected positions—The zero mode remains left-localized and the bound states are centered near the defect. Strikingly, around $g = c$, the topological mode is no longer localized at the left boundary; instead, it relocates to the defect bond, becoming collocated with the defect modes $E_{\pm}$. At this point, the defect modes $E_{\pm}$ form a second-order exceptional point, whereas the relocated topological mode is identified by continuous tracking in g ; although these states strongly overlap spatially in finite lattices, they remain linearly independent in general. For $g > c$, the zero mode returns to its conventional position at the left edge.

We note that when the non-Hermitian defect is instead placed at the far right boundary—where no topological state is expected in the Hermitian limit, given our choice of couplings and lattice size—the same mechanism drives the topological zero mode completely across the chain. When the defect is placed at the far right boundary, the mode instead relocates to the right edge, demonstrating complete left-right flipping within a single one-dimensional lattice. This demonstrates that full left-right flipping can occur even in the one-dimensional SSH chain, consistent with the two-dimensional honeycomb example discussed below. Moreover, this behavior cannot be attributed to a geometric truncation of the lattice. In a cut geometry, one would expect the original left-edge topological state to persist while an additional right-edge state is created. In contrast, in our case, the original mode itself migrates, confirming that the relocation is a genuine non-Hermitian effect rather than a trivial boundary truncation.

3. Wave-matching explanation at $g = c$

For a semi-infinite lattice, the relocation at $g = c$ follows directly from wave matching at $E_T = 0$. Substituting $E = 0$ into Eq. (1) enforces $\psi_{2\ell} = 0$ for all $\ell = 1, 2, \dots, \frac{N-1}{2}$, independent of the defect. At $g = c$, the first line of Eq. (2) reduces to

$$E_T \psi_{2m} = k \psi_{2m-1} \Rightarrow \psi_{2m-1} = 0,$$

which by backward recurrence forces $\psi_n = 0$ for all $n \leq 2m$, including the edge amplitude ψ_1 . Thus, no edge localization is possible at the left boundary when $g = c$. Meanwhile, the second line of Eq. (2) at $E_T = 0$ with $\psi_{2m} = 0$ reduces to the same form as Eq. (1a), which admits a decaying solution anchored at ψ_{2m+1} . Our numerical inspection shows that as g increases from 0 to c , $|\psi_1|^2$ decreases while $|\psi_{2m+1}|^2$ increases, reaching extrema at $g = c$; for $g > c$, this trend

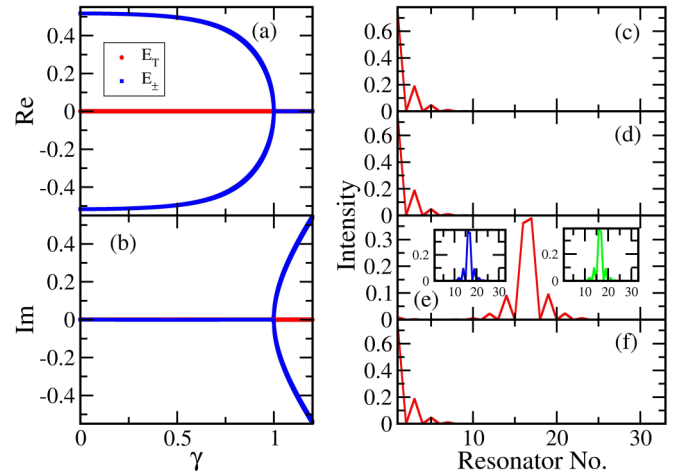


FIG. 3. PT -symmetric defect in an SSH lattice. (a), (b) Real and imaginary parts of the eigenvalues of the topological state E_T (red) and defect-induced bound states $E_{\pm}$ (blue) vs the gain-loss parameter γ ($c = 1$, $k = 0.5$). At $\gamma = c$, the defect states coalesce at a second-order exceptional point, while the topological state becomes purely imaginary. (c), (d) Topological edge state for $\gamma = 0$ and $\gamma = 0.2$, localized at the left boundary. (e) At $\gamma = c$, the topological mode relocates to the PT defect and becomes exponentially localized there, coexisting with the defect modes $E_{\pm}$ (insets). (f) For $\gamma > c$, the zero mode reappears at the left edge. In contrast to the asymmetric-coupling case, here the topological eigenvalue E_T acquires a purely imaginary component near the EP, reflecting broken PT symmetry. Note that because the system lacks global PT symmetry (odd site number), all eigenvalues are in principle complex for $\gamma \neq 0$; eigenvalues appearing real in panels (a) and (b) have imaginary parts below the resolution of the plotted scale.

reverses, recovering the conventional left-edge zero mode with $E_T = 0$.

B. PT -symmetric defect

We next consider the anomalous relocation of the topological mode in a system with a single PT -symmetric (PTS) defect, realized by adding local gain and loss, as shown in Fig. 1(d). The inclusion of imaginary onsite potentials $+i\gamma$ and $-i\gamma$ on sites ψ_{2m+1} and ψ_{2m} , respectively, modifies the corresponding two equations in the system of Eq. (1) to

$$\begin{aligned} E\psi_{2m} &= k\psi_{2m-1} + c\psi_{2m+1} - i\gamma\psi_{2m}, \\ E\psi_{2m+1} &= c\psi_{2m} + k\psi_{2m+2} + i\gamma\psi_{2m+1}, \end{aligned} \quad (3)$$

while all other equations remain as in Eq. (1). This local defect breaks the global PT symmetry of the lattice. In finite lattices with an odd number of sites, global PT symmetry is absent and all eigenvalues are, strictly speaking, complex for any nonzero γ . However, when the defect is placed deep in the bulk of a long lattice, the imaginary part of the topological eigenvalue remains exponentially small over a broad range of γ , rendering the mode effectively real on the scale of Fig. 3. Near $\gamma \approx c$, the imaginary part becomes dominant and the mode behaves as a purely imaginary state to leading order.

1. Defect-bound modes and topological zero mode

Figure 3 summarizes the effect of a PTS defect in a topological SSH lattice with $\mathcal{N} = 33$ sites, $k = 0.5$, and $c = 1$. In the Hermitian case ($\gamma = 0$), the expected left-edge topological mode with $E_T = 0$ is observed [Figs. 3(a)–3(c)]. As γ increases, two defect-induced bound states $E_{\pm}$ emerge, analogous to the asymmetric-coupling case, while the topological mode begins to shift toward the defect [Figs. 3(d)–3(f)]. For $\gamma < c$, the zero mode remains localized at the left edge. At $\gamma \approx c$, however, its maximum amplitude relocates to the defect sites, with negligible intensity on the boundary [Fig. 3(e)]. In this regime, three localized states coexist near the defect: the two defect-bound modes $E_{\pm}$ and the relocated topological mode E_T , all with purely imaginary eigenvalues. Although these states may approach degeneracy under parameter tuning, in general, they retain distinct eigenvalues and eigenvectors. For $\gamma > c$, the topological mode reappears at the left edge [Fig. 3(f)].

2. Analytical explanation at $\gamma = c$

To analyze this behavior, we place the PT defect at the rightmost dimer of a lattice with odd site number $\mathcal{N}$. The boundary equations become

$$E\psi_{\mathcal{N}-1} = k\psi_{\mathcal{N}-2} + c\psi_{\mathcal{N}} - i\gamma\psi_{\mathcal{N}-1} \quad (\text{a}),$$

$$E\psi_{\mathcal{N}} = c\psi_{\mathcal{N}-1} + i\gamma\psi_{\mathcal{N}} \quad (\text{b}). \quad (4)$$

At $\gamma = c$, assume $E_T = 0$. From Eq. (4b), we obtain $i\psi_{\mathcal{N}-1} = \psi_{\mathcal{N}}$. Substituting this into Eq. (4a) requires $\psi_{\mathcal{N}-2} = 0$, which by backward iteration forces all even sites up to the boundary to vanish. Combined with Eq. (1a), which enforces $\psi_{2\ell} = 0$, this yields the trivial solution $\psi_n = 0$. Hence, E_T cannot remain zero at $\gamma = c$. Because the bulk topological invariant is unchanged by a single local defect, the topological edge state remains continuously connected to the zero-energy mode of the Hermitian SSH lattice. Moreover, the local non-Hermitian perturbation does not break the chiral (sublattice) symmetry that pins the real part of the topological eigenvalue. As a result, while gain and loss can introduce an imaginary component to E_T , its real part remains fixed at zero, so that the eigenvalue evolves along the imaginary axis when it becomes complex. This argument holds as long as the defect does not close the bulk gap or globally break the protecting symmetry. Thus, the real part of E_T must remain zero, implying that E_T is purely imaginary. From Eq. (4b), one finds

$$\psi_{\mathcal{N}-1} = \frac{E_T - ic}{c} \psi_{\mathcal{N}}, \quad \psi_{\mathcal{N}-2} = \frac{E_T^2}{kc} \psi_{\mathcal{N}},$$

and so on, generating a decaying profile peaked at $\psi_{\mathcal{N}}$. This construction shows that the relocated topological state at $\gamma = c$ is centered at the defect site, with the property that $\psi_{2\ell+1}$ are real while $\psi_{2\ell}$ are purely imaginary—a $\pi/2$ phase shift between sublattices, which is a hallmark of non-Hermitian topological zero modes.

We emphasize that a $\pi/2$ phase difference between sublattice amplitudes is not unique to the PT -symmetric defect, but is a generic property of chiral-symmetric zero modes in SSH-type lattices, including the odd-site SSH chain even in the absence of gain and loss. In this sense, the PT defect does not generate the $\pi/2$ phase structure; rather, it relo-

cates the preexisting topological zero mode while preserving its internal chiral phase relation. Near the exceptional point, one might be tempted to interpret the relocated state as a newly generated defect-bound mode distinct from the original edge state. However, this interpretation is inconsistent with the structure of the eigenstate itself: the relocated mode remains exponentially localized and retains the characteristic $\pi/2$ sublattice phase relation of the SSH zero mode. These features demonstrate that the exceptional point does not destroy the topological state and replace it with a trivial defect mode, but instead relocates the preexisting topological mode while preserving its internal structure. In contrast, relocation induced by asymmetric-coupling defects does not enforce a fixed quadrature phase relation, and the detailed phase profile depends on microscopic defect parameters. Similar considerations apply in the two-dimensional honeycomb lattice, where relocated edge states inherit their phase structure from the underlying chiral symmetry rather than exhibiting a universal sublattice phase shift. This reasoning also extends to the two-dimensional honeycomb lattice, which likewise possesses chiral (sublattice) symmetry: Relocated edge states inherit their internal phase structure from the underlying chiral zero modes, although in two dimensions this may manifest as a spatially varying phase texture rather than a single uniform $\pi/2$ phase difference.

Although the analytical derivation places the PT dimer at the boundary for algebraic simplicity, the essential ingredient is local and does not depend on a physical edge. Near $\gamma \approx c$, the defect matching equations in the PT dimer subspace become nearly singular, which strongly constrains the continuation of the zero-mode solution across the defect bond. As a result, the eigenstate reorganizes and becomes pinned at the defect location. This mechanism does not imply formation of a higher-order exceptional point involving the edge state; rather, the EP remains confined to the two-site defect subspace, while the topological mode hybridizes with it. The same local constraint operates when the defect is placed in the bulk, explaining why Fig. 3 shows relocation and acquisition of a predominantly imaginary eigenvalue near $\gamma \approx c$. Away from this parameter window (including deep in the broken phase), the constraint is relaxed and the zero mode is not generically forced to remain pinned.

C. Extension to 2D lattices

The relocation of the topological edge state is not limited to the SSH chain described above. As a two-dimensional example, we consider a bearded-zigzag honeycomb lattice with periodic boundary conditions along the y direction, as shown schematically in Fig. 1(a) [38]. For equal couplings $k = c$, this geometry supports topological edge states localized on the left boundary. Introducing alternating gain and loss on the rightmost sites [Fig. 1(b), inset] induces relocation: For a given transverse momentum q , the edge mode shifts to the opposite boundary when $\gamma \approx 2c \cos q$, as confirmed in Fig. 1(b). In this case, left- and right-edge states coexist at different wave numbers, showing that the relocation principle naturally extends beyond one dimension. For completeness, we note that a single pointlike two-site PT defect placed in the bulk produces only a local response and does not generically

relocate the edge state deep into the bulk. Crucially, the non-Hermitian perturbation also acts as a momentum-selective filter: Within the continuum of left-edge modes, relocation occurs primarily for those momenta satisfying the matching condition $\gamma \approx 2c \cos q$. In finite ribbons, residual hybridization between opposite edges broadens this condition into a narrow window of allowed q , ensuring that relocation remains observable without requiring perfect tuning. This mechanism effectively isolates a subset of edge modes around a target momentum, providing a powerful handle for mode control in two dimensions. By adjusting γ , different momentum channels can be reanchored, enabling selective access to individual branches of the edge-state continuum and offering routes toward single-mode topological lasing, edge-state routing, and direct spectroscopic probing of edge dispersions.

III. CONCLUSIONS

We have shown that embedding a single non-Hermitian defect in an otherwise Hermitian topological lattice can anomalously relocate an edge state into the bulk at the defect site. Two representative defects—an asymmetric coupling and a PT -symmetric dimer—both realize this effect, which occurs when the non-Hermiticity matches the stronger lattice coupling, corresponding to an EP. The phenomenon is robust against disorder, independent of system size, and generalizes to two-dimensional (2D) lattices such as graphene ribbons.

Our analysis applies equally to abstract SSH chains and to arrays of coupled optical ring resonators in a whispering-gallery mode, where couplings and gain/loss can be engineered with high precision.

Unlike the non-Hermitian skin effect, which drives all bulk states to one side, the relocation reported here involves a single topological zero mode while the bulk remains unaffected. Nor is it related to algebraic “scale-free” localization in disordered chains: Our effect arises in a clean lattice and follows directly from wave matching at the defect.

These results introduce non-Hermitian defects as a tool for programmable control of topological states, with potential applications in photonic routing, topological switches, and robust quantum sensors.

ACKNOWLEDGMENTS

H.R. acknowledges partial support from the Army Research Office (Grant No. W911NF-20-1-0276) and the National Science Foundation (Grant No. PHY-2012172).

The views and conclusions expressed here are those of the author and do not necessarily reflect the official policies of the funding agencies.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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