

Parent Lindbladians for matrix product density operators

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Understanding quantum phases of matter is a fundamental goal in physics. For pure states, the representatives of phases are the ground states of locally interacting Hamiltonians, which are also renormalization fixed points (RFPs). These RFP states are exactly described by tensor networks. Extending this framework to mixed states, matrix product density operators (MPDOs) that are RFPs are believed to encapsulate mixed-state phases of matter in one dimension, where nontrivial topological phases have already been shown to exist. However, to better motivate the physical relevance of those states, and in particular the physical relevance of the recently found non trivial phases, it remains an open question whether such MPDO RFPs can be realized as steady states of local Lindbladians. In this work, we resolve this question by analytically constructing parent Lindbladians for MPDO RFPs. These Lindbladians are local, frustration free, and exhibit minimal steady-state degeneracy. Interestingly, we find that parent Lindbladians possess a rich structure that distinguishes them from their Hamiltonian counterparts. In particular, we uncover an intriguing connection between the noncommutativity of the Lindbladian terms and the fact that the corresponding MPDO RFP belongs to a nontrivial phase.

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I. INTRODUCTION

In quantum many-body systems, short-ranged interactions strongly constrain the entanglement of physically relevant states. This fact provides a natural foundation for theoretical tools that exploit the underlying entanglement structure. Among these tools, tensor networks have been remarkably successful in revealing a variety of phenomena in strongly interacting many-body quantum systems [1]. The tensor-network framework makes the spatial structure of the underlying correlations explicit while allowing for the efficient description of physically relevant many-body quantum states. For instance, matrix product states (MPS) [2], the prototypical class of pure tensor-network states in one-dimensional (1D), can approximate ground states of gapped local Hamiltonians using a number of parameters that scale only polynomially with system size [3]. Analogously, matrix-product density operators (MPDO) are the mixed-state counterparts of MPS. They can provide an efficient description of Gibbs states of local Hamiltonians [4,5] and, via the bulk-boundary corre-

spondence of tensor networks [6], they appear as boundary states of two-dimensional (2D) systems.

Beyond serving as a successful variational class [7], tensor-network states also emerge as exact solutions to the ground state problem of local Hamiltonians. Specifically, every projected entangled-pair state [1] (PEPS)—the generalization of MPS to higher dimensions—can be realized as the exact ground state of a local Hamiltonian. The resulting so-called parent Hamiltonians are local, have the minimal possible ground-state degeneracy, and, in 1D, are also gapped [1,2]. The parent Hamiltonian construction thus endows pure tensor-network states—MPS in particular—with a direct physical interpretation.

Parent Hamiltonians also play a central role in the classification of quantum phases of matter [1]. PEPS, together with their parent Hamiltonians, provide representatives for all known (nonchiral) gapped topological phases of matter in 2D, as they encompass string-net models [8–13]. In particular, this holds true even if PEPS are restricted to renormalization fixed point (RFP) states [1]. This fact becomes both rigorous and explicit in 1D. There, RFP MPS emerge in the limit of a real-space renormalization flow. Moreover, the complete phase classification can be conducted directly [14,15] and the associated MPS RFPs provide representatives for all phases.

Recently, there has been an increasing interest in classifying quantum phases of matter for mixed states [16–30], where phases—not present in pure states—can emerge. Notably, for 1D MPDO, nontrivial phases exist even with short-range correlations [31,32]. This contrasts sharply with pure states,

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TABLE I. Properties of parent Lindbladian for MPDO renormalization fixed point, in comparison with parent Hamiltonian for MPS.

	Parent Hamiltonian H	Parent Lindbladian $\mathcal{L}$ (RFP)
Geometrically local	$H = \sum_i h_i$	$\mathcal{L} = \sum_i \mathcal{L}_i$
Local term construction	$h_i = \tau_i(\mathbb{I} - VV^\dagger)$	$\mathcal{L}_i = \tau_i(\mathcal{T} \circ \mathcal{S} - \mathbb{I})$
Frustration free	$h_i \psi_{\text{MPS}}\rangle = 0 \ \forall i \quad (H \geq 0)$	$\mathcal{L}_i(\rho_{\text{MPDO}}) = 0 \ \forall i$
Degeneracy	Minimal ground-state degeneracy	Minimal steady-state degeneracy
Degeneracy implies long-range correlation	✓	Not always
Commuting for RFP?	✓	Not possible for a class of RFP

where nontrivial phases with short-range entanglement appear only after imposing symmetry protection [14,15]. Although a full classification of mixed-state phases is still unresolved, MPDO RFPs are expected to capture all possible phases, analogously to their pure-state counterparts. However, it has remained unclear whether MPDO RFPs admit a physical interpretation analogous to parent Hamiltonians for MPS.

In this paper, we resolve this question by establishing the open system analog of the parent Hamiltonian construction for MPDO RFPs. Since this concerns mixed quantum states, in place of the Hamiltonian ground state, we consider steady states of a Markovian open-system evolution, described by a Lindblad generator [33]. When is a given MPDO the steady state of a local Lindbladian? In analogy to the pure state case, we call this a parent Lindbladian of the MPDO. We systematically answer this question, giving an explicit analytical parent Lindbladian construction for the case when the MPDO is an RFP (see Table I). We show that the constructed parent Lindbladians satisfy many important properties: They are local, frustration free, and with minimal steady-state degeneracy. In contrast to parent Hamiltonians, parent Lindbladians are not always commuting at the RFP and a nontrivial steady-state degeneracy does not necessarily imply long-range correlation (see Table I for a summary and Sec. II for further discussion).

The problem of constructing a local Lindbladian with a unique (specified) tensor-network state as its fixed point has been considered before, most notably as a numerical algorithm under certain assumptions [34], in dissipative state preparation [35–37] and in the context of thermal states [38–42]. In particular, Ref. [37] includes a simple construction to leverage the parent Hamiltonian of a given injective MPS to a parent Lindbladian with the specified MPS as its unique fixed point. Although this method can be used to construct a parent Lindbladian for the purification of a mixed state, it fails to give a parent Lindbladian for the mixed state itself, which is the main objective here. For thermal states, Ref. [41] gives a construction for a parent Lindbladian for the case of stabilizer Hamiltonians. Our construction contains thermal states of nearest-neighbor commuting Hamiltonians with zero correlation length but also includes states that are not necessarily full-rank, such as a generalization of (non-full-rank) thermal states and mixed states arising as boundaries of 2D topologically ordered systems.

Our results endow MPDO, which are RFP, with a direct physical meaning—they arise as the exact steady states of local dissipative dynamics. Moreover, we identify the class of MPDO whose parent Lindbladian is rapid mixing, thereby allowing efficient mixed-state preparation. We envision our

the parent Lindbladian construction can serve as a tool for classifying mixed-state quantum phases.

The remainder of the paper is organized as follows. In Sect. II, we summarize the results, outlining the core components for constructing parent Lindbladians and discussing their key properties. Section III provides an overview of the essential concepts of MPS and MPDO, offering the necessary background for the parent Lindbladian construction. The classification of MPDO RFPs is divided into two cases: simple and nonsimple. In Sec. IV, we present a general framework for constructing parent Lindbladians for MPDO RFPs, with explicit treatments of the simple and nonsimple cases detailed in Secs. V and VI, respectively. Finally, we conclude in Sec. VII by highlighting open questions and directions for future research.

II. SUMMARY OF RESULTS

We consider N spins in 1D and a density operator, ρ , which is an RFP MPDO (as defined below). Our goal is to find a Lindbladian $\mathcal{L}^{(N)}$ such that (1) $\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i$, where $\mathcal{L}_i$'s are local Lindbladians themselves (i.e., act on the site i and its r neighbors, where r is a fixed integer); (2) $\mathcal{L}_i(\rho) = 0$ for all i ; and (3) the convex set of states satisfying $\mathcal{L}^{(N)}(\rho) = 0$ has a finite cardinality of extreme points independent of the system size N , and are RFP MPDOs. We will call such cardinality steady-state degeneracy (SSD). We will call $\mathcal{L}$ the parent Lindbladian.

We recall that a Lindbladian is a superoperator that generates the time evolution of a density matrix capturing Markovian dynamics in open quantum systems [33]. An MPDO, specified by a tensor M , is an RFP if there exist two local quantum channels [i.e., completely positive (CP) and trace-preserving (TP) maps] $\mathcal{T}$ and $\mathcal{S}$ that respectively convert a one-site tensor and a two-site tensor into each other [43],

$$\begin{array}{c} \text{---} \circlearrowleft M \text{---} \end{array} \xrightarrow{\mathcal{T}} \begin{array}{c} \text{---} \circlearrowleft MM \text{---} \end{array} \xrightarrow{\mathcal{S}} \begin{array}{c} \text{---} \circlearrowleft M \text{---} \end{array} \quad (1)$$

As expected physically, at the RFP, there is no (finite) characteristic length scale of the correlations in the system. This is made apparent by the fact that the RFP of an MPDO has zero correlation length [43], in the sense that every two-point correlation function $\langle O_i O_j \rangle$ is independent of the distance $|i - j|$ between the two operators. Furthermore, the conditional mutual information $I(A : C|B)$ vanishes for any consecutive regions A, B, C when the complement of ABC is nonempty.

The two quantum channels of the RFP— $\mathcal{T}$ and $\mathcal{S}$ —allow for the straightforward construction of a local parent Lindbladian with desirable properties. For this, we first construct a local channel acting on two neighboring sites

$$\mathcal{E} = \mathcal{T} \circ \mathcal{S}, \quad (2)$$

and denote $\mathcal{E}_i = \tau_i(\mathcal{E})$, where τ_i translates the sites by amount i . The parent Lindbladian is then constructed as a summation of these terms,

$$\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i, \quad (3)$$

$$\mathcal{L}_i = \mathcal{E}_i - \mathbb{1}, \quad (4)$$

where $\mathcal{L}_i$ is understood as $\mathcal{L}_i \otimes \mathbb{1}_{\text{rest}}$, acting trivially outside the support of $\mathcal{L}_i$. By construction, $\mathcal{L}_i$ is a valid Lindbladian operator, and the parent Lindbladian $\mathcal{L}$ is local. Conditions (1) and (2) are obviously satisfied, and we will show in Secs. V and VI that (3) is also fulfilled.

We note that the above construction can be understood as a generalization of the MPS parent Hamiltonian. In that case, the role of $\mathcal{S}, \mathcal{T}$ is played an isometry V and its hermitian conjugate $V^\dagger$. That is, $V^\dagger$ is the isometry that brings two-site MPS tensor to one-site MPS tensor during renormalization, and V brings one-site tensor to two-site tensor. The parent Hamiltonian $H = \sum_i h_i$ is then formed by choosing for each local term $h = \mathbb{1} - VV^\dagger$, which is completely analogous to our Lindbladian construction.

While the construction is intuitive, the major contributions of the paper are to characterize several key theoretical properties of the parent Lindbladian:

(1) *Local and frustration free*: The parent Lindbladian is a summation of geometrically local terms, $\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i$. A global steady state that satisfies $\mathcal{L}^{(N)}(\rho) = 0$ is also in the space of steady states of each local term, $\mathcal{L}_i(\rho) = 0$, making it frustration free.

(2) *Minimal steady-state degeneracy*: The SSD of our constructed parent Lindbladian is minimal; no other local Lindbladian construction admits lower steady-state degeneracy.

(3) *Steady-state degeneracy does not necessarily imply long-range correlations*: Unlike MPS, we show the MPDO, whose parent Lindbladian hosts two or more steady states, does not necessarily have long-range correlations. For example, the parent Lindbladian of the state $\rho = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N})$ has two degenerate steady states, but ρ has only short-range correlations.

(4) *Steady-state degeneracy does not classify mixed-state quantum phases*: Unlike MPS, we show the MPDOs, whose local parent Lindbladians host the same minimal steady-state degeneracy, may belong to different mixed-state quantum phases. For example, the parent Lindbladians for state $\rho = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N})$ and $\rho = \frac{1}{2}(|0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N})$ both admit exactly two linearly independent steady states. Nevertheless, they belong to different phases.

(5) *(Non)commutativity of the parent Lindbladian*: The parent Lindbladian cannot always be chosen to be commuting ($[\mathcal{L}_i, \mathcal{L}_j] = 0$) even at the RFP if we require it to be local and have minimal steady-state degeneracy. In contrast, the

parent Hamiltonian at the MPS RFP can always be chosen as commuting, local, and with minimal ground-state degeneracy.

(6) *rapid mixing dynamics*: We identify a class of MPDO RFP whose parent Lindbladian is commuting, and thus rapid mixing, in the sense that, starting from any initial state, the Lindbladian evolution brings it to the unique steady-state (or space of steady states) with vanishing error after time $t = O(\text{poly log } N)$. For example, the parent Lindbladian for $\rho = \frac{1}{2^N}(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N})$ is rapid mixing and can bring product state $\rho = |0\rangle\langle 0|^{\otimes N}$ to it after $t = O(\text{poly log } N)$ with vanishing error. More interestingly, we will show in Sec. VI that the parent Lindbladian for the nontrivial density matrix [31] $\rho_{\text{CZX}}^{(N)} = \frac{1}{2^N}(\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)})$ is also rapid mixing.

The distinction of MPDO RFP between the so-called simple and nonsimple [43] will play an important role in our parent Lindbladian construction, which applies to both cases. The simple MPDO RFPs include, but are not limited to, Gibbs states of local commuting Hamiltonians ($e^{-\beta H}$, where $H = \sum_i h_i$ and $[h_i, h_j] = 0$) with zero correlation length. However, the simple MPDOs under consideration can be non-full-rank mixed states, which can be understood as special limiting cases of Gibbs states. In that sense, our result goes beyond the usual Gibbs states of commuting Hamiltonians. More interestingly, the nonsimple MPDO can correspond to boundaries of 2D topologically ordered systems. For this class, we will employ a systematic construction that exploits tools from C^* -weak Hopf algebras [44–47].

III. PRELIMINARIES

In this section, we provide a minimal review of MPS and MPDO and set up the notation. For a comprehensive review, we refer the reader to Ref. [1]. Regarding MPS, we introduce the concept of RFPs and the construction of parent Hamiltonians. Parent Hamiltonians enable the demonstration that MPS RFPs provide an exhaustive classification of MPS phases [15]. Specifically, every MPS belongs to the same phase as its corresponding fixed point.

After establishing a unified structure for all MPS RFPs, we next introduce MPDOs and review the full characterization of their associated RFPs [43], highlighting their significantly richer structures. In particular, we will treat simple and nonsimple MPDO RFPs separately, as their distinct structures will lead to different parent Lindbladian constructions. This framework sets the stage for the parent Lindbladian construction, which we present in the following sections.

A. Matrix product state, renormalization fixed points, and parent hamiltonian

1. Basics of MPS

Consider a rank-3 tensor A with graphical representation,

$$(A^i)_{\alpha\beta} = \begin{array}{c} i \\ | \\ \alpha - \boxed{A} - \beta \end{array} \in \mathbb{C} \quad (5)$$

where $i = 1, 2, \dots, d$ labels the Hilbert space and $\alpha, \beta = 1, 2, \dots, D$ label the auxiliary (virtual) space. d is the local Hilbert space dimension and D is the bond dimension. A^i can

be viewed as a $D \times D$ matrix, $A^i \in \mathcal{M}_D$, where $\mathcal{M}_D$ denotes the matrix algebra of $D \times D$ -dimensional complex matrices.

Given the tensor A , one can concatenate N copies to form a uniform MPS,

$$|\psi^{(N)}(A)\rangle = \sum_{\{i\}} \text{Tr}(A^{i_1} A^{i_2} \dots A^{i_N}) |i_1 i_2 \dots i_N\rangle. \quad (6)$$

Graphically,

$$|\psi^{(N)}(A)\rangle = \text{---} \left[\begin{array}{c} | \\ \boxed{A} \\ \text{---} \end{array} \right] \text{---} \left[\begin{array}{c} | \\ \boxed{A} \\ \text{---} \end{array} \right] \dots \left[\begin{array}{c} | \\ \boxed{A} \\ \text{---} \end{array} \right] \text{---} \quad (7)$$

When restricted to a finite bond dimension D , MPS provides an efficient description of the underlying many-body states while incorporating by construction the entanglement area law.

The MPS tensor has redundant degrees of freedom, in the sense that two different tensors may generate the same MPS $|\psi^{(N)}(A)\rangle$ for any N . One can use such redundant degrees of freedom, along with a finite number of blocking steps (combining n neighboring tensors to form a larger tensor with physical dimension d^n) to bring A^i into a block-diagonal form where each block exhibits a desired structure. Before showing this canonical form of the tensor, we first need to define the concept of injectivity [43]:

Definition 1 (Normal and injective tensors). An MPS tensor A is called *injective* if the matrices A^i span the whole set of $\mathcal{M}_D$ matrices. An MPS tensor A is a *normal* tensor if it becomes injective after blocking.

In other words, A is injective if any $D \times D$ matrix X can be expanded as $X = \sum_{i=1}^d c_i A^i$, where $c_i \in \mathbb{C}$.

For any tensor $\bar{B}$, it is always possible to obtain another tensor A that generates the same MPS state for any N , and A is in the following *canonical form*,

$$A^i = \bigoplus_{a=1}^g (\mu_a \otimes A_a^i), \quad (8)$$

$$\text{where } \mu_a = \text{diag}\{\mu_{a,1}, \mu_{a,2}, \dots, \mu_{a,n_a}\}. \quad (9)$$

Moreover, the tensors $\{A_a\}_{a=1,\dots,g}$ can be taken to form a *basis of normal tensors* (BNT): (1) Each A_a is a normal tensor; (2) There exists some N_0 such that for all $N > N_0$, $|\psi^{(N)}(A_a)\rangle$ are linearly independent. That is, a BNT explicitly separates the (normal) tensors generating proportional MPS.

The MPS state generated by A can, therefore, be written as a summation,

$$|\psi^{(N)}(A)\rangle = \sum_{a=1}^g \left(\sum_{q=1}^{n_a} \mu_{a,q}^N \right) |\psi^{(N)}(A_a)\rangle. \quad (10)$$

2. Renormalization fixed point

In statistical mechanics, for phase classification, one defines the local coarse-graining transformation and the corresponding renormalization flow. The limit of renormalization flow, if it exists, is the renormalization fixed point. Different renormalization fixed points typically correspond to different phases.

In the context of MPS, coarse-graining is defined naturally by blocking neighboring MPS tensors and applying a local

isometry to remove short-range entanglement, while keeping the long-range entanglement intact [37]. Such a renormalization flow is reversible, in the sense that there exists a local isometry to fine-grain the state. The renormalization fixed point is then defined as the limit of the renormalization flow [43].

Definition 2 (Renormalization fixed point). A tensor A is called an RFP if it satisfies

$$A^{i_1}A^{i_2} = \sum_j V_{(i_1 i_2), j} A^j, \quad (11)$$

where V is an isometry $V^\dagger V = \mathbb{1}$. Graphically,

$$\begin{array}{c} | \\ \text{---} \square A \text{---} \end{array} \begin{array}{c} | \\ \text{---} \square A \text{---} \end{array} = \begin{array}{c} | \quad | \\ \text{---} \square V \text{---} \\ | \\ \text{---} \square A \text{---} \end{array} . \quad (12)$$

The RFP of an MPS has zero correlation length, in the sense that every two-point correlation function is independent of the distance $|i - j|$ between the two operators. In the context of MPS, the phase classification is done conveniently using the RFP. One can show that an MPS and its RFP are in the same phase using the parent Hamiltonian [15]. The phase classification problem is then reduced to the phase classification for the RFP; MPS RFPs provide representatives for all 1D phases.

At RFP, $|\psi^{(N)}(A)\rangle$ is a summation of linearly independent states $|\psi^{(N)}(A_a)\rangle$, each being a direct product of states that are entangled only between neighboring sites. Specifically, by unitarily rotating the basis on each local site, $|\psi^{(N)}(A_a)\rangle$ takes the form of

$$|\psi^{(N)}(A_a)\rangle = |\omega(\Lambda_a)\rangle^{\otimes N} \equiv \text{---}\langle\!\!-\!\!\Lambda_a\rangle\text{---}\langle\!\!-\!\!\Lambda_a\rangle\text{---}\dots\dots\text{---}\langle\!\!-\!\!\Lambda_a\rangle\text{---}, \quad (13)$$

where the matrix Λ_a is diagonal, positive, and $\text{Tr}(\Lambda_a^2) = 1$. Thus,

$$|\psi^{(N)}(A)\rangle = \sum_{a=1}^g \left(\sum_{q=1}^{n_a} \mu_{a,q}^N \right) |\omega(\Lambda_a)\rangle^{\otimes N}, \quad (14)$$

where now $|\mu_{a,q}| = 1$.

3. Parent Hamiltonian

Given an MPS, one can always construct a local Hamiltonian such that this MPS is a ground state. A particularly useful construction is the parent Hamiltonian, which enjoys many nice properties. Given a uniform MPS generated by tensor A , one defines the vector space G_L as the space spanned by all states $|\psi^{(L)}(A)\rangle_X$, generated by L consecutive tensors with arbitrary boundary condition X ,

$$G_L = \{|\psi^{(L)}(A)\rangle_X | \forall X \in \mathcal{M}_D\} \quad (15)$$

$$|\psi^{(L)}(A)\rangle_X = \sum_{\{i\}} \text{Tr}(XA^{i_1} \cdots A^{i_L}) |i_1 \cdots i_L\rangle. \quad (16)$$

L is called the interaction length, and we choose large enough L such that its orthogonal subspace $G_L^\perp$ in the L -site Hilbert space is not a null space, which is always possible. Consider any positive definite operator $h \geq 0$ such that $\ker(h) = G_L$

(one simple choice of h is the projector to $G_L^\perp$). A parent Hamiltonian is constructed as [48]

$$H^{(N)} = \sum_{i=1}^N h_i, \quad (17)$$

where $h_i = \tau_i(h)$ and τ_i translates the sites by i . At the RFP, the choice of h as the projector onto $G_L^\perp$ can be rewritten as

$$h = \mathbb{1} - VV^\dagger, \quad (18)$$

where V is the isometry from Eq. (12).

Definition 3 (Frustration freeness). A Hamiltonian is *frustration free* if it can be written as a sum of terms $H^{(N)} = \sum_{i=1}^N h_i$, such that the lowest energy state of the full Hamiltonian is the lowest energy state of each individual term. Namely,

$$H^{(N)}|\psi\rangle = E_0|\psi\rangle \Leftrightarrow h_i|\psi\rangle = e_0^i|\psi\rangle, \quad (19)$$

where E_0 is the ground state energy of $H^{(N)}$ and e_0^i is the ground state energy of h_i .

By construction, the parent Hamiltonian is translational invariant and frustration free. The ground state of the full Hamiltonian satisfies $H^{(N)}|\psi\rangle = 0$. Since each individual term h_i is positive semidefinite, $\langle\psi|h_i|\psi\rangle \geq 0$. Therefore, $h_i|\psi\rangle = 0$ holds for each individual state, verifying frustration freeness.

Frustration freeness plays an important role in proving that the ground-state degeneracy of a parent Hamiltonian is minimal when choosing sufficiently large L . In particular, the ground state degeneracy is g , the number of BNT elements in tensor A , and the ground state space is $\text{Span}\{|\psi^{(N)}(A_a)\rangle\}$ [1,48]. Using the parent Hamiltonian of MPS at the RFP, one can establish the phase classification of MPS: The number of elements in the BNT completely specifies the different phases [15]. This number coincides with the ground-state degeneracy of the corresponding parent Hamiltonian.

Finally, the parent Hamiltonian is commuting $[h, \tau_j(h)] = 0$ for all j if the tensor is RFP. A nearest-neighbor parent Hamiltonian has $L = 2$. One has the following theorem [43].

Theorem 1. A tensor A is an RFP iff for all $N > 2$, $|\psi^{(N)}(A)\rangle$ is a ground state of a nearest-neighbor commuting parent Hamiltonian.

Let us summarize a few key properties of MPS that will be contrasted later in the case of MPDO with nontrivial distinctions: the overall structure of the noninjective MPS RFPs does not depend on the details of each BNT element. Moreover, the corresponding parent Hamiltonian is always commuting and straightforward to construct. However, as we will demonstrate in subsequent sections, the additional structure inherent to density matrices in the mixed-state case leads to a richer set of properties.

B. Matrix product density operator and renormalization fixed points

Analogously, one may ask whether mixed states on a one-dimensional chain also allow a tensor-network description.

Consider a rank-4 tensor M with graphical representation,

$$(M^{ij})_{\alpha\beta} = \alpha \text{---} \textcircled{M} \text{---} \beta \quad \begin{array}{c} i \\ \uparrow \\ \downarrow \\ j \end{array} \in \mathbb{C} \quad (20)$$

where $i, j = 1, 2, \dots, d$ are physical space indices and $\alpha, \beta = 1, 2, \dots, D$ are auxiliary space indices. A tensor M generates a valid matrix-product density operator (MPDO)

$$\rho^{(N)}(M) = \sum_{\{i,j\}} \text{Tr}(M^{i_1 j_1} M^{i_2 j_2} \dots M^{i_N j_N}) |i_1 i_2 \dots i_N\rangle \langle j_1 j_2 \dots j_N|, \quad (21)$$

or graphically,

$$\rho^{(N)}(M) = \textcircled{M} \textcircled{M} \dots \textcircled{M}^N, \quad (22)$$

if $\rho^{(N)} = (\rho^{(N)})^\dagger \geq 0$. More generally, one can include a boundary condition matrix B in MPDO,

$$\rho^{(N)}(M)_B := \textcircled{B} \textcircled{M} \textcircled{M} \dots \textcircled{M}^N. \quad (23)$$

Given M , deciding whether $\rho^{(N)}(M)$ is a positive operator for all system sizes is, in fact, an undecidable problem [49,50]. Nevertheless, as we will see, it is possible to bypass this difficulty by assuming an RFP condition.

What kind of mixed states admit an MPDO description? There are at least two physical scenarios where MPDOs appear:

- (1) Gibbs states of local Hamiltonians [5,51,52].
- (2) Boundaries of two-dimensional topologically ordered systems [53,54].

Although a complete classification of mixed-state quantum phases is still ongoing, it is believed that the renormalization fixed points of MPDOs capture mixed-state phases of matter. In the following, we introduce RFPs for MPDO, distinguishing between simple and nonsimple RFP characterizations.

1. Renormalization fixed point

One natural way to generalize the definition of renormalization to MPDO is to define coarse-graining channels which map two consecutive tensors to one and also maintain long-range correlation. Unlike the renormalization flow for MPS, which is always reversible, the existence of another quantum channel that can reverse the coarse-graining is not guaranteed. We require the existence of such a channel to restore reversibility. Unlike MPS, devising an explicit algorithm to find the coarse-graining quantum channel of a given MPDO tensor M is still challenging (see a recent progress in Ref. [55]). Nevertheless, one can still define the renormalization fixed point for MPDO as the limit of renormalization flow [43].

Definition 4 (RFP for MPDO). A tensor M generating MPDO is called an RFP if there exist two quantum channels,

$\mathcal{T}$ and $\mathcal{S}$, that act on the physical space and fulfill the following condition:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} M \xrightarrow{\mathcal{T}} \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \text{---} \text{---} \end{array} M \quad , \quad (24)$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \quad \uparrow \\ \text{---} \text{---} \text{---} \end{array} M \xrightarrow{\mathcal{S}} \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} M$$

namely,

$$\mathcal{T}(M_{\alpha\beta}) = \sum_{\gamma} M_{\alpha\gamma} \otimes M_{\gamma\beta},$$

$$\mathcal{S}\left(\sum_{\gamma} M_{\alpha\gamma} \otimes M_{\gamma\beta}\right) = M_{\alpha\beta}. \quad (25)$$

One can show this definition agrees with Definition 2 when the MPDO is pure.

Next, we would like to characterize the RFP for MPDO. By treating indices i, j as a compound index (namely, by vectorizing the MPDO), M can be viewed as a rank-3 tensor, and one can apply the previous results for matrix-product state to bring M^{ij} into the canonical form with a BNT as in Eqs. (8) and (9), namely,

$$M^{ij} = \bigoplus_{a=1}^g (\mu_a \otimes M_a^{ij}), \quad (26)$$

$$\text{where } \mu_a = \text{diag}\{\mu_{a,1}, \mu_{a,2}, \dots, \mu_{a,n_a}\}. \quad (27)$$

We call this the *horizontal canonical form* and denote the BNT as $\{M_a\}$.

Consider evaluating a two-point correlation function, which can be evaluated via the tensor-network transfer matrix. The transfer matrices E for MPS and MPDO are defined as

$$\text{MPS : } E = \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} \begin{array}{c} A^* \\ A \end{array} \quad , \quad \text{MPDO : } E = \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} M \quad (28)$$

While all BNT element A_a contribute to the transfer matrix for MPS, some BNT element M_a for an MPDO may lead to nilpotent operators, meaning that $\text{Tr}_R(\rho^{(N)}(M_a)) = 0$ when we trace $R \leq D$ consecutive sites, and N is sufficiently large. Therefore, such BNT elements do not contribute to the two-point correlation function when the separation of the two operators is larger than R . This motivates the following definition.

Definition 5 (Simple). A tensor generating an MPDO is *simple* if none of the elements in its BNT is nilpotent.

Below, we give two examples of noninjective MPDO tensors; one is simple, and the other is nonsimple.

Example 1 (Classical GHZ state). Consider a tensor M with $d = D = 2$ which has components $M_{00}^{00} = M_{11}^{11} = 1$, and all other components vanishing. The corresponding MPDO is

$$\rho^{(N)}(M) = |0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N}. \quad (29)$$

This is an example of a simple tensor. There are two elements in BNT, each with bond dimension 1: $M_{(a=0)}^{00} = 1$ and $M_{(a=1)}^{11} = 1$. None of them is nilpotent. Similar to the Greenberger-Horne-Zeilinger (GHZ) state, this MPDO has long-range correlations. Indeed, $\rho^{(N)}(M)$ can be obtained from the pure GHZ state by tracing out one site.

Example 2 (Boundary state of the Toric Code). Consider a $d = D = 2$ tensor M with components $M_{00}^{00} = M_{00}^{11} = M_{11}^{00} = -M_{11}^{11} = 1/2$ and all other components vanishing. The corresponding MPDO is

$$\rho^{(N)}(M) = \frac{1}{2^N} (\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N}). \quad (30)$$

This is an example of a nonsimple tensor. There are two elements in BNT, each with bond dimension 1: $M_{(a=0)}^{00} = \frac{1}{2}$, $M_{(a=0)}^{11} = \frac{1}{2}$ and $M_{(a=1)}^{00} = \frac{1}{2}$, $M_{(a=1)}^{11} = -\frac{1}{2}$. $M_{(a=0)}$ generates $\mathbb{1}^{\otimes N}$ and $M_{(a=1)}$, which is nilpotent, generates $\sigma_z^{\otimes N}$. After tracing any single site, the MPDO is proportional to a maximally mixed state and thus does not have long-range correlations.

The simple and nonsimple MPDO RFP tensors have different structures, which we review below.

2. Simple RFP characterization

Theorem 2. If a tensor M is in canonical form, generates an MPDO, is an RFP, and is simple, then each BNT element M_a takes the following form after a local unitary rotation:

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} M_a = \bigoplus_{k \in S_a} \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} l_k \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} r_k, \quad (31)$$

with

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} \eta_{k,k'} = \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} r_k \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ \uparrow \\ \text{---} \text{---} \text{---} \end{array} l_{k'} \geq 0, \quad \text{for } k, k' \in S_a, \quad (32)$$

and the matrix R defined below is a rank-1 matrix

$$R_{k,k'} := \text{Tr}(\eta_{k,k'}) = a_k b_{k'}, \quad (33)$$

with normalization $\sum_{k \in S_a} a_k b_k = 1$.

Different elements of a BNT are supported on orthogonal physical subspaces. They further satisfy $\eta_{k,k'} = 0$ if $k \in S_a, k' \in S_{a'}$ and $a \neq a'$.

We emphasize that the direct sum in Eq. (31) is in the physical space [unlike Eq. (8), where the direct sum is in the auxiliary space]. The local Hilbert space for each site is decomposed into a direct sum $\mathcal{H} = \bigoplus_a \mathcal{H}_a$. Moreover, each $\mathcal{H}_a$ is further decomposed into another direct sum $\mathcal{H}_a = \bigoplus_{k \in S_a} \mathcal{H}^{(k)}$ with $\mathcal{H}^{(k)} = \mathcal{H}_l^{(k)} \otimes \mathcal{H}_r^{(k)}$. Denoting $d_l^{(k)}, d_r^{(k)}$ the Hilbert space dimension of $\mathcal{H}_l^{(k)}, \mathcal{H}_r^{(k)}$ respectively, the physical Hilbert space dimension of a single site is $d = \sum_a \sum_{k \in S_a} d_l^{(k)} d_r^{(k)}$.

By concatenating M , the MPDO generated by simple RFP tensor M is thus a summation of mixed states, each of which is a direct product,

$$\rho^{(N)}(M) = \bigoplus_{a=1}^g n_a \bigoplus_{k_1, \dots, k_N \in S_a} \eta_{k_1, k_2} \otimes \eta_{k_2, k_3} \otimes \dots \otimes \eta_{k_N, k_1}. \quad (34)$$

We use the direct sum to emphasize different summands are supported on orthogonal subspaces. In fact, each summand labeled by a in the above MPDO is the generalized Gibbs state of nearest-neighboring commuting Hamiltonian with zero correlation length,

$$\rho^{(N)} \propto e^{-\sum_{i=1}^N h_i}, \quad (35)$$

where $h_i = \tau_i(h)$, $[h_i, h_j] = 0$. By “generalized” here we mean that each h includes the limiting case where $h = h_1 + \beta h_2$ with $h_2 \geq 0$, $[h_1, h_2] = 0$ and $\beta \rightarrow \infty$, so that $e^{-h} = P_2 e^{-h_1}$, with P_2 the projector onto the ground space of h_2 . Therefore, the generalized Gibbs state can be rank-deficient. Vice versa, the generalized Gibbs state of nearest-neighboring commuting Hamiltonian with zero correlation length can be rewritten in the form of Eq. (34) [43,51,56].

The MPDO generated by the simple RFP tensor satisfies several properties physically expected for an RFP. First, it has zero correlation length $E^2 = E$. Second, the conditional mutual information $I(A : C|B)$ vanishes for any consecutive regions A, B, C when the complement of ABC is nonempty, which directly leads to the above local Hilbert space decomposition [57].

Example 3 (Gibbs state). We provide two examples of simple MPDO RFPs in Appendix A. The first one is a direct product of Werner states, therefore with a single k in Eq. (31): There is no further decomposition for the local Hilbert space at each site. The second one is constructed by two Werner states with different parameters, and has two k 's: The local Hilbert space is a direct sum $\mathcal{H} = \mathcal{H}^{(k=1)} \oplus \mathcal{H}^{(k=2)}$. For both cases, we show that they are indeed Gibbs states by constructing the Gibbs state Hamiltonians explicitly.

Finally, using the characterization of simple RFP, one can explicitly construct the renormalization channels $\mathcal{T}$ and $\mathcal{S}$. When the MPDO is injective, with only one horizontal BNT element, the channel $\mathcal{T}$ that transforms two sites to three sites and the channel $\mathcal{S}$ that transforms three sites to two sites read

$$\mathcal{T}(X) = \bigoplus_{k_1, k_2} \left[\frac{1}{a_{k_1} b_{k_2}} \text{Tr}_{1r, 2l} (Q_{k_1} \otimes Q_{k_2}(X) Q_{k_1}^\dagger \otimes Q_{k_2}^\dagger) \otimes \left(\bigoplus_{k'} (\eta_{k_1, k'} \otimes \eta_{k', k_2}) \right) \right], \quad (36)$$

and

$$\mathcal{S}(X) = \bigoplus_{k_1, k_2, k_3} \left[\frac{1}{a_{k_1} b_{k_3}} \text{Tr}_{1r, 2, 3l} (Q_{k_1} \otimes Q_{k_2} \otimes Q_{k_3}(X) Q_{k_1}^\dagger \otimes Q_{k_2}^\dagger \otimes Q_{k_3}^\dagger) \otimes \eta_{k_1, k_3} \right]. \quad (37)$$

Here, Q_k is an isometry $Q_k Q_k^\dagger = \mathbb{1}$ that projects onto the Hilbert subspace $\mathcal{H}^{(k)}$ and satisfies $\sum_k Q_k^\dagger Q_k = \mathbb{1}$. After applying the projector Q_k , the symbol Tr_{1r} denotes taking partial trace on $\mathcal{H}_r^{(k)}$, and “1” labels the site of the action. Similarly for Tr_{2l} the partial trace is taken on $\mathcal{H}_l^{(k)}$ on the second site. Intuitively, the quantum channels perform project, partial trace, and replace. Graphically,

$$\mathcal{T} = \bigoplus_{k_1, k_2, k'} \frac{1}{a_{k_1} b_{k_2}} \quad \begin{array}{c} \text{Diagram showing two blue boxes } \eta_{k_1, k'} \text{ and } \eta_{k', k_2} \text{ connected by a line. Below them are two white boxes } Q_{k_1} \text{ and } Q_{k_2} \text{ with arrows indicating the flow of information.} \end{array}, \quad (38)$$

and

$$\mathcal{S} = \bigoplus_{k_1, k_2, k_3} \frac{1}{a_{k_1} b_{k_3}} \quad \begin{array}{c} \text{Diagram showing a blue box } \eta_{k_1, k_3} \text{ connected by a line. Below it are three white boxes } Q_{k_1}, Q_{k_2}, \text{ and } Q_{k_3} \text{ with arrows indicating the flow of information.} \end{array}, \quad (39)$$

where we employ the vectorized notation, using the upward arrows to denote the output Hilbert space degrees of freedom and the downward arrows to denote the input Hilbert space degrees of freedom. Q_k in the graph thus denotes $Q_k \otimes Q_k^*$.

We would like to comment that the channels $\mathcal{T}$ and $\mathcal{S}$ introduced above convert a two-site tensor and a three-site tensor into each other. By blocking the tensor once more, we can define $\mathcal{T}$ and $\mathcal{S}$ that go between one-site tensor and two-site tensor; however, we will keep the form between two-site and three-site tensor due to its simplicity.

3. Nonsimple RFP characterization

In this section, we review the more general nonsimple MPDO RFPs and their characterization [43]. The nonsimple RFPs have several nontrivial features since they can emerge on the boundary of topologically ordered models in 2D. We will first briefly state the general characterization, then use the language of C^* -weak Hopf algebra to write down explicit RFP tensors M and the corresponding MPDOs.

For the general characterization, we need to use the canonical form with a BNT in the vertical, instead of a horizontal, direction. Starting from the rank-4 tensor M , we now treat indices α, β as a compound index and view M as a rank-3 tensor. One can then apply the results for MPS (with the horizontal and vertical directions exchanged) to bring $M_{\alpha\beta}$ into canonical form with a corresponding BNT. We call this the vertical canonical form and denote the BNT as $\{M_\lambda\}$. Moreover, it is proven that [43] if an MPDO tensor is in the horizontal canonical form, it is automatically also in the vertical canonical form.

Consider the one-site tensor M and the two-site tensor K formed by concatenating two copies of M horizontally,

$$\text{Diagram of } K := \text{Diagram of } M \text{ followed by } M. \quad (40)$$

Up to isometries acting on the physical indices, they can be written in the vertical canonical form,

$$M_{(\alpha\beta)} = \bigoplus_{\lambda} \mu_{\lambda} \otimes M_{(\alpha\beta), \lambda}, \quad (41)$$

$$K_{(\alpha\beta)} = \bigoplus_{\kappa} \nu_{\kappa} \otimes K_{(\alpha\beta), \kappa}, \quad (42)$$

where $\{M_\lambda\}$ and $\{K_\kappa\}$ are BNTs for M and K , respectively, and μ_λ, ν_κ are diagonal positive matrices. We define $m_\lambda = \text{Tr}(\mu_\lambda)$ and $n_\kappa = \text{Tr}(\nu_\kappa)$. Again, we emphasize the direct sum and direct product are in the physical space (vertical direction).

Theorem 3. A tensor M generating an MPDO is an RFP iff the elements in BNTs $\{M_\lambda\}$ and $\{K_\kappa\}$ for one-site tensor M and two-site tensor K are related by a unitary U_λ acting on the

physical index

$$K_{(\alpha\beta),\lambda} = U_\lambda M_{(\alpha\beta),\lambda} U_\lambda^\dagger, \quad (43)$$

for any λ and $m_\lambda = n_\lambda$ [43].

The multiplication in Eq. (43) is over the physical indices (along the vertical direction). In other words, at the RFP, the one-site tensor M and two-site tensor K share the same vertical BNT up to a unitary transformation. This feature can be used to construct the channels $\mathcal{T}$ and $\mathcal{S}$, and thereby the parent Lindbladian for the nonsimple case in later sections.

We comment that, in practice, it is more convenient to generate the nonsimple MPDO RFP using C^* -weak Hopf algebras, as we introduce in Sec. VIA; this construction also makes the MPDOs manifest as boundaries of 2D topologically ordered systems. However, such constructions do not exhaust all nonsimple RFPs [58]. We will explicitly present a counterexample and construct its parent Lindbladian in Sec. VIE.

C. Lindblad master equation

Here we give a short introduction to the Lindbladian formalism. The Lindbladian operator $\mathcal{L}$ is a linear map that generates a continuous dynamical semigroup of quantum channels $\mathcal{E}_{\text{evol},t}$ via $\mathcal{E}_{\text{evol},t} = e^{\mathcal{L}t}$. Physically, the Lindblad master equation [59] governs the evolution of an open quantum system weakly coupled to the environment,

$$\dot{\rho} := \mathcal{L}(\rho) = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right), \quad (44)$$

where the Hamiltonian part H governs the unitary part of the evolution, and $\{L_k\}$ are called jump operators, governing the dissipative part of the evolution. An initial density matrix ρ_{init} evolves as $\rho(t) = e^{\mathcal{L}t}(\rho_{\text{init}})$. The evolution operator $e^{\mathcal{L}t}$ is completely positive and trace preserving for all $t \geq 0$.

One can prove that the eigenvalues of $\mathcal{L}$ have a nonpositive real part with at least one zero eigenvalue. In particular, the parent Lindbladian we construct takes the form of $\mathcal{L}^{(N)} = \sum_{i=1}^N (\mathcal{E}_i - \mathbb{1})$, subsequently has $H = 0$ and only jump operators, and all the nonzero eigenvalues have a strictly negative real part [60].

Given a Lindbladian operator, a steady state is a density matrix ρ_∞ corresponding to the zero eigenvalue, satisfying $\mathcal{L}(\rho_\infty) = 0$. Namely, it is a fixed point of the evolution operator $e^{\mathcal{L}t}(\rho_\infty) = \rho_\infty$. The set of steady states forms a convex set, and all steady states can be written as a convex combination of extreme points. In case there is a finite number of such points, we will call it SSD. If $\mathcal{L}$ has a unique steady state, any initial state ρ_{init} will evolve toward the steady state after infinite time evolution, $\lim_{t \rightarrow \infty} e^{\mathcal{L}t}(\rho_{\text{init}}) = \rho_\infty$. If the steady state is not unique, the initial state will evolve toward a state in the convex set of steady states.

D. Mixed-state quantum phases

For completeness, here we provide a definition of the mixed-state quantum phases via Lindbladian evolution [16].

Definition 6. A state ρ_0 can be driven fast to another state ρ_1 , and we write $\rho_0 \rightarrow \rho_1$, if there exists a local and time-dependent Lindbladian $\mathcal{L}$ acting on the original system and

a locality preserving ancillary system, such that for any $t \geq \text{poly log } N$,

$$\|e^{t\mathcal{L}}(\rho_0 \otimes \sigma_0) - \rho_1 \otimes \sigma_1\|_1 \leq \epsilon_N, \quad (45)$$

with σ_0 and σ_1 respectively the initial and final states of the ancillas, and $\epsilon_N \rightarrow 0$ in the thermodynamic limit $N \rightarrow \infty$.

Two states belong to the same phase if there exist two local Lindbladians described above such that $\rho_0 \rightarrow \rho_1$ and $\rho_1 \rightarrow \rho_0$.

In the above definition, including the ancillary system is necessary for the phase equivalence to be transitive.

In particular, if an MPDO $\rho^{(N)}$ admits a rapid mixing parent Lindbladian that brings a product state ρ_0 to $\rho^{(N)}$, then it is in the trivial phase. Showing the reverse direction $\rho^{(N)} \rightarrow \rho_0$ is easy by designing a parent Lindbladian for the product state.

We comment that we emphasize the rapid mixing property of Lindbladian instead of the gapped property, because rapid mixing, by definition, ensures fast convergence to the steady state and guarantees the smoothness of observables along the evolution. In contrast, the presence of a spectral gap alone does not provide such a guarantee [16,61].

IV. PARENT LINDBLADIAN CONSTRUCTIONS

After reviewing the characterization of the MPDO renormalization fixed point tensor, we are now ready to construct its local parent Lindbladian. We use the channels $\mathcal{T}$ and $\mathcal{S}$ of the RFP to form a local quantum channel

$$\mathcal{E} = \mathcal{T} \circ \mathcal{S}, \quad (46)$$

and denote $\mathcal{E}_i = \tau_i(\mathcal{E})$, where τ_i translates the sites by amount i . We then construct the parent Lindbladian as

$$\mathcal{L}_i = \mathcal{E}_i - \mathbb{1}, \quad (47)$$

$$\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i. \quad (48)$$

In this construction, the jump operators of the local Lindbladians $\mathcal{L}_i$ are identified with the Kraus operators of $\mathcal{E}_i$ (note that $\sum_k L_k^\dagger L_k = \mathbb{1}$ due to the trace-preserving property of $\mathcal{E}_i$), and the Hamiltonian part of $\mathcal{L}_i$ is zero. The MPDO generated by M is a steady state of $\mathcal{L}^{(N)}$ by construction, $\mathcal{L}^{(N)}(\rho^{(N)}(M)) = 0$, and the Lindbladian is geometrically local.

One may wonder whether we can find an easier solution by first constructing the parent Lindbladian for the purification of the mixed state and then taking a partial trace over the purifying system. There are two problems with this approach. First, not every MPDO allows for a local purification, as pointed out in Ref. [50]. The second is that the quantum channel for partial trace cannot be generated by a Lindbladian. Therefore, this procedure fails to give a parent Lindbladian for the mixed state itself.

Next, we define several key properties for parent Lindbladians.

Definition 7 (Frustration freeness). A Lindbladian is *frustration free* if it can be written as a sum of terms $\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i$, such that every steady state of the full Lindbladian is the steady state of each individual term. Namely,

$$\mathcal{L}^{(N)}(\rho) = 0 \Leftrightarrow \mathcal{L}_i(\rho) = 0. \quad (49)$$

Definition 8 (Minimal steady-state degeneracy). A parent Lindbladian $\mathcal{L}^{(N)}$ of an MPDO $\rho^{(N)}(M)$ reaches *minimal* SSD if there does not exist any other local Lindbladian with a lower-dimensional steady-state subspace which includes $\rho^{(N)}(M)$.

While the parent Lindbladian construction is straightforward, proving its properties, such as minimal steady-state degeneracy and frustration freeness, is challenging. The rest of the text is devoted to proving the properties summarized in Sec. II for the parent Lindbladian defined in Eq. (48). There are at least two main differences compared to the parent Hamiltonian.

First, unlike the parent Hamiltonian, parent Lindbladian is not automatically frustration free. Although the eigenvalues of $\mathcal{L}_i$ always have a nonpositive real part, it is possible that a global steady-state is not a steady-state of each local term, as shown in the following counterexample,

Example 4. Consider the following Lindbladian acting on a single qubit:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{L}_1(X) &= \text{Tr}(X)|0\rangle\langle 0| - X, \\ \mathcal{L}_2(X) &= \text{Tr}(X)|1\rangle\langle 1| - X.\end{aligned}\quad (50)$$

The steady state of $\mathcal{L}$ is $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$, and it is not a steady state of $\mathcal{L}_1$ or $\mathcal{L}_2$.

However, if the parent Lindbladian is commuting, then frustration freeness is guaranteed.

Lemma 1. A Lindbladian $\mathcal{L} = \sum_i \mathcal{L}_i$ with $[\mathcal{L}_i, \mathcal{L}_j] = 0$ and $\mathcal{L}_i = \mathcal{E}_i - \mathbb{1}$ is frustration free. That is, if $\mathcal{L}(\rho) = 0$ then $\mathcal{L}_i(\rho) = 0$ for all i .

Proof. Using the local Lindbladian term, we define the projectors,

$$P_i = \lim_{t \rightarrow \infty} e^{t\mathcal{L}_i}, \quad P = \lim_{t \rightarrow \infty} e^{t\sum_i \mathcal{L}_i} = \prod_i P_i, \quad (51)$$

where we used $[\mathcal{L}_i, \mathcal{L}_j] = 0$. The limit is well defined because $\mathcal{L}_i$ does not have pure-imaginary eigenvalues by construction. P_i is a projector $P_i^2 = P_i$ but not necessarily Hermitian. Using $P_i^2 = P_i$, one can show that P_i as a linear map is nondefective: For each eigenvalue, the geometric multiplicity equals the algebraic multiplicity. Therefore, P_i is diagonalizable.

The commutation of the Lindbladian $[\mathcal{L}_i, \mathcal{L}_j] = 0$ leads to the commutation of the projectors $[P_i, P_j] = 0$, and subsequently, they are simultaneously diagonalizable. Furthermore, the eigenvalues of P_i can only be 0 or 1. For a steady state, $\mathcal{L}(\rho_\infty) = 0$ leads to $P(\rho_\infty) = 1$, and thus $P_i(\rho_\infty) = 1$ for all i , equivalently, $\mathcal{L}_i(\rho_\infty) = 0$. This finishes the proof. ■

Second, let the linear space of kernels of $\mathcal{L}_i$, which is equivalent to the linear space of fixed points of $\mathcal{E}$, be denoted as $\mathcal{F}_\mathcal{E}$. Define the vector space $G_L(M)$ as the space spanned by all states $\rho^{(L)}(M)_X$, which are states of L consecutive sites generated by M with arbitrary boundary condition X ,

$$G_L(M) = \{\rho^{(L)}(M)_X | \forall X \in \mathcal{M}_D\}, \quad (52)$$

where

$$\rho^{(L)}(M)_X = \sum_{\{ij\}} \text{Tr}(X M^{i_1 j_1} \dots M^{i_L j_L}) |i_1 \dots i_L\rangle \langle j_1 \dots j_L|, \quad (53)$$

and L is the interaction length of $\mathcal{E}$. Unlike parent Hamiltonians where $G_L = \ker(h)$, for a parent Lindbladian, we generally have

$$G_L(M) \subseteq \mathcal{F}_\mathcal{E}. \quad (54)$$

This is because the fixed-point space of a quantum channel has a specific structure (see Theorem 6.14 of Ref. [62]) that might not be satisfied by $G_L(M)$ (see Example A2).

Because of the above two differences, the techniques for proving parent Hamiltonian ground-state degeneracy cannot be directly applied to determine the steady-state degeneracy of parent Lindbladians. In the following, we give explicit constructions of the channels $\mathcal{E}$ and show the above-constructed parent Lindbladian is still frustration free and reaches the minimal steady-state degeneracy, along with other properties such as commuting, rapid mixing, and the nature of its correlations. We discuss the simple MPDO and nonsimple MPDO in the next two sections, respectively.

V. PARENT LINDBLADIAN CONSTRUCTIONS: SIMPLE MPDO

In this section, we explicitly construct the parent Lindbladians for simple MPDOs and prove their properties. We begin by considering the injective MPDO case, where the MPDO has only one BNT element in horizontal canonical form, and then extend to the noninjective MPDO case. The parent Lindbladian for injective MPDOs is commuting, has a unique steady state, and exhibits rapid mixing. For noninjective MPDOs, we prove a no-go theorem stating that the parent Lindbladian cannot be commuting. We also show that the steady-state degeneracy is minimal and equals g , the number of BNT elements in the horizontal canonical form of the tensor M .

A. Injective MPDO

Using the channels $\mathcal{T}$ and $\mathcal{S}$ in Eq. (37), the local channel $\mathcal{E}_i = \tau_i(\mathcal{T} \circ \mathcal{S})$ acting on three consecutive sites $i, i+1, i+2$ reads

$$\begin{aligned}\mathcal{E}_i(X) &= \bigoplus_{k_1, k_3} \left[\frac{1}{a_{k_1} b_{k_3}} \text{Tr}_{1r, 2, 3l} (Q_{k_1} \otimes \mathbb{1} \otimes Q_{k_3}(X) Q_{k_1}^\dagger \right. \\ &\quad \left. \otimes \mathbb{1} \otimes Q_{k_3}^\dagger) \otimes \left(\bigoplus_{k'_2} (\eta_{k_1, k'_2} \otimes \eta_{k'_2, k_3}) \right) \right].\end{aligned}\quad (55)$$

Graphically,

$$\mathcal{E}_i = \bigoplus_{k_1, k_3, k'_2} \frac{1}{a_{k_1} b_{k_3}} \left[\begin{array}{c} \text{Diagram showing } Q_{k_1}, Q_{k_3}, \text{ and } \eta_{k_1, k'_2}, \eta_{k'_2, k_3} \text{ tensors with input/output arrows} \end{array} \right], \quad (56)$$

where we employ the vectorized notation, using the upward arrows to denote the output Hilbert space degrees of freedom and the downward arrows to denote the input Hilbert space degrees of freedom.

Using the graphical notation, it is easy to see the channels $\mathcal{E}_i$ commute. The product of neighboring channels is

$$\mathcal{E}_i \circ \mathcal{E}_{i+1} = \bigoplus_{k_1, k_4, k'_2, k'_3} \frac{1}{a_{k_1} b_{k_4}} \quad (57)$$

and similarly for $\mathcal{E}_{i+1} \circ \mathcal{E}_i$, leading to

$$\mathcal{E}_i \circ \mathcal{E}_{i+1} = \mathcal{E}_{i+1} \circ \mathcal{E}_i, \quad (58)$$

and subsequently $[\mathcal{L}_i, \mathcal{L}_{i+1}] = 0$. The nonoverlapping Lindbladians trivially commute. Therefore, the parent Lindbladian is commuting for injective MPDO, which leads to frustration freeness.

Frustration freeness allows for a simple proof of the steady-state uniqueness. Using frustration freeness, a global steady state ρ with $\mathcal{L}(\rho) = 0$ also satisfies $\mathcal{L}_i(\rho) = 0$, or equivalently, $\mathcal{E}_i(\rho) = \rho$ for all i . Therefore, it must satisfy

$$\mathcal{E}_1 \circ \mathcal{E}_2 \circ \dots \circ \mathcal{E}_N(\rho) = \rho. \quad (59)$$

The channel $\mathcal{E}_1 \circ \mathcal{E}_2 \circ \dots \circ \mathcal{E}_N$ is nothing but the replacement channel, which can be shown graphically, for example, when $N = 4$,

$$\mathcal{E}_1 \circ \mathcal{E}_2 \circ \mathcal{E}_3 \circ \mathcal{E}_4 = \bigoplus_{k'_1, k'_2, k'_3, k'_4} \quad (60)$$

This quantum channel takes the trace of the input state and replaces it with $\rho^{(N)}(M)$. The unique steady state of the parent Lindbladian is then $\rho_\infty = \rho^{(N)}(M)$.

B. Rapid mixing dynamics

The parent Lindbladian of a simple MPDO has one additional nice property: it is a rapid mixing. That is, starting from any initial ρ_{init} and evolving for time t , the distance between the evolved state $e^{\mathcal{L}t}(\rho_{\text{init}})$ and the unique steady-state ρ_∞ vanishes when $t = O(\text{poly log } N)$ in the thermodynamic limit $N \rightarrow \infty$ [16]. The rapid mixing dynamics is a direct consequence of the fact that the parent Lindbladian is commuting.

Lemma 2. A local Lindbladian $\mathcal{L}^{(N)} = \sum_{i=1}^N \mathcal{L}_i$, where $[\mathcal{L}_i, \mathcal{L}_j] = 0$ and $\mathcal{L}_i$ does not have pure-imaginary eigenvalues, has rapid mixing dynamics.

Note that in our formalism, $\mathcal{L}_i$ does not have pure-imaginary eigenvalues by construction.

Proof. Since the Lindbladian is commuting, we define the projectors P_i, P as in Eq. (51). Consider the completely

bounded trace norm of the superoperator $e^{\mathcal{L}t} - P$,

$$\begin{aligned} \|e^{\mathcal{L}t} - P\|_\diamond &= \left\| \prod_i e^{\mathcal{L}_i t} - \prod_i P_i \right\|_\diamond \\ &\leq \sum_{i=1}^N \|e^{\mathcal{L}_1 t} e^{\mathcal{L}_2 t} \dots e^{\mathcal{L}_{i-1} t} (e^{\mathcal{L}_i t} - P_i) P_{i+1} \dots P_N\|_\diamond \\ &\leq \sum_{i=1}^N \|e^{\mathcal{L}_i t} - P_i\|_\diamond, \end{aligned} \quad (61)$$

where we use the triangle inequality, submultiplicativity of the completely bounded trace norm, and $\|\mathcal{E}\|_\diamond = 1$ for a quantum channel $\mathcal{E}$.

For each local term, one can use the Jordan decomposition of the superoperator to show that the norm is bounded by

$$\|e^{\mathcal{L}_i t} - P_i\|_\diamond \leq e^{-\gamma_i t}, \quad (62)$$

for some constant γ_i since $\mathcal{L}_i$ acts locally on a Hilbert space of fixed dimensions. Subsequently,

$$\|e^{\mathcal{L}t} - P\|_\diamond \leq N e^{-\gamma t}, \quad (63)$$

where $\gamma = \min(\gamma_1, \gamma_2, \dots, \gamma_N)$.

The error is then bounded by

$$\begin{aligned} \epsilon &:= \|e^{\mathcal{L}t}(\rho_{\text{init}}) - \rho_\infty\|_1 \\ &= \|e^{\mathcal{L}t}(\rho_{\text{init}}) - P(\rho_{\text{init}})\|_1 \leq \|e^{\mathcal{L}t} - P\|_1 \|\rho_{\text{init}}\|_1 \\ &= \|e^{\mathcal{L}t} - P\|_1 \leq \|e^{\mathcal{L}t} - P\|_\diamond \leq N e^{-\gamma t}. \end{aligned} \quad (64)$$

The norm in $\|e^{\mathcal{L}t} - P\|_1$ is the induced trace norm of the superoperator, which lower bounds the completely bounded trace norm. One can take $t = (\log N)^{1+\delta}$ for any positive δ and show $\epsilon \rightarrow 0$ when $N \rightarrow \infty$. This justifies that $\mathcal{L}$ is a rapid mixing Lindbladian. ■

C. Noninjective simple MPDO

A natural guess for generalizing the local quantum channel for a noninjective MPDO is given by

$$\begin{aligned} \mathcal{E}_i^{(1)}(X) &= \bigoplus_a \bigoplus_{k_1, k_3 \in S_a} \frac{1}{a_{k_1} b_{k_3}} \text{Tr}_{1r, 2, 3l} \\ &\times (Q_{k_1} \otimes P_a \otimes Q_{k_3}(X) Q_{k_1}^\dagger \otimes \mathbb{1} \otimes Q_{k_3}^\dagger) \\ &\otimes \left(\bigoplus_{k'_2 \in S_a} (\eta_{k_1, k'_2} \otimes \eta_{k'_2, k_3}) \right), \end{aligned} \quad (65)$$

where $P_a = \sum_{k \in S_a} Q_k^\dagger Q_k$ is the projector to Hilbert subspace corresponding to BNT element a and satisfy $\sum_a P_a = \mathbb{1}$. By direct inspection, one can see $\mathcal{E}_i^{(1)}$ as a linear map is CP and $\rho^{(N)}(M)$ in Eq. (34) is a fixed point of this CP map. However, a valid quantum channel must also be TP, while

$$\begin{aligned} \text{Tr}(\mathcal{E}_i^{(1)}(X)) &= \text{Tr} \left(\left(\sum_a P_a \otimes P_a \otimes P_a \right) X \right) \\ &\neq \text{Tr}(X). \end{aligned} \quad (66)$$

To form a CPTP map, denoting $P = \sum_a P_a \otimes P_a \otimes P_a$ and $P^\perp = \mathbb{1} - P$, we construct the local quantum channel $\mathcal{E}_i$ as

$$\mathcal{E}_i = \mathcal{E}_i^{(I)} + \mathcal{E}_i^{(II)}, \quad (67)$$

$$\text{where } \mathcal{E}_i^{(II)}(X) = \text{Tr}(P^\perp X) \rho_0, \quad (68)$$

and ρ_0 is a density matrix satisfying

$$\sum_a (P_a \otimes P_a \otimes P_a) \rho_0 (P_a \otimes P_a \otimes P_a) = \rho_0. \quad (69)$$

Note that this condition is stronger than $P \rho_0 P = \rho_0$. One can check that the second piece $\mathcal{E}_i^{(II)}$ is still a CP map, and the combined map $\mathcal{E}_i$ is indeed trace preserving.

By construction, the second piece $\mathcal{E}_i^{(II)}$ satisfies $\mathcal{E}_i^{(II)} \circ \mathcal{E}_i = 0$, which leads to $\mathcal{E}_i \circ \mathcal{E}_i = \mathcal{E}_i^{(I)} \circ \mathcal{E}_i$. Using the graphical representation, one can show that $\mathcal{E}_i^{(I)}$ is idempotent, $\mathcal{E}_i^{(I)} \circ \mathcal{E}_i^{(I)} = \mathcal{E}_i^{(I)}$. Combining the above properties, one can show the equivalence

$$\mathcal{E}_i(X) = X \Leftrightarrow \mathcal{E}_i^{(I)}(X) = X. \quad (70)$$

The fixed point space of local channel $\mathcal{E}_i$ is the same as the fixed point space of the CP map $\mathcal{E}_i^{(I)}$.

The proof of frustration freeness is more involved, since $\mathcal{E}_i$ is no longer commuting. As we will show later, this is not merely an artifact of our choice of $\mathcal{E}_i^{(II)}$; in fact, noninjective simple MPDO RFPs never admit a commuting local parent Lindbladian (see Sec. VD). To prove the parent Lindbladian is frustration free, we make a convenient choice of ρ_0 as

$$\rho_0 = \frac{1}{g} \bigoplus_{a=1}^g \bigoplus_{k_1, k_2, k_3 \in S_a} \eta_{k_1, k_2} \otimes \eta_{k_2, k_3} \otimes \eta_{k_3, k_1}. \quad (71)$$

Under this condition, $\mathcal{E}_i^{(I)}(\rho_0) = \rho_0$ and

$$\mathcal{E}_i^{(I)} \circ \mathcal{E}_i^{(II)} = \mathcal{E}_i^{(II)}. \quad (72)$$

Combined with the properties

$$\mathcal{E}_i^{(I)} \circ \mathcal{E}_i^{(I)} = \mathcal{E}_i^{(I)} \quad (73)$$

$$\mathcal{E}_i^{(I)} \circ \mathcal{E}_{i+1}^{(I)} = \mathcal{E}_{i+1}^{(I)} \circ \mathcal{E}_i^{(I)}, \quad (74)$$

(while keeping in mind that $\mathcal{E}_i \circ \mathcal{E}_{i+1} \neq \mathcal{E}_{i+1} \circ \mathcal{E}_i$) one can prove frustration freeness using a “flow diagram” method. We leave the proof for Appendix C.

With frustration freeness, steady-state degeneracy can be shown using a method similar to simple MPDOs. A global steady state satisfies $\mathcal{E}_i(\rho) = \rho$, or equivalently, $\mathcal{E}_i^{(I)}(\rho) = \rho$ for all i . Consequently, it must satisfy

$$\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_N^{(I)}(\rho) = \rho. \quad (75)$$

The CP map $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_N^{(I)}$ can similarly be found using graphical representation, where now

$$\mathcal{E}_i^{(I)} = \bigoplus_a \bigoplus_{k_1, k_3, k'_2 \in S_a} \frac{1}{a k_1 b k_3} \quad (76)$$

Let's take $N = 4$ as an example,

$$\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)} \circ \mathcal{E}_4^{(I)} = \bigoplus_a \bigoplus_{k'_1, k'_2, k'_3, k'_4 \in S_a} \quad (77)$$

and the generalization to arbitrary N is straightforward. Therefore, the global steady-states take the form of

$$\rho_\infty^{(N)} = \bigoplus_{a=1}^g c_a \bigoplus_{k_1, \dots, k_N \in S_p} \eta_{k_1, k_2} \otimes \eta_{k_2, k_3} \otimes \dots \otimes \eta_{k_N, k_1}, \quad (78)$$

with $c_a \in \mathbb{C}$. A direct inspection shows all the states in the above form are steady states of $\mathcal{L}$. This concludes that the SSD is g , the number of horizontal BNT elements, i.e.,

$$(\text{SSD}) = g. \quad (79)$$

Moreover, the minimally possible steady-state degeneracy is g , as proven below.

Theorem 4. Consider a simple MPDO $\rho^{(N)}(M)$ generated by the RFP tensor M with g BNT elements in the horizontal canonical form. A local Lindbladian $\mathcal{L}^{(N)}$ that has $\rho^{(N)}(M)$ as its steady state must have at least g independent steady states. In particular, the MPDO $\rho^{(N)}(M_a)$ generated by each BNT element is a steady state.

Proof. Consider the vectorized state $|\rho(M)\rangle\rangle$ of $\rho(M)$ and define an operator in the vectorized space $H_{\mathcal{L}} = \hat{\mathcal{L}}^\dagger \hat{\mathcal{L}}$, where $\hat{\mathcal{L}}$ is the representation of $\mathcal{L}$ in the vectorized space. By construction, $H_{\mathcal{L}}$ is positive semi-definite, satisfies $\langle\langle \rho(M) | H_{\mathcal{L}} | \rho(M) \rangle\rangle = 0$, and it is k -local (but not geometrically local) because each term in the expansion $H_{\mathcal{L}} = \sum_{ij} \hat{\mathcal{L}}_i^\dagger \hat{\mathcal{L}}_j$ is only supported on a finite number of sites.

Now, if there are g BNT elements in the horizontal canonical form of M , one can decompose $|\rho(M)\rangle\rangle = \sum_{a=1}^g |\rho(M_a)\rangle\rangle$, where $\langle\langle \rho(M_a) | \rho(M_{a'}) \rangle\rangle = 0$ for $a \neq a'$. We construct another vectorized state

$$\begin{aligned} |\rho'(M)\rangle\rangle &= 2|\rho(M_{a=1})\rangle\rangle - |\rho(M)\rangle\rangle \\ &= |\rho(M_{a=1})\rangle\rangle - |\rho(M_{a=2})\rangle\rangle - \dots - |\rho(M_{a=g})\rangle\rangle. \end{aligned}$$

Consider any region A and its complement $\bar{A}$. By construction,

$$\text{Tr}_{\bar{A}}(|\rho(M)\rangle\rangle \langle\langle \rho(M) |) = \text{Tr}_{\bar{A}}(|\rho'(M)\rangle\rangle \langle\langle \rho'(M) |),$$

where we use the fact that different BNT elements are supported on orthogonal physical subspaces locally (Theorem 2). This shows that each term $\hat{\mathcal{L}}_i^\dagger \hat{\mathcal{L}}_j$ in $H_{\mathcal{L}}$, being k local, cannot distinguish $|\rho(M)\rangle\rangle$ and $|\rho'(M)\rangle\rangle$, and therefore $|\rho'(M)\rangle\rangle$ also fulfills $\langle\langle \rho'(M) | H_{\mathcal{L}} | \rho'(M) \rangle\rangle = 0$. Equivalently, $\hat{\mathcal{L}}|\rho'(M)\rangle\rangle = 0$ and back to the unvectorized notation, $\mathcal{L}(\rho'(M)) = 0$.

Since both $\rho(M)$ and $\rho'(M)$ are steady states of $\mathcal{L}$, their summation, $\rho(M_{a=1})$, is also a steady state of $\mathcal{L}$. This can be applied to any BNT element a and finishes the proof that each $\rho(M_a)$ is a steady state of $\mathcal{L}$. The steady-state degeneracy is at least g . ■

This shows that our parent Lindbladian construction reaches the minimal steady-state degeneracy. Below we show

no purely imaginary eigenvalues. Specifically, starting from an initial state $\rho_{\text{init}}^{(N)}$ which is a product state, the connected correlation function with $O(t) = \text{Tr}(\mathcal{E}_{\text{evol},t}(\rho_{\text{init}}^{(N)})O)$ satisfies

$$(O_A O_B)_c(t) \equiv (O_A O_B)(t) - O_A(t)O_B(t) = 0, \quad (88)$$

when the separation between region A and B is larger than $2L$ for any operators O_A, O_B . Consequently, the steady state after infinite time evolution $\rho_{\infty}^{(N)} = \mathcal{E}_{\text{evol},\infty}(\rho_{\text{init}}^{(N)})$ is still short-range correlated.

Now, consider initial states as product states,

$$\rho_{\text{init}}^{(N)} = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N, \quad (89)$$

where ρ_i is an arbitrary density matrix at site i . By assumption, the steady state is in the space spanned by $\{\rho(M_a)\}$. Suppose there exists a product state whose steady state is a superposition $c_1 \rho(M_a) + c_2 \rho(M_{a'})$ with $a \neq a'$. By Lemma 3, this state has long-range correlations and is in contradiction to Eq. (88). Therefore, the steady state of a product initial state does not contain superposition.

Since all possible product states $\rho_{\text{init}}^{(N)}$ form a complete basis for the operators, all the states in the space of steady states shall be reached, and there must exist two initial states, $\rho_{\text{init},a}^{(N)}$ and $\rho_{\text{init},a'}^{(N)}$, such that their corresponding steady states are the MPDOs generated by different BNT elements,

$$\begin{aligned} \mathcal{E}_{\text{evol},\infty}(\rho_{\text{init},a}^{(N)}) &= \rho(M_a), \\ \mathcal{E}_{\text{evol},\infty}(\rho_{\text{init},a'}^{(N)}) &= \rho(M_{a'}), \end{aligned} \quad (90)$$

which do not have long-range correlations. Denote $\rho_{\text{init},a}^{(N)} = \rho_{1,a} \otimes \cdots \otimes \rho_{N,a}$ and similarly for $\rho_{\text{init},a'}^{(N)}$, we next interpolate between $\rho_{i,a}$ and $\rho_{i,a'}$ by $\rho_i(s)$ where $0 \leq s \leq 1$, and $\rho_i(s)$ is continuous in s , $\rho_i(0) = \rho_{i,a}$ and $\rho_i(1) = \rho_{i,a'}$. For example, one can choose

$$\rho_i(s) = (1-s)\rho_{i,a} + s\rho_{i,a'}, \quad (91)$$

and build an interpolated product initial state

$$\rho_{\text{init}}^{(N)}(s) = \rho_1(s) \otimes \rho_2(s) \otimes \cdots \otimes \rho_N(s). \quad (92)$$

The evolution operator $\mathcal{E}_{\text{evol},\infty}$ as a linear map is continuous, so there must exist a $s' \in (0, 1)$ such that $\mathcal{E}_{\text{evol},\infty}(\rho_{\text{init}}^{(N)}(s')) = \frac{1}{2}(\rho^{(N)}(M_a) + \rho^{(N)}(M_{a'}))$. This is since $\mathcal{E}_{\text{evol},\infty}$ preserves positivity and $\rho^{(N)}(M_a), \rho^{(N)}(M_{a'})$ are supported on orthogonal spaces, thus the convex-combination coefficients are indeed nonnegative for all s . Again, by Lemma 3 this state has long-range correlations and contradicts Eq. (88). ■

The long-range correlations in $\rho^{(N)}(M)$ not only lead to a no-go theorem for a commuting local parent Lindbladian but also forbid rapid mixing dynamics [63,64].

VI. PARENT LINDBLADIAN CONSTRUCTIONS: NONSIMPLE MPDO

In this section, we explicitly construct the parent Lindbladian of nonsimple MPDOs and prove their properties. In principle, $\mathcal{T}$ and $\mathcal{S}$ can be constructed using the vertical canonical forms of one-site tensor M and two-site tensor K , subject to the condition in Theorem 3, which we demonstrate explicitly for two examples in Sec. VIE and Appendix E. For

the purpose of proving the properties of the parent Lindbladian, we adopt a more convenient algebraic approach based on the C^* -weak Hopf algebra, which implicitly uses the horizontal canonical form.

We begin with a lightweight review of the characterization of nonsimple MPDO RFPs using C^* -weak Hopf algebra, employing the graphical notation. Given a C^* -weak Hopf algebra A , one can generate an MPDO RFP, which corresponds to the boundary of a 2D topologically ordered system, and construct the renormalization channels $\mathcal{T}$ and $\mathcal{S}$. We leave the rigorous algebraic treatments behind the graphical notations in Appendix B. Next, we construct the parent Lindbladian using $\mathcal{T}$ and $\mathcal{S}$. In the special case of C^* -Hopf algebras, the parent Lindbladian is commuting and exhibits rapid mixing, though the steady state is not unique. For a general C^* -weak Hopf algebra A , the SSD depends on the boundary condition: with open boundary condition SSD equals $\dim(A)$, while with periodic boundary condition SSD equals g , the number of BNT elements in the horizontal canonical form.

Finally, we present a case study of a nonsimple MPDO RFP that lies beyond the framework of C^* -weak Hopf algebra, and construct its parent Lindbladian via the vertical canonical form. The conclusion regarding SSD remains unchanged in this case.

A. Nonsimple RFP characterization: C^* -weak Hopf algebras

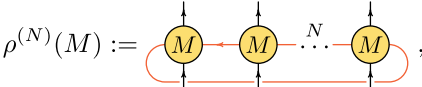
This subsection is devoted to recalling some elementary facts for the construction of RFP MPDOs from the algebraic point of view in Refs. [44–46]; see also Appendix B for rigorous definitions and statements. The central algebraic structures in this context, introduced by Böhm and Szlachányi in Ref. [65], are known as C^* -weak Hopf algebras, and are, in essence, complex finite-dimensional vector spaces A such that both A and their dual vector spaces A^* are C^* -algebras, and which are furthermore compatible in the sense that their categories of $*$ -algebra representations encompass unitary (multi)fusion categories [66,67]. For example, any group algebra $A = \mathbb{C}G$ of a finite group G can be promoted to a C^* -Hopf algebra (see details in Example B1).

In the context of tensor networks and for our purposes here, let us recall that, given any C^* -weak Hopf algebra A , one can systematically construct a rank-four tensor



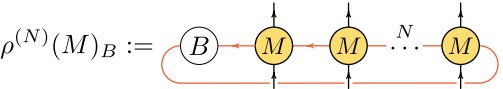
$$(93)$$

generating RFP MPDOs for all system sizes $N \in \mathbb{N}$:



$$\rho^{(N)}(M) := \text{[Diagram]}, \quad (94)$$

in the sense of Definition 4 or, more generally,



$$\rho^{(N)}(M)_B := \text{[Diagram]}, \quad (95)$$

for certain admissible boundary conditions B ; see [45]. Furthermore, the tensor M is in the horizontal canonical form by

construction, for which we label the BNT as $a = 1, \dots, g$ and let

$$\begin{array}{c} \uparrow \\ \textcircled{M} \\ \downarrow \end{array} = \sum_{a=1}^g \begin{array}{c} \uparrow \\ \textcircled{M}^a \\ \downarrow \end{array} \quad (96)$$

stand for its corresponding block decomposition; here, the labels a correspond to projecting onto the a -th block at the virtual level. Furthermore, there exists a rank-four tensor M' with the same block decomposition, and two rank-two tensors Ω and Ξ satisfying

$$\begin{array}{c} \textcircled{\Omega} \textcircled{M'} \\ \textcircled{M} \end{array} = \sum_a \begin{array}{c} \textcircled{a} \\ \textcircled{a} \end{array} \quad (97)$$

and, on the other hand,

$$\sum_a \begin{array}{c} \textcircled{a} \\ \textcircled{a} \end{array} \textcircled{\Xi} = \textcircled{\Omega} \quad (98)$$

See Appendix B for explicit constructions of the tensors M, M', Ω and Ξ . Essentially, Eq. (97) is a consequence of the horizontal canonical form. Moreover, Ω is invertible and is proportional to the identity matrix in each block a , and therefore it commutes at the virtual level with M (and M') by construction:

$$\begin{array}{c} \textcircled{\Omega} \textcircled{M} \\ \textcircled{M} \end{array} = \begin{array}{c} \textcircled{M} \textcircled{\Omega} \\ \textcircled{M} \end{array}; \quad (99)$$

in other words, we say Ω is central with respect to M . By using Eqs. (97) and (98) and the property of Ω , one can easily derive that

$$\begin{array}{c} \textcircled{\Omega} \textcircled{M'} \textcircled{\Xi} \textcircled{M'} \\ \textcircled{M} \end{array} = \sum_a \begin{array}{c} \textcircled{a} \\ \textcircled{a} \end{array} \quad (100)$$

Below we show the example of MPDO tensor when $A = \mathbb{C}\mathbb{Z}_2$. See Appendix E for a more advanced example where A is the Fibonacci C^* -weak Hopf algebra.

Example 6 (Boundary state of the Toric Code). Let us reconsider the boundary state of the toric code introduced in Example 2. This MPDO tensor is in fact generated using $\mathbb{C}\mathbb{Z}_2$ group C^* -algebra, where the tensors M and M' are given by

$$\begin{array}{c} \uparrow \\ \textcircled{M} \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \textcircled{M'} \\ \downarrow \end{array} = \frac{\mathbb{1}}{2} \otimes |e\rangle\langle e| + \frac{\sigma_z}{2} \otimes |g\rangle\langle g|$$

where the first tensor factor corresponds to the endomorphism in the physical space and the second tensor factor to one in the virtual space, and the rank-two tensors acting in the virtual space are

$$\begin{array}{c} \textcircled{\Omega} \\ \textcircled{\Xi} \end{array} = \frac{\mathbb{1}}{2}.$$

We show the explicit derivation using C^* -Hopf algebra in Appendix D.

Any tensor M satisfying Eqs. (97)–(99) defines RFP MPDOs in the sense of Definition 4. Indeed, on the one hand, the fine-graining quantum channel $\mathcal{T}$ is given by the following diagrammatic formula using the vectorized notation:

$$\mathcal{T} = \begin{array}{c} \textcircled{\Omega} \textcircled{M} \textcircled{M} \\ \textcircled{M'} \end{array}. \quad (101)$$

On the other hand, the coarse-graining quantum channel $\mathcal{S}$ is given by a sum of the form

$$\mathcal{S} = \mathcal{S}^{(I)} + \mathcal{S}^{(II)}, \quad (102)$$

where the first summand is defined by the expression

$$\mathcal{S}^{(I)} = \begin{array}{c} \textcircled{\Omega} \textcircled{M} \textcircled{M'} \textcircled{\Xi} \textcircled{M'} \\ \textcircled{M} \end{array}, \quad (103)$$

which is completely positive and trace-nonincreasing, but not necessarily trace preserving, and the second summand is such that

$$\begin{array}{c} \textcircled{S^{(II)}} \\ \textcircled{M} \textcircled{M} \end{array} \equiv 0, \quad (104)$$

i.e., acts nontrivially only on the orthogonal of the space spanned by $\rho^{(2)}(M)_X$ where X is an arbitrary boundary condition matrix, and makes $\mathcal{S}$ trace preserving. We will make a particular choice of $\mathcal{S}^{(II)}$ in the next subsection.

It can be deduced that the effect of the fine-grained map followed by the coarse-grained map is trivial, i.e., $\mathcal{S} \circ \mathcal{T}$ leaves the tensor M invariant; and the opposite combination, the quantum channel $\mathcal{T} \circ \mathcal{S}$ is an idempotent map that leaves the two-site tensor invariant and stabilizes the RFP MPDOs. This makes $\mathcal{T} \circ \mathcal{S}$ the main candidate for the local stabilizers on the Lindbladian construction in the next subsections.

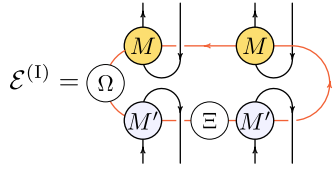
B. Parent Lindbladian

With the notations of Sec. VI A, let us consider the quantum channel $\mathcal{E} := \mathcal{T} \circ \mathcal{S}$, which can be decomposed in the form

$$\mathcal{E} = \mathcal{E}^{(I)} + \mathcal{E}^{(II)}, \quad (105)$$

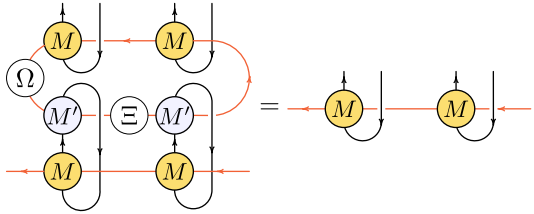
$$\text{where } \mathcal{E}^{(k)} := \mathcal{T} \circ \mathcal{S}^{(k)}, \quad k = \text{I, II}. \quad (106)$$

In diagrammatic notation, it is easy to check that the first term takes the form



$$\mathcal{E}^{(I)} = \Omega \text{ (diagram) }, \quad (107)$$

and, moreover, that the RFP MPDOs with arbitrary boundary conditions are invariant under the action of $\mathcal{E}^{(I)}$ [and hence of $\mathcal{E}$ due to Eq. (104)], since



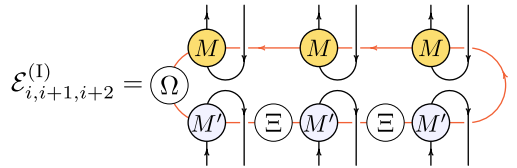
$$\text{(diagram)} = \text{(simplified diagram)} \quad (108)$$

as an immediate consequence of Eq. (100).

Let us denote $\mathcal{E}_i = \tau_i(\mathcal{E})$ where τ_i translates the operator by i sites and define the linear maps

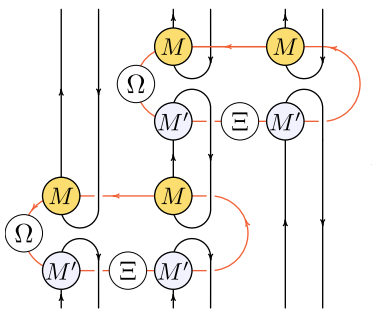
$$\mathcal{E}_{i,\dots,i+\ell}^{(k)} = \mathcal{E}_i^{(k)} \circ \mathcal{E}_{i+1}^{(k)} \circ \dots \circ \mathcal{E}_{i+\ell}^{(k)}, \quad (109)$$

for $k = \text{I, II}$, where the operators are tensorized with the identity map elsewhere. For example, for $\ell = 2$ it is easy to check that it takes the form

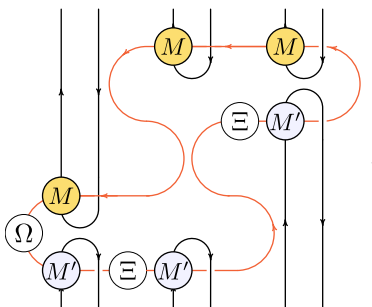


$$\mathcal{E}_{i,i+1,i+2}^{(I)} = \Omega \text{ (diagram) }. \quad (110)$$

This is because $\mathcal{E}_i^{(I)} \circ \mathcal{E}_{i+1}^{(I)}$ is given by the expression



and by virtue of Eq. (97), this is equal to



which corresponds exactly to the expression introduced in Eq. (110). Similarly, one can also show $\mathcal{E}_{i+1}^{(I)} \circ \mathcal{E}_i^{(I)}$ is given by the same expression in Eq. (110), and therefore the two linear maps $\mathcal{E}_{i+1}^{(I)}$ and $\mathcal{E}_i^{(I)}$ commute,

$$\mathcal{E}_i^{(I)} \circ \mathcal{E}_{i+1}^{(I)} = \mathcal{E}_{i+1}^{(I)} \circ \mathcal{E}_i^{(I)}. \quad (111)$$

Due to this commutation, the definition of $\mathcal{E}_{i,\dots,i+\ell}^{(I)}$ is independent of the order of the products.

Furthermore, in the particular case of C^* -Hopf algebras it holds that $\mathcal{E} = \mathcal{E}^{(I)}$ is already trace preserving and hence $\mathcal{E}^{(II)} \equiv 0$; this includes the case of group C^* -algebras. Hence, the parent Lindbladian is commuting for the MPDO RFP generated by C^* -Hopf algebra, and subsequently, the parent Lindbladian is frustration free and is rapid mixing (the same proof in Sec. VB applies). Define the quantum channels for open boundary condition (OBC)

$$\mathcal{E}_{1,2,\dots,N}^{(I)\text{OBC}} := \mathcal{E}_{1,2,\dots,N}^{(I)}, \quad (112)$$

and periodic boundary condition (PBC),

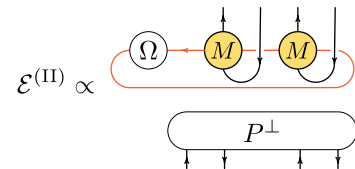
$$\mathcal{E}_{1,2,\dots,N}^{(I)\text{PBC}} := \mathcal{E}_N^{(I)} \circ \mathcal{E}_{1,2,\dots,N}^{(I)}, \quad (113)$$

where $\mathcal{E}_N^{(I)}$ acts on sites N and 1 , closing the boundary. The global steady state of the parent Lindbladian $\mathcal{L}$ is, therefore, the fixed point of the channel

$$\mathcal{E}_{1,2,\dots,N}^{(I)\text{O/PBC}}(\rho) = \rho, \quad (114)$$

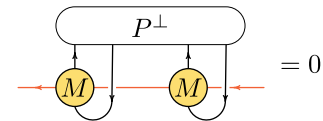
depending on the boundary condition.

However, one cannot expect that $\mathcal{E} = \mathcal{E}^{(I)}$ for C^* -weak Hopf algebras in general; see, e.g., Appendix E. Similar to the noninjective simple case, there are degrees of freedom in the choice of $\mathcal{E}^{(II)}$. Here, we make a convenient choice of,



$$\mathcal{E}^{(II)} \propto \text{(diagram)} \quad (115)$$

where the normalization constant can be determined by Ω , and $P^\perp$ is a projector onto the orthogonal space of the space generated by two-site MPDO,



$$\text{(diagram)} = 0. \quad (116)$$

By construction, $\mathcal{E}^{(II)}$ satisfies the properties,

$$\mathcal{E}^{(II)} \circ \mathcal{E} = 0, \quad (117)$$

$$\mathcal{E}^{(I)} \circ \mathcal{E}^{(II)} = \mathcal{E}^{(II)}, \quad (118)$$

and along with the fact that $\mathcal{E}^{(I)} \circ \mathcal{E}^{(I)} = \mathcal{E}^{(I)}$ one can show $\mathcal{E}(X) = X$ if and only if $\mathcal{E}^{(I)}(X) = X$, same as Eq. (70). These properties also allow us to prove the frustration freeness, where we show the proof in Appendix C. Being frustration free, the global steady state of the parent Lindbladian is, again,

case, the transfer matrix for a horizontal BNT element $E_a = 0$ unless $a = e$. In other words, all BNT elements are simple except the one associated with the vacuum sector. Biconnected C^* -weak Hopf algebras include unitary fusion categories, which are the cases of primary interest (see precise definition in Appendix B).

Theorem 6. Consider a nonsimple MPDO $\rho^{(N)}(M)$ generated by RFP tensor M with g BNT elements in the horizontal canonical form, constructed from a biconnected C^* -weak Hopf algebra. A local Lindbladian $\mathcal{L}^{(N)}$ that has $\rho^{(N)}(M)$ as its steady state must have at least g independent steady states.

Proof. Using the horizontal canonical form, one can decompose the state into $\rho^{(N)}(M) = \sum_{a=1}^g \rho^{(N)}(M_a)$. Let's construct an operator

$$\begin{aligned} \rho'^{(N)}(M) &= \rho^{(N)}(M) - 2\rho^{(N)}(M_1) \\ &= -\rho^{(N)}(M_1) + \rho^{(N)}(M_2) + \cdots + \rho^{(N)}(M_g), \end{aligned} \quad (131)$$

where $a = 1$ is not the vacuum sector e . Since $E_a = 0$ for all nonvacuum sectors, the reduced density matrix for $\rho^{(N)}(M_a)$ vanishes when $a \neq e$. Therefore, when tracing out one site, the reduced density matrix for $\rho^{(N)}(M)$ and $\rho'^{(N)}(M)$ are the same and equal to the reduced density matrix for $\rho^{(N)}(M_e)$. Any term $\mathcal{L}_i$ in a local Lindbladian $\mathcal{L}$ cannot distinguish $\rho^{(N)}(M)$ and $\rho'^{(N)}(M)$; if $\rho^{(N)}(M)$ is a steady state of $\mathcal{L}$, then $\rho'^{(N)}(M)$ must also be a steady state. Finally, there are $(g-1)$ independent ways to construct $\rho^{(N)}(M) - 2\rho^{(N)}(M_a)$, $a \neq e$. We conclude that there are at least g steady states of a local Lindbladian that has $\rho^{(N)}(M)$ as its steady state. ■

Therefore, under the periodic boundary condition for $\mathcal{L}$, our constructed parent Lindbladian reaches the minimal steady-state degeneracy.

Finally, we would like to point out the implications of the above results for the particular case of C^* -Hopf algebras.

Theorem 7. An MPDO RFP generated by C^* -Hopf algebras exhibits only short-range correlations and admits a commuting local rapid mixing parent Lindbladian with minimal degenerate steady states under periodic boundary conditions. The minimal degeneracy equals g , the number of BNT elements in the horizontal canonical form of the MPDO tensor, which is larger than 1.

The fact that this class of RFPs only exhibits short-range correlations is a direct consequence of the existence of rapid mixing local parent Lindbladian [63,64]. It can also be understood by noting that the transfer matrix $E = \sum_a E_a = E_e$ is a rank-one matrix. This demonstrates that steady-state degeneracy does not necessarily imply the existence of long-range correlations for MPDOs. Moreover, it shows that steady-state degeneracy alone cannot classify mixed-state quantum phases. For example, MPDO RFPs generated by C^* -Hopf algebras and noninjective simple RFPs may share the same steady-state degeneracy, yet they belong to different phases. This is demonstrated in the following example.

Example 7 (Boundary state of the Toric Code). As a continuation of Example 6, let us note that the local quantum channel acting on two sites is given by the explicit formula

$$\mathcal{E}(X) = \frac{1}{4}(\text{Tr}(X)\mathbb{1}_2^{\otimes 2} + \text{Tr}(\sigma_z^{\otimes 2}X)\sigma_z^{\otimes 2}). \quad (132)$$

The corresponding local Lindbladian is rapid mixing. It is easy to verify that it brings the product state $|0\rangle\langle 0|^{\otimes N}$ to $(\mathbb{1}^{\otimes N} + \sigma_z^{\otimes N})/2^N$ and the product states $|0\rangle\langle 0|^{\otimes(N-1)} \otimes |1\rangle\langle 1|$ to $(\mathbb{1}^{\otimes N} - \sigma_z^{\otimes N})/2^N$. The steady states of the global Lindbladian take the form of

$$\rho_\infty^{(N)} = \frac{1}{2^N} \mathbb{1}_2^{\otimes N} + c \sigma_z^{\otimes N}, \quad (133)$$

with $c \in (-2^{-N}, 2^{-N})$. There are two independent steady states, independent of the boundary condition. Indeed, $g = \dim(A) = 2$ for $A = \mathbb{C}\mathbb{Z}_2$. However, unlike Example 4, the coefficient of $\mathbb{1}_2^{\otimes N}$ is fixed to $1/2^N$ for $\rho_\infty^{(N)}$ to be a valid density matrix. This is a general feature for nonsimple MPDO. This state is in the trivial phase because the parent Lindbladian drives a product state fast to it, and therefore, it is in a different phase from the classical GHZ state.

We provide a more complicated example for the parent Lindbladian construction in Appendix E for the Fibonacci C^* -weak Hopf algebra $A = \text{Fib}$. In this example, $g = 2$ while $\dim(A) = 13$.

E. Beyond C^* -weak Hopf algebra: a case study

Having reached general conclusions for nonsimple RFPs constructed from C^* -weak Hopf algebras, in this final section, we present a case study of a nonsimple RFP beyond the framework of C^* -weak Hopf algebras [31,58]. The density matrix under concern is the CZX density matrix associated with a nontrivial 3-cocycle.

Example 8 (CZX density matrix). Consider the density matrix of even system size N [31]

$$\rho_{\text{CZX}}^{(N)} = \frac{1}{2^N} (\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)}), \quad (134)$$

where $U_{\text{CZX}}^{(N)}$ is a nontrivial representation of $\mathbb{Z}_2$ group element associated with a nontrivial 3-cocycle,

$$U_{\text{CZX}}^{(N)} = \prod_{i=1}^N CZ_{i,i+1} \prod_{i=1}^N (\sigma_x)_i. \quad (135)$$

This density matrix can be represented by an MPDO tensor $\tilde{M}$ with $d = 2$, $D = 3$,

$$\begin{aligned} \tilde{M}^{00} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \tilde{M}^{11} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \tilde{M}^{01} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \tilde{M}^{10} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned} \quad (136)$$

This is an example of a nonsimple RFP with two elements in the horizontal BNT, the first with bond dimension $D = 1$ generating $\mathbb{1}_2^{\otimes N}$ and the second with $D = 2$ generating $U_{\text{CZX}}^{(N)}$. After tracing any single site, the MPDO is proportional to a maximally mixed state and does not have long-range correlations. Nevertheless, $\rho_{\text{CZX}}^{(N)}$ is in the nontrivial phase due to multipartite entanglement [31].

One can prove that this MPDO is an RFP after blocking two sites by showing the vertical canonical form verifies Theorem 3 (see details in Appendix F). However, this RFP lacks the C^* -weak Hopf algebra structure, as carefully analyzed

in Ref. [58]. In particular, the counit is absent in A and A is not cosemisimple, therefore the previous constructions of renormalization channels $\mathcal{T}$ and $\mathcal{S}$ from C^* -weak Hopf algebra do not apply. Nevertheless, one can still construct the renormalization channels from the vertical canonical form, as described below.

Example 9 (Parent Lindbladian for the CZX density matrix). Using the vertical canonical form of M and K , one can construct the channel $\mathcal{T}$ and $\mathcal{S}$ as follows. Denote M as the two-site tensor (blocking two sites of $\tilde{M}$) and K as the four-site tensor (blocking four sites of $\tilde{M}$) with vertical canonical form as in Eqs. (41) and (42),

$$UM_{(\alpha\beta)}U^\dagger = \bigoplus_{\lambda=1}^2 \mu_\lambda \otimes M_{(\alpha\beta),\lambda}, \quad (137)$$

$$VK_{(\alpha\beta)}V^\dagger = \bigoplus_{\lambda=1}^2 \nu_\lambda \otimes M_{(\alpha\beta),\lambda}, \quad (138)$$

that verifies the conditions in Theorem 3, $m_\lambda := \text{Tr}[\mu_\lambda] = n_\lambda := \text{Tr}[\nu_\lambda]$. There are two elements in the vertical BNT, each with physical dimension $d_\lambda = 2$, and both μ_1, μ_2 are one-dimensional, both ν_1 and ν_2 are four-dimensional. Using the vertical canonical form, the channel $\mathcal{T}$ bringing M to K is

$$\mathcal{T}(X) = V^\dagger \left[\bigoplus_{\lambda=1}^2 \left(\frac{\nu_\lambda}{m_\lambda} \otimes P_\lambda U X U^\dagger P_\lambda^\dagger \right) \right] V, \quad (139)$$

where

$$P_1 = (\mathbb{1}_2 \mid 0_{2 \times 2}), \quad P_2 = (0_{2 \times 2} \mid \mathbb{1}_2) \quad (140)$$

are the isometries projecting to the first and second BNT, respectively.

Similarly, the four-site MPDO K to two-site MPDO M channel $\mathcal{S}$ is

$$\mathcal{S}(X) = U^\dagger \left[\bigoplus_{\lambda=1}^2 \text{Tr}_1(P'_\lambda V X V^\dagger (P'_\lambda)^\dagger) \right] U, \quad (141)$$

where

$$P'_1 = (\mathbb{1}_8 \mid 0_{8 \times 8}), \quad P'_2 = (0_{8 \times 8} \mid \mathbb{1}_8) \quad (142)$$

and Tr_1 denotes tracing over the multiplicity degrees of freedom. The explicit forms of $M_\lambda, \mu_\lambda, \nu_\lambda, U, V$ are given in Appendix F. Knowing the renormalization channels $\mathcal{T}, \mathcal{S}$, the parent Lindbladian is constructed by the standard algorithm as in Sec. IV.

This parent Lindbladian is commuting and therefore rapid mixing. The steady states of the global Lindbladian take the form of

$$\rho_\infty^{(N)} = \frac{1}{2^N} (\mathbb{1}_2^{\otimes N} + c U_{\text{CZX}}^{(N)}), \quad c \in [-1, 1], \quad (143)$$

hence the steady-state degeneracy equals 2. We leave the detailed derivation in Appendix F.

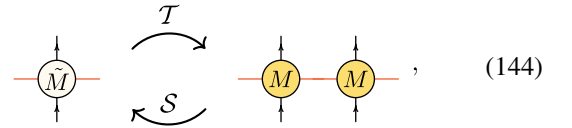
VII. OUTLOOK

In this paper, we systematically construct the parent Lindbladians for the renormalization fixed-points of MPDO. Our constructed parent Lindbladian satisfies the important properties of local, frustration free, and reaches minimal steady-state

degeneracy. In comparison to pure states, the parent Lindbladian has unique properties that are distinct from the parent Hamiltonian.

There are at least three interesting questions that deserve further study. The first open question concerns the parent Lindbladians with symmetries. Given the construction of our parent Lindbladians, they naturally inherit the symmetries present in the state. As such, this construction could be potentially useful in exploring symmetry-protected topological phases for mixed states [16,17,19–29]. However, in this work, we have focused exclusively on topological phases without symmetry protection. Investigating parent Lindbladians with symmetries is left for future study.

Second, can we construct the parent Lindbladian for a general MPDO that is not at RFP? A natural generalization is still taking $\mathcal{E} = \mathcal{T} \circ \mathcal{S}$ and $\mathcal{L} = \mathcal{E} - \mathbb{1}$, but with $\mathcal{T}, \mathcal{S}$ being the renormalization quantum channels for a general MPDO,



where the tensor $\tilde{M}$ after coarse-graining is not the same as M . We note that the parent Hamiltonian construction $h = \mathbb{1} - VV^\dagger$ applies to general MPS that is not at RFP, and it is natural to expect the above generalization to also apply to general MPDOs. One difficulty, however, is that there is no universal algorithm to find the renormalization quantum channels $\mathcal{T}$ and $\mathcal{S}$ when away from MPDO RFPs; furthermore, such channels do not exist for certain MPDOs. We note a recent progress in Ref. [55], and it is reasonable to expect that our construction can be applied to the MPDO where renormalization channels exist and are known.

The third question comes from an interesting numerical observation that the parent Lindbladians constructed from the vertical canonical form and from the C^* -weak Hopf algebra (which is associated with the horizontal canonical form) appears identical for the examples studied (see Appendix E). This finding hints at a potential duality between the horizontal and vertical canonical forms at MPDO RFPs, requiring further study.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [68].

APPENDIX A: EXAMPLE OF SIMPLE MPDO RFP

In this Appendix, we give two examples of simple MPDO RFP, which has the structure as in Theorem 3. We construct their Gibbs state Hamiltonian and parent Lindbladian and discuss their properties.

Example A1 (Product of Werner state). Consider an MPDO RFP as a product of Werner state $W(b)$, where $W(b)$ is supported on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $b \in [0, 1]$,

$$\rho^{(N)} = W(b) \otimes W(b) \otimes \cdots \otimes W(b),$$

$$W(b) = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} - b(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)). \quad (\text{A1})$$

The MPDO tensor M only has a single BNT element and a single $k = 1$, and we identify $\eta_{1,1} = W(b)$. Therefore, the Hilbert space for each site is decomposed as $\mathcal{H} = \mathcal{H}_l \otimes \mathcal{H}_r$ with $d_l = 2, d_r = 2$ and $d = d_l d_r = 4$. The explicit MPDO tensor representation is injective and has bond dimension $D = 4$,

$$\begin{aligned} \begin{array}{|c|} \hline r_1 \\ \hline \end{array} \text{---} 1 &= \frac{1}{2} \mathbb{1}, & 1 \text{---} \begin{array}{|c|} \hline l_1 \\ \hline \end{array} &= \frac{1}{2} \mathbb{1}, \\ \begin{array}{|c|} \hline r_1 \\ \hline \end{array} \text{---} 2 &= \frac{\sqrt{b}}{2} \sigma_x, & 2 \text{---} \begin{array}{|c|} \hline l_1 \\ \hline \end{array} &= -\frac{\sqrt{b}}{2} \sigma_x, \\ \begin{array}{|c|} \hline r_1 \\ \hline \end{array} \text{---} 3 &= \frac{\sqrt{b}}{2} \sigma_y, & 3 \text{---} \begin{array}{|c|} \hline l_1 \\ \hline \end{array} &= -\frac{\sqrt{b}}{2} \sigma_y, \\ \begin{array}{|c|} \hline r_1 \\ \hline \end{array} \text{---} 4 &= \frac{\sqrt{b}}{2} \sigma_z, & 4 \text{---} \begin{array}{|c|} \hline l_1 \\ \hline \end{array} &= -\frac{\sqrt{b}}{2} \sigma_z. \end{aligned} \quad (\text{A2})$$

When $b \in [0, 1)$, $\eta_{1,1} = W(b)$ is full-rank, and this MPDO is a Gibbs state of nearest-neighboring commuting Hamiltonian $\rho^{(N)} = e^{-H^{(N)}}$, with Hamiltonian

$$H^{(N)} = \sum_{i=1}^N h_i, \quad (\text{A3})$$

where $h_i = \tau_i(h)$ and h is supported on two consecutive sites as

$$h = \mathbb{1}_{\mathcal{H}_l} \otimes (-\log \eta_{1,1}) \otimes \mathbb{1}_{\mathcal{H}_r}. \quad (\text{A4})$$

The identity $e^{\mathbb{1} \otimes A} = \mathbb{1} \otimes e^A$ is used in the derivation.

The channel $\mathcal{E}$ in parent Lindbladian is simply a replacement channel,

$$\mathcal{E}(X) = \text{Tr}_{1r,2,3l}(X) \otimes (\eta_{11} \otimes \eta_{11}). \quad (\text{A5})$$

Example A2 (Variation of Werner state). Consider an MPDO RFP generated by tensor M that has a single BNT element and two k 's, $k = 1$ and $k = 2$, with components

$$\begin{aligned} \begin{array}{|c|} \hline r_k \\ \hline \end{array} \text{---} 1 &= \frac{1}{2\sqrt{2}} \mathbb{1}, & 1 \text{---} \begin{array}{|c|} \hline l_k \\ \hline \end{array} &= \frac{1}{2\sqrt{2}} \mathbb{1}, \\ \begin{array}{|c|} \hline r_k \\ \hline \end{array} \text{---} 2 &= \frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_x, & 2 \text{---} \begin{array}{|c|} \hline l_k \\ \hline \end{array} &= -\frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_x, \\ \begin{array}{|c|} \hline r_k \\ \hline \end{array} \text{---} 3 &= \frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_y, & 3 \text{---} \begin{array}{|c|} \hline l_k \\ \hline \end{array} &= -\frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_y, \\ \begin{array}{|c|} \hline r_k \\ \hline \end{array} \text{---} 4 &= \frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_z, & 4 \text{---} \begin{array}{|c|} \hline l_k \\ \hline \end{array} &= -\frac{\sqrt{b_k}}{2\sqrt{2}} \sigma_z, \end{aligned} \quad (\text{A6})$$

where $b_1, b_2 \in [0, 1]$. The Hilbert space for each site is decomposed as $\mathcal{H} = (\mathcal{H}_l^{(1)} \otimes \mathcal{H}_r^{(1)}) \oplus (\mathcal{H}_l^{(2)} \otimes \mathcal{H}_r^{(2)})$ with $d_l^{(1)} = d_r^{(1)} = d_l^{(2)} = d_r^{(2)} = 2$, and $d = \sum_k d_l^{(k)} d_r^{(k)} = 8$. The tensor is injective with bond dimension $D = 4$. Explicitly,

$$\alpha \text{---} \begin{array}{|c|} \hline M \\ \hline \end{array} \text{---} \beta = \left((l_1)_\alpha \otimes (r_1)_\beta \quad (l_2)_\alpha \otimes (r_2)_\beta \right). \quad (\text{A7})$$

By construction,

$$\begin{aligned} \eta_{1,1} &= \frac{1}{2} W(b_1), & \eta_{2,2} &= \frac{1}{2} W(b_2), \\ \eta_{1,2} &= \eta_{2,1} = \frac{1}{2} W(\sqrt{b_1 b_2}), \end{aligned} \quad (\text{A8})$$

and $\text{Tr}(\eta_{k,k'}) = 1/2$ for $k, k' = 1, 2$, satisfying $\sum_k \text{Tr}(\eta_{k,k}) = 1$.

When $b_1, b_2 \in [0, 1)$, $\eta_{k,k'}$ are full-rank for $k, k' = 1, 2$, and this MPDO is a Gibbs state of nearest-neighboring commuting Hamiltonian $\rho^{(N)} = e^{-H^{(N)}}$, with Hamiltonian

$$H^{(N)} = \sum_{i=1}^N h_i, \quad (\text{A9})$$

where $h_i = \tau_i(h)$ and h is supported on two consecutive sites as

$$h = \bigoplus_{k_1, k_2} \mathbb{1}_{\mathcal{H}_l^{(k_1)}} \otimes (-\log \eta_{k_1, k_2}) \otimes \mathbb{1}_{\mathcal{H}_r^{(k_2)}}. \quad (\text{A10})$$

The identities $e^{\mathbb{1} \otimes A} = \mathbb{1} \otimes e^A$ and $e^{A \oplus B} = e^A \oplus e^B$ are used to derive

$$\begin{aligned} B_{i,i+1} &:= e^{-\tau_i(h) \otimes \mathbb{1}_{\text{rest}}} \\ &= \left[\bigoplus_{k_i, k_{i+1}} \mathbb{1}_{\mathcal{H}_l^{(k_i)}} \otimes \eta_{k_i, k_{i+1}} \otimes \mathbb{1}_{\mathcal{H}_r^{(k_{i+1})}} \right] \otimes \mathbb{1}_{\text{rest}}, \end{aligned} \quad (\text{A11})$$

leading to

$$\begin{aligned} \rho^{(N)}(M) &= \bigoplus_{k_1, \dots, k_N} \eta_{k_1, k_2} \otimes \eta_{k_2, k_3} \otimes \cdots \otimes \eta_{k_N, k_1} \\ &= \prod_{i=1}^N B_{i,i+1}. \end{aligned} \quad (\text{A12})$$

Indeed, the MPDO is a Gibbs state of commuting local Hamiltonian.

Finally, the channel $\mathcal{E}$ in parent Lindbladian is constructed as in Eq. (55), with isometries

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and η 's constructed above. This is an example for $G_L \subseteq \mathcal{F}_\mathcal{E}$ commented in Sec. IV. In this example, $L = 3$, and G_L has dimension $\dim(G_{L=3}) = D^2 = 16$ since the MPDO is injective. On the other hand, the fixed point space of $\mathcal{E}$ is

$$\mathcal{F}_\mathcal{E} = \bigoplus_{k_1, k_3} \mathcal{M}_{d_l^{(k_1)}} \otimes \left(\bigoplus_{k_2} \eta_{k_1, k_2} \otimes \eta_{k_2, k_3} \right) \otimes \mathcal{M}_{d_r^{(k_3)}}, \quad (\text{A13})$$

where $\mathcal{M}_d$ is the full matrix algebra of $d \times d$ matrix. Therefore, the fixed point space has dimension $\dim(\mathcal{F}_\mathcal{E}) = \sum_{k_1, k_3} (d_l^{(k_1)})^2 (d_r^{(k_3)})^2 = 64$, and $G_L \subseteq \mathcal{F}_\mathcal{E}$. Due to this mismatch, the techniques for proving ground state degeneracy for parent Hamiltonian do not directly apply to parent Lindbladian. For example, we cannot use the intersection property in Ref. [48].

APPENDIX B: CONSTRUCTION OF RFP MPDOS USING C^* -WEAK HOPF ALGEBRAS

This Appendix is devoted to introducing and recalling the rigorous definitions and statements for the construction of RFP MPDOs based on C^* -weak Hopf algebras. We also refer the reader to Refs. [44–46] for detailed proofs.

1. Algebraic framework

Let us first recall that a coalgebra is a complex vector space C endowed with a linear map $\Delta : C \rightarrow C \otimes C$, called comultiplication, which is comultiplicative, i.e.,

$$(\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta, \quad (\text{B1})$$

and a linear functional $\epsilon : C \rightarrow \mathbb{C}$, called counit, which is compatible in the sense that

$$(\epsilon \otimes \text{Id}) \circ \Delta = \text{Id} = (\text{Id} \otimes \epsilon) \circ \Delta. \quad (\text{B2})$$

For successive applications of the comultiplication,

$$\Delta^{(1)} := \Delta, \quad \Delta^{(n+1)} := (\Delta \otimes \text{Id}^{\otimes n}) \circ \Delta^{(n)} \quad (\text{B3})$$

for all $n \in \mathbb{N}$. In general, it is more appropriate to employ Sweedler's notation, in which we denote

$$x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(n+1)} := \Delta^{(n)}(x), \quad (\text{B4})$$

a shorthand notation for sums $\sum_i x_{1,i} \otimes \cdots \otimes x_{n+1,i}$; note that the terms $x_{(i)}$ above are not individual tensor factors. It

turns out that representations of coalgebras (more concretely, comodules) are in one-to-one correspondence with tensors generating MPS [44], and the action of the comultiplication is understood as a notion of system size extensivity for these kinds of states.

Definition B1. A C^* -weak Hopf algebra A is a complex finite-dimensional C^* -algebra endowed with the structure of a coalgebra, such that the comultiplication is a $*$ -algebra map, i.e., for all $x \in A$:

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \Delta(x^*) = \Delta(x)^*, \quad (\text{B5})$$

where the product and the $*$ -operation is defined on $A \otimes A$ componentwise, the counit is *weakly multiplicative*, i.e., for all $x, y, z \in A$:

$$\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z), \quad (\text{B6})$$

the unit is *weakly comultiplicative*, i.e.,

$$1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)} 1_{(1')} \otimes 1_{(2')}, \quad (\text{B7})$$

where the prime symbol is intended to distinguish between different coproduct instances of $1 \in A$, and there exists a linear map, $S : A \rightarrow A$, called antipode, which is antimultiplicative, anticomultiplicative, and satisfies for all $x \in A$:

$$S(x_{(1)})x_{(2)} = \epsilon(1_{(1)}x)1_{(2)}, \quad (\text{B8})$$

$$x_{(1)}S(x_{(2)}) = \epsilon(x1_{(2)})1_{(1)}. \quad (\text{B9})$$

A C^* -Hopf algebra is a C^* -weak Hopf algebra with

$$\Delta(1) = 1 \otimes 1.$$

The axioms of C^* -weak Hopf algebras are such that the notion is self-dual, i.e., the dual vector space $A^* = \text{Hom}(A, \mathbb{C})$ is endowed with the structure of a C^* -weak Hopf algebra [69], and such that category of $*$ -algebra representations are unitary (multi)fusion category [66,67]. First, the monoidal product is given by $\Phi_1 \boxtimes \Phi_2 := (\Phi_1 \otimes \Phi_2) \circ \Delta$, and there exists a *trivial* $*$ -representation in the sense of relaxed monoidal categories, denoted $(\mathcal{H}_\epsilon, \Phi_\epsilon)$ [70]. Second, for any finite $*$ -representation $(\mathcal{H}, \Phi)$ there exists a $*$ -representation of A , given by the expression

$$\bar{\Phi}(x) = (\Phi \circ J)(x)^t,$$

on the dual Hilbert space, for certain linear map $J : A \rightarrow A$ [70,71]; this generalizes the notion of dual group-representations. We note that, in general, $J \neq S$. Furthermore, it is a semisimple category and the set $\text{Irr} A$ of equivalence classes of irreducible representations is finite, also known as the set of *sectors* of A . The unusual feature of the trivial $*$ -representation in this setting is that it can be decomposable [69,70]; if this is not the case, A is said to be *connected* or *pure*, and if both trivial $*$ -representations of A and A^* are indecomposable, A is said to be *biconnected*. We note that categories of $*$ -representation of biconnected C^* -weak Hopf algebras encompass unitary fusion categories [66].

Example B1. Let G be a finite group. Then, the group algebra $A = \mathbb{C}G$, endowed with the linear extensions of the maps defined by

$$\Delta(e_i) = e_i \otimes e_i, \quad \varepsilon : e_i \mapsto 1, \quad S(e_i) = e_i^* = e_i^{-1},$$

for all $e_i \in G$, defines a C^* -Hopf algebra.

2. Tensor-network constructions

In terms of tensor networks, given a C^* -weak Hopf algebra A and two faithful $*$ -representations Φ and Ψ of A and A^* , respectively, one can define the rank-four tensor

$$\begin{array}{c} \uparrow \\ \text{---} \bullet \text{---} \\ \downarrow \end{array} = \sum_{i=1}^{\dim A} \Phi(e_i) \otimes \Psi(e^i), \quad (\text{B10})$$

where $e_1, \dots, e_d \in A$ stands for a basis of A and $e^1, \dots, e^d \in A^*$ stands for the dual basis. Here, the first tensor factor corresponds to an endomorphism at the physical level and the second one corresponds to an endomorphism to the virtual level.

For any $x \in A$, there is $b(x) \in \Psi(A^*) \subset \text{End}(\mathcal{K})$ satisfying the following:

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \quad \downarrow \end{array} \xrightarrow{b(x)} \begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \quad \downarrow \end{array} = (\Phi^{\otimes N} \circ \Delta^{(N-1)})(x). \quad (\text{B11})$$

Moreover, provided a decomposition of the $*$ -representation $\Psi = \Psi_1^{\otimes n_1} \oplus \dots \oplus \Psi_g^{\otimes n_g}$ in terms of irreducible $*$ -representations $\Psi_1, \dots, \Psi_g$, the rank-four tensor introduced above can be decomposed as a sum of the rank-four tensors denoted

$$\begin{array}{c} \text{---} \bullet \text{---} \\ \uparrow \quad \downarrow \end{array} = \sum_{i=1}^{\dim A} \Phi(e_i) \otimes \Psi_a^{\otimes n_a}(e^i); \quad (\text{B12})$$

or, from another point of view, these tensors induce a BNT. In particular, the tensor is in horizontal Canonical Form. The duality manifests as the existence of another tensor,

$$\begin{array}{c} \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \end{array} = \sum_{i=1}^{\dim A} (\Phi \circ J)(e_i) \otimes \Psi(e^i). \quad (\text{B13})$$

3. Pulling-through identities

In this context, it is fundamental to recall the existence of a positive idempotent $\Omega \in A$ satisfying a pulling-through identity, which can be regarded as a microscopical description of the topological order phenomena. The following result summarizes the fundamental properties needed in the framework.

Theorem B1. Let A be a C^* -weak Hopf algebra. Then, there exists $\Omega \in A$ such that it is a positive idempotent which is invariant under both S and J :

$$\Omega^2 = \Omega = \Omega^* = S(\Omega) = J(\Omega), \quad (\text{B14})$$

and it is a cocentral element, i.e., it satisfies

$$\Omega_{(2)} \otimes \Omega_{(1)} = \Omega_{(1)} \otimes \Omega_{(2)}, \quad (\text{B15})$$

or, equivalently, for any $n \in \mathbb{N}$ and any shift permutation σ of $\{1, \dots, n\}$, it holds that

$$\Omega_{(\sigma(1))} \otimes \dots \otimes \Omega_{(\sigma(n))} = \Omega_1 \otimes \dots \otimes \Omega_n, \quad (\text{B16})$$

and moreover, there exists an invertible positive element $\xi \in A$ such that $J(\xi) = \xi$ and, for all $x, y \in A$,

$$x \xi^{\frac{1}{2}} J(\Omega_{(1)}) \otimes \Omega_{(2)} = \xi^{\frac{1}{2}} J(\Omega_{(1)}) \otimes \Omega_{(2)} x, \quad (\text{B17})$$

$$J(\Omega_{(1)}) \xi^{\frac{1}{2}} y \otimes \Omega_{(2)} = J(\Omega_{(1)}) \xi^{\frac{1}{2}} \otimes y \Omega_{(2)}, \quad (\text{B18})$$

known as pulling-through identities. Let $\omega \in A^*$ stand for the dual analog of $\Omega \in A$, then

$$\omega(\xi^{\frac{1}{2}} \Omega_{(1)} \xi^{\frac{1}{2}} J(x)) \Omega_{(2)} = x \quad (\text{B19})$$

for all $x \in A$.

See Refs. [45,46] for a proof.

Remark B1. (1) If the C^* -weak Hopf algebra A is bi-connected,

$$\Omega = \mathcal{D}^{-2}(d_1 x_1 + \dots + d_g x_g),$$

where $x_1, \dots, x_g \in A \cong A^{**}$ stand for the characters of the irreducible $*$ -representations of A^* , $d_1, \dots, d_g > 0$ are the associated quantum dimensions or Frobenius-Perron dimensions, and $\mathcal{D}^2 := d_1^2 + \dots + d_g^2$ is the total quantum dimension.

(2) In the particular case of C^* -Hopf algebras, see Definition B1, the element introduced above $\Omega \in A$ coincides with the well-known Haar integral [45], $J = S$, and $\xi = \mathcal{D}1$.

(3) More concretely, for group algebras $A = \mathbb{C}G$,

$$\Omega = |G|^{-1} \sum_{g \in G} g, \quad \omega = \delta_e,$$

where δ_e is the dual basis element of $e \in G$.

4. Renormalization fixed point MPDOs

Proposition B1. The tensor

$$\begin{array}{c} \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \end{array} := \begin{array}{c} \uparrow \\ \text{---} \bullet \text{---} \\ \downarrow \end{array} b(\omega) \quad (\text{B20})$$

defines RFP MPDOs in the sense of Definition 4.

Moreover, let us define the rank-four tensor introduced in Sec. VIA by the expression

$$\begin{array}{c} \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \end{array} := \begin{array}{c} \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \end{array} \begin{array}{c} \Phi(\xi)^{1/2} \\ \Phi(\xi)^{1/2} \end{array}, \quad (\text{B21})$$

the rank-two tensor Ω as the one given by

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} := b(\Omega), \quad (\text{B22})$$

and the rank-two tensor Ξ is defined by

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} := \Psi(\hat{\xi})^{-1} \quad (\text{B23})$$

where $\hat{\xi} \in A^*$ stands for the dual analog of ξ . By the above definitions, Eq. (97) is simply a consequence of the

pulling-through identity Eqs. (B19) and (98) is translated to

$$\text{Tr}(\Psi_a(\hat{\xi}^{-1})) = d_a/\mathcal{D}^2.$$

On the one hand, the explicit expression of the fine-graining quantum channel $\mathcal{T}$ is the following:

$$\begin{aligned} \mathcal{T}(X) &:= \text{Tr}((\Phi \circ J)(\xi^{\frac{1}{2}} \Omega_{(2)} \xi^{\frac{1}{2}})X) \\ &\quad b(\omega)^{\otimes 2}(\Phi^{\otimes 2} \circ \Delta)(\Omega_{(1)}), \end{aligned} \quad (\text{B24})$$

for any $X \in \text{End}(\mathcal{H})$. We refer the reader to Ref. [45] for a proof of the facts that it is completely positive, trace preserving and, in addition, it satisfies the fine-graining condition

$$\mathcal{T}(b(\omega)\Phi(x)) = b(\omega)^{\otimes 2}(\Phi^{\otimes 2} \circ \Delta)(x).$$

On the other hand, the coarse-graining quantum channel $\mathcal{S} : \text{End}(\mathcal{H} \otimes \mathcal{H}) \rightarrow \text{End}(\mathcal{H})$ is given by

$$\mathcal{S} := \mathcal{S}^{(\text{I})} + \mathcal{S}^{(\text{II})},$$

where the first summand is defined by the expression

$$\begin{aligned} \mathcal{S}^{(\text{I})}(Y) &:= \text{Tr}((\Phi^{\otimes 2} \circ \Delta \circ J)(\xi^{\frac{1}{2}} \Omega_{(2)} \xi^{\frac{1}{2}})Y) \\ &\quad b(\omega)\Phi(\Omega_{(1)}) \end{aligned} \quad (\text{B25})$$

and is supported on the symmetric subspace, satisfying the coarse-graining condition

$$\mathcal{S}^{(\text{I})}(b(\omega)^{\otimes 2}(\Phi^{\otimes 2} \circ \Delta)(x)) = b(\omega)\Phi(x)$$

for all $x \in A$, and the second part is given by the expression on the complementary subspace,

$$\mathcal{S}^{(\text{II})}(Y) := \text{Tr}(Y - (\Phi^{\otimes 2} \circ \Delta)(1)Y)\sigma_0, \quad (\text{B26})$$

where $\sigma_0 \in \text{End}(\mathcal{H} \otimes \mathcal{H})$ is any mixed state; as commented above, this part is completely positive by construction and completes the first part making $\mathcal{S}$ a trace-preserving linear map. We typically consider $\sigma_0 := \omega(\Omega)^{-1}b(\omega)\Phi(\Omega)$ for simplicity. Note that

$$\mathcal{S}^{(\text{II})}(b(\omega)^{\otimes 2}(\Phi^{\otimes 2} \circ \Delta)(x)) = 0.$$

APPENDIX C: PROOF OF FRUSTRATION FREENESS

In this Appendix, we prove the frustration freeness of a general parent Lindbladian (not necessarily commuting) using a “flow diagram” method. Define

$$\mathcal{E}^{(N)} = \frac{1}{N} \sum_{i=1}^N \mathcal{E}_i, \quad (\text{C1})$$

which is related to parent Lindbladian by $\mathcal{L}^{(N)} = N(\mathcal{E}^{(N)} - \mathbb{1})$, and the steady state of $\mathcal{L}^{(N)}$ corresponds to the fixed point of $\mathcal{E}^{(N)}$. The proof is inspired by the observation that if ρ is a fixed point of $\mathcal{E}^{(N)}$, then it is also a fixed point of $(\mathcal{E}^{(N)})^m$ for any positive integer m . One can rewrite the fixed-point ρ into a summation

$$\begin{aligned} \rho &= \frac{1}{N} \mathcal{E}_1^{(\text{I})}(\rho) + \frac{1}{N} \mathcal{E}_1^{(\text{II})}(\rho) + \frac{1}{N} \mathcal{E}_2^{(\text{I})}(\rho) + \frac{1}{N} \mathcal{E}_2^{(\text{II})}(\rho) + \cdots \\ &\quad + \frac{1}{N} \mathcal{E}_N^{(\text{I})}(\rho) + \frac{1}{N} \mathcal{E}_N^{(\text{II})}(\rho), \end{aligned} \quad (\text{C2})$$

and ask how each of these terms gets modified by successive applications of $\mathcal{E}^{(N)}$ and draw the corresponding flow diagram. We note that all $\mathcal{E}_i^{(\text{I})}$ and $\mathcal{E}_i^{(\text{II})}$ are CP maps and each term in the decomposition is positive semi-definite.

Since the simple injective MPDOs and the nonsimple MPDOs generated by C^* -Hopf algebra are guaranteed to be frustration free due to the commutation of their parent Lindbladians, we only need to consider the remaining two scenarios: (1) simple noninjective simple MPDOs and (2) nonsimple MPDOs generated by a C^* -weak Hopf algebra which is not a C^* -Hopf algebra. For noninjective simple MPDOs, by choosing ρ_0 in $\mathcal{E}_i^{(\text{II})}$ as Eq. (71), the pieces in the channels satisfy

$$\mathcal{E}_i^{(\text{I})}(\rho_0) = \rho_0, \quad (\text{C3})$$

$$\mathcal{E}_i^{(\text{I})} \circ \mathcal{E}_i^{(\text{II})} = \mathcal{E}_i^{(\text{II})}, \quad (\text{C4})$$

$$\mathcal{E}_i^{(\text{II})} \circ \mathcal{E}_i^{(\text{I})} = 0, \quad (\text{C5})$$

$$\mathcal{E}_i^{(\text{I})} \circ \mathcal{E}_i^{(\text{I})} = \mathcal{E}_i^{(\text{I})}, \quad (\text{C6})$$

$$\mathcal{E}_i^{(\text{I})} \circ \mathcal{E}_{i+1}^{(\text{I})} = \mathcal{E}_{i+1}^{(\text{I})} \circ \mathcal{E}_i^{(\text{I})}. \quad (\text{C7})$$

For MPDO generated by C^* -weak Hopf algebra, by choosing $\mathcal{E}^{(\text{II})}$ as Eq. (115), the pieces in the channels also satisfy Eqs. (C3)–(C7). We will use these properties to draw the flow diagram. We organize the terms with different structures into separate boxes. For every action of $\mathcal{E}^{(N)}$, ρ is rewritten, flowing between boxes. Since $\mathcal{E}^{(N)}$ is a quantum channel, the summation of the trace of terms in different boxes is equal to one. By iterating this process for large enough m , we prove below that all the terms of a fixed-point ρ in Eq. (C2) will flow to the final box, in the form of

$$\rho = \mathcal{E}_1^{(\text{I})} \circ \mathcal{E}_2^{(\text{I})} \circ \cdots \circ \mathcal{E}_N^{(\text{I})}(\rho). \quad (\text{C8})$$

Using Eqs. (C5)–(C7), one can then show such a fixed-point state is frustration free: $\mathcal{E}_i(\rho) = \mathcal{E}_i^{(\text{I})}(\rho) = \rho$, leading to $\mathcal{L}_i(\rho) = 0$. In the following, we separate the treatments of the two cases.

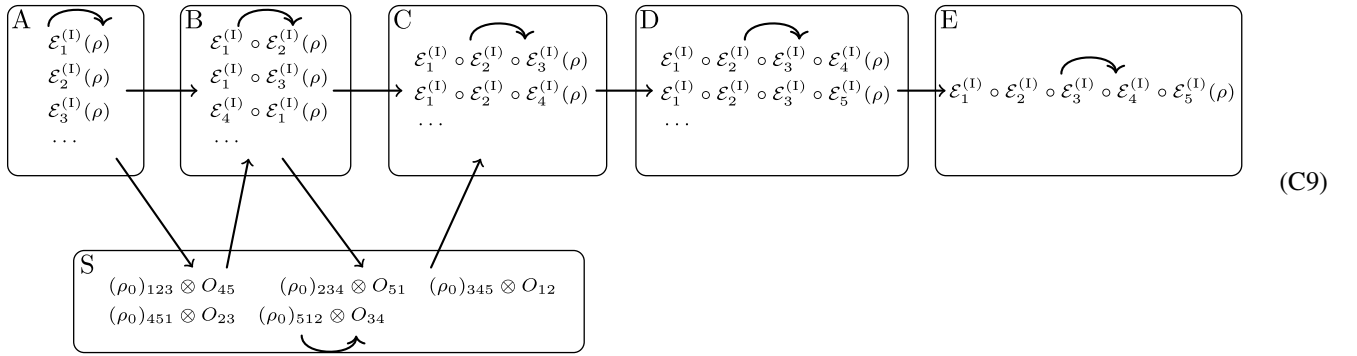
1. Noninjective simple MPDO

Below, we prove the frustration freeness of parent Lindbladian for noninjective simple MPDO RFP with system size $N = 5$ as an explicit example. Generalization to larger N is straightforward.

Proposition C1. The parent Lindbladian for noninjective simple MPDO RFP with system size $N = 5$ is frustration free.

Proof. We first decompose fixed-point ρ as in Eq. (C2). Under an action of $\mathcal{E}^{(N)}$, the decomposition terms of ρ will flow to each other, which includes the following

arrows,



We group all possible terms in boxes labeled by “A,” “B,” “C,” “D,” “E,” and “S.” The terms in box B come from two neighboring actions of the form $\mathcal{E}_i^{(I)} \circ \mathcal{E}_j^{(I)}$, where “neighboring” means the support of $\mathcal{E}_i^{(I)}$ and $\mathcal{E}_j^{(I)}$ overlap; namely, j can be $i-2, i-1, i+1, i+2$. The terms in box C come from three neighboring actions of the form $\mathcal{E}_i^{(I)} \circ \mathcal{E}_j^{(I)} \circ \mathcal{E}_k^{(I)}$, where “neighboring” means the support of $\mathcal{E}_i^{(I)}$ and $\mathcal{E}_j^{(I)}$ overlap, and the support of $\mathcal{E}_j^{(I)}$ and $\mathcal{E}_k^{(I)}$ overlap. For example, when considering a sufficiently large N , say $N > 8$, the term $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)$ belongs to box C, while $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_6^{(I)}(\rho)$ belongs to box B. Similarly for boxes “D” and “E.” Finally, the terms in box “S” come from an action of $\mathcal{E}^{(II)}$, taking the form of $\mathcal{E}_i^{(II)}(\rho) = (\rho_0)_{i,i+1,i+2} \otimes O_{i+3,i+4}$. $\mathcal{E}^{(II)}$ is a completely-positive linear map and ρ is a density matrix, leading to $O_{i+3,i+4}$ being positive semi-definite.

We justify the following facts:

(1) *Either arrow $A \Rightarrow B$ or arrow $A \Rightarrow S$ must exist.* Without loss of generality, let’s start from term $\mathcal{E}_1^{(I)}(\rho)$. When acting $\mathcal{E}_2$, $\mathcal{E}_2^{(I)}$ will bring it to term $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}$ in box B, and $\mathcal{E}_2^{(II)}$ will bring it to term $(\rho_0)_{234} \otimes O_{51}$ in box S. Since $\mathcal{E}_2$ is a CPTP map, at least one of these two terms is nonzero. Therefore, an arrow must exist for either box A to box B or box A to box S.

(2) *The arrow $S \Rightarrow B$ must exist.* Without loss of generality, let’s start from term $((\rho_0)_{123} \otimes O_{45})$, and decompose $O_{45} = \sum_{aa'} (P_a \otimes \mathbb{1}) O_{45} (P_{a'} \otimes \mathbb{1})$. Since $O_{45} \geq 0$, for each a , $(P_a \otimes \mathbb{1}) O_{45} (P_a \otimes \mathbb{1}) \geq 0$ and $\text{Tr}(O_{45}) = \sum_a \text{Tr}((P_a \otimes \mathbb{1}) O_{45} (P_a \otimes \mathbb{1})) > 0$. Therefore, there exists at least one a such that $(P_a \otimes \mathbb{1}) O_{45} (P_a \otimes \mathbb{1}) \geq 0$, and for this a we have a lower bound for the trace $\text{Tr}((P_a \otimes \mathbb{1}) O_{45} (P_a \otimes \mathbb{1})) \geq \frac{1}{g} \text{Tr}(O_{45})$, where g is the number of elements in the BNT. For this a and with explicit choice of ρ_0 in Eq. (71), $(P_a \otimes P_a \otimes P_a \otimes P_a \otimes \mathbb{1})((\rho_0)_{123} \otimes O_{45})(P_a \otimes P_a \otimes P_a \otimes P_a \otimes \mathbb{1}) \geq 0$, leading to $(\mathbb{1} \otimes P_a \otimes P_a \otimes P_a \otimes \mathbb{1})((\rho_0)_{123} \otimes O_{45})(\mathbb{1} \otimes P_a \otimes P_a \otimes P_a \otimes \mathbb{1}) \geq 0$, and subsequently,

$$\mathcal{E}_2^{(I)}((\rho_0)_{123} \otimes O_{45}) \geq 0.$$

The action $\frac{1}{N} \mathcal{E}_2^{(I)}$ will bring the term $(\rho_0)_{123} \otimes O_{45}$ from box S to box B. Therefore, the arrows from box S to box B must exist.

Furthermore, by noting that $\text{Tr}(P_a \otimes P_a \otimes P_a(\rho_0)P_a \otimes P_a \otimes P_a) = \frac{1}{g} \text{Tr}(\rho_0)$ by construction, we can put a lower bound for the trace,

$$\text{Tr}(\mathcal{E}_2^{(I)}((\rho_0)_{123} \otimes O_{45})) \geq \frac{1}{g^2} \text{Tr}((\rho_0)_{123} \otimes O_{45}),$$

where the lower bound $1/g^2$ is a constant.

(3) *Either arrow $B \Rightarrow C$ or arrows $B \Rightarrow S \Rightarrow C$ must exist.* Without loss of generality, let’s start from term $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)$ in box C and consider the action of $\frac{1}{N} \mathcal{E}_3$. If $\mathcal{E}_3^{(I)} \circ \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho) \geq 0$, then $\frac{1}{N} \mathcal{E}_3$ will bring $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)$ from box B to box C.

If, however, that $\mathcal{E}_3^{(I)} \circ \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho) = 0$, we then consider $\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho))$, which is term $(\rho_0)_{345} \otimes O_{12}$ in box B. Due to the structure of $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}$, $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)$ satisfies

$$\sum_a (P_a \otimes P_a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}) (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)) (P_a \otimes P_a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}) = \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho),$$

and therefore by acting $\mathcal{E}_3^{(II)}$ which does not act on site 1 and 2, $\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho))$ can be written as a sum of direct product,

$$\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)) = \sum_a (O_a)_{12} \otimes (\rho_0)_{345},$$

where $O_a \geq 0$ and $(P_a \otimes P_a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1})(O_a)_{12}(P_a \otimes P_a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}) = (O_a)_{12}$. There exists at least one a such that $O_a \geq 0$, and for this a we have a lower bound for the trace $\text{Tr}((P_a \otimes P_a \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1})(O_a \otimes \rho_0)) \geq \frac{1}{g} \text{Tr}(\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)))$. For this a and with explicit choice of ρ_0 in Eq. (71), $(P_a \otimes P_a \otimes P_a \otimes P_a \otimes P_a)(\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)))(P_a \otimes P_a \otimes P_a \otimes P_a \otimes P_a) \geq 0$, therefore by acting on another $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}$,

$$\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)) \geq 0.$$

Using Eq. (C4), we see this term is in box C. Therefore, either arrow from box B to box C or from box B to box C via box S must exist. Furthermore, we have a lower bound for

the trace

$$\begin{aligned} & \text{Tr}(\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho))) \\ & \geq \frac{1}{g^2} \text{Tr}(\mathcal{E}_3^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho))), \end{aligned}$$

where the lower bound $1/g^2$ is a constant.

(4) *There are no arrows from boxes C, D, E to box S.* At $N = 5$, the terms in box C are already in the desired space. Specifically, taking $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)$ as an example, it already satisfies

$$\begin{aligned} & \sum_a (P_a \otimes P_a \otimes P_a \otimes P_a \otimes P_a)(\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) \\ & (P_a \otimes P_a \otimes P_a \otimes P_a \otimes P_a) = \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho), \end{aligned}$$

because the support of $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}$ covers all the five sites. When acting $\mathcal{E}_4$ only $\mathcal{E}_4^{(I)}$ leads to nonzero result and brings it to term $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)} \circ \mathcal{E}_4^{(I)}(\rho)$ in box D, while acting $\mathcal{E}_4^{(II)}$ gives zero. Similarly, the terms in box D are already in the desired space, and further actions of $\mathcal{E}_i$ can only bring the terms to themselves or the term in box E. Finally, acting $\mathcal{E}_i$ to terms in box E bring them to themselves. Therefore, no arrows are from boxes C, D, and E to box S.

With the above arrows, it is clear that the trace of terms in box E will keep increasing as long as there exist terms in previous boxes, and subsequently, a steady state can only be supported in box E, taking the form of Eq. (C8). Therefore, the parent Lindbladian is frustration free. ■

The above proof can be generalized to arbitrary N by repeating the argument in point 3. For example, starting from $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)$, if $\mathcal{E}_4^{(I)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) = 0$, then one can similarly show

$$\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)} \circ \mathcal{E}_4^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) \geq 0,$$

justifying either arrow $C \Rightarrow D$ or arrows $C \Rightarrow S \Rightarrow D$ must exist.

2. MPDO from C^* -weak Hopf algebras

In this Appendix, we prove the frustration freeness of parent Lindbladian for nonsimple MPDO RFPs generated by C^* -weak Hopf algebras. We make a particular choice for $\mathcal{E}^{(II)}$ as in Eq. (115), which takes the explicit form of

$$\mathcal{E}^{(II)}(X) := \text{Tr}(P^\perp X) \rho_0,$$

$$\rho_0 := \frac{1}{\omega(\Omega)} b(\omega)^{\otimes 2} (\Phi^{\otimes 2} \circ \Delta)(\Omega), \quad (\text{C10})$$

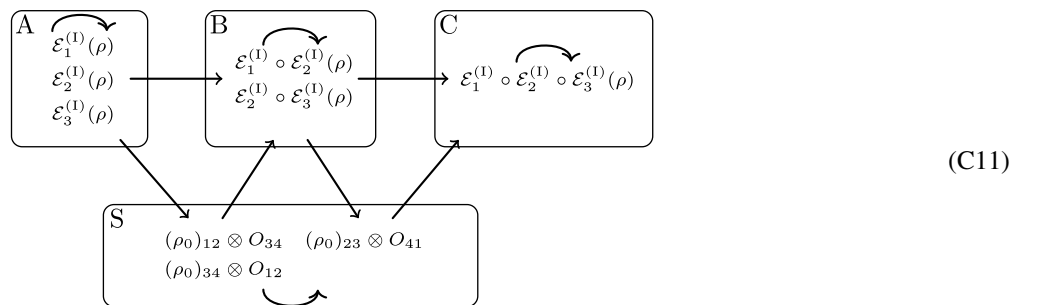
where $P^\perp := \mathbb{1} - (\Phi^{\otimes 2} \circ \Delta)(1)$ and ρ_0 satisfies Eq. (C3).

a. Open boundary condition

Let us first consider system size $N = 4$ and open boundary conditions as an explicit example, where under open boundary condition $\mathcal{E}^{(N)}$ is defined as $\mathcal{E}^{(N)} = \frac{1}{N-1} \sum_{i=1}^{N-1} \mathcal{E}_i$.

Proposition C2. The parent Lindbladian for nonsimple MPDO RFPs generated by C^* -Weak Hopf algebras under the open boundary condition with a system size of $N = 4$ is frustration free.

Proof. We first decompose the fixed-point ρ as in Eq. (C2). Under an action of $\mathcal{E}^{(N)}$, the decomposition terms of ρ flow to each other as



We group all possible terms in five boxes labeled by “A,” “B,” “C,” “S.” The terms in box B come from two neighboring actions of the form $\mathcal{E}_i^{(I)} \circ \mathcal{E}_j^{(I)}$, where “neighboring” means the support of $\mathcal{E}_i^{(I)}$ and $\mathcal{E}_j^{(I)}$ overlap; namely, j can be $i - 1, i + 1$. The terms in box C come from three neighboring actions of $\mathcal{E}^{(I)}$. Finally, the terms in box “S” come from an action of $\mathcal{E}^{(II)}$ on a density matrix and therefore $O_{i,i+1} \geq 0$.

We justify the following facts.

- (1) *Either arrow $A \Rightarrow B$ or arrow $A \Rightarrow S$ must exist.* The argument is the same as for noninjective simple MPDOs.
- (2) *The arrow $S \Rightarrow B$ must exist.* Without loss of generality, let’s start from the term $(\rho_0)_{12} \otimes O_{34}$ and act $\mathcal{E}_2^{(I)}$, and show

$$\mathcal{E}_2^{(I)} \circ ((\rho_0)_{12} \otimes O_{34}) \geq 0. \quad (\text{C12})$$

This expression can be simplified to

$$\mathcal{E}_2^{(I)} \circ ((\rho_0)_{12} \otimes O_{34}) \equiv$$

by virtue of Eq. (97).

Now, let us recall the definition of the tensors M' and Ξ in Eqs. (B21) and (B23). One can also prove that the element $\xi \in A$ can be factorized in the form $\xi = \xi_L \xi_R$ for some positive invertible elements $\xi_L, \xi_R \in A$, which in particular have the properties of $\xi_L x = \hat{\xi}_L(x_{(1)})x_{(2)}$, $x\xi_R = \hat{\xi}_R(x_{(1)})x_{(2)}$; we refer the reader to Lemma 3.5 in Ref. [45] for a proof. Using these properties,

$$\begin{aligned} \Xi \circ M' &= \frac{\Psi(\hat{\xi})^{-1}}{\Phi(\xi)^{1/2}} \circ \frac{\Phi(\xi)^{1/2}}{\Phi(\xi)^{1/2}} = \frac{\Phi(g)^{-\frac{1}{2}}}{\Phi(g)^{-\frac{1}{2}}} \circ \frac{\Phi(g)^{-\frac{1}{2}}}{\Phi(g)^{-\frac{1}{2}}} = \frac{\Phi(\xi_L)^{-1}}{\Phi(\xi_L)^{-1}} \circ \frac{\Phi(\xi_L)^{-1}}{\Phi(\xi_L)^{-1}} \end{aligned} \quad (C13)$$

where $g := \xi_L \xi_R^{-1}$, i.e., the previous expression takes the form

$$\mathcal{E}_2^{(I)} \circ ((\rho_0)_{12} \otimes O_{34}) =$$

$$= ((\mathcal{T} \otimes \text{Id}) \circ \mathcal{T})((\Phi(\xi_L^{-1}) \otimes \text{Id}) O_{34} (\Phi(\xi_L^{-1})^\dagger \otimes \text{Id})),$$

and since $\mathcal{T}$ is completely positive, we conclude that the term in Eq. (C12) is positive. Moreover, it is nonzero since its trace is of the form $\text{Tr}(\Phi(\xi_L)^{-2} \otimes \text{Id}) O_{34}$ by the previous equation and the fact that $\mathcal{T}$ is trace preserving, and ξ_L is positive definite, leading to that $\text{Tr}(\Phi(\xi_L)^{-2} \otimes \text{Id}) O_{34} > 0$. Therefore, the arrow from box S to box B must exist.

Furthermore, using the von Neumann trace inequality, the trace is lower bounded by $\text{Tr}(\Phi(\xi_L)^{-2} \otimes \text{Id}) O_{34} \geq r \text{Tr}(O_{34})$ where $r = \lambda_{\min}(\Phi(\xi_L)^{-2})$ is the minimal eigenvalue of $\Phi(\xi_L)^{-2}$, which is a constant given a C^* -weak Hopf algebra. Since $\mathcal{T}$ is trace preserving, we can put a lower bound for the trace

$$\text{Tr}(\mathcal{E}_2^{(I)} \circ ((\rho_0)_{12} \otimes O_{34})) \geq r \text{Tr}((\rho_0)_{12} \otimes O_{34}),$$

where r is a constant.

(3) *Either arrow $B \Rightarrow C$ or arrows $B \Rightarrow S \Rightarrow C$ must exist.* Without loss of generality, let's start with the term $\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)$ in box C and consider the action of $\mathcal{E}_1$. If $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho) \gneq 0$, then $\frac{1}{N} \mathcal{E}_1$ will bring $\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)$ from box B to box C.

If, however, that $\mathcal{E}_1^{(I)} \circ (\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) = 0$, we then need to consider $\mathcal{E}_1^{(II)} \circ (\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho))$, which is term $(\rho_0)_{12} \otimes O_{34}$ in box B. We need to show by acting another $\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}$,

$$\mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)} \circ (\mathcal{E}_1^{(II)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) \gneq 0,$$

which, by noting that $[\mathcal{E}_3^{(I)}, \mathcal{E}_1^{(II)}] = 0$ because these two linear maps do not overlap and using $[\mathcal{E}_i^{(I)}, \mathcal{E}_j^{(I)}] = 0$, $\mathcal{E}_i^{(I)} \circ \mathcal{E}_i^{(I)} = \mathcal{E}_i^{(I)}$, amounts to

$$\mathcal{E}_2^{(I)} \circ (\mathcal{E}_1^{(II)} \circ \mathcal{E}_2^{(I)} \circ \mathcal{E}_3^{(I)}(\rho)) = \mathcal{E}_2^{(I)} \circ ((\rho_0)_{12} \otimes O_{34}) \gneq 0. \quad (C14)$$

This is true based on the proof in point 3; using Eq. (C4), we see this term is in box C.

When starting from $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)}(\rho)$, we would need to consider the action of $\mathcal{E}_3$ and finish the proof using a similar equation as Eq. (C13) with Ξ on the right of M' . Therefore, either arrow from box B to box C or from box B to box C via box S must exist.

(4) *There are no arrows from box C to box S.* This is a direct result of Eq. (C5).

With the above arrows, it is clear that the trace of terms in box C will keep increasing as long as there exist terms in previous boxes, and subsequently, a steady state can only be supported in box C, taking the form of $\rho = \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho)$. Therefore, the parent Lindbladian is frustration free. ■

The above proof can be generalized to arbitrary N with the open boundary condition by repeating the argument in point 3.

b. Periodic boundary condition

Finally, for the periodic boundary condition, one more step is needed to close the boundary. Let's start from a term $\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho) \gtrsim 0$ in the second to last box and act $\mathcal{E}_N$ on it. If $\mathcal{E}_N^{(I)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho)) \gtrsim 0$, the resulting term is in the final box. If, however, that $\mathcal{E}_N^{(I)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho)) = 0$, we need to consider the term $\mathcal{E}_N^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho))$. By acting $\mathcal{E}_{N-1}^{(I)} \circ \mathcal{E}_1^{(I)}$, one can show

$$\mathcal{E}_{N-1}^{(I)} \circ \mathcal{E}_1^{(I)} \circ (\mathcal{E}_N^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho))) \gtrsim 0, \quad (C15)$$

which is a term in the final box. In the following, we provide the proof.

By assumption, the C^* -weak Hopf algebra is biconnected with a unique vacuum sector e . Denote $\tilde{\rho} = \mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho)$ and the corresponding boundary condition matrix as $B = b(x)$ where $x \in A$ is positive semi-definite. W.l.o.g., we choose $B = b(x)$ to be block diagonal in the same basis as Ψ , i.e., $B = \bigoplus_a B_a$. Recalling Eq. (127) and note that Ω_a is proportional to identity, if $\mathcal{E}_N^{(I)}(\tilde{\rho}) = 0$, then $\text{Tr}(B_a \Xi_a) = \text{Tr}(\Xi_a^{1/2} B_a \Xi_a^{1/2}) = 0$ for any a (for example, in Fibonacci C^* -WHA, choosing $x = e_{2,11}$ would lead to $\text{Tr}(B_a \Xi_a) = 0$ for all a). In this case we consider $\mathcal{E}_N^{(II)}(\tilde{\rho})$, which amounts to partially tracing $\tilde{\rho}$ and replacing the traced degrees of freedom by ρ_0 , diagrammatically.

$$\mathcal{E}_N^{(II)}(\tilde{\rho}) = \frac{1}{\omega(\Omega)} \text{Diagram: A circle with } \Omega \text{ inside, connected to a chain of } M \text{ boxes. The chain starts with } B \text{ and ends with } \Omega \text{, with } N \text{ boxes in between.} \text{Diagram: A circle with } \Omega \text{ inside, connected to a chain of } M \text{ boxes. The chain starts with } B \text{ and ends with } \Omega \text{, with } N \text{ boxes in between.}$$

The partial trace amounts to replacing the boundary condition matrix $B \rightarrow B' = EBE$ where E is the transfer matrix. By construction, the transfer matrix $E = \Psi(\omega)$ is rank-one and is positive semi-definite. Therefore, $B' \propto E \gtrsim 0$ and in particular, $B'_e \propto E_e \gtrsim 0$.

We next act $\mathcal{E}_{N-1}^{(I)} \circ \mathcal{E}_1^{(I)}$. Diagrammatically,

$$\mathcal{E}_{N-1}^{(I)} \circ \mathcal{E}_1^{(I)} \circ (\mathcal{E}_N^{(II)}(\tilde{\rho})) = \text{Diagram: A complex diagram showing the action of } \mathcal{E}_{N-1}^{(I)} \text{ and } \mathcal{E}_1^{(I)} \text{ on } \mathcal{E}_N^{(II)}(\tilde{\rho}). \text{ It involves multiple layers of } M \text{ boxes, } \Omega \text{ circles, and } B' \text{ boxes, with } N-2 \text{ boxes in between.} \text{Diagram: A complex diagram showing the action of } \mathcal{E}_{N-1}^{(I)} \text{ and } \mathcal{E}_1^{(I)} \text{ on } \mathcal{E}_N^{(II)}(\tilde{\rho}). \text{ It involves multiple layers of } M \text{ boxes, } \Omega \text{ circles, and } B' \text{ boxes, with } N-2 \text{ boxes in between.}$$

where we use Lemma 3.5 in Ref. [45] to move the position of Ξ and Eq. (97). The boundary condition matrix is replaced by $B' \rightarrow B'' = \Xi^{1/2} B' \Xi^{1/2}$ where $\Xi^{1/2}$ is positive definite (and each block Ξ_a is positive definite). In particular, $B'' = \Xi_e^{1/2} B'_e \Xi_e^{1/2} \gtrsim 0$. At least for the vacuum sector $\text{Tr}(\Xi_e^{1/2} B'_e \Xi_e^{1/2}) > 0$, and therefore $\mathcal{E}^{(I)}$ acting on this state must be nonzero. With this, we conclude

$$\mathcal{E}_{N-1}^{(I)} \circ \mathcal{E}_1^{(I)} \circ (\mathcal{E}_N^{(II)} \circ (\mathcal{E}_1^{(I)} \circ \mathcal{E}_2^{(I)} \circ \dots \circ \mathcal{E}_{N-1}^{(I)}(\rho))) \gtrsim 0.$$

This finishes the proof.

APPENDIX D: EXAMPLE OF $\mathbb{C}\mathbb{Z}_2$

In this Appendix, we show explicitly the construction in Example 6 of the MPDO tensor for the boundary state of the toric code, using $\mathbb{C}\mathbb{Z}_2$ group C^* -algebra. The MPDO tensor is constructed by Eqs. (B10) and (B20). Hence, we need to identify the basis $\{e_i\}$ of A and the dual basis $\{e^i\}$ of A^* , their faithful $*$ -representations Φ , Ψ , and $b(\omega)$.

Since $A = \mathbb{C}\mathbb{Z}_2$ is a group algebra, its basis elements are identified with the group elements, and we denote the two basis elements as e_1, e_z ; namely, the dimension of A is 2. They obey the multiplication rule of $\mathbb{Z}_2$ group, and their faithful

representation is the direct sum of the two irreps of $\mathbb{Z}_2$ group,

$$\Phi(e_1) = \mathbb{1}_2, \quad \Phi(e_z) = \sigma_z. \quad (\text{D1})$$

For the dual algebra A^* , its multiplication rule is induced by the comultiplication rule in A , $\Delta(e_g) = e_g \otimes e_g$. The multiplication of A^* is thus $e^i \cdot e^j = \sum_k \delta_{ijk} e^k$, and the faithful representation of A^* is

$$\Psi(e^1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Psi(e^z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{D2})$$

Finally, the boundary condition matrix $b(\omega)$ is found using $\omega = e^1$ and by definition $\text{Tr}(b(\omega)\Phi(x)) = \omega(x)$ for any $x \in A$. The result is $b(\omega) = \mathbb{1}_2/2$.

Assemble everything, we obtain the MPDO tensor



$$M = \frac{\mathbb{1}_2}{2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\sigma_z}{2} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{D3})$$

where the first tensor factor corresponds to the endomorphism in the physical space and the second tensor factor to one in the virtual space, as given in Example 6. The parent Lindbladian of this MPDO is given in Example 7.

APPENDIX E: EXAMPLE OF FIBONACCI

In this Appendix, we construct explicitly the MPDO RFP for the Fibonacci weak Hopf algebra A_{Fib} and its parent Lindbladian as a more complicated example. A_{Fib} is not a Hopf algebra because $\Delta(1) \neq 1 \otimes 1$. This necessitates the second piece $\mathcal{E}^{(\text{II})}$ in the parent Lindbladian construction. We will show the parent Lindbladian construction from two different ways (horizontal canonical form and C^* -weak Hopf algebra) and see that they give the same Lindbladian.

1. Fibonacci MPDO construction

To construct the MPDO RFP using Eqs. (B10) and (B20), we need to identify the basis $\{e_i\}$ of A and its dual basis $\{e^i\}$ of A^* , their faithful $*$ -representations Φ and Ψ , and $b(\omega)$. The Fibonacci C^* -weak Hopf algebra is defined as the direct sum $\mathcal{M}_2(\mathbb{C}) \oplus \mathcal{M}_3(\mathbb{C})$ of full matrix C^* -algebras with complex coefficients. The basis elements of A_{Fib} is identified with the matrix units $e_{1,ij}$, $i, j = 1, 2$ in $\mathcal{M}_2(\mathbb{C})$ and $e_{2,ij}$, $i, j = 1, 2, 3$ in $\mathcal{M}_3(\mathbb{C})$. Namely, the dimension of A is 13. The multiplication rule comes from the matrix multiplication, and the faithful representation Φ is simply the 5×5 matrix as a direct sum of the two irreducible $*$ -representations labeled by $a = 1$ and $a = \tau$. For example,

$$\Phi(e_{1,12}) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix}. \quad (\text{E1})$$

To obtain the MPDO tensor, we still need the faithful representation Ψ of A^* . Let's use I to denote the combined index (a, ij) , where $a = 1, 2$. Ψ can be found by noticing that A_{Fib} is self-dual, and therefore there exists a "natural basis" $\{\tilde{e}^I\}$, which is related to the dual basis $\{e^I\}$ by a transformation

$\tilde{e}^I = \sum_J R^{IJ} e^J$, such that the faithful representation in the natural basis reads

$$\Psi(\tilde{e}^I) = \Phi(e_I). \quad (\text{E2})$$

The transformation matrix R , in general, is not unique. It is the solution of a consistency relation between multiplication and comultiplication. Specifically, the multiplication and comultiplication on A can be specified using the basis elements

$$e_I \cdot e_J = \sum_K \lambda_{IJ}^K e_K, \\ \Delta(e_K) = \sum_{IJ} \Lambda_K^{IJ} e_I \otimes e_J.$$

By definition of dual algebra, they induce the comultiplication and multiplication in A^* ,

$$e^I \cdot e^J = \sum_K \Lambda_K^{IJ} e^K, \\ \Delta(e^K) = \sum_{IJ} \lambda_{IJ}^K e^I \otimes e^J.$$

As a natural basis, $\tilde{e}^I \cdot \tilde{e}^J = \sum_K \lambda_{IJ}^K \tilde{e}^K$ and therefore the consistency relation requires

$$\sum_{I'J'} R^{II'} R^{JJ'} \Lambda_S^{I'J'} = \sum_K \lambda_{IJ}^K R^{KS}. \quad (\text{E3})$$

For A_{Fib} , the multiplication rule λ_{IJ}^K comes from matrix multiplication and the comultiplication rule Λ_K^{IJ} is given explicitly in Example 2.5 of Ref. [45,72], and one possible solution for transformation R of Eq. (E3) is [47]

$$\begin{aligned} \tilde{e}^{1,11} &= e^{1,11}, & \tilde{e}^{1,12} &= e^{2,11}, & \tilde{e}^{1,21} &= e^{2,22}, \\ \tilde{e}^{1,22} &= \zeta^4 e^{1,22} + \zeta^2 e^{2,33}, \\ \tilde{e}^{2,11} &= e^{1,12}, & \tilde{e}^{2,12} &= e^{2,12}, & \tilde{e}^{2,13} &= e^{2,13}, \\ \tilde{e}^{2,21} &= \zeta^{-2} e^{2,21}, & \tilde{e}^{2,22} &= e^{1,21}, & \tilde{e}^{2,23} &= \zeta^{-1} e^{2,31}, \\ \tilde{e}^{2,31} &= \zeta^{-1} e^{2,23}, & \tilde{e}^{2,32} &= e^{2,32}, \\ \tilde{e}^{2,33} &= \zeta^2 e^{1,22} - \zeta^2 e^{2,33}, \end{aligned} \quad (\text{E4})$$

where $\zeta^2 = 2/(1 + \sqrt{5})$.

Finally, the boundary condition matrix $b(\omega)$ can be found by using character χ_a of Irrep class $a \in \text{Irr}(A)$ (namely, $\chi_a = \text{Tr} \circ \Phi_a$, which is the dual analog of x_a),

$$\begin{aligned} \chi_1 &= e^{1,11} + e^{1,22}, \\ \chi_\tau &= e^{2,11} + e^{2,22} + e^{2,33}, \\ \omega &= \frac{1}{\mathcal{D}^2} (\delta_1 \chi_1 + \delta_\tau \chi_\tau), \end{aligned}$$

where $\delta_1 = 1$, $\delta_\tau = \zeta^{-2}$ are the quantum dimensions of each Irrep, $\mathcal{D}^2 = \sum_a \delta_a^2 = 1 + \zeta^{-4}$ is the total quantum dimension, and by definition $\text{Tr}(b(\omega)\Phi(x)) = \omega(x)$ for any $x \in A$. The result is

$$b(\omega) = \frac{\delta_1}{\mathcal{D}^2} \mathbb{1}_2 \oplus \frac{\delta_\tau}{\mathcal{D}^2} \mathbb{1}_3.$$

In fact, one can show that in general, $b(\omega) = \oplus_a (\delta_a / \mathcal{D}^2) \mathbb{1}_a$.

Assemble everything, we obtain the MPDO tensor

$$\begin{aligned} \alpha \text{---} \textcircled{M} \text{---} \beta &= \sum_{I=1}^{13} [b(\omega)]_{ii'} \Phi_{i'j}(e_I) \Psi_{\alpha\beta}(e^I) \\ &= \sum_{IJ} (R^{-1})^{IJ} [b(\omega)]_{ii'} \Phi_{i'j}(e_I) \Phi_{\alpha\beta}(e_J). \end{aligned} \quad (\text{E5})$$

Using the representation Φ and transformation R in Eq. (E4), the part without $b(\omega)$ is

$$\begin{aligned} \begin{array}{c} 1 \\ | \\ 1 \text{---} \bullet \text{---} 1 \\ | \\ 1 \end{array} &= \begin{array}{c} 3 \\ | \\ 1 \text{---} \bullet \text{---} 2 \\ | \\ 3 \end{array} = \begin{array}{c} 4 \\ | \\ 2 \text{---} \bullet \text{---} 1 \\ | \\ 4 \end{array} = \begin{array}{c} 2 \\ | \\ 2 \text{---} \bullet \text{---} 2 \\ | \\ 2 \end{array} \\ &= \begin{array}{c} 5 \\ | \\ 2 \text{---} \bullet \text{---} 2 \\ | \\ 5 \end{array} = \begin{array}{c} 2 \\ | \\ 3 \text{---} \bullet \text{---} 3 \\ | \\ 1 \end{array} = \begin{array}{c} 4 \\ | \\ 3 \text{---} \bullet \text{---} 4 \\ | \\ 3 \end{array} = \begin{array}{c} 5 \\ | \\ 3 \text{---} \bullet \text{---} 5 \\ | \\ 3 \end{array} \\ &= \begin{array}{c} 1 \\ | \\ 4 \text{---} \bullet \text{---} 4 \\ | \\ 2 \end{array} = \begin{array}{c} 4 \\ | \\ 5 \text{---} \bullet \text{---} 4 \\ | \\ 5 \end{array} = \begin{array}{c} 2 \\ | \\ 5 \text{---} \bullet \text{---} 5 \\ | \\ 2 \end{array} = 1, \\ &\begin{array}{c} 3 \\ | \\ 4 \text{---} \bullet \text{---} 5 \\ | \\ 5 \end{array} = \begin{array}{c} 5 \\ | \\ 5 \text{---} \bullet \text{---} 3 \\ | \\ 4 \end{array} = \zeta, \\ &\begin{array}{c} 3 \\ | \\ 4 \text{---} \bullet \text{---} 3 \\ | \\ 4 \end{array} = - \begin{array}{c} 5 \\ | \\ 5 \text{---} \bullet \text{---} 5 \\ | \\ 5 \end{array} = \zeta^2, \end{aligned} \quad (\text{E6})$$

which agrees with the explicit components given in Ref. [45].

Using the explicit expression, one can also compute the transfer matrix

$$E = \text{---} \textcircled{M} \text{---} = \begin{pmatrix} \frac{2}{5+\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{10}(5+\sqrt{5}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{E7})$$

and verifies that $E^2 = E$ and $E = \Psi(\omega)$. The eigenvalues of E are 1, 0, 0, 0, 0, which indicates that $\rho(M)$ is a short-range correlated state, similar to the boundary state of the toric code.

2. Parent Lindbladian construction: vertical canonical form

To construct the parent Lindbladian, we need the renormalization channels $\mathcal{T}$ and $\mathcal{S}$. In this section, we construct these two channels from the vertical canonical form Eqs. (41) and (42) and Theorem 3.

The one-site tensor $M_{(\alpha\beta)}$, taking the form of a direct sum, is already in the vertical canonical form. We write explicitly

$$M_{(\alpha\beta)} = \mu_1 M_{(\alpha\beta),1} \oplus \mu_2 M_{(\alpha\beta),2}, \quad (\text{E8})$$

where $M_{(\alpha\beta),1}$ is the upper-left 2×2 block of $M_{(\alpha\beta)}$ and $M_{(\alpha\beta),2}$ is the lower-right 3×3 block of $M_{(\alpha\beta)}$ with proper

normalization constants. Namely, there are two BNT elements. The corresponding multiplicity matrices are constants $\mu_1 = \frac{1}{\sqrt{5}}$ and $\mu_2 = \frac{5+\sqrt{5}}{10}$, which takes care of the normalization [viewing $M_{(\alpha\beta),\lambda}$ as matrix-product vector (MPV)].

To obtain the vertical canonical form for two-site MPDO $K_{(\alpha\beta)}$, we need a basis rotation U on the physical indices. The result is

$$\begin{aligned} U^\dagger K_{(\alpha\beta)} U &= \alpha \text{---} \textcircled{M} \text{---} \textcircled{M} \text{---} \beta \\ &= \left[\bigoplus_{\kappa=1}^2 \nu_\kappa \otimes M_{(\alpha\beta),\kappa} \right] \oplus \mathbf{0}_{12}, \\ \text{where } \nu_1 &= \begin{pmatrix} \frac{\sqrt{5}+1}{10} & 0 \\ 0 & \frac{\sqrt{5}-1}{10} \end{pmatrix}, \\ \nu_2 &= \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{\sqrt{5}+1}{10} \end{pmatrix}, \end{aligned} \quad (\text{E9})$$

and $\mathbf{0}_{12}$ denotes a 12×12 zero matrix. The explicit form of U will be specified at the end of the section. Indeed, the vertical BNT of K is still $\{M_{(\alpha\beta),\lambda}\}$, as stated in Theorem 3. The trace relation is also satisfied: $m_1 = \mu_1 = \text{Tr}(\nu_1) = \frac{1}{\sqrt{5}}$ and $m_2 = \mu_2 = \text{Tr}(\nu_2) = \frac{5+\sqrt{5}}{10}$.

Using the vertical canonical form of M and K , one can construct the channel $\mathcal{T}$ and $\mathcal{S}$ as follows. The one-site MPDO to two-site MPDO channel $\mathcal{T}$ is

$$\mathcal{T}(X) = U \left[\left(\frac{\nu_1}{m_1} \otimes P_1 X P_1^\dagger \right) \oplus \left(\frac{\nu_2}{m_2} \otimes P_2 X P_2^\dagger \right) \oplus \mathbf{0}_{12} \right] U^\dagger, \quad (\text{E10})$$

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are the isometries projecting to the first and second BNT, respectively. By direct inspection, one can see $\mathcal{T}(M_{(\alpha\beta)}) = K_{(\alpha\beta)}$.

The two-site MPDO to one-site MPDO channel $\mathcal{S}$ is

$$\mathcal{S} = \mathcal{S}^{(\text{I})} + \mathcal{S}^{(\text{II})},$$

$$\mathcal{S}^{(\text{I})}(X) = \text{Tr}_1(P'_1 U^\dagger X U (P'_1)^\dagger) \oplus \text{Tr}_1(P'_2 U^\dagger X U (P'_2)^\dagger),$$

$$\mathcal{S}^{(\text{II})}(X) = \text{Tr}(P'_0 U^\dagger X U (P'_0)^\dagger) \rho_0, \quad (\text{E11})$$

where we use $\mu_\lambda = m_\lambda$ in this example. The isometries P'_1, P'_2, P'_0 project to the blocks corresponding to the first BNT, the second BNT, and the zero matrix, respectively. Their corresponding dimensions are $4 \times 25, 9 \times 25$ and 12×25 . Tr_1 denotes tracing over the multiplicity degrees of freedom. The term $\mathcal{S}^{(\text{II})}$ is added to make the map trace preserving, and ρ_0 can be any density matrix supported on one site. By direct inspection one can see $\mathcal{S}(K_{(\alpha\beta)}) = M_{(\alpha\beta)}$.

Combining $\mathcal{T}$ and $\mathcal{S}$, the local quantum channel $\mathcal{E}$ is

$$\begin{aligned}\mathcal{E} &= \mathcal{E}^{(I)} + \mathcal{E}^{(II)}, \\ \mathcal{E}^{(I)}(X) &= U \left[\frac{v_1}{m_1} \otimes \text{Tr}_1(P'_1 U^\dagger X U (P'_1)^\dagger) \right. \\ &\quad \left. \oplus \frac{v_2}{m_2} \otimes \text{Tr}_1(P'_2 U^\dagger X U (P'_2)^\dagger) \oplus \mathbf{0}_{12} \right] U^\dagger, \\ \mathcal{E}^{(II)}(X) &= U \left[\frac{v_1}{m_1} \otimes (P_1 \rho_0 P_1^\dagger) \text{Tr}(P'_0 U^\dagger X U (P'_0)^\dagger) \right. \\ &\quad \left. \oplus \frac{v_2}{m_2} \otimes (P_2 \rho_0 P_2^\dagger) \text{Tr}(P'_0 U^\dagger X U (P'_0)^\dagger) \oplus \mathbf{0}_{12} \right] U^\dagger.\end{aligned}\quad (\text{E12})$$

By direct inspection, one can see the fixed-point space of channel $\mathcal{E}$ is the fixed-point space of $\mathcal{E}^{(I)}$,

$$\mathcal{F}_{\mathcal{E}} = U \left[\left(\bigoplus_{\lambda=1}^2 v_\lambda \otimes \mathcal{M}_{d_\lambda} \right) \oplus \mathbf{0}_{12} \right] U^\dagger. \quad (\text{E13})$$

where $\mathcal{M}_d$ is the full $d \times d$ dimensional matrix algebra and $d_{\lambda=1} = 2, d_{\lambda=2} = 3$. The dimension of the fixed point space is therefore $d_{\lambda=1}^2 + d_{\lambda=2}^2 = 13$. This finishes the explicit construction of parent Lindbladian. We note again that the construction is based on the fact that at RFP, M and K share the same vertical BNT; the coarse-graining and fine-graining channels only act on the multiplicity degrees of freedom, without changing the BNT part.

Although this construction is intuitive, proving frustration freeness and finding global steady states is difficult. In the next section, we present another construction using C^* -weak Hopf algebra, which is more convenient for finding the global steady states. Numerically, we verify that these two constructions give the same CP maps $\mathcal{E}^{(I)}$.

Finally, we provide the explicit form of the transformation matrix U . The unitary matrix U is the concatenation the following 25 column vectors (where $|v_i\rangle_j$ is the j -th component of the vector $|v_i\rangle$),

$$\begin{aligned}|v_1\rangle_{14} &= 1, & |v_2\rangle_{18} &= \frac{\zeta}{\sqrt{\zeta^2 + 1}}, & |v_2\rangle_{25} &= \frac{1}{\sqrt{\zeta^2 + 1}}, \\ |v_3\rangle_1 &= 1, & |v_4\rangle_7 &= 1, & |v_5\rangle_3 &= 1, & |v_6\rangle_9 &= 1, \\ |v_7\rangle_{10} &= 1, & |v_8\rangle_{12} &= 1, & |v_9\rangle_{16} &= 1, \\ |v_{10}\rangle_{22} &= 1, & |v_{11}\rangle_{15} &= 1, & |v_{12}\rangle_{24} &= 1, \\ |v_{13}\rangle_{18} &= \frac{1}{\sqrt{\zeta^2 + 1}}, & |v_{13}\rangle_{25} &= \frac{-\zeta}{\sqrt{\zeta^2 + 1}},\end{aligned}\quad (\text{E14})$$

with all other components equal to zero, and $|v_{14}\rangle$ to $|v_{25}\rangle$ being arbitrary basis vectors in the orthogonal space of $|v_1\rangle$ to $|v_{13}\rangle$,

3. Parent Lindbladian construction: C^* -weak Hopf algebras

In terms of C^* -weak Hopf algebras, the quantum channels $\mathcal{T}$ and $\mathcal{S}$ are given in Eqs. (B24)–(B26), from which we can

write down $\mathcal{E} = \mathcal{T} \circ \mathcal{S}$. The first piece is

$$\begin{aligned}\mathcal{E}^{(I)}(X) &= \mathcal{T} \circ \mathcal{S}^{(I)}(X) \\ &= b(\omega)^{\otimes 2} (\Phi^{\otimes 2} \circ \Delta)(\Omega_{(1)}) \\ &\quad \cdot \text{Tr}((\Phi^{\otimes 2} \circ \Delta \circ J)(\xi^{\frac{1}{2}} \Omega_{(2)} \xi^{\frac{1}{2}}) X),\end{aligned}\quad (\text{E15})$$

where we use Eq. (B19), and the second piece is

$$\begin{aligned}\mathcal{E}^{(II)}(X) &= \mathcal{T} \circ \mathcal{S}^{(II)}(X) \\ &= b(\omega)^{\otimes 2} (\Phi^{\otimes 2} \circ \Delta)(\Omega_{(1)}) \\ &\quad \cdot \text{Tr}((\Phi \circ J)(\xi^{\frac{1}{2}} \Omega_{(2)} \xi^{\frac{1}{2}}) \rho_0) \text{Tr}(P^\perp X),\end{aligned}\quad (\text{E16})$$

where $P^\perp = \mathbb{1} - (\Phi^{\otimes 2} \circ \Delta)(1)$ and ρ_0 is an arbitrary density matrix.

To explicitly construct $\mathcal{E}$, we need the explicit form of $1, \Omega, \xi$ and the map J . The comultiplication Δ is given explicitly in Ref. [45]. The explicit form of 1 comes from its definition as the unit element in A ,

$$1 = e_{1,11} + e_{1,22} + e_{2,11} + e_{2,22} + e_{2,33}, \quad (\text{E17})$$

and the explicit form of Ω is

$$\begin{aligned}\Omega &= \frac{1}{\mathcal{D}^2} (d_1 x_1 + d_\tau x_\tau) \\ &= \frac{1}{\mathcal{D}^2} \left(e_{1,11} + \frac{1}{\zeta^2} e_{1,12} + \frac{1}{\zeta^2} e_{1,21} + \frac{1}{\zeta^4} e_{1,22} \right),\end{aligned}\quad (\text{E18})$$

where $d_1 = 1, d_\tau = \zeta^{-2}, \mathcal{D}^2 = 1 + \zeta^{-4}$. The form of ξ can be determined using $\xi^{-1} = \omega(\Omega_{(1)})\Omega_{(2)}$, and the result reads

$$\xi = \mathcal{D}^4 (e_{1,11} + \zeta^4 e_{1,22} + \zeta^2 e_{2,11} + \zeta^2 e_{2,22} + \zeta^4 e_{2,33}). \quad (\text{E19})$$

Finally, the linear map J can be obtained from the pulling-through identity Eq. (B18), which after applying J to the first site, becomes $(J(x)\xi^{\frac{1}{2}} \otimes 1) \cdot \Delta(\Omega) = (\xi^{\frac{1}{2}} \otimes x) \cdot \Delta(\Omega)$. The result is

$$\begin{aligned}J(e_{1,11}) &= e_{1,11}, & J(e_{1,12}) &= e_{1,21}, & J(e_{1,21}) &= e_{1,12}, \\ J(e_{1,22}) &= e_{1,22}, & J(e_{2,11}) &= e_{2,22}, & J(e_{2,12}) &= e_{2,12}, \\ J(e_{2,13}) &= e_{2,32}, & J(e_{2,21}) &= e_{2,21}, & J(e_{2,22}) &= e_{2,11}, \\ J(e_{2,23}) &= e_{2,31}, & J(e_{2,31}) &= e_{2,23}, & J(e_{2,32}) &= e_{2,13}, \\ J(e_{2,33}) &= e_{2,33}.\end{aligned}\quad (\text{E20})$$

Using the above data, one can explicitly construct the quantum channel $\mathcal{E}$.

By direct inspection, one can see the fixed-point space of channel $\mathcal{E}$ is the fixed-point space of $\mathcal{E}^{(I)}$, which is the span

$$\mathcal{F}_{\mathcal{E}} = \left\{ \sum_{l=1}^{13} c_l b(\omega)^{\otimes 2} (\Phi^{\otimes 2} \circ \Delta)(e_l) \mid c_l \in \mathbb{C} \right\}. \quad (\text{E21})$$

The dimension of the fixed-point space is 13.

4. A hint of duality

Using two explicit parent Lindbladian constructions, we numerically find that $\mathcal{E}^{(I)}$ from Eqs. (E12) and (E15) are identical [68]. This result is intriguing because the first construction is derived from the vertical canonical form of the

MPDO, while the second stems from the algebraic structure, which implicitly relies on the horizontal canonical form. This observation suggests a potential duality between the vertical and horizontal canonical forms. One might speculate whether this duality arises from the self-duality of the Fibonacci C^* -weak Hopf algebra. To explore this further, we examine a nonself-dual case, the MPDO RFP constructed from the $\mathbb{CS}_3$ Hopf algebra. Interestingly, we find that $\mathcal{E}$ from the two constructions remains identical. A deeper understanding of this apparent duality is left for future work.

APPENDIX F: EXAMPLE OF ρ_{CZX}

In this Appendix, we provide details of the vertical canonical form analysis of ρ_{CZX} and study the properties of its parent Lindbladian. This is an example of the nonsimple MPDO beyond the C^* -weak Hopf algebra [58].

1. Vertical canonical form

We now show explicitly that this MPDO is an RFP after blocking two sites by using the vertical canonical form. Define M as the tensor obtained by blocking two sites of $\tilde{M}$. By a unitary transformation U on the physical degrees of freedom,

$$M_{(\alpha\beta)} \rightarrow M_{(\alpha\beta)}^U = U M_{(\alpha\beta)} U^\dagger$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (F1)$$

one can bring M to the vertical canonical form with two BNT elements,

$$M_{(\alpha\beta)}^U = \mu_1 M_{(\alpha\beta),1} \oplus \mu_2 M_{(\alpha\beta),2}, \quad (F2)$$

with $\mu_1 = \mu_2 = \frac{\sqrt{3}}{4}$,

$$M_{(11),1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{(22),1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$M_{(23),1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_{(32),1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$M_{(33),1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (F3)$$

and

$$M_{(11),2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{(22),2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$M_{(23),2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad M_{(32),2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$M_{(33),2} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (F4)$$

One can verify that these two BNT elements are indeed independent by computing the leading eigenvalue of the mixed transfer matrix.

Next, consider K as the tensor obtained by blocking two sites of M (which is, four sites of $\tilde{M}$). By a unitary

transformation V on the physical degrees of freedom,

$$K_{(\alpha\beta)} \rightarrow K_{(\alpha\beta)}^V = V K_{(\alpha\beta)} V^\dagger, \quad (F5)$$

one can bring K to the vertical canonical form with the same two BNT elements,

$$K_{(\alpha\beta)}^V = v_1 M_{(\alpha\beta),1} \oplus v_2 M_{(\alpha\beta),2}, \quad (F6)$$

with $v_1 = v_2 = \frac{\sqrt{3}}{16} \mathbb{1}_4$. The coefficients verify $m_\lambda = \mu_\lambda = \text{Tr}[v_\lambda] = \frac{\sqrt{3}}{4}$. The explicit form of the unitary V is

$$V_{1,1} = V_{2,16} = V_{3,3} = V_{4,14} = 1,$$

$$V_{5,5} = V_{6,12} = V_{7,7} = -V_{8,10} = 1,$$

$$V_{9,2} = V_{10,15} = V_{11,4} = -V_{12,13} = 1,$$

$$V_{13,6} = V_{14,11} = V_{15,8} = V_{16,9} = 1. \quad (F7)$$

We would like to comment that the one-site MPDO tensor $\tilde{M}$ contains only one BNT element, and therefore $\tilde{M}$ itself is not an RFP tensor. On the other hand, after blocking two sites, we see that M is an RFP tensor by showing that M and K share the same set of BNT. This is expected, since we require N to be even, and the above MPDO representation is valid only for even N .

2. Steady states of parent Lindbladian

Using the vertical canonical form, the renormalization channels $\mathcal{T}$, $\mathcal{S}$ are constructed as in Example 9 and with them one constructs the parent Lindbladian. In this section, we analyze the properties of this parent Lindbladian.

Following the general treatment, we build the local channel $\mathcal{E}$ (that acts on four original sites) and the Lindbladian

$$\mathcal{E} = \mathcal{T} \circ \mathcal{S}, \quad \mathcal{E}_i = \tau_i(\mathcal{E}),$$

$$\mathcal{L}_i = \mathcal{E}_i - \mathbb{1}, \quad \mathcal{L}^{(N)} = \sum_{i=1}^{N/2} \mathcal{L}_{2i-1}. \quad (F8)$$

Let us first examine the fixed-point space of the local channel $\mathcal{E}$,

$$\mathcal{F}_{\mathcal{E}} = V^\dagger \left[\bigoplus_{\lambda=1}^2 v_\lambda \otimes \mathcal{M}_{d_\lambda} \right] V, \quad (F9)$$

where $\mathcal{M}_{d_\lambda}$ is the full $d_\lambda \times d_\lambda$ dimensional matrix algebra, and $d_{\lambda=1} = d_{\lambda=2} = 2$. The dimension of the fixed point space is therefore $d_{\lambda=1}^2 + d_{\lambda=2}^2 = 8$. Explicitly,

$$\mathcal{F}_{\mathcal{E}} = \mathbb{V} \oplus \mathbb{W},$$

$$\mathbb{V} = \text{span}\{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z,$$

$$\sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z\},$$

$$\mathbb{W} = \text{span}\{\sigma_x^{\otimes 4} CZ^{(4)}(|m\rangle\langle m| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2$$

$$\otimes |n\rangle\langle n|)|m, n = 0, 1\}, \quad (F10)$$

where $CZ^{(4)} = \prod_{i=1}^4 (CZ)_{i,i+1}$ and note that $\mathbb{V} \cap \mathbb{W} = \{0\}$. In particular, the vector space $\mathbb{W}$ allows a simple MPO

representation same as U_{CZX} ,

$$\begin{aligned} W &= G_{L=4}(O) \\ O^{12} &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\ O^{21} &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \end{aligned} \quad (\text{F11})$$

where $G_L(O)$ is defined as in Eq. (52). As a side remark, we note that the vector space $G_{L=4}(\tilde{M})$ has dimension 5 and is a subspace of the fixed-point space,

$$G_{L=4}(\tilde{M}) \subseteq \mathcal{F}_{\mathcal{E}}. \quad (\text{F12})$$

To find the space of steady state of the global Lindbladian, we examine the commutation relation of local channels. We verify numerically that

$$[\mathcal{E}_i, \mathcal{E}_{i+2}] = 0. \quad (\text{F13})$$

Therefore, the parent Lindbladian is commuting and thus frustration free. The global space of steady state is then the intersection of the fixed-point spaces $\cap_{i=1}^N \mathcal{F}_{\mathcal{E}_i}$. Let's first consider

$$\mathcal{F}_{\mathcal{E}_i} \cap \mathcal{F}_{\mathcal{E}_{i+2}} = (V_i \oplus W_i) \cap (V_{i+2} \oplus W_{i+2}), \quad (\text{F14})$$

where $V_i = V \otimes \mathcal{M}_2 \otimes \mathcal{M}_2$, $V_{i+2} = \mathcal{M}_2 \otimes \mathcal{M}_2 \otimes V$, and similar for W_i, W_{i+2} . To simplify, we note that the following two intersections are empty: $V_i \cap W_{i+2} = \{0\}$ and $W_i \cap V_{i+2} = \{0\}$, which can be proven by considering a nonzero vector in this intersection and then tracing out site $i+2$, leading to a contradiction. Along with the conditions $V_i \cap W_i = \{0\}$ and $V_{i+2} \cap W_{i+2} = \{0\}$, the above expression can be simplified to

$$\mathcal{F}_{\mathcal{E}_i} \cap \mathcal{F}_{\mathcal{E}_{i+2}} = (V_i \cap V_{i+2}) \oplus (W_i \cap W_{i+2}). \quad (\text{F15})$$

The first term can be computed by considering a nonzero vector in the intersection $v_1 = v_2$, $v_1 \in V_i$, $v_2 \in V_{i+2}$, multiplying σ_z on site $i+2$, and then tracing out site $i+2$. Since v_1 can only take $\mathbb{1}_2$ on site $i+2$, this procedure leads to $\text{Tr}_{i+2}[v_2(\sigma_z)_{i+2}] = 0$, enforcing v_2 to take $\mathbb{1}_2$ on site $i+2$. One can apply similar procedures on other sites to obtain other constraints, and the final result is

$$\begin{aligned} V_{i,i+2} &:= V_i \cap V_{i+2} \\ &= \text{span}\{\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \\ &\quad \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z, \\ &\quad \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \\ &\quad \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z\}. \end{aligned} \quad (\text{F16})$$

Next, we consider the second term $W_i \cap W_{i+2}$. We note that by blocking two sites of MPO tensor $O_{\alpha\beta}^{ij}$, it becomes injective when viewed as an MPV (by grouping i, j indices). One can then apply the intersection property of MPV [1] and obtain

$$\begin{aligned} W_{i,i+2} &:= W_i \cap W_{i+2} \\ &= (G_{L=4}(O) \otimes \mathcal{M}_4) \cap (\mathcal{M}_4 \otimes G_{L=4}(O)) \\ &= G_{L=6}(O). \end{aligned} \quad (\text{F17})$$

The above procedure can be performed iteratively until we reach the system size N , and for W , using the closure property

when crossing the boundary. We conclude

$$\rho_{\infty}^{(N)} = \frac{1}{2^N} (\mathbb{1}_2^{\otimes N} + c U_{\text{CZX}}^{(N)}), \quad c \in [-1, 1], \quad (\text{F18})$$

where N is even.

We note that the above-constructed parent Lindbladian is rapid mixing, which is a direct result of commuting local terms. Nevertheless, it cannot drive a product state fast to ρ_{CZX} , as shown in the following evolution analysis.

a. Evolution

Starting from a trivial state $\rho = \frac{1}{2^N} (\mathbb{1}^{\otimes N} + \sigma_x^{\otimes N})$, we numerically verify that: when $N = 4, 6, 8$, the resulting steady state is

$$\rho_{\infty}^{(N)} = \frac{1}{2^N} \left(\mathbb{1}_2^{\otimes N} + \frac{1}{2^{N/2-1}} U_{\text{CZX}}^{(N)} \right). \quad (\text{F19})$$

We conjecture that this relation holds for general N . In the following, we prove that the parent Lindbladian constructed above cannot drive a fully separable state to ρ_{CZX} .

Proposition F1. Consider any initial state as a fully separable state

$$\rho_{\text{init}}^{(N)} = \sum_s q_s (\rho_{1,s} \otimes \rho_{2,s} \otimes \cdots \otimes \rho_{N,s}), \quad (\text{F20})$$

where N is an even integer and $\sum_s q_s = 1$. The corresponding steady state $\rho_{\infty}^{(N)}$ under the evolution of CZX parent Lindbladian takes the form of Eq. (F18), with

$$|c| \leq \frac{1}{2^{N/2-1}}. \quad (\text{F21})$$

Proof. To begin, note that the coefficient c in Eq. (F18) can be computed via

$$\begin{aligned} c &= \text{Tr}[U_{\text{CZX}}^{(N)} \rho_{\infty}^{(N)}] = \text{Tr}[U_{\text{CZX}}^{(N)} \mathcal{E}_{\infty}(\rho_{\text{init}}^{(N)})] \\ &= \text{Tr}[\mathcal{E}_{\infty}^*(U_{\text{CZX}}^{(N)}) \rho_{\text{init}}^{(N)}], \end{aligned} \quad (\text{F22})$$

where $\mathcal{E}_{\infty} := \lim_{t \rightarrow \infty} e^{\mathcal{L}t}$, and $\mathcal{E}_{\infty}^*$ is its adjoint. Since the CZX parent Lindbladian $\mathcal{L}$ is a commuting Lindbladian, $\mathcal{E}_{\infty} = \prod_i \mathcal{E}_{i,\infty}$ where $\mathcal{E}_{i,\infty} := \lim_{t \rightarrow \infty} e^{\mathcal{L}_i t}$. Similarly, $\mathcal{E}_{\infty}^*$ can also be decomposed as $\mathcal{E}_{\infty}^* = \prod_i \mathcal{E}_{i,\infty}^*$ where $\mathcal{E}_{i,\infty}^*$ is the adjoint of $\mathcal{E}_{i,\infty}$.

The local infinite time evolution channel is a projector on the space of local fixed point (on four physical sites),

$$\mathcal{E}_{i,\infty}(X) = \sum_{m=1}^8 \text{Tr}[O_m^\dagger X] O_m, \quad (\text{F23})$$

where $\text{Tr}[O_m^\dagger O_n] = \delta_{mn}$, and $\{O_m\}$ are the basis operators of local fixed point space in Eq. (F10) up to normalization,

$$\begin{aligned} O_1 &= \frac{1}{4} \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, & O_2 &= \frac{1}{4} \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z, \\ O_3 &= \frac{1}{4} \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, & O_4 &= \frac{1}{4} \sigma_z \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_z, \\ O_5 &= \sigma_x^{\otimes 4} \text{CZ}^{(4)}(|0\rangle\langle 0| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |0\rangle\langle 0|), \\ O_6 &= \sigma_x^{\otimes 4} \text{CZ}^{(4)}(|0\rangle\langle 0| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |1\rangle\langle 1|), \\ O_7 &= \sigma_x^{\otimes 4} \text{CZ}^{(4)}(|1\rangle\langle 1| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |0\rangle\langle 0|), \\ O_8 &= \sigma_x^{\otimes 4} \text{CZ}^{(4)}(|1\rangle\langle 1| \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes |1\rangle\langle 1|). \end{aligned} \quad (\text{F24})$$

Subsequently, its adjoint takes the form of

$$\mathcal{E}_{i,\infty}^*(X) = \sum_{m=1}^8 \text{Tr}[O_m X] O_m^\dagger. \quad (\text{F25})$$

Note that $O_m = O_m^\dagger$ for $m = 1, \dots, 4$ and $O_5^\dagger = O_8$, $O_6^\dagger = O_7$, the explicit form of $\mathcal{E}_{i,\infty}^*$ is therefore the same as $\mathcal{E}_{i,\infty}$, and thus U_{CZX} stays invariant under the global adjoint map,

$$\mathcal{E}_\infty^*(U_{\text{CZX}}) := \left(\prod_{i=1}^{N/2} \mathcal{E}_{2i-1,\infty}^* \right) U_{\text{CZX}} = U_{\text{CZX}}. \quad (\text{F26})$$

Now, consider a pure single-site initial state with no summation over s , $\rho_i = |v_i\rangle\langle v_i|$ where $|v_i\rangle = a_i|0\rangle + b_i|1\rangle$, $|a_i|^2 + |b_i|^2 = 1$. The MPU representation of U_{CZX} reads

$$U^{01} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad U^{10} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad (\text{F27})$$

and we define a transfer matrix

$$E_{U,i} := \text{Diagram of a node } U \text{ with a self-loop } \rho_i = \begin{pmatrix} a_i^* b_i & a_i^* b_i \\ a_i b_i^* & -a_i b_i^* \end{pmatrix}. \quad (\text{F28})$$

Then,

$$c = \text{Tr}[E_{U,1}E_{U,2}\cdots E_{U,N}], \quad (\text{F29})$$

and its absolute value can be bound by

$$\begin{aligned} |c| &\leq \|E_{U,1}\|_1 \|E_{U,2} \cdots E_{U,N}\|_\infty \\ &\leq \|E_{U,1}\|_1 \prod_{i=2}^N \|E_{U,i}\|_\infty. \end{aligned} \quad (\text{F30})$$

Using the explicit form of $E_{U,i}$, one can show $\|E_{U,i}\|_\infty = \sqrt{2}|a_i||b_i| \leq 1/\sqrt{2}$, where the inequality comes from $|a_i|^2 + |b_i|^2 = 1$; and similarly $\|E_{U,1}\|_1 = 2\sqrt{2}|a_1||b_1| \leq \sqrt{2}$. This leads to

$$|c| \leq \frac{1}{2^{N/2-1}}. \quad (\text{F31})$$

Finally, we justify that the extreme value of c must occur when there is no summation over s and each ρ_i in ρ_{init} is a pure state. To see this, consider

$$\begin{aligned}
c + 1 &= \text{Tr}[(\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)})\rho_{\text{init}}^{(N)}] \\
&= \text{Tr}[\mathcal{E}_{\infty}^*(\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)})\rho_{\text{init}}^{(N)}] \\
&= \sum_s q_s \sum_{\{m_{1s}\}} p_{m_{1s}} \cdots p_{m_{N_s}} \text{Tr}[(\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)}) \\
&\quad \cdot (|\psi_{m_{1s}}\rangle\langle\psi_{m_{1s}}| \otimes \cdots \otimes |\psi_{m_{N_s}}\rangle\langle\psi_{m_{N_s}}|)], \quad (\text{F32})
\end{aligned}$$

where we decompose each

$$\rho_{i,s} = \sum_{m_{is}} p_{m_{is}} |\psi_{m_{is}}\rangle\langle\psi_{m_{is}}|,$$

with $p_{m_{is}} \geq 0$, $\sum_{m_{is}} p_{m_{is}} = 1$. Since $(\mathbb{1}_2^{\otimes N} + U_{\text{CZX}}^{(N)})$ is positive semi-definite, each trace term in the last line of Eq. (F32) must

be non-negative. Therefore,

$$c \leq \max_{\{\psi_{m_{1s}}\}} \langle \psi_{m_{1s}} | \cdots \langle \psi_{m_{N_s}} | U_{\text{CZX}}^{(N)} | \psi_{m_{1s}} \rangle \cdots | \psi_{m_{N_s}} \rangle, \quad (\text{F33})$$

and

$$c \geq \min_{\{|\psi_{m_{is}}\rangle\}} \langle \psi_{m_{1s}} | \cdots \langle \psi_{m_{Ns}} | U_{\text{CZX}}^{(N)} | \psi_{m_{1s}} \rangle \cdots | \psi_{m_{Ns}} \rangle. \quad (\text{F34})$$

This finishes the proof. $\square$

The above proposition and proof can be generalized to the initial state after blocking a finite number of l sites.

Proposition F2. Consider any initial state as a fully separable state

$$\rho_{\text{init}}^{(N)} = \sum_s q_s (\rho_{1,s} \otimes \rho_{2,s} \otimes \cdots \otimes \rho_{N/l,s}), \quad (\text{F35})$$

where each $\rho_{i,s}$ supports on l consecutive sites of qubits, N is an even integer, N/l is an integer and $\sum_s q_s = 1$. The corresponding steady state $\rho_\infty^{(N)}$ under the evolution of CZX parent Lindbladian takes the form of Eq. (F18), with

$$|c| \leq \frac{1}{2^{N/2l-1}}. \quad (\text{F36})$$

Proof. Most of the proof runs parallel to the previous one. In particular, we now consider the transfer matrix of a block of l consecutive sites,

$$E_{U^{(l)}} := \begin{array}{c} \text{---} \rho \text{---} \\ \text{---} U \text{---} U \text{---} \dots \text{---} U \text{---} \\ \text{---} \end{array} \quad (\text{F37})$$

where we omit the position label i , and we would like to show that $\|E_{U^{(i)}}\|_\infty \leq 1/\sqrt{2}$ and $\|E_{U^{(i)}}\|_1 \leq \sqrt{2}$.

To start, denote $U^{(l)}$ as the horizontal blocking of l sites of U , and apply a Hadamard gate H on its right, $U^{(l)} \rightarrow U^{(l)}H$. Note that $\|U^{(l)}H\|_p = \|U^{(l)}\|_p$, the Schatten norm is unchanged. The explicit form of $U^{(l)}$ (after applying H) reads

$$\begin{aligned}
U^{(l)} = & \sqrt{2}|0\rangle\langle 0| \otimes \sigma_1^+ \otimes O^{(1)} \otimes \sigma_l^+ \\
& + \sqrt{2}|0\rangle\langle 1| \otimes \sigma_1^+ \otimes O^{(2)} \otimes \sigma_l^- \\
& + \sqrt{2}|1\rangle\langle 0| \otimes \sigma_1^- \otimes O^{(3)} \otimes \sigma_l^+ \\
& + \sqrt{2}|1\rangle\langle 1| \otimes \sigma_1^- \otimes O^{(4)} \otimes \sigma_l^-, \quad (\text{F38})
\end{aligned}$$

where the first register is the virtual bond degree of freedom, and $O^{(r)}$ is a summation of strings of $\sigma^\pm$,

$$O^{(r)} = \sum_k O_k f_r(k), \quad (\text{F39})$$

where $k = (k_2, k_3, \dots, k_{n-1})$ with $k_i = 0, 1$ and denoting $\sigma^1 := \sigma^-, \sigma^0 := \sigma^+$,

$$\begin{aligned}
O_k &= \sigma_2^{k_2} \otimes \sigma_3^{k_3} \otimes \cdots \otimes \sigma_{l-1}^{k_{l-1}}, \\
f_1(k) &= (-1)^{k_2 k_3 + k_3 k_4 + \cdots + k_{l-2} k_{l-1}}, \\
f_2(k) &= (-1)^{k_2 k_3 + k_3 k_4 + \cdots + k_{l-2} k_{l-1} + k_{l-1}}, \\
f_3(k) &= (-1)^{k_2 k_3 + k_3 k_4 + \cdots + k_{l-2} k_{l-1} + k_2}, \\
f_4(k) &= (-1)^{k_2 k_3 + k_3 k_4 + \cdots + k_{l-2} k_{l-1} + k_2 + k_{l-1}}.
\end{aligned} \tag{F40}$$

Now, consider the pure state $\rho = |\psi\rangle\langle\psi|$ and write

$$|\psi\rangle = \sum_{i,j=0}^1 |i\rangle_1 \otimes |\psi_{i,j}\rangle \otimes |j\rangle_L, \quad (\text{F41})$$

where we single out the first and last physical qubit as above. Thus, we can write

$$E_{U^l} = \sum_k E_{U^l,k}, \quad (\text{F42})$$

where $E_{U^l,k}$ are 2×2 matrices with components

$$\begin{aligned} E_{U^l,k}(0,0) &= \sqrt{2}\langle\psi_{0,0}|O_k|\psi_{1,1}\rangle f_1(k), \\ E_{U^l,k}(0,1) &= \sqrt{2}\langle\psi_{0,1}|O_k|\psi_{1,0}\rangle f_2(k), \\ E_{U^l,k}(1,0) &= \sqrt{2}\langle\psi_{1,0}|O_{\bar{k}}|\psi_{0,1}\rangle f_3(\bar{k}), \\ E_{U^l,k}(1,1) &= \sqrt{2}\langle\psi_{1,1}|O_{\bar{k}}|\psi_{0,0}\rangle f_4(\bar{k}). \end{aligned} \quad (\text{F43})$$

Note that in the third and fourth lines, we have changed the index of summation and replaced $k \rightarrow \bar{k} = (1, 1, \dots, 1) - k$. Expand $|\psi_{i,j}\rangle$ in the computational basis

$$|\psi_{i,j}\rangle = \sum_k a_k^{ij} |k\rangle = \sum_{\bar{k}} b_{\bar{k}}^{ij} |\bar{k}\rangle. \quad (\text{F44})$$

Note that $\langle k''|O_k|k'\rangle = \delta_{k',\bar{k}''}\delta_{k,k''}$, the matrix $E_{U^l,k}$ is simplified to

$$E_{U^l,k} = \sqrt{2} \begin{pmatrix} y_k f_1(k) & z_k f_2(k) \\ \bar{y}_k f_3(\bar{k}) & \bar{y}_k f_4(\bar{k}) \end{pmatrix}, \quad (\text{F45})$$

where $y_k = \bar{a}_k^{00} b_k^{11}$ and $z_k = \bar{a}_k^{01} b_k^{10}$. Using

$$\|E_{U^l}\|_p \leq \sum_k \|E_{U^l,k}\|_p, \quad (\text{F46})$$

we need to compute the singular values of $E_{U^l,k}$. When $f_1(k)f_3(\bar{k}) + f_2(k)f_4(\bar{k}) = 0$, the singular values are degenerate,

$$\begin{aligned} \lambda_1 &= \lambda_2 = \sqrt{2}\sqrt{|y_k|^2 + |z_k|^2} \\ &\leq \frac{1}{\sqrt{2}}(|a_k^{00}|^2 + |b_k^{11}|^2 + |a_k^{01}|^2 + |b_k^{10}|^2), \end{aligned} \quad (\text{F47})$$

and when $f_1(k)f_3(\bar{k}) + f_2(k)f_4(\bar{k}) = \pm 2$, the singular values are

$$\begin{aligned} \lambda_1 &= \sqrt{2}(|y_k| + |z_k|), \\ \lambda_2 &= \sqrt{2}||y_k| - |z_k||, \\ \lambda_{1,2} &\leq \frac{1}{\sqrt{2}}(|a_k^{00}|^2 + |b_k^{11}|^2 + |a_k^{01}|^2 + |b_k^{10}|^2). \end{aligned} \quad (\text{F48})$$

Thus,

$$\begin{aligned} \|E_{U^l}\|_\infty &\leq \frac{1}{\sqrt{2}} \sum_k (|a_k^{00}|^2 + |b_k^{11}|^2 + |a_k^{01}|^2 + |b_k^{10}|^2) \\ &= \frac{1}{\sqrt{2}} \langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}. \end{aligned} \quad (\text{F49})$$

Similarly,

$$\|E_{U^l}\|_1 \leq \sqrt{2}. \quad (\text{F50})$$

This finishes the proof. $\blacksquare$

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