

Breakdown of bulk-radiation correspondence in radiative photonic lattices

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(Received 3 June 2025; accepted 3 October 2025; published 7 November 2025)

The topological characteristics of energy bands in crystalline systems are encapsulated in the Berry curvature of the bulk Bloch states. In photonic crystal slabs, far-field emission from guided resonances naturally provides a noninvasive way to probe the embedded wavefunctions, raising the question of how the information carried by escaping photons relates to the band topology. We develop a non-Hermitian model to describe the guided and leaky modes of photonic crystal slabs with long-range couplings and non-local responses. Within this framework, radiation Berry curvature is defined from the far-field polarization and compared to the conventional bulk Berry curvature of the crystal Bloch modes. We investigate this bulk-radiation correspondence in the vicinity of the Γ point of the square lattice and the K point of the honeycomb lattice. The results show that the comparability between the bulk topology and the radiation topology is not universal; the validity is contingent upon the specific bulk Bloch states. Notably, the correspondence completely breaks down surrounding the far-field singularities, while it can hold in smooth regions under special symmetry conditions, e.g., inversion symmetry. Besides, net Berry curvature concentration is captured at the valleys of the nonlocal honeycomb lattice, facilitating further exploration on generalized topological phases in photonic lattices beyond the regimes with localized couplings and Hermiticity.

DOI: [10.1103/1gbs-rdxm](https://doi.org/10.1103/1gbs-rdxm)

I. INTRODUCTION

Berry curvature is a fundamental concept in quantum mechanics that describes the local geometric properties of a system's parameter space [1]. In the context of band theory, it plays a crucial role in relating local band features to global topological invariants and physical observables [2–7]. In recent years, the concept has been extended to various engineered wave systems, capturing their band characteristics, topological phases, and responses to external perturbations [8–12]. In photonic crystals based on Hermitian tight-binding models, these topological properties have demonstrated remarkable potential in manipulating electromagnetic waves, enabling applications such as robust transport, beam steering, and the realization of exotic states of light [8,9,11,13–18]. Recently, the concept of topology has been generalized to open systems [19,20], further expanding the possibilities of

topological photonics with non-Hermiticity surpassing the tight-binding regime [21].

Beyond band topology, photonic lattices also host other kinds of topological features, such as polarization vortices in the radiating field. Polarization vortices often manifest as bound states in the continuum (BICs) with nontrivial topological charges [22–24]. BICs are modes that remain localized even though they coexist with a continuous spectrum of radiating waves that can carry energy away [25]. They are known as intrinsic defects for nonlocal photonic crystal slabs and are commonly regarded as the topological characteristics in such materials [26–32]. In contrast, conventional Berry curvature from crystal Bloch modes is less discussed in these systems, despite its significance in understanding the full topological nature of the bulk. The limited discussion mainly arises from the difficulty of accessing photonic Berry curvature [33], whereas the presence of photonic BICs is directly measurable [34–38].

The concept of far-field radiation topology generalizes the photonic Berry curvature to the far field, resulting in a curvature term defined by the unit polarization vectors in momentum space [39–43]. The idea of representing bulk Berry curvature using far-field polarization vectors was first introduced for cavity polariton systems [39,40,44,45], and later adapted to lattice structures [41,43]. Since escaping photons naturally carry information about the bulk states,

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the far-field radiation topology is proposed as an effective surrogate to access the band topology in radiative photonic structures [39,42,43]. Particularly, the comparability between the far-field Berry curvature pattern and the bulk Berry curvature pattern is called the “bulk-radiation correspondence.” Despite the great potential of this approach, systematic discussion on the validity and generality of the bulk-radiation correspondence in lattice prototypes is still missing. Moreover, the current focus has been primarily on two-band systems, while investigations into multiband systems remain scarce.

To better understand the bulk-radiation correspondence in radiative photonic lattices, we develop a model to describe the guided and leaky modes of photonic crystal slabs with long-range couplings and nonlocal responses. The model facilitates the presentation and analysis of multiband systems. On this basis, we conduct analytical calculations under a generic multiband configuration. The result shows that the comparability between the bulk and far-field Berry curvature is not universal; the validity is contingent upon the particular states. Further, we apply our model to two foundational two-dimensional lattice prototypes. Nonzero Berry curvatures and nontrivial topological charges are simultaneously found around the high symmetry points of interest. Accordingly, we separately study the correspondence on BIC and non-BIC bands. The observations obtained from numerical results are consistent with those indicated by the analytical results.

The paper is organized as follows. In Sec. II, the diffractive and radiative coupling model is introduced to construct the non-Hermitian Hamiltonian. Within this framework, we conduct step-by-step derivations to analyze a generic multiband configuration in Sec. III. In Secs. IV and V, the model is applied to the Γ point of a square lattice and the K point of a honeycomb lattice, respectively. We comment on the physical insights provided by this breakdown of bulk-radiation correspondence, followed by a detailed analysis of the symmetry constraints and parameter settings in our simulations in Sec. VI. Final conclusions are given in Sec. VII.

II. THE DIFFRACTIVE AND RADIATIVE COUPLING MODEL OF PHOTONIC LATTICES

The band structure for two-dimensional photonic crystal slabs can be understood by considering guided modes folded at diffraction orders (m, n) , where m, n are integers [46]. For a lattice with periodicity a_1 and a_2 , the diffraction orders are located at multiples of the reciprocal lattice vector $\mathbf{G} = mG_1\hat{\mathbf{k}}_1 + nG_2\hat{\mathbf{k}}_2$, where $G_{1,2} = 2\pi/a_{1,2}$ defines the Brillouin zone. In the case of a rectangular lattice, for example, $a_{1,2} = a_{x,y}$, $\hat{\mathbf{k}}_{1,2} = \hat{\mathbf{k}}_{x,y}$, and $G_{1,2} = G_{x,y}$. We indicate the guided modes in the vicinity of $\mathbf{G}$ with in-plane wave vectors $(k_x, k_y) \in 1^{st}BZ$ as $|m, n\rangle$. These guided modes constitute the basis for the effective model. Here, we only consider a single “unfolded” guided mode—for instance, the fundamental TE guided mode propagating in the uncorrugated slab made of the average effective index. The thickness of the slab is chosen to ensure operation in the single-mode regime, and all other guided TE modes lie at much higher energies.

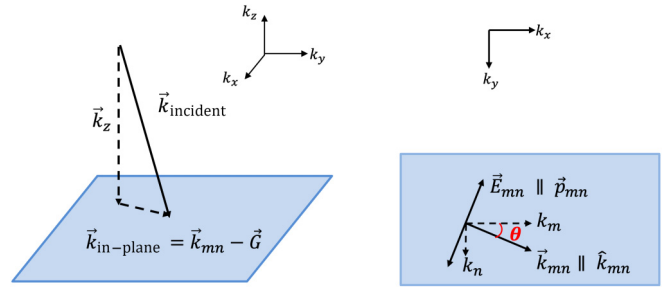


FIG. 1. The geometric representation of the basis sets $\mathbf{k}_{m,n}$ and $\mathbf{E}_{m,n}$, the wave vector $\mathbf{k}$ of guided resonances, and far-field electric field components $\mathbf{E}$ are resulted from their linear combinations.

The dispersion relation of the mode $|m, n\rangle$ in the vicinity of Γ point corresponding to $\mathbf{G}$ with a group velocity v_g is

$$\omega_{m,n} = \omega_0 + \frac{v_g}{\hbar}(|\mathbf{k}_{m,n}| - |\mathbf{G}|), \quad (1)$$

where ω_0 is the mode frequency at the first Γ point. The guided modes are then coupled via a diffractive mechanism and the radiation continuum [37,47–49]. The coefficients determining the coupling parameters can be obtained from a perturbative method based on the periodic potential profile acting on the photonic modes.

For TE modes, the unit polarization vector $\hat{\mathbf{p}}_{\mathbf{m},n}$ associated with the guided mode $|m,n\rangle$ is perpendicular to the corresponding unit in-plane wave vector $\mathbf{k}_{\mathbf{m},n}$. We define the angle $\theta_{m,n}$ on the k_x - k_y plane as

$$\theta_{m,n} = \arg[k_x + mG_x + i(k_y + nG_y)], \quad (2)$$

such that

$$\hat{\mathbf{k}}_{m,n} = \cos \theta_{m,n} \cdot \hat{\mathbf{k}}_x + \sin \theta_{m,n} \cdot \hat{\mathbf{k}}_y, \quad (3)$$

$$\hat{\mathbf{p}}_{m,n} = -\sin \theta_{m,n} \cdot \hat{\mathbf{k}}_x + \cos \theta_{m,n} \cdot \hat{\mathbf{k}}_y, \quad (4)$$

see Fig. 1.

The coupling via the radiation continuum is achieved through far-field interference, resulting in the loss of photons at a rate γ_r . The loss exchange depends on the product between the polarization components $\hat{\mathbf{p}}_{\mathbf{m},\mathbf{n}}$ and $\hat{\mathbf{p}}_{\mathbf{m}',\mathbf{n}'}$ [37,47,48], and the projection angle between two guided modes can be represented as

$$\varphi_{m,n,m',n'} = \theta_{m,n} - \theta_{m',n'}. \quad (5)$$

The effective Hamiltonian can thereby be written as

$$\begin{aligned} \hat{H}_{\text{rad}}(k_x, k_y) = & \sum_{m,n} (\omega_{m,n} + i\gamma_r) |m, n\rangle \langle m, n| \\ & + \sum_{m,n,m',n'} [t_{m,n,m',n'} + i\gamma_r \cos(\varphi_{m,n,m',n'})] \\ & \times |m, n\rangle \langle m', n'|. \end{aligned} \quad (6)$$

The linear combinations of the basis sets have to be in alignment since $\mathbf{k} \cdot \mathbf{E} = 0$ and $\mathbf{k}_{m,n} \cdot \mathbf{E}_{m,n} = 0$ always hold. Thus, given the eigenstate of $\hat{H}_{\text{rad}}$ as $|\psi\rangle = \sum_{m,n} c_{m,n} |m, n\rangle$, the unit vector that characterizes the corresponding far-field radiation can be written as

$$|E\rangle = \sum_{m,n} c_{m,n} (-\sin \theta_{m,n} \cdot \hat{\mathbf{k}}_x + \cos \theta_{m,n} \cdot \hat{\mathbf{k}}_y). \quad (7)$$

A 1D version of this model was introduced in Refs. [37,47] for polaritons, where the radiative coupling of counterpropagating guided modes leads to the formation of symmetry-protected BIC. A 2D version of this model was introduced in Ref. [49], where the interplay between radiative coupling and diffractive coupling of three guided modes leads to the formation of tunable Friedrich–Wintgen BICs at a highly oblique angle.

III. THE BULK-RADIATION CORRESPONDENCE UNDER N-BAND CONFIGURATION

Since we have a radiative model, i.e., a non-Hermitian system, the Hamiltonian admits left and right eigenstates

$$\hat{H}_{\text{rad}}|\psi^R\rangle = \varepsilon |\psi^R\rangle, \quad (8)$$

$$\hat{H}_{\text{rad}}^\dagger|\psi^L\rangle = \varepsilon^* |\psi^L\rangle, \quad (9)$$

where in general $|\psi^R\rangle \neq |\psi^L\rangle$, and they satisfy the bi-orthogonality condition $\langle\psi_i^L|\psi_j^R\rangle = \delta_{ij}$ [50]. Using combinations of these eigenvectors, four different Berry curvatures can be calculated from

$$B_{\alpha\beta} = -i(\langle\partial_\mu\psi^\alpha|\partial_\nu\psi^\beta\rangle - \langle\partial_\nu\psi^\alpha|\partial_\mu\psi^\beta\rangle) \quad (10)$$

for $\alpha, \beta = L, R$, with $\partial_{\mu,\nu}$ being different operators in k space. Since all four Berry curvatures carry equivalent information [51] and give rise to associated results, we restrict ourselves to B_{RR} in the following discussions. Moreover, right eigenvectors are the quantities directly accessible in the radiation channel and, therefore, more experimentally relevant. The band index is omitted in the following for simplicity of notation. The complex entry for forming nonzero bulk Berry curvatures can be purely provided by the non-Hermitian part of Eq. (6), e.g., the model shown in Sec. IV. Besides, the elicitation of the bulk Berry curvature can be further promoted by having a complex coupling scheme in the Hermitian part of Eq. (6), e.g., the model shown in Sec. V.

Now consider an $N \times N$ Hamiltonian written in the basis of N guided modes $|m, n\rangle$, noted as $\mathbf{G}_j$. The eigenstate for $\hat{H}|\psi\rangle = \varepsilon|\psi\rangle$ can be generally expressed as

$$|\psi\rangle = \sum_{j=1}^N c_j |\mathbf{G}_j\rangle \equiv \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}. \quad (11)$$

The bulk Berry curvature calculated from right-right eigenvectors of the bulk states is

$$B^b = -i(\langle\partial_\mu\psi|\partial_\nu\psi\rangle - \langle\partial_\nu\psi|\partial_\mu\psi\rangle). \quad (12)$$

Thus, for a general N -components vector, the bulk Berry curvature can be explicitly written as

$$B^b = -i \sum_{j=1}^N (\partial_\mu c_j^* \partial_\nu c_l - \partial_\nu c_j^* \partial_\mu c_l). \quad (13)$$

Following the framework introduced in Sec. II, the E field for a general N -components vector can be correspondingly

expressed as in Eq. (7)

$$|E\rangle = \begin{pmatrix} -\sum_{j=1}^N \sin\theta_j c_j \\ \sum_{j=1}^N \cos\theta_j c_j \end{pmatrix}. \quad (14)$$

The concept of far-field radiation topology generalizes the concept of the Berry curvature to the far field, resulting in a curvature term defined by the unit vectors of the electric field components on the k plane as

$$B^f = -i(\langle\partial_\mu E|\partial_\nu E\rangle - \langle\partial_\nu E|\partial_\mu E\rangle). \quad (15)$$

The electric field components can be extracted from the texture of the Stokes parameters in k space.

For the far-field Berry curvature, substituting Eq. (15) with expressions in Eq. (14), we get

$$\begin{aligned} B^f = & -i \sum_{j,l=1}^N \cos(\theta_j - \theta_l) [\partial_\mu c_j^* \partial_\nu c_l - \partial_\nu c_j^* \partial_\mu c_l] \\ & + c_j^* c_l (\partial_\mu \theta_j \partial_\nu \theta_l - \partial_\nu \theta_j \partial_\mu \theta_l) \\ & + -i \sum_{j,l=1}^N \sin(\theta_j - \theta_l) [\partial_\mu c_j^* \partial_\nu \theta_l - \partial_\nu c_j^* \partial_\mu \theta_l] \\ & - c_j^* (\partial_\mu \theta_j \partial_\nu c_l - \partial_\nu \theta_j \partial_\mu c_l). \end{aligned} \quad (16)$$

Straightforwardly, the relation $B^f = B^b$ does not always hold. The term in curly brackets is the same appearing in the bulk Berry curvature B^b in Eq. (13). The establishment of the equation is subject to specific conditions, e.g., when the sum has only contributions from $j = l$. It might also hold roughly for the case where the redundant terms are sufficiently small. In real experiments, these conditions would correspond to particular choices of system settings, e.g., material properties, lattice prototypes, and site geometries. For completeness, one can analogously repeat the derivation with left eigenstates to obtain the biorthogonal curvatures B_{LR}^b and B_{LR}^f , which leads to an expression of the same form as the one in Eq. (16).

Thus, the bulk-radiation correspondence is not a universal concept in photonic lattices with nonlocal responses. It cannot be applied directly without considering specific settings; the validity depends on the states themselves. Aside from the measurement that has already been validated [40], which is specific to continuous two-band systems [39], further generalization requires analysis of the targeted bands. Given the limitations of deriving a broader general conclusion based solely on Eq. (16), we numerically construct some specific states in Secs. IV and V to substantiate and reinforce our arguments. We focus on the square and the honeycomb lattice as foundational prototypes in two-dimensional crystal structures.

IV. BULK-RADIATION CORRESPONDENCE AROUND THE Γ POINT OF SQUARE LATTICES

We first apply our model to study the topological properties around the Γ point of a square array with a lattice constant a . A minimal model with four diffracted orders $(m, n) = \{(+1, 0), (-1, 0), (0, +1), (0, -1)\}$ is used to obtain the far-field spectrum, with four guided resonances lying in the radiative continuum. Following the framework built in

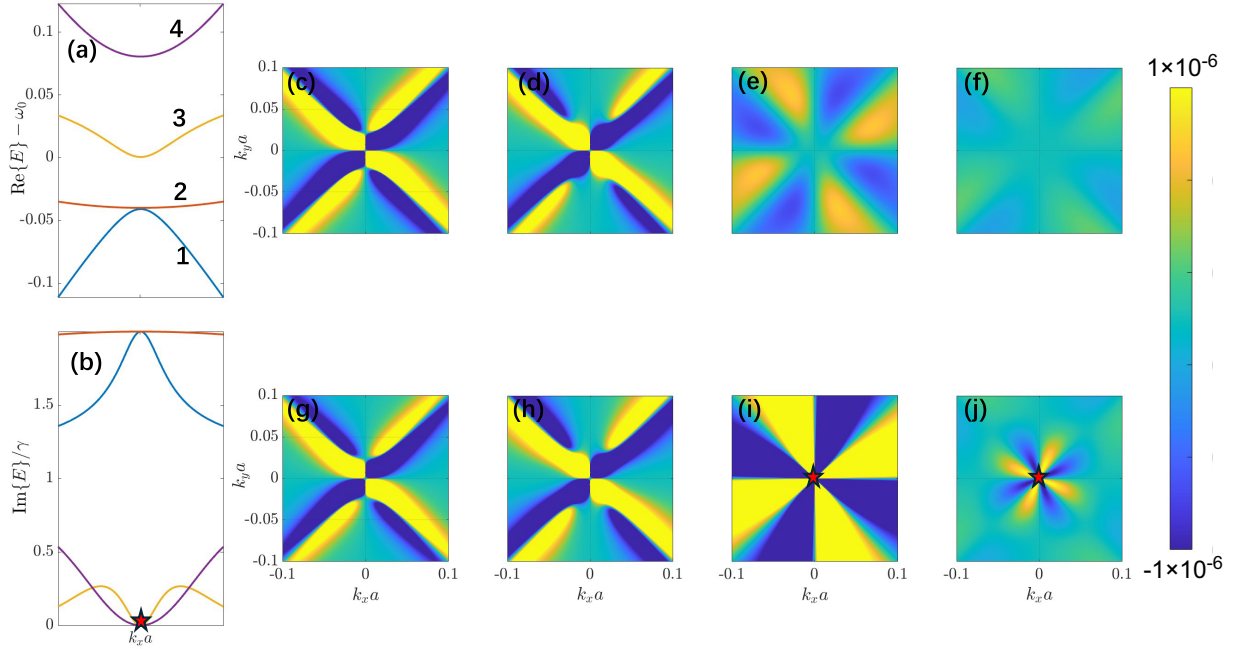


FIG. 2. Bulk-radiation correspondence for $u_x = 0.041$, $u_y = 0.04$, $v = 0.02$, $\gamma_r = 0.002$, representing the quasisymmetric case. Slightly rotational symmetry breaking is introduced here to avoid the degeneracies, which can cause the failure of the adopted algorithm [52]. The stars indicate BIC modes. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(f) Bulk Berry curvature for bands 1–4, respectively. (g)–(j) Radiation Berry curvature for bands 1–4, respectively.

Sec. II, the Hermitian part of the Hamiltonian can be written as

$$\hat{H} = \begin{pmatrix} \omega_{1,0} & u_x & v & v \\ u_x & \omega_{-1,0} & v & v \\ v & v & \omega_{0,1} & u_y \\ v & v & u_y & \omega_{0,-1} \end{pmatrix}, \quad (17)$$

where $\omega_{m,n}$ is defined in Eq. (1), while the couplings v , u_x , and u_y are introduced to provide different kinds of in-plane mode mixing. v stands for the coupling between diffracted orders that mix G_x and G_y , while u_x and u_y stand for the couplings whose diffracted order only involves G_x or G_y crossing the Γ point. The non-Hermitian part can be correspondingly expressed as

$$\hat{\gamma} = i\gamma_r \times \begin{pmatrix} 1 & \cos \varphi_{1,0,-1,0} & \cos \varphi_{1,0,0,1} & \cos \varphi_{1,0,0,-1} \\ \cos \varphi_{-1,0,1,0} & 1 & \cos \varphi_{-1,0,0,1} & \cos \varphi_{-1,0,0,-1} \\ \cos \varphi_{0,1,1,0} & \cos \varphi_{0,1,-1,0} & 1 & \cos \varphi_{0,1,0,-1} \\ \cos \varphi_{0,-1,1,0} & \cos \varphi_{0,-1,-1,0} & \cos \varphi_{0,-1,0,1} & 1 \end{pmatrix}, \quad (18)$$

where $\varphi_{m,n,m',n'}$ is defined from Eqs. (5) and (2). We notice that the loss exchange is most effective for diffracted orders that involve the same G_x or G_y for which the electric field is co-polarized and $\varphi_{m,n,m',n'} = \pi$. The full non-Hermitian model of a square photonic lattice is thus obtained from Eqs. (17) and (18) as

$$\hat{H}_{\text{rad}} = \hat{H} + \hat{\gamma} \approx \begin{pmatrix} k_x + i\gamma_r & u_x - i\gamma_r & v & v \\ u_x - i\gamma_r & -k_x + i\gamma_r & v & v \\ v & v & k_y + i\gamma_r & u_y - i\gamma_r \\ v & v & u_y - i\gamma_r & -k_y + i\gamma_r \end{pmatrix}, \quad (19)$$

having expanded around the Γ point in the last step.

The simulation results of the bulk-radiation correspondence around the Γ point for two representative parameter settings are shown in Figs. 2 and 3. Co-occurrence of two different topological features, i.e., nontrivial topological charges and nonzero Berry curvatures, is captured on this platform. The BICs are identified by the purely real eigen-energies found at the Γ point. This is consistent with the symmetry

constraints for BICs given by group theory [53]. We separately discuss the BIC bands and the non-BIC bands in the following, since the existence of nonremovable singularities would bring in fundamental differences between the two cases.

For non-BIC bands, the degree of comparability is determined by the ratio between u_x and u_y . When $u_x \approx u_y$, the Hamiltonian is quasi- C_4 symmetric. Comparable patterns are

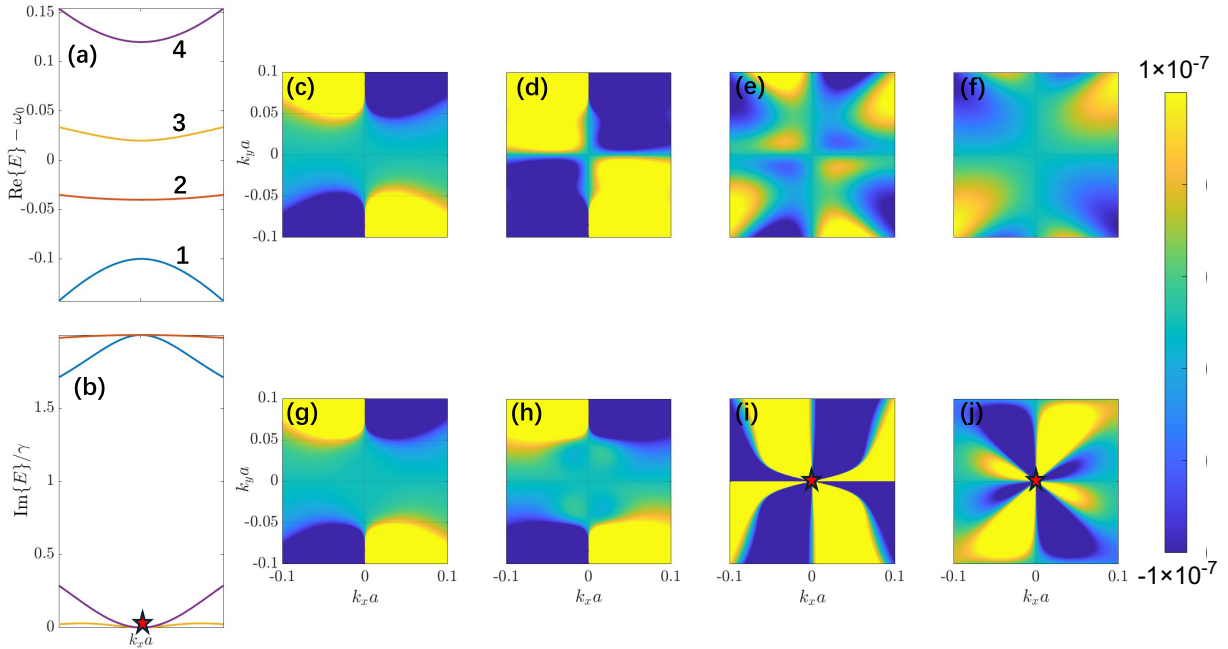


FIG. 3. Bulk-radiation correspondence for $u_x = 0.04$, $u_y = 0.1$, $v = 0.02$, $\gamma_r = 0.002$, representing the asymmetric case. The stars indicate BIC modes. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(f) Bulk Berry curvature for bands 1–4, respectively. (g)–(j) Radiation Berry curvature for bands 1–4, respectively.

observed in the vicinity of Γ , e.g., for bands 1 and 2 in Fig. 2. Here, we define the vicinity as $k_x a < 0.1$ and $k_y a < 0.1$. In contrast, the validity decreases when the difference between u_x and u_y becomes larger, e.g., for bands 1 and 2 in Fig. 3.

For BIC bands, the polarization vectors cannot be defined at the Γ point with the existence of BIC points, so are the far-field Berry curvatures. In other words, $|\partial_\alpha E\rangle$ and its transpose are not effective expressions at the BIC points. Due to the presence of nonremovable singularities, the original polarization distribution as well as the associated derivatives around the singularity is entirely altered. As a result, the bulk-radiation correspondence always breaks down surrounding the field singularities, e.g., for bands 3 and 4 in Figs. 2 and 3.

V. BULK-RADIATION CORRESPONDENCE AROUND THE K POINT OF HONEYCOMB LATTICES

We now apply our model to the K point of the honeycomb lattice. The lattice, with periodicity a , has reciprocal lattice vectors $\mathbf{G}_1 = \frac{2\pi}{3a}\hat{\mathbf{k}}_x + \frac{2\pi}{\sqrt{3}a}\hat{\mathbf{k}}_y$ and $\mathbf{G}_2 = \frac{2\pi}{3a}\hat{\mathbf{k}}_x - \frac{2\pi}{\sqrt{3}a}\hat{\mathbf{k}}_y$. We consider a minimal model with three diffracted orders $(m, n) = \{(0, +1), (0, -1), (+2, +1)\}$, resulting in three guided modes of wave vector

$$\mathbf{k}_{m,n} = (k_x + \mathbf{G}_{m,n} \cdot \hat{\mathbf{k}}_x)\hat{\mathbf{k}}_x + (k_y + \mathbf{G}_{m,n} \cdot \hat{\mathbf{k}}_y)\hat{\mathbf{k}}_y, \quad (20)$$

where $\mathbf{G}_{m,n} = m\mathbf{G}_1 + n\mathbf{G}_2 + \mathbf{K}$, and $k_{x,y}$ are relative to the K point located at $\mathbf{K} = (\frac{2\pi}{3a}, \frac{2\pi}{3\sqrt{3}a})$. Similarly as in Sec. IV, we consider coupling between these diffracted orders. A complex coupling scheme is introduced, which in essence reflects a different symmetry-breaking mechanism. Further discussion is included in Sec. VI.

The real part of the Hamiltonian can be written as

$$\hat{H} = \begin{pmatrix} \omega_{0,1} & u & t^* \\ u^* & \omega_{2,1} & u \\ t & u^* & \omega_{0,-1} \end{pmatrix}, \quad (21)$$

where $\omega_{m,n}$ is defined in Eq. (1) with Eq. (20), while the couplings u , t provide in-plane mode mixing between the three diffracted orders. Although $u \gtrsim t$ for a generic setting, we keep $u = t$ since a weak perturbation on the potential profile is considered. The non-Hermitian part can be correspondingly expressed as

$$\hat{\gamma} = i\gamma_r \begin{pmatrix} 1 & \cos \varphi_{0,1,2,1} & \cos \varphi_{0,1,0,-1} \\ \cos \varphi_{2,1,0,1} & 1 & \cos \varphi_{2,1,0,-1} \\ \cos \varphi_{0,-1,0,1} & \cos \varphi_{0,-1,2,1} & 1 \end{pmatrix}, \quad (22)$$

where again $\varphi_{m,n,m',n'}$ can be computed explicitly from Eqs. (5) and (2) with Eq. (20). Notice that we consider uniform losses γ_r to highlight the key physics, and a more complete treatment accounting for their mode-dependence is discussed in Ref. [54]. The full non-Hermitian model is obtained from Eqs. (21) and (22) as

$$\begin{aligned} \hat{H}_{\text{rad}} &= \hat{H} + \hat{\gamma} \\ &\approx \begin{pmatrix} -k_y + i\gamma_r & u - \frac{i\gamma_r}{2} & u^* - \frac{i\gamma_r}{2} \\ u^* - \frac{i\gamma_r}{2} & \frac{\sqrt{3}}{2}k_x + \frac{k_y}{2} + i\gamma_r & u - \frac{i\gamma_r}{2} \\ u - \frac{i\gamma_r}{2} & u^* - \frac{i\gamma_r}{2} & -\frac{\sqrt{3}}{2}k_x - \frac{k_y}{2} + i\gamma_r \end{pmatrix}, \end{aligned} \quad (23)$$

having expanded around the K point in the last step. The simulation results are shown from Figs. 4–6.

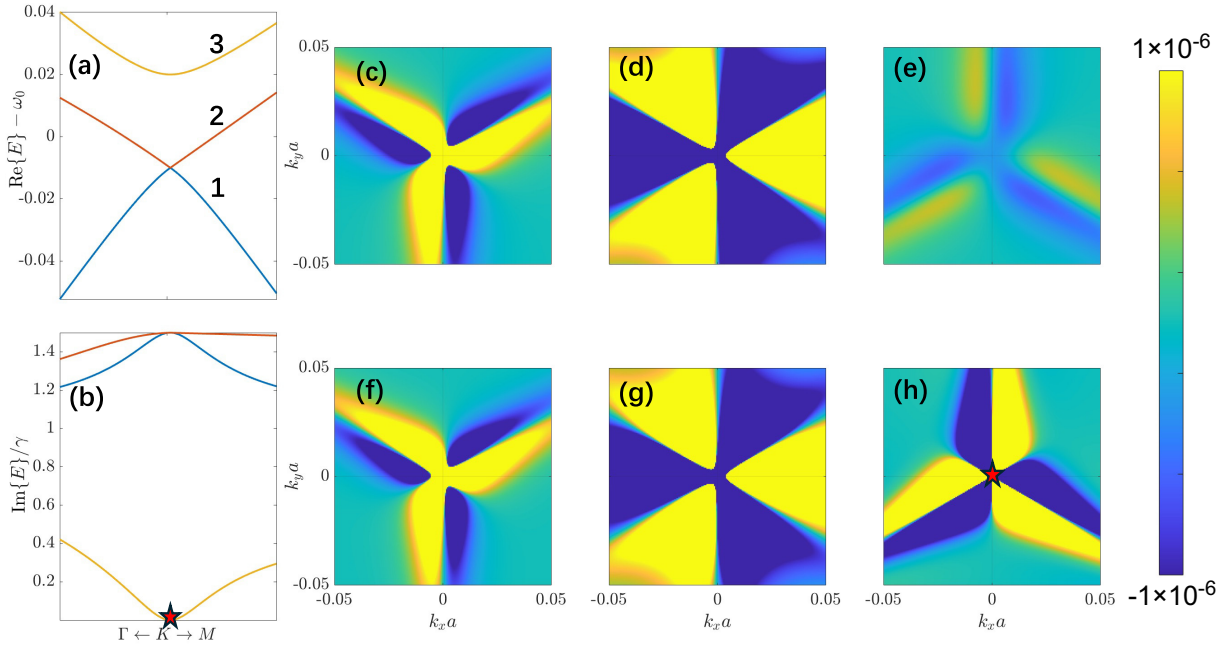


FIG. 4. Bulk-radiation correspondence for $u = 0.01 + 0.000\,01i$, $\gamma_r = 0.002$, representing the quasisymmetric case. Slightly rotational symmetry breaking is introduced here to avoid the degeneracies, which can cause the failure of the adopted algorithm [52]. The star indicates a BIC mode. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(e) Bulk Berry curvature for bands 1–3, respectively. (f)–(h) Radiation Berry curvature for bands 1–3, respectively.

Similarly to the square lattice case, two topological features are simultaneously found around the K point. We separately discuss the case where u is a real number and the case where u is complex.

When u is purely real, the Hamiltonian is intrinsically C_3 symmetric. Splitting patterns are observed in the vicinity of

K . Here, we define the vicinity as $k_x a < 0.05$ and $k_y a < 0.05$. The two Berry curvature patterns for the non-BIC bands are straightforwardly comparable, e.g., for bands 1 and 2 in Fig. 4, while the degree of validity can completely break down for BIC bands, e.g., for band 3 in Fig. 4. These are in alignment with the results of Sec. IV.

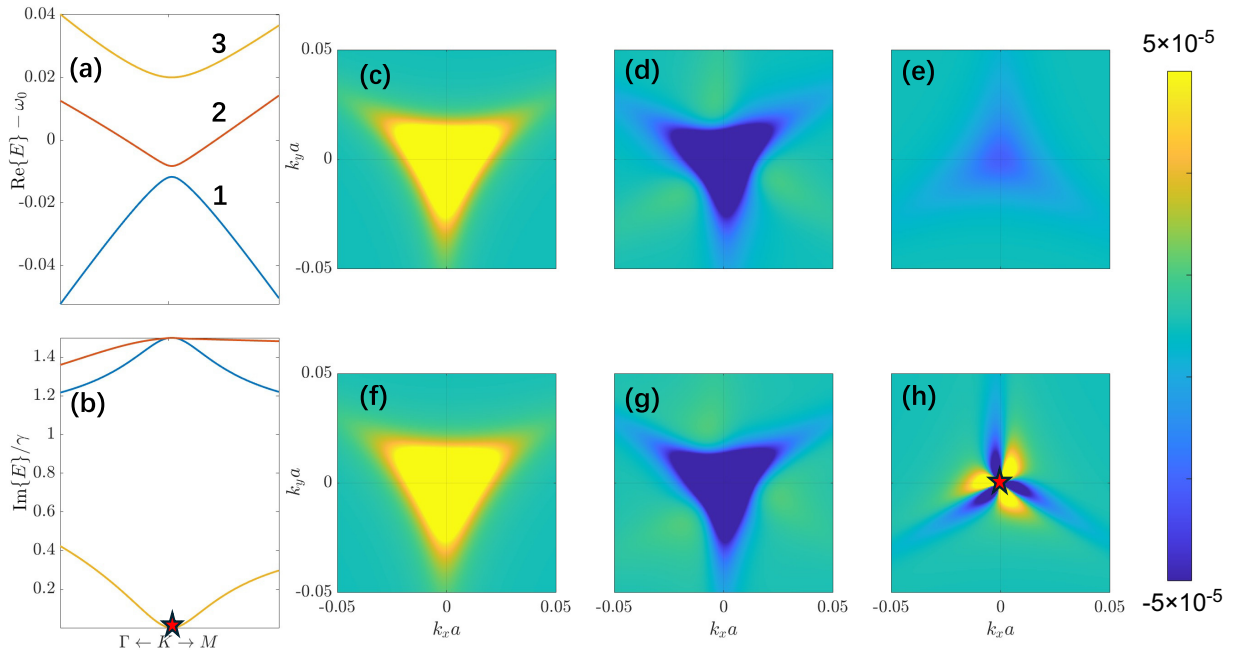


FIG. 5. Bulk-radiation correspondence for $u = 0.01 + 0.001i$, $\gamma_r = 0.002$, representing a regime of weak asymmetry. The stars indicate BIC modes. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(e) Bulk Berry curvature for bands 1–3, respectively. (f)–(h) Radiation Berry curvature for bands 1–3, respectively.

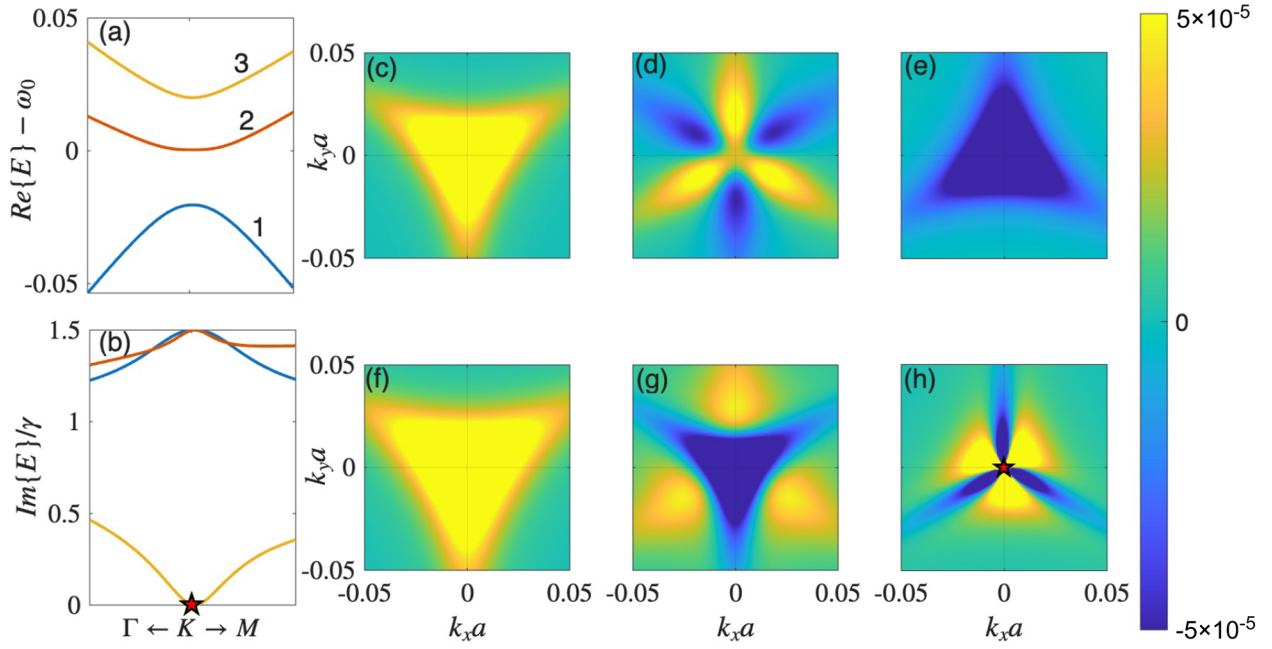


FIG. 6. Bulk-radiation correspondence for $u = 0.01 + 0.006i$, $\gamma_r = 0.002$, representing a regime of strong asymmetry. The stars indicate BIC modes. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(e) Bulk Berry curvature for bands 1–3, respectively. (f)–(h) Radiation Berry curvature for bands 1–3, respectively.

When u is a complex number, inversion symmetry is broken and some unique features can be captured on this platform. Particularly, net Berry curvature accumulation is found at the valleys of the nonlocal honeycomb lattice under the complex setting. The comparability around the K point, as reported in [43], is successfully reproduced using our method, e.g., for non-BIC bands in Fig. 5. On top of that, we identify a regime where the comparability breaks down, e.g., for non-BIC bands in Fig. 6. The former case holds when the real part of u is way larger than the imaginary part of u , while the latter case stands for the setting as the imaginary part of u increases and becomes comparable to the real part of u .

Additionally, the flattening process of the two non-BIC bands is unequal as we promote the imaginary part of parameter u , e.g., vertical comparisons for the bulk Berry curvature patterns of bands 1, 2 from Figs. 4–6. That is to say, richer band geometry exhibiting nontrivial topological landscape is unblocked in systems of this kind. This feature distinguishes it from the existing valley degree of freedom, making it possible to uncover more diverse Berry curvature effects.

VI. DISCUSSION AND EXPERIMENTAL PROPOSALS

A. Breakdown of bulk-radiation correspondence: physical insights

The breakdown of bulk-radiation correspondence as exemplified in Secs. IV and V lies in two aspects: dimensional mismatch of the bulk-radiation projection and unavoidable leakage from guided resonances in the photonic crystal slab under consideration.

First, it is not fully legitimate to ignore the multiband nature of photonic lattices without any valid justification, especially for those with $a < \lambda$. For example, the models in

Secs. IV and V cannot be effectively treated as two-band systems. The nonlocal nature of the photonic Hamiltonian, the vectorial aspect, and the inclusion of complex eigenfrequencies in the master equation give rise to an intrinsic dense eigenvalue problem. The full Hamiltonian is thus harder to separate into invariant subspaces with assigned dimensionality, even just locally. This results in more complicated connections between the states both within the band and among the bands, reflecting on the local geometric properties of the parameter space, e.g., the distorted Dirac cone with nonequal flattening processes discussed in Sec. V.

Unlike the case discussed in Ref. [39], the multiband nature leads to dimensional mismatch upon projecting the bulk Hilbert space to the far-field synthetic parameter space. This dimensional mismatch results in loss, distortion, or blurring of local geometric features of the original manifolds. There are occasions when the projection is correctly done so the main features can be kept, but that is not guaranteed in general. These occasions could mostly relate to effective decoupling of the local Hamiltonian, e.g., those with block-diagonal structures accidentally formed from the wavefunction coupling, or those with inversion symmetry thus being more inherently sparselike.

Besides, radiation loss is unavoidable in the photonic crystal slabs considered in the model built in Sec. II. The escaping photons from guided resonances interfere in the far field, and the energy redistribution resulting from far-field interference will reversely regulate the near-field photon behaviors. In our model, the far-field regulation manifests as coupling between spatially coherent modes via the radiative continuum. That is to say, the information of the bulk Bloch states carried by the leaky modes will be further deformed in the far field as a result of intensity modulation. One manifestation is that the highest degree of destructive interference happens at the BIC points,

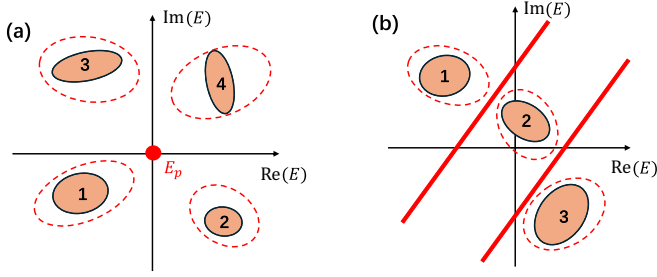


FIG. 7. Schematic illustration of the types of gaps for the complex energy spectrum. The circles labeled with numbers represent complex energy clusters from different bands. The dashed lines indicate possible deformations. (a) A non-Hermitian Hamiltonian with its bands exhibiting the point gap, where bands cannot cross the E_p point in the real-imaginary plane. (b) A non-Hermitian Hamiltonian with its bands exhibiting line gaps, where bands cannot cross entire lines in the real-imaginary plane.

corresponding to a complete loss of all the information about the bulk Bloch states.

Thus, the bulk-radiation correspondence is not universal in radiative photonic lattices. The two-band regime is not complete, and far-field interference plays an essential role. This is consistent with the analytical results obtained in Sec. III. Additionally, we infer that the bulk-radiation correspondence also fails in nonlattice planar photonic structures with more than two decoupled degrees of freedom, where effective two-level treatment no longer applies.

B. Non-Hermitian AZ symmetry constraints

In Hermitian systems, topological physics is governed by three fundamental symmetries: time-reversal symmetry, particle-hole symmetry, and chiral symmetry. Based on different combinations of these three, the AZ symmetry scheme classifies topological insulators into ten categories, which are further grouped into two major classes [55–59].

For open systems, the topological phases are enriched by the non-Hermiticity. The conjugates of the time-reversal symmetry and the particle-hole symmetry are introduced as new degrees of freedom, further expanding the categories and classes in the AZ symmetry table [19]. For simplicity, we denote the five relevant symmetries as follows: time-reversal symmetry and its conjugate (T_+ , P_+), particle-hole symmetry and its conjugate (P_- , T_-), and chiral symmetry (C) in the following discussions. To fully locate the non-Hermitian topological classification table, the concepts of “point gap” and “line gap” shall also be considered [19,20]. A schematic illustration is presented in Fig. 7.

For the square lattice case discussed in Sec. IV, we identify the following fundamental symmetry

$$P_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (24)$$

that is the conjugate of time-reversal symmetry, both when $u_x = u_y$ and $u_x \neq u_y$. Therefore, the Hamiltonian always falls into the $AI^\dagger$ class [19]. Furthermore, this Hamiltonian exhibits

a line gap in both the symmetric $u_x = u_y$ and the asymmetric $u_x \neq u_y$ cases, and is thereby topologically trivial. This explains why only alternating positive and negative Berry curvature patterns are observed on this platform.

For the honeycomb lattice case discussed in Sec. V, we cannot identify any of the five symmetries regardless of the choice of the parameter u . The Hamiltonian around the K point always falls into class A. The Hamiltonian has a point gap in the quasisymmetric case $u \in \mathbb{R}$. However, it exhibits a line gap as we promote the imaginary part $u \in \mathbb{C}$, i.e., entering the asymmetric regime. This could potentially lead to locally nontrivial topological features.

C. Realistic meanings of coupling parameter settings

Different settings of the coupling parameters can have various practical implications. For example, they may be related to varied material properties or distinct site geometries. In the following, we qualitatively discuss the possible meanings of the parameters included in Secs. IV and V, showcasing potential experimental implementations based on our observations.

In Sec. IV, the parameters u_x and u_y relate to the couplings between counterpropagating guided modes along the two diagonals across the first Brillouin zone. As the couplings cross the Γ point, the relation between u_x and u_y largely relies on the symmetry of the site geometry with respect to the two diagonals in the square unit cell. For instance, u_x and u_y are equivalent if a symmetric shape is adopted, e.g., a circle or a square oriented along x and y , which corresponds to the cases shown in Fig. 2. In contrast, u_x and u_y become nonequal if an asymmetric shape is adopted, e.g., an ellipse or a rectangle, which corresponds to the cases shown in Fig. 3. The degree of asymmetry determines the u_x/u_y ratio, which determines the degree of validity of the bulk radiation correspondence. More details on the relation between lattice geometry and coupling parameters are given in Appendix A, where we also compare the energy dispersions from the effective model and full-wave simulations. Furthermore, the contrast between $u_{x,y}$ and v could reflect material properties. For example, directional dipolar couplings along k_x and k_y become dominant in plasmonic cases, where $u_{x,y}$ is supposed to be way larger than the parameter v . By increasing the scale difference between the parameters, we get comparable patterns as measured in Ref. [41] and numerically calculated from T-matrix method in Ref. [60] (see Fig. 8).

In Sec. V, the parameter u is not directly determined by the shape but by the difference between the two sites in the honeycomb unit cell. For example, u is purely real if the two sites are identical. The bulk-radiation correspondence can thereby be regarded as valid for non-BIC bands regardless of the shape, which was shown in Fig. 4. However, when the two sites are not identical, e.g., different in size, the parameter u enters the complex regime. The bulk-radiation correspondence gradually collapses as the imaginary part is increased, e.g., which corresponds to the change shown from Figs. 5 and 6. Moreover, the complex coupling brings in different AZ symmetry constraints, leading to net Berry concentration at the K point (with the K' point exhibiting opposite Berry curvature accumulation). Furthermore, we reach this regime by having complex couplings in the Hermitian part of the

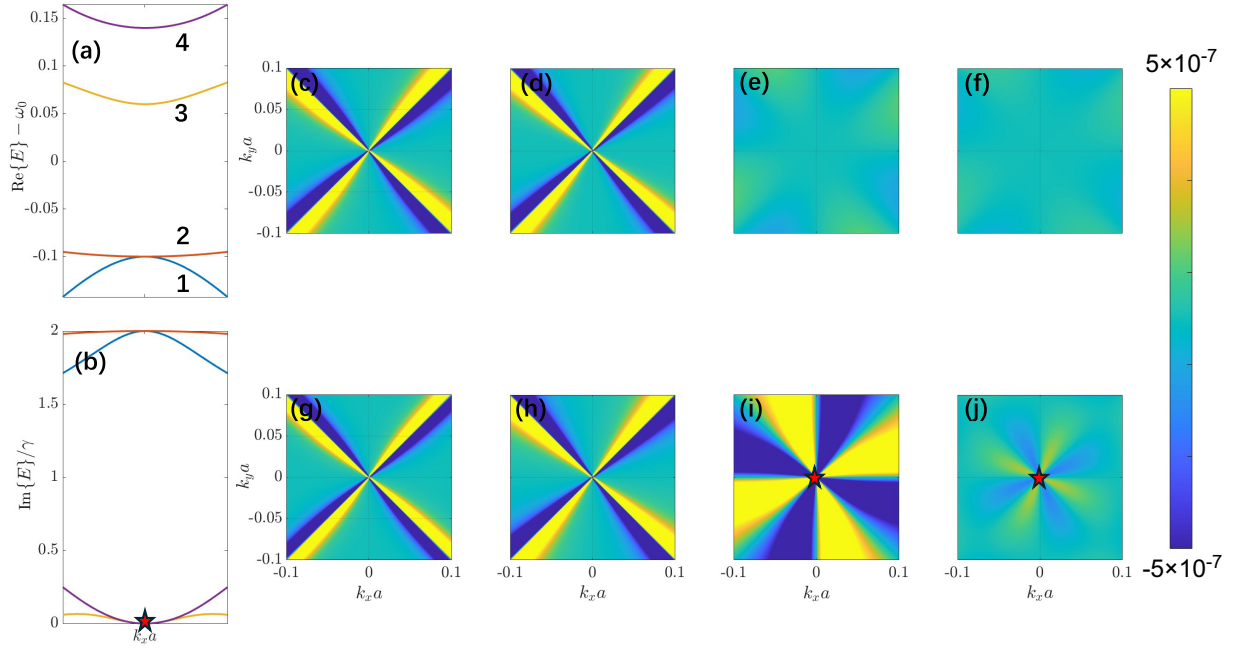


FIG. 8. Bulk-radiation correspondence for the square lattice described by the Hamiltonian in Eq. (19), having used the parameters $u_x = 0.1$, $u_y = 0.1$, $v = 0.02$, $\gamma_r = 0.002$, representing the symmetric case in plasmonic lattices. The stars indicate BIC modes. (a), (b) Real and imaginary parts of the energy spectrum, with curves of the same color symbolizing the same bands. (c)–(f) Bulk Berry curvature for bands 1–4, respectively. (g)–(j) Radiation Berry curvature for bands 1–4, respectively.

Hamiltonian shown in Eq. (23). Thus, we predict that one can also reach this regime by employing magnetic materials and applying an external magnetic field.

VII. CONCLUSION

By developing an effective model to describe the guided and leaky modes of nonlocal radiative photonic lattices, we construct the bulk and the radiation Berry curvatures. We analyze the comparability between bulk and radiation topology with both analytical and numerical methods. The analytical results straightforwardly convey that the bulk-radiation correspondence is not universal, and the validity depends on the particular bulk Bloch states. The numerical results further exemplify specific cases where the bulk-radiation correspondence holds and collapses.

Notably, the correspondence completely breaks down surrounding the field singularities on the BIC bands, while it can hold on non-BIC bands under special symmetry conditions, e.g., C_4 symmetry for the Hamiltonian deduced in Sec. IV and C_3 symmetry for the Hamiltonian deduced in Sec. V. Besides, for the square lattice, both the bulk and radiation Berry curvature show alternating positive and negative regions around Γ , whose integral vanishes, giving the same zero net Berry flux. Conversely, a net Berry curvature concentration is observed for the nonlocal honeycomb under particular symmetry constraints in the vicinity of K or K' . This net Berry curvature concentration paves the way for further exploration on generalized topological phases of photonic lattices in regimes with long-range couplings and non-Hermiticity.

While only far-field curvature is directly measurable, the authenticity of the simulated bulk Berry curvature can be indirectly probed by phenomena like wave packet dynamics [61].

Comparing these indirect signatures with far-field measurements may experimentally resolve this ambiguity between bulk and radiation topology. Additionally, incorporating nonlinear effects in this model, relevant for polariton condensates and lasing, offers another direction for future studies, potentially enriching the interplay between topology, interactions, and radiative losses in non-Hermitian photonic lattices.

ACKNOWLEDGMENTS

This work was partially supported by the French National Research Agency (ANR) under the project POLAROID (Grant No. ANR-24-CE24-7616-01). The work in Aalto is supported by the Academy of Finland under Project no. 13354165. G.S. also received support from the MUR–Italian Ministry of Research–under the Rita Levi Montalcini program. H.S. acknowledges Project No. 2022/45/P/ST3/00467 cofunded by the Polish National Science Centre and the European Union Framework Programme for Research and Innovation Horizon 2020 under the Marie Skłodowska-Curie Grant agreement No. 945339.

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

TABLE I. Values of the parameters used to reproduce the results in Fig. 9.

	$u_x 10^{-3}$	$u_y 10^{-3}$	$v 10^{-3}$	a_x (μm)	a_y (μm)	l_x (μm)	l_y (μm)
BBNN	-3.5	-3.5	1.1	0.4	0.4	0.12	0.12
BNBN	-13	4.5	2	0.396	0.404	0.12	0.28
BNNB	0.014	0.014	1.4	0.4	0.4	0.208	0.208
NBNB	-4.5	14	1.4	0.396	0.404	0.2	0.28
NNBB	5.6	5.6	2.4	0.4	0.4	0.28	0.28

APPENDIX A: FULL-WAVE SIMULATIONS

We show a comparison between the effective model and the full-wave simulations for the Γ point on the square array, utilizing realistic lattice parameters. The simulations are performed with the S^4 code developed by Stanford [62]. We consider a photonic crystal slab composed of TiO_2 with a thickness of $h = 0.1 \mu\text{m}$. The glass layer is modeled as semi-infinite, and the refractive indices are $n_{\text{TiO}_2} = 2.4$ and $n_{\text{glass}} = 1.46$. The permittivity of the TiO_2 has a small imaginary part 10^{-5} . The lattice has spacing a_x and a_y along the two directions. The lattice sites are rectangular holes etched in

the TiO_2 layer with sides l_x and l_y . We obtain five different configurations, depending on the order of the BICs (B) and the non-BICs state (N) at the Γ point. We see a good agreement between the energy dispersions calculated from the effective model in Eq. (19) and the absorption spectra simulated with rigorous coupled wave analysis (RCWA). The polarization phases calculated with the two methods are in excellent agreement. In Table I, we provide the geometrical parameters of the lattice used for the full-wave simulations and the couplings of the effective model that fit these RCWA simulations.

For the honeycomb lattice case, more detailed simulations are provided in Ref. [54].

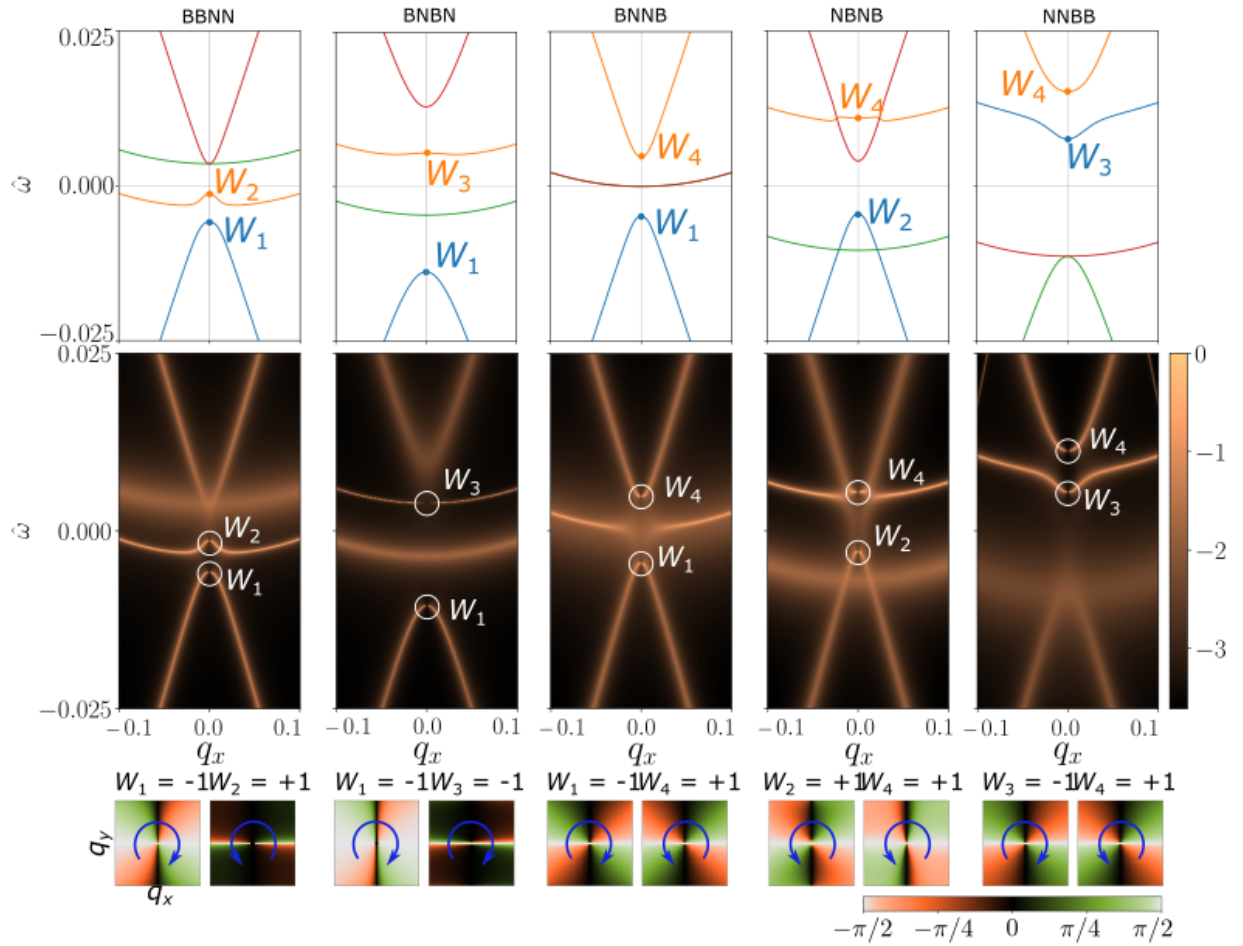


FIG. 9. Top: Different band configurations obtained from the effective model in Eq. (19) of the main text for $\gamma_r = 0.0030$ and different values of the couplings u_x, u_y, v , as reported in Table I. Middle: Absorption spectra of a photonic crystal slab simulated in RCWA, with different geometrical parameters of the lattice, as reported in Table I. White circles show the BICs. Bottom: Polarization phase in the k_x - k_y plane for the BIC modes only, calculated with the RCWA; the polarization calculated with the effective model is in excellent agreement (not shown). The blue arrow indicates the direction of rotation of each vortex.

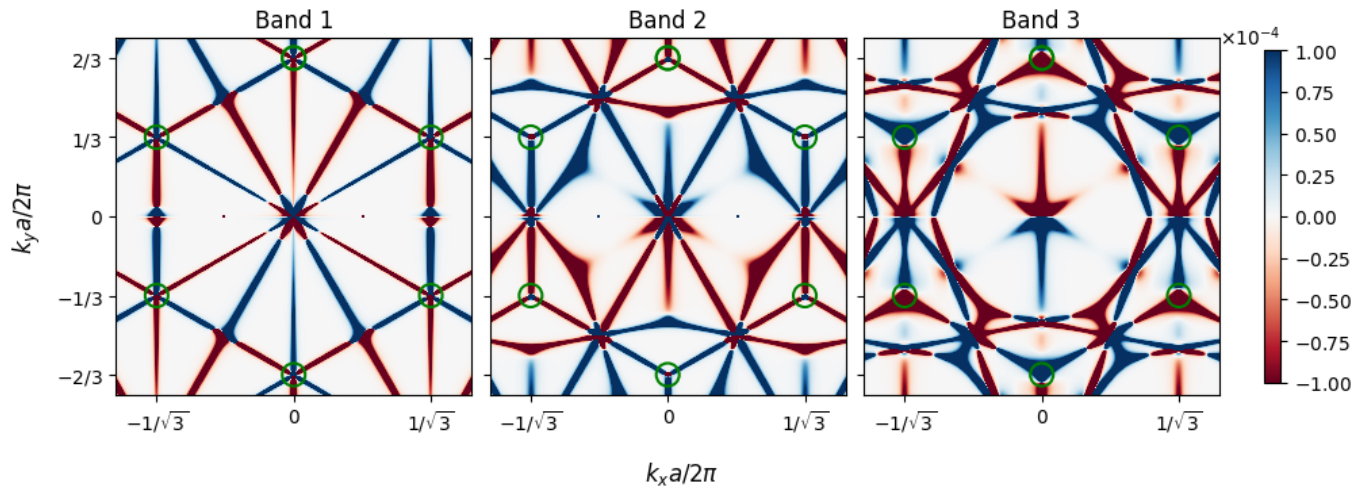


FIG. 10. The bulk Berry curvature calculated from Legume for a slab (refractive index $n = 3$) of thickness $d = 0.2a$ with circular holes arranged in a honeycomb geometry (periodicity a) immersed in air. The radii of the two holes in the unit cell are $r_A = 0.1a$ and $r_B = 0.15a$, respectively. The K and K' points are highlighted with green circles.

APPENDIX B: BULK BERRY CURVATURE ON THE ENTIRE BRILLOUIN ZONE

We hereby employ a full Maxwell solver for photonic crystal slabs based on the guided-mode expansion method, namely, the Legume package [63,64], as a justification for the results in Sec. V. Based on the original package, we extract the eigenstates and calculate the corresponding bulk Berry curvature across the first Brillouin zone. We simulate a slab of circular air holes arranged in a honeycomb geometry with

lattice periodicity a . The slab has a refractive index of $n = 3$ and a thickness $d = 0.2a$. The radii of the two holes within each unit cell are $r_A = 0.1a$ and $r_B = 0.15a$. The configuration slightly breaks the inversion symmetry in real space, corresponding to the weak asymmetric situation studied in Sec. V (see Fig. 6). The symmetry breaking setting results in locally nontrivial topological landscape at the valleys, with K and K' points highlighted with green circles taking opposite concentration (see Fig. 10).

- [1] M. V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. R. Soc. Lond. A* **392**, 45 (1984).
- [2] D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, *Rev. Mod. Phys.* **82**, 1959 (2010).
- [3] D. Yoshioka, *The Quantum Hall Effect* (Springer, Berlin, 2002).
- [4] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall effect in a two-dimensional periodic potential, *Phys. Rev. Lett.* **49**, 405 (1982).
- [5] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized Hall conductance as a topological invariant, *Phys. Rev. B* **31**, 3372 (1985).
- [6] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).
- [7] C. L. Kane and E. J. Mele, Z_2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [8] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Robust optical delay lines with topological protection, *Nat. Phys.* **7**, 907 (2011).
- [9] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Photonic Floquet topological insulators, *Nature (London)* **496**, 196 (2013).
- [10] C. L. Kane and T. C. Lubensky, Topological boundary modes in isostatic lattices, *Nat. Phys.* **10**, 39 (2014).
- [11] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg *et al.*, Topological photonics, *Rev. Mod. Phys.* **91**, 015006 (2019).
- [12] G. Ma, M. Xiao, and C. T. Chan, Topological phases in acoustic and mechanical systems, *Nat. Rev. Phys.* **1**, 281 (2019).
- [13] S. Raghu and F. D. M. Haldane, Analogs of quantum-Hall-effect edge states in photonic crystals, *Phys. Rev. A* **78**, 033834 (2008).
- [14] F. D. M. Haldane and S. Raghu, Possible realization of directional optical waveguides in photonic crystals with broken time-reversal symmetry, *Phys. Rev. Lett.* **100**, 013904 (2008).
- [15] S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljačić, Experimental observation of large Chern numbers in photonic crystals, *Phys. Rev. Lett.* **115**, 253901 (2015).
- [16] H. Wang, S. K. Gupta, B. Xie, and M. Lu, Topological photonic crystals: A review *Front. Optoelectron* **13**, 50 (2020).
- [17] M. Segev and M. A. Bandres, Topological photonics: Where do we go from here? *Nanophoton.* **10**, 425 (2021).
- [18] M. S. Rider, S. J. Palmer, S. R. Pocock, X. Xiao, P. Arroyo Huidobro, and V. Giannini, A perspective on topological nanophotonics: Current status and future challenges, *J. Appl. Phys.* **125**, 120901 (2019).
- [19] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Symmetry and topology in non-Hermitian physics, *Phys. Rev. X* **9**, 041015 (2019).

- [20] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).
- [21] K. Yokomizo, T. Yoda, and S. Murakami, Non-Hermitian waves in a continuous periodic model and application to photonic crystals, *Phys. Rev. Res.* **4**, 023089 (2022).
- [22] C. W. Hsu, B. Zhen, A. D. Stone, J. D. Joannopoulos, and M. Soljačić, Bound states in the continuum, *Nat. Rev. Mater.* **1**, 16048 (2016).
- [23] S. I. Azzam and A. V. Kildishev, Photonic bound states in the continuum: From basics to applications, *Adv. Opt. Mater.* **9**, 2001469 (2021).
- [24] K. L. Koshelev, Z. F. Sadrieva, A. A. Shcherbakov, Y. S. Kivshar, and A. A. Bogdanov, Bound states in the continuum in photonic structures, *Phys.-Usp.* **66**, 494 (2023).
- [25] J. von Neumann and E. P. Wigner, Über merkwürdige diskrete eigenwerte, in *The Collected Works of Eugene Paul Wigner: Part A: The Scientific Papers*, edited by A. S. Wightman (Springer, Berlin, 1993), Vol. 1, pp. 291–293.
- [26] B. Zhen, C. W. Hsu, L. Lu, A. D. Stone, and M. Soljačić, Topological nature of optical bound states in the continuum, *Phys. Rev. Lett.* **113**, 257401 (2014).
- [27] T. Fösel, V. Peano, and F. Marquardt, L lines, C points and Chern numbers: Understanding band structure topology using polarization fields, *New J. Phys.* **19**, 115013 (2017).
- [28] K. Y. Bliokh, M. A. Alonso, and M. R. Dennis, Geometric phases in 2D and 3D polarized fields: Geometrical, dynamical, and topological aspects, *Rep. Prog. Phys.* **82**, 122401 (2019).
- [29] W. Liu, W. Liu, L. Shi, and Y. Kivshar, Topological polarization singularities in metaphotonics, *Nanophoton.* **10**, 1469 (2021).
- [30] T. Yoda and M. Notomi, Generation and annihilation of topologically protected bound states in the continuum and circularly polarized states by symmetry breaking, *Phys. Rev. Lett.* **125**, 053902 (2020).
- [31] J. Wang, L. Shi, and J. Zi, Spin Hall effect of light via momentum-space topological vortices around bound states in the continuum, *Phys. Rev. Lett.* **129**, 236101 (2022).
- [32] C.-Y. Ji, W. Lan, P. Fu, G. Wang, C. Gu, Y. Wang, J. Li, Y. Yao, and B. Liu, Probing phase transition of band topology via radiation topology, *Photon. Res.* **12**, 1150 (2024).
- [33] P. Kokhanchik, H. Sigurdsson, B. Piętko, J. Szczytko, and P. G. Lagoudakis, Photonic Berry curvature in double liquid crystal microcavities with broken inversion symmetry, *Phys. Rev. B* **103**, L081406 (2021).
- [34] A. Kodigala, T. Lepetit, Q. Gu, B. Bahari, Y. Fainman, and B. Kanté, Lasing action from photonic bound states in continuum, *Nature (London)* **541**, 196 (2017).
- [35] H. M. Doeleman, F. Monticone, W. den Hollander, A. Alù, and A. F. Koenderink, Experimental observation of a polarization vortex at an optical bound state in the continuum, *Nat. Photon.* **12**, 397 (2018).
- [36] R. Heilmann, G. Salerno, J. Cuerda, T. K. Hakala, and P. Törmä, Quasi-BIC mode lasing in a quadrumer plasmonic lattice, *ACS Photon.* **9**, 224 (2022).
- [37] V. Ardizzone, F. Riminucci, S. Zanotti, A. Gianfrate, M. Efthymiou-Tsironi, D. Suárez-Forero, F. Todisco, M. De Giorgi, D. Trypogeorgos, G. Gigli *et al.*, Polariton Bose–Einstein condensate from a bound state in the continuum, *Nature (London)* **605**, 447 (2022).
- [38] K. Arjas, J. M. Taskinen, R. Heilmann, G. Salerno, and P. Törmä, High topological charge lasing in quasicrystals, *Nat. Commun.* **15**, 9544 (2024).
- [39] O. Bleu, D. Solnyshkov, and G. Malpuech, Measuring the quantum geometric tensor in two-dimensional photonic and exciton-polariton systems, *Phys. Rev. B* **97**, 195422 (2018).
- [40] A. Gianfrate, O. Bleu, L. Dominici, V. Ardizzone, M. De Giorgi, D. Ballarini, G. Lerario, K. West, L. Pfeiffer, D. Solnyshkov *et al.*, Measurement of the quantum geometric tensor and of the anomalous Hall drift, *Nature (London)* **578**, 381 (2020).
- [41] J. Cuerda, J. M. Taskinen, N. Källman, L. Grabitz, and P. Törmä, Observation of quantum metric and non-Hermitian Berry curvature in a plasmonic lattice, *Phys. Rev. Res.* **6**, L022020 (2024).
- [42] Y.-M. R. Hu, E. A. Ostrovskaya, and E. Estrecho, Generalized quantum geometric tensor in a non-Hermitian exciton-polariton system, *Opt. Mater. Express* **14**, 664 (2024).
- [43] X. Yin, Y. Chen, X. Zhang, Z. Zhang, S. Noda, and C. Peng, Observation of Berry curvature in non-Hermitian system from far-field radiation, *Nat. Commun.* **16**, 2796 (2025).
- [44] L. Polimeno *et al.*, Tuning of the Berry curvature in 2D perovskite polaritons, *Nat. Nanotechnol.* **16**, 1349 (2021).
- [45] K. Łempicka-Mirek, M. Król, H. Sigurdsson, A. Wincukiewicz, P. Morawik, R. Mazur, M. Muszyński, W. Piecek, P. Kula, T. Stefaniuk, M. Kamińska, L. De Marco, P. G. Lagoudakis, D. Ballarini, D. Sanvitto, J. Szczytko, and B. Piętko, Electrically tunable Berry curvature and strong light-matter coupling in liquid crystal microcavities with 2D perovskite, *Sci. Adv.* **8**, eabq7533 (2022).
- [46] J. D. Joannopoulos, S. G. Johnson, J. N. Winn, and R. D. Meade, *Molding the Flow of Light* (Princeton University Press, Princeton, 2008).
- [47] L. Lu, Q. Le-Van, L. Ferrier, E. Drouard, C. Seassal, and H. S. Nguyen, Engineering a light–matter strong coupling regime in perovskite-based plasmonic metasurface: Quasi-bound state in the continuum and exceptional points, *Photon. Res.* **8**, A91 (2020).
- [48] H. Sigurdsson, H. C. Nguyen, and H. S. Nguyen, Dirac exciton–polariton condensates in photonic crystal gratings, *Nanophoton.* **13**, 3503 (2024).
- [49] R. Mermét-Lyauoz, C. Symonds, F. Berry, E. Drouard, C. Chevalier, G. Trippé-Allard, E. Deleporte, J. Bellessa, C. Seassal, and H. S. Nguyen, Taming Friedrich-Wintgen interference in a resonant metasurface: Vortex laser emitting at an on-demand tilted angle, *Nano Lett.* **23**, 4152 (2023).
- [50] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, *Adv. Phys.* **69**, 249 (2020).
- [51] H. Shen, B. Zhen, and L. Fu, Topological band theory for non-Hermitian Hamiltonians, *Phys. Rev. Lett.* **120**, 146402 (2018).
- [52] T. Fukui, Y. Hatsugai, and H. Suzuki, Chern numbers in discretized Brillouin zone: Efficient method of computing (spin) Hall conductances, *J. Phys. Soc. Jpn.* **74**, 1674 (2005).
- [53] G. Salerno, R. Heilmann, K. Arjas, K. Aronen, J.-P. Martikainen, and P. Törmä, Loss-driven topological transitions in lasing, *Phys. Rev. Lett.* **129**, 173901 (2022).
- [54] V. A. Nguyen, H. S. Nguyen, Z. Yuan, D. X. Nguyen, C. Dang, S. T. Ha, X. Letartre, Q. Le-Van, and H. S. Nguyen, First-principles-informed non-Hermitian Hamiltonian for guided resonances in photonic crystal slabs, *arXiv:2507.20033*.

- [55] A. Altland and M. R. Zirnbauer, Novel symmetry classes in mesoscopic normal-superconducting hybrid structures, *Phys. Rev. B* **55**, 1142 (1997).
- [56] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions, *Phys. Rev. B* **78**, 195125 (2008).
- [57] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, Topological insulators and superconductors: Tenfold way and dimensional hierarchy, *New J. Phys.* **12**, 065010 (2010).
- [58] C.-K. Chiu, H.-T. Hsu, J. C. Y. Teo, and S. Ryu, Classification of topological insulators and superconductors in the presence of reflection symmetry, *Phys. Rev. B* **88**, 075142 (2013).
- [59] E. Cornfeld and S. Carmeli, Tenfold topology of crystals: Unified classification of crystalline topological insulators and superconductors, *Phys. Rev. Res.* **3**, 013052 (2021).
- [60] J. Cuerda, J. M. Taskinen, N. Källman, L. Grabitz, and P. Törmä, Pseudospin-orbit coupling and non-Hermitian effects in the quantum geometric tensor of a plasmonic lattice, *Phys. Rev. B* **109**, 165439 (2024).
- [61] Y.-M. R. Hu, E. A. Ostrovskaya, and E. Estrecho, Quantum geometric tensor and wavepacket dynamics in two-dimensional non-Hermitian systems, *Phys. Rev. Res.* **7**, L012067 (2025).
- [62] V. Liu and S. Fan, S^4 : A free electromagnetic solver for layered periodic structures, *Comput. Phys. Commun.* **183**, 2233 (2012).
- [63] M. Minkov, I. A. Williamson, L. C. Andreani, D. Gerace, B. Lou, A. Y. Song, T. W. Hughes, and S. Fan, Inverse design of photonic crystals through automatic differentiation, *ACS Photon.* **7**, 1729 (2020).
- [64] S. Zanotti, M. Minkov, D. Nigro, D. Gerace, S. Fan, and L. C. Andreani, Legume: A free implementation of the guided-mode expansion method for photonic crystal slabs, *Comput. Phys. Commun.* **304**, 109286 (2024).