

## Thick liquid crystalline cholesteric shells

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We numerically investigate the phase behavior of thick shells of cholesteric liquid crystals with tangential anchoring at the shell boundary. For achiral liquid crystals, we demonstrate a thickness-dependent transition from a configuration featuring four disclination lines connecting the inner and outer surfaces to a defect-free bulk state, where each surface is topologically isolated and features two boojums. Incorporating chirality stabilizes defect arrangements, including a mixed state combining boojums and disclination lines, as well as blue phases at high chirality, and we show that shell thickness strongly modulates these transitions. We exploit the metastable features of the observed phases to obtain an elastically induced rearrangement of the shell surfaces during a cholesteric hysteresis cycle, stabilizing an alternative configuration that minimizes the free energy at low chirality. Finally, we consider the case of shells featuring multiple inner isotropic cores, that induce a topological and morphological reorganization, where some inclusions form a connected cluster with the outer shell, while the remaining inclusions become isolated. Our work paves the way for exploring dynamic behaviors under external fields, mixed anchoring conditions, or active flows.

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### I. INTRODUCTION

Liquid crystals (LCs) are a fascinating phase of matter of enduring interest in physics, serving as a versatile platform for studying and exploiting topological phases [1–7]. Theoretically, LCs offer a unique opportunity to investigate topological phenomena in experimentally accessible systems, where the interplay between molecular chirality, confinement, and defect structures gives rise to complex and often surprising behavior. From an applied perspective, LCs exhibit unique optical properties arising from their striking interaction with electromagnetic fields that have positioned LCs at the forefront of applications in nanotechnology, from photonic crystals [8–10], display devices and laser media [11,12] to smart nanomaterials [12–15] and sensing devices [16]. This diverse behavior has motivated a wealth of research, highlighting the tunability of liquid crystals as a platform for creating materials with engineered optical properties.

Confinement plays a crucial role in shaping the optical properties of LC [4,17–23]. By restricting LCs within specific geometries, the topological characteristics of the system—and thus its defect structures—can be precisely controlled. This is particularly impactful because the characteristic sizes of LC defects and distortions often match the wavelength of visible light, allowing LCs to act as exceptional anisotropic media

with remarkable optical properties. A compelling demonstration of the interplay between topology and geometry is provided by liquid crystalline spherical droplets and shells with tangential anchoring. Both conform to the Poincaré-Hopf [17,24,25] theorem, which dictates that the total topological charge on any spherical surface must equal the Euler characteristic of the sphere [3,17,26,27], i.e.,  $+2$ . In LC droplets, this constraint is satisfied by the formation of boojums, namely,  $+1$  defects positioned at the poles. In contrast, thin spherical shells accommodate this requirement through a configuration of four  $+1/2$  defects arranged at the vertices of a regular tetrahedron [28,29].

Chirality introduces a further feature to the study of LCs, playing a central role in stabilizing topological phases [30–40]. Blue phases (BPs) are a remarkable example [9,41–48], characterized by networks of skyrmions that arise from topological frustration as the system accommodates helical twisting in multiple spatial directions by patching together double-twist cylinders. When confined, a cholesteric liquid crystal (CLC) must balance its intrinsic helical twist with the constraints imposed by finite geometry and its topology. This competition between topological and geometric frustration gives rise to phases, enabling precise tuning of LC properties by adjusting the geometry and size of the confining domain. This intricate interplay has been the focus of various studies. For example, de Pablo and co-workers investigated confined CLC droplets and blue phases [49], revealing that confinement not only broadens the typically narrow stability range of BPs but also alters their topological and optical characteristics. Moreover, some of the authors of the present study demonstrated numerically that CLCs confined to the surface of an infinitesimally thin shell exhibit fascinating phases that are robust regardless of the anchoring conditions on the sphere [50,51]. In such configurations, BPs appear as arrays of 2D

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skyrmions distributed across the spherical surface, forming skyrmionic quasi-crystals and amorphous patterns, further expanding the range of phases accessible to confined CLCs.

A particularly interesting setup is the one of *thick* spherical shells of LC with degenerate tangential anchoring. This approach enables the systematic modulation of the competing effects of chirality-induced topological frustration and geometrical confinement, offering a flexible platform to explore the resulting phases and defect structures. Experimental studies on thick shells of CLCs, such as those in Refs. [52,53], have revealed a variety of topological defect arrangements. These include configurations beyond the classic bipolar shell and the four  $+1/2$  defect arrangements, such as the stabilization of a Frank-Pryce-like structure featuring a single  $+2$  defect, as well as intermediate defect configurations. While the equilibrium properties of nematic and cholesteric shells have been previously studied through experiments and simulations [4,52–54], the dynamics of thick shells—particularly under deformable boundary conditions—remain poorly investigated. Our work bridges this gap by introducing a computational framework that captures the coupled hydrodynamic of surface tension and elastic effects in thick cholesteric shells.

Inspired by prior studies on confined cholesteric liquid crystals, in this article we numerically investigate the interplay between topological and geometrical confinement in thick shells of CLCs, to build a phase diagram of their topological features at varying the shell thickness and LC chirality. In this work, we consider the hydrodynamic coupling between the nematic order parameter and the underlying liquid crystalline backflow, critical when studying LC systems featuring metastability [50]. To this end, we use lattice Boltzmann simulations to study the phase behavior of CLCs confined in a thick soft spherical shell modeled as dynamical phase fields, accounting for the intrinsic dynamics of the shell boundary mediated by surface tension.

Our findings reveal how the constraints imposed by the Gauss-Bonnet theorem shape the defect configurations in shells of varying thickness. This requirement, which ensures a total topological charge of  $+2$  on any spherical surface, governs the structures that can emerge under different geometric and chiral conditions. For sufficiently thick shells, we identify distinct phases depending on the chirality, including the bipolar structure,  $+1/2$  disclinations connecting the inner and outer surfaces, and an intermediate phase that combines features of both. Interestingly, the transition from the boojum configuration to the four disclination lines can be induced either by reducing the shell thickness or increasing the chirality. In the latter case, the transition exhibits a metastable region occupied by intermediate states and pronounced hysteresis effects. For thin shells, the most stable configuration is characterized by four disclination lines, with a direct transition to BP.

## II. THE MODEL

To model thick cholesteric shells, we adopt a hydrodynamic approach based on coarse-grained order parameters for nematic ordering and concentration fields. In particular, we consider a multiphase field approach [55,56] and intro-

duce two scalar phase fields  $\phi_i(\mathbf{r}, t)$ , with  $i = 1, 2$ , and the nematic  $Q$ -tensor  $\mathbf{Q}(\mathbf{r}, t)$ , whose principle eigenvector  $\mathbf{n}$ —the so-called director field—defines the local direction of alignment of the LC. Finally, we consider the incompressible velocity field  $\mathbf{v}(\mathbf{r}, t)$ . The equilibrium of the system is defined by the free energy

$$\mathcal{F} = \int d\mathbf{r} (f^\phi + f^{\text{LC}} + f^{\text{anch}}), \quad (1)$$

where  $f^\phi$  is the concentration free energy density,  $f^{\text{LC}}$  is the liquid crystal free energy accounting for the isotropic-nematic transition and elastic properties of the LC, and  $f^{\text{anch}}$  defines the anchoring properties of the LC at the shell interface. In particular,

$$f^\phi = \sum_{i=1}^2 \frac{a}{4} \phi_i^2 (\phi_i - \phi_0)^2 + \frac{k_\phi}{2} \sum_{i=1}^2 (\nabla \phi_i)^2 + \epsilon \phi_1^2 \phi_2^2. \quad (2)$$

For  $a, k_\phi > 0$ ,  $f^\phi$  describes phase separating fields with two minima at  $\phi_i = \phi_0$  and  $\phi_i = 0$  that, at equilibrium, arrange into spherical droplets with surface tension  $\sigma = \sqrt{8ak_\phi/9}$ , and interface thickness  $\xi_\phi = \sqrt{2k_\phi/a}$ . The interaction term proportional to  $\epsilon$  gives rise to a repulsive contribution ( $\epsilon > 0$ ) that allows for the stabilization of the emulsion. The LC free energy density  $f^{\text{LC}} = f^{\text{bulk}} + f^{\text{el}}$ , where

$$f^{\text{bulk}} = A_0 \left[ \frac{1}{2} \left( 1 - \frac{\chi(\bar{\phi})}{3} \right) \text{Tr} \mathbf{Q}^2 - \frac{\chi(\bar{\phi})}{3} \text{Tr} \mathbf{Q}^3 + \frac{\chi(\bar{\phi})}{4} (\text{Tr} \mathbf{Q}^2)^2 \right], \quad (3)$$

$$f^{\text{el}} = f^{\text{sb}} + f^{\text{tw}} = \frac{L}{2} [(\nabla \cdot \mathbf{Q})^2 + (\nabla \times \mathbf{Q} + 2q_0 \mathbf{Q})^2]. \quad (4)$$

In Eq. (3), the bulk constant  $A_0 > 0$ , and  $\chi(\bar{\phi})$  is a temperaturelike parameter that drives the isotropic-nematic transition that occurs for  $\chi(\bar{\phi}) > \chi_{cr} = 2.7$  [1], where we have denoted the global concentration field with  $\bar{\phi} = \phi_1 + \phi_2$ . We choose  $\chi = \chi_0 + \chi_s \bar{\phi}$ , with  $\chi_0 > 2.7$  and  $\chi_s < 0$  to confine the LC within the region where  $\bar{\phi} = 0$ . The elastic free energy density in Eq. (4) proportional to the elastic constant  $L$  captures the energy cost of elastic deformations in the single elastic constant approximation [1]. It has been split into a splay-bend contribution  $f^{\text{sb}}$  and a twist contribution  $f^{\text{tw}}$ , which stabilizes helical structures when the chiral wave number  $q_0$  is nonzero. For  $q_0 > 0$ , the equilibrium configuration in unconfined geometries features a right-handed helix with pitch  $p_0 = 2\pi/q_0$ . Finally, the anchoring contribution to the free energy

$$f^{\text{anch}} = W \sum_{i=1}^2 [\partial_\alpha \phi_i Q_{\alpha\beta} \partial_\beta \phi_i] \quad (5)$$

defines the orientation of the LC at the shell's interface. We choose  $W > 0$  to impose tangential anchoring to the surface.

The dynamics of the fields is governed by the following set of coupled partial differential equations:

$$D_t \phi_i = M \nabla^2 \mu_i, \quad (6)$$

$$D_t \mathbf{Q} = \mathbf{S}(\mathbf{W}, \mathbf{Q}) + \gamma^{-1} \mathbf{H}, \quad (7)$$

$$\rho D_t \mathbf{v} = \nabla \cdot (\boldsymbol{\sigma}^{\text{hydro}} + \boldsymbol{\sigma}^\phi + \boldsymbol{\sigma}^{\text{LC}}), \quad (8)$$

where the operator  $D_t = \partial_t + \mathbf{v} \cdot \nabla$  denotes the material derivative. Equation (6) is a Cahn-Hilliard equation for the conserved concentration field  $\phi_i$ , with  $M$  the mobility parameter, and  $\mu_i = \delta\mathcal{F}/\delta\phi_i$  for  $i = 1, 2$  denoting the chemical potential. Equation (7) is the Beris-Edwards equation ruling the dynamics of the  $Q$ -tensor. The operator  $S(\mathbf{W}, \mathbf{Q})$  denotes the corotational derivative and defines the dynamical response of the LC to straining and shearing. Its explicit form, reported in Appendix A 1, depends on both the velocity gradient  $\mathbf{W} = \nabla \mathbf{v}$  and the  $Q$ -tensor configuration. The coefficient  $\gamma$  is the rotational viscosity measuring the relative importance of advection with respect to relaxation, and the molecular field  $\mathbf{H} = -\frac{\delta\mathcal{F}}{\delta\mathbf{Q}} + (\mathbf{I}/3)\text{Tr}\frac{\delta\mathcal{F}}{\delta\mathbf{Q}}$ . Finally, Eq. (8) is the Navier-Stokes equation for the incompressible velocity field ( $\nabla \cdot \mathbf{v} = 0$ ) with constant density  $\rho$ . Here, the stress tensor has been divided into (1) a hydrodynamic contribution  $\boldsymbol{\sigma}^{\text{hydro}} = -P\mathbf{I} + \eta\nabla\mathbf{v}$  accounting for the hydrodynamic pressure  $P$  ensuring incompressibility, and viscous effects, proportional to the viscosity  $\eta$ ; (2) an interface contribution  $\boldsymbol{\sigma}^\phi$ ; and (3) a LC contribution  $\boldsymbol{\sigma}^{\text{LC}}$  accounting for elastic and flow-aligning effects and whose explicit expressions are reported in Appendix A 2. Unlike mesoscopic methods that rely on molecular pseudopotentials [54], our Landau-de Gennes mean-field framework provides an optimal balance of spatial and temporal resolution and computational efficiency while naturally incorporating hydrodynamic couplings essential for capturing the system's evolution. In addition, hydrodynamics actually helps overcoming the energy barrier typical of metastable states, and hop over free energy barriers and kinetic traps often displayed by chiral liquid crystals.

### Dimensionless numbers

Thick LC shells can be obtained by encapsulating an isotropic droplet of the phase field  $\phi_1$  with radius  $R_1$  ( $\phi_1 = \phi_0$  for  $r < R_1$ ) into an inverted larger droplet of the phase field  $\phi_2$  with radius  $R_2 > R_1$  ( $\phi_2 = \phi_0$  for  $r > R_2$ ). This effectively realizes a shell of the total concentration field characterized by  $\bar{\phi} = 0$  where the LC is in the nematic phases, for  $R_1 < r < R_2$ .

In the following, we will fix the radius of the external droplet  $R_2$  and vary the radius  $R_1$  of the internal core. A suitable dimensionless number to characterize the shell thickness is therefore

$$r = \frac{R_2 - R_1}{R_2}, \quad (9)$$

which ranges between 0 and 1. For the limit case of a thin shell,  $R_1 \approx R_2$  and  $r \approx 0$ ; conversely for the case of a full droplet, in absence of an internal core,  $R_1 = 0$  and  $r = 1$ .

The cholesteric power of the LC will be expressed in terms of the reduced chirality

$$\kappa = \sqrt{\frac{108q_0^2L}{A_0\chi_0}}, \quad (10)$$

proportional to the ratio  $\ell_c/p_0$ , where the coherence length  $\ell_c = \sqrt{L/A_0}$  is a measure of how rapidly nematic order decays in proximity of a topological defect.

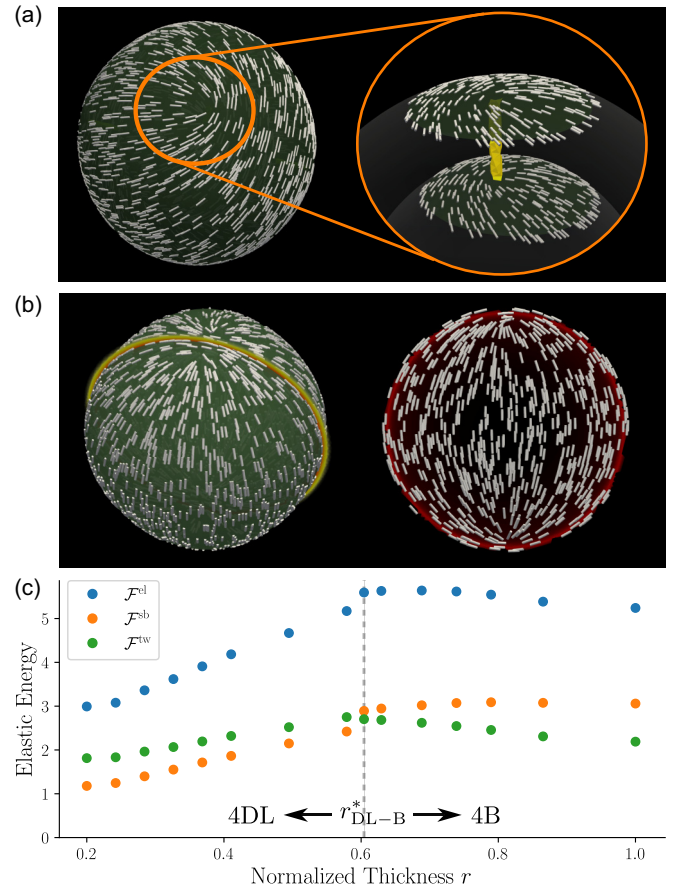


FIG. 1. Nematic shells. (a) Thin LC shells are characterized by four  $+1/2$  topological defects. The defects on the inner and outer surface of the shell are terminal points of an open disclination line (yellow line in the zoom). (b) Full nematic LC droplets arrange into a bipolar configuration, featuring two antipodal *boojums*. These defective structures consist of a  $+1$  surface defect that escapes in the third dimension in the droplet bulk, leaving the droplet's interior defect free. Right panel shows the nematic arrangement in the droplet's cross section along the plane highlighted in yellow on the left. (c) Elastic energy  $\mathcal{F}^{\text{el}}$  at varying the normalized shell thickness  $r$ . Dashed gray line marks the transition point between 4DL (four disclination lines connecting the inner and outer surfaces) and 4B (four boojums) configurations at  $r_{\text{DL-B}}^* \approx 0.6$ .

### III. NONCHIRAL NEMATIC SHELLS

We begin our discussion by analyzing the topological properties and defect configurations of nematic shells with variable thickness  $r$  in the absence of chirality ( $\kappa = 0$ ).

In the limiting case of an infinitesimally thin shell ( $r = 0$ ), the LC behaves as a two-dimensional system, and the equilibrium configuration is characterized by four defects of charge  $s = +1/2$  arranged in the so-called baseball ball configuration, summing to a total topological charge of  $+2$ , as required by the Poincaré-Hopf theorem [see Fig. 1(a)]. While the topological constraint allows for any defect configuration with a total charge equal to the Euler characteristic of the sphere  $\chi_E = 2$ , the equilibrium state minimizes the free energy, as described by Eq. (1), for the given geometry. For instance, the topological requirement could also be satisfied by two

integer defects of charge  $s = +1$ . However, such a configuration is energetically unfavorable because the energy of a two-dimensional defect scales with the square of its charge,  $\mathcal{F}^{\text{def}} \sim s^2$ . Consequently, while topologically equivalent, a configuration with four half-integer defects is energetically preferred, as it reduces the defect energy by half compared to the two-integer-defect arrangement. When the shell thickness  $r$  increases to finite values, the topology of the system changes. Now, the topological constraint must be satisfied on both the internal and external surfaces of the LC shell. For thin shells, the baseball ball pattern of four defects is reproduced across concentric spherical surfaces, from the outer to the inner shell surface. This leads to the stabilization of four disclination lines connecting the  $+1/2$  defects on the outer and inner surfaces of the shell, as illustrated in the magnified view in Fig. 1(a).

For the limit case of a full LC droplet ( $r = 1$ ), the topological requirement of a total charge of  $+2$  on the droplet surface remains unchanged. However, in this case, the energetically favored configuration must minimize the elastic energy of the whole droplet, not just the surface. This leads to a drastic change in the defect arrangement. The four disclination lines give way to the *bipolar* configuration, i.e., two antipodal *boojums* on the droplet surface. These singular structures consist of a surface  $+1$  defect that escapes into the third dimension within the droplet's bulk, leaving it free of disclination lines [see Fig. 1(b)]. In this regard, it is important to remember that the radial symmetry of  $+1$  defects allows the director to rotate away from the plane of the defect, so that the singular behavior in the bulk *softly and suddenly vanishes away* [27,57]. This is in stark contrast to semi-integer defects, where a continuous out-of-plane rotation of the director around the singularity would introduce a discontinuity along a ray extending from the singularity—commonly referred to as Volterra's cut.

Interestingly, we observe a configuration analogous to the bipolar arrangement for very thick shells ( $r \lesssim 1$ ). In these cases, the topological requirement on the inner shell surface is satisfied by the emergence of two additional boojums, while the nematic pattern in the bulk remains devoid of disclination lines [see Figs. 2(a) and 2(b) for reference].

Having described the two limiting cases, we now address the transition between these states as the shell thickness  $r$  varies. Specifically, we find that the system undergoes a transition from a configuration with four disclination lines connecting the inner and outer surfaces (4DL) to one characterized by four boojums (4B), where the defect lines vanish entirely from the bulk. This transition occurs at a critical shell thickness of  $r_{\text{DL-B}}^* \approx 0.6$ . We rationalize this transition as a balance between the elastic energy contributions from the bulk and the surfaces. For configurations with disclination lines, the  $+1/2$  surface defects are energetically favored over integer defects due to their lower surface energy. However, the energy of the disclination lines scales linearly with their length, making them increasingly costly as the shell thickness grows. In contrast, boojums, despite their higher surface energy due to the  $+1$  charge, completely eliminate disclination lines from the bulk. This results in a weakly deformed nematic director field within the bulk, significantly reducing the bulk elastic energy for  $r > r_{\text{DL-B}}^*$ . This is evidenced by the behavior of the splay-bend contribution to the elastic energy, which stabilizes

at a constant value beyond the transition [see Fig. 1(c)]. This is primarily attributed to localized splay deformations near the boojums. Meanwhile, the twist contribution to the elastic energy decreases as the shell thickness increases, further minimizing the total elastic energy.

#### IV. CHOLESTERIC LC SHELLS

We now turn our attention to cholesteric liquid crystalline shells. In the presence of chirality, we observe a wide range of configurations that depend on both the chirality of the liquid crystal and the shell thickness. We begin by describing their topological properties and then discuss their stability.

**4B.** As outlined in the previous section, the topological constraint of a total  $+2$  topological charge can be satisfied by the formation of two boojums on both the inner and outer surfaces of the shell [Fig. 2(a)]. When chirality is introduced, the internal arrangement of the liquid crystal adapts to incorporate a twisting deformation with a pitch determined by the equilibrium cholesteric wave number  $q_0$  [Fig. 2(b)]. Under cross-polarized light oriented parallel to the axis defined by the four boojums, the transmitted light intensity is enhanced along this axis and displays a characteristic modulation in the transverse direction, reflecting the twisted liquid crystal pattern.

**4DL.** In the presence of chirality, the 4DL structure retains its defining feature of four disclination lines connecting semi-integer defects on the inner and outer surfaces of the shell. Due to the imposed equilibrium twist, the disclination lines adopt a helical configuration [Fig. 2(c)]. In particular, at sufficiently high chirality, the liquid crystal in the bulk forms skyrmionic structures that emerge between each pair of disclination lines, as illustrated in the cross section shown in Fig. 2(d). Cross-polarizer experiments reveal a distinctive quadrupolar pattern, with shadowed regions corresponding to the positions of the four disclinations and intensity modulations indicative of the internal twisted arrangement.

**2B-2DL.** Not encountered in absence of chirality, the 2B-2DL structure is a mixed structure resulting from the combination of the 4B and 4DL configuration. It features two boojums (one on the internal and one on the external surface) and two disclination lines connecting semi-integer surface defects on the internal and external surface [Fig. 2(e)]. The resulting configuration in the bulk is characterized by an escaped configuration in the region occupied by the two boojums and a skyrmionic structure separating the two disclination lines. The cross-polarizer textures reveal a strong luminosity in correspondence of the skyrmionic structure [Fig. 2(f)]. Notably, we observe that the inner core of the shells exhibiting a 2B-2DL structure is *slightly* displaced from the center, shifting toward the shell surface along the direction of the disclination lines.

**BP.** The BP is a topological phase that emerges at sufficiently high chirality and is characterized by a three-dimensional network of disclination lines [Fig. 2(g)], which accommodate an array of double-twist cylinders in the director field [Fig. 2(h)]. In our simulations, the disclination network does not exhibit periodicity, resulting in a structure that resembles an unconfined blue phase of type

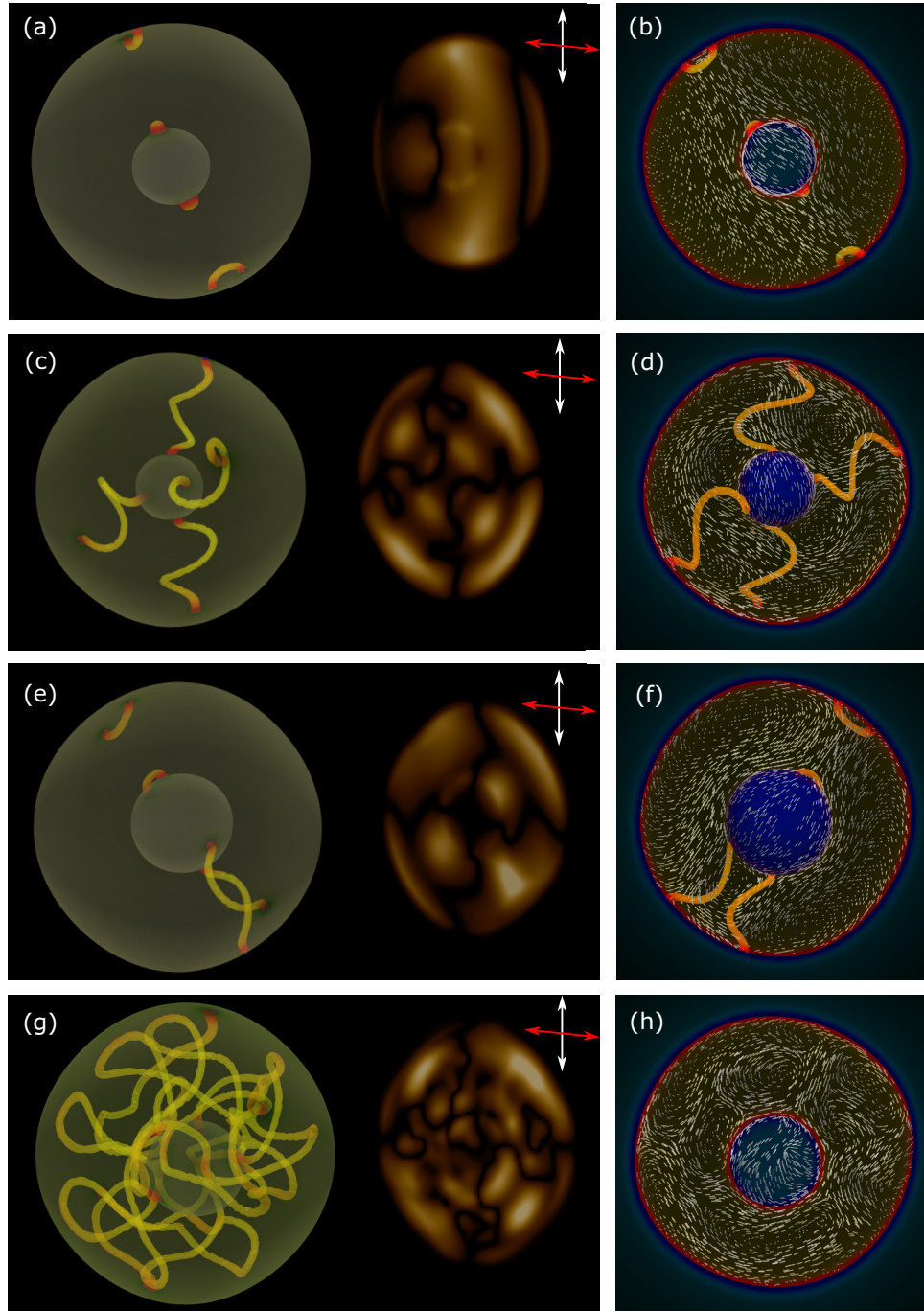


FIG. 2. CLC shells. Typical defect configurations, cross-polarizer textures, and bulk arrangement of director field for (a), (b) 4B state ( $r = 0.74; \kappa = 0.2$ ); (c), (d) 4DL state ( $r = 0.76; \kappa = 0.6$ ); (e), (f) 2B – 2DL state ( $r = 0.66; \kappa = 0.64$ ); and (g), (h) BP state ( $r = 0.68; \kappa = 0.84$ ). The red-white arrows at the top right corner of the panels on the left define the cross-polarizer orientation. Right panels [(b), (d), (f), and (h)] show the director field (white rods) on the inner shell surface and on the central slice parallel to the cross-polarizer plane. In all panels, the full three-dimensional disclination lines are shown as yellow-orange curves, except in panel (h), where they are omitted for clarity due to the complexity of the BP disclination network.

III, commonly referred to as *blue fog* [41,45] due to its diffuse and irregular optical appearance under cross-polarizers.

#### Phase behavior of CLC shells

We now analyze the stability of the four states discussed in the previous section as a function of shell thickness  $r$

and reduced chirality  $\kappa$ . The complete phase diagram of the system is shown in Fig. 3.

In Sec. III, we described the transition from the 4DL configuration (green region in Fig. 3) to the 4B configuration (red region) as the shell thickness exceeds the critical value  $r_{DL-B}^*$ . For low chirality ( $\kappa^* \simeq 0.2$ ), this transition remains unchanged. This is because for chirality to affect the

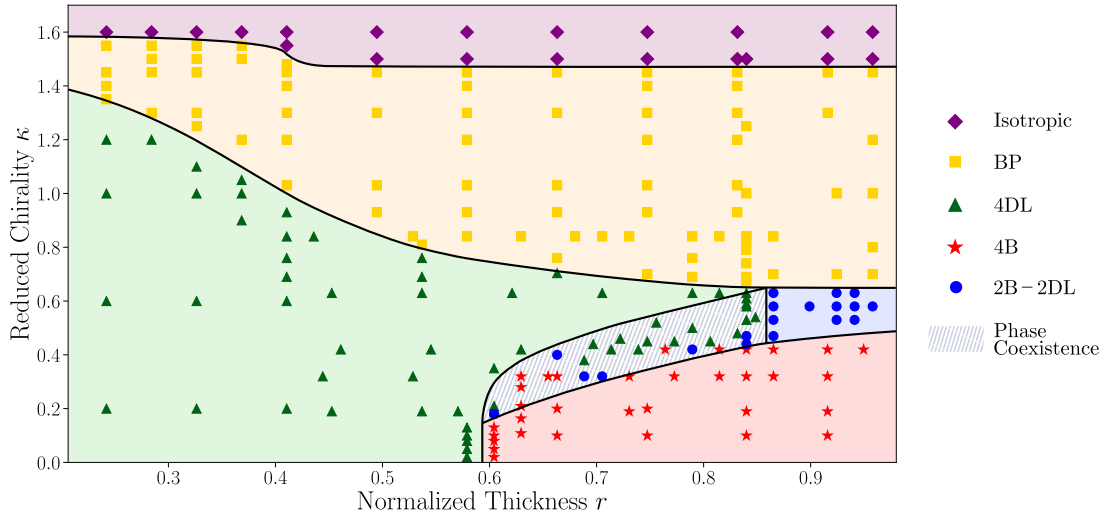


FIG. 3. Phase diagram. Phase diagram at varying normalized shell thickness  $r$  and reduced chirality  $\kappa$ .

transition, the equilibrium pitch must be comparable to the shell thickness, i.e.,  $p_0 \approx R_2 r_{\text{DL-B}}^*$ , which corresponds to  $\kappa = 0.22$ , in agreement with our observations.

As chirality increases beyond  $\kappa^*$ , the stability range of the 4DL configuration extends to thicker shells. This occurs because the topological frustration introduced by chirality forces the liquid crystal in the bulk to a chiral arrangement, leading to helical disclination lines that accommodate skyrmionic structures [Figs. 2(c) and 2(d)]. Notably, the transition between 4DL and 4B regions in the phase diagram of Fig. 3 is now separated by a phase coexistence region (shaded in Fig. 3), where 4DL, and 4B, and 2B-2DL configurations are observed. The 2B-2DL configuration stabilizes for thicker shells ( $r \gtrsim 0.82$ ) and intermediate chirality values ( $0.4 \lesssim \kappa \lesssim 0.6$ ) colored blue in Fig. 3. For higher chirality values ( $\kappa > \kappa_{\text{BP}}^* = 0.6$ ), BP states emerge for thick shells (yellow region in the phase diagram of Fig. 3). This roughly corresponds to the transition points between a helical state and the BP in an unconfined geometry [48,49]. Interestingly, the frustration of the helical LC pattern induced by the shell thickness at small  $r$ , allows for the stabilization of 4DL states even at large chirality. This effect is particularly evident for  $r < r_{\text{DL-B}}^*$ , consistent with the requirement that the pitch  $p_0$  of the blue phase's double-twist cylinders matches the shell thickness. Finally, for  $\kappa > \kappa_{\text{ISO}} = 1.5$ , the BP structure transitions to the isotropic phase (purple region Fig. 3), independently of the thickness, analogously to what happens in bulk and under spherical confinement [48,49].

To better characterize the transitions between the various phases described above, Fig. 4 shows the elastic energy  $\mathcal{F}^{\text{el}}$  and the total length of the disclination lines  $\ell$  normalized by the shell thickness  $\Delta R$  as functions of chirality  $\kappa$  for a shell with reduced thickness  $r = 0.84$  (details on the numerical implementation of disclination line detection provided in Appendix C 1). At low chirality, the shell adopts the 4B configuration, where disclination lines are effectively absent. As chirality increases, the system transitions to the region characterized by phase coexistence, where it alternates between the 2B-2DL and 4DL configurations. This transition is marked by a significant rise in the total length of the disclination lines.

In this region, we perform up to 10 independent simulations for each parameter point, starting from different random initial conditions. Among these, the configuration attaining the lowest final free energy is reported in the phase diagram of Fig. 3 as the representative state. With further increases in chirality, the disclination lines lengthen as they develop helical structures that extend beyond the shell thickness  $\Delta R$ . When the system enters BP, the length of the disclination lines  $\ell$  continues to grow due to the formation of the network of double-twist cylinders and their shrinkage as  $\kappa$  increases. This effect is further accompanied by the monotonous increase in the elastic energy. Finally, as chirality exceeds  $\kappa_{\text{ISO}}$ , the system transitions to the isotropic phase, causing the elastic energy to drop.

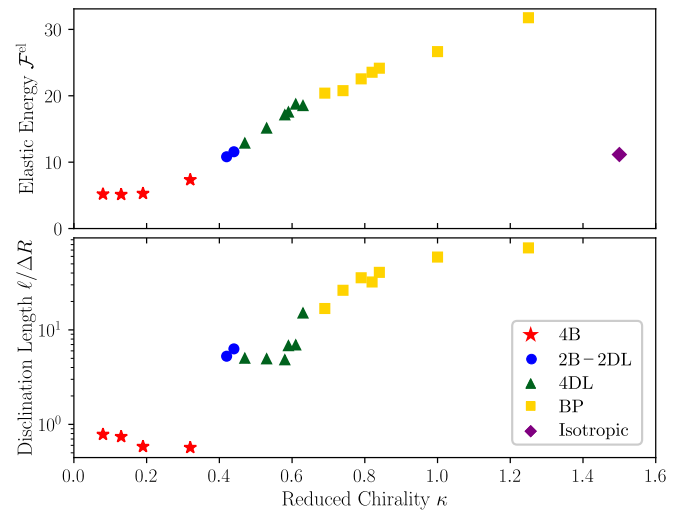


FIG. 4. Characterization as a function of  $\kappa$  for  $r = 0.84$ . The top panel shows the total elastic energy  $\mathcal{F}^{\text{el}}$ , plotted as a function of the reduced chirality  $\kappa$ . The bottom panel displays the total disclination line length  $\ell$ , normalized by the shell thickness  $\Delta R = R_2 - R_1$ , with the y axis plotted on a logarithmic scale. Data points in both panels are color coded according to the phase diagram in Fig. 3, as indicated by the legend in the bottom panel.

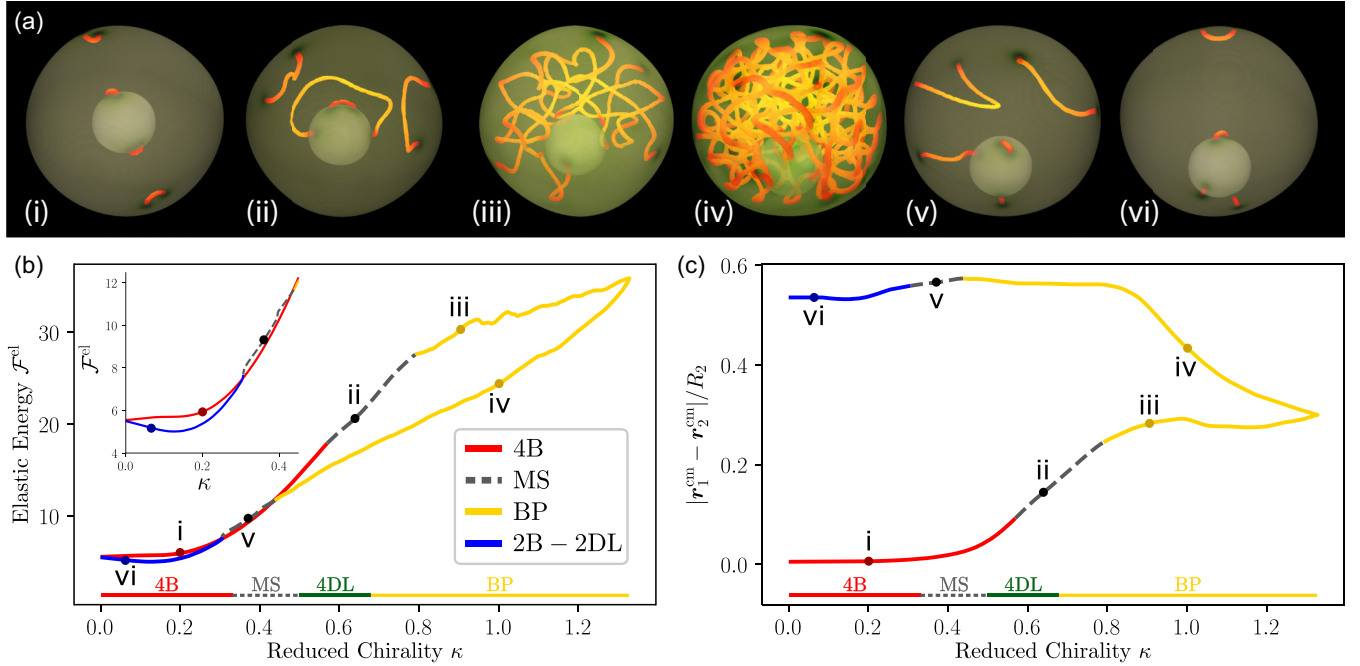


FIG. 5. Hysteresis test at  $r = 0.71$ . (a) Gallery of configurations observed during the hysteresis test, with disclination lines highlighted in orange. (b) Elastic free energy,  $\mathcal{F}^{\text{el}}$ , as a function of reduced chirality,  $\kappa$ . The inset provides a closer view of the low-chirality region during the hysteresis test. (c) Relative displacement of the center of mass of the internal droplet,  $\mathbf{r}_1^{\text{cm}}$ , with respect to the center of mass of the external droplet,  $\mathbf{r}_2^{\text{cm}}$ , normalized by the radius of the external droplet,  $R_2$ . The curves in panels (b) and (c) are color coded to correspond to the configurations described in the legend of panel (b). Colored dots in panels (b) and (c), labeled with Roman numerals, indicate the configurations shown in panel (a). The colored bars at the bottom of panels (b) and (c) denote the observed phases, as reported in the phase diagram of Fig. 3 for  $r = 0.71$ . The dashed region labeled as MS refers to the metastability region observed in panels (a)(ii) and (a)(v).

## V. HYSTERESIS TEST

To further investigate the nature of the transitions between configurations, we perform a hysteresis test by quasi-statically increasing the chirality of the CLC in a shell with reduced thickness  $r = 0.71$ . Starting from a chirality-free initial state in the 4B configuration [Fig. 5(a)(i)], we tracked the changes in configuration by plotting the elastic free energy  $\mathcal{F}^{\text{el}}$  as a function of reduced chirality in Fig. 5(b). Notably, the 4B configuration remained stable up to  $\kappa \simeq 0.57$ , well beyond the transition point  $\kappa_{\text{PC-DL}}(r = 0.71) \simeq 0.5$  between phase coexistence and 4DL state, therefore indicating strong metastability within the phase coexistence region identified in the phase diagram. Beyond  $\kappa \simeq 0.57$ , the system transitioned into spurious metastable configurations, as shown in Figs. 5(a)(ii) and 5(a)(v). This state is labeled in Figs. 5(b) and 5(c) as MS. Here, the two boojums on the external surface split into pairs of semi-integer surface defects connected by disclination lines. On the inner shell, one boojum remained stable, while the other split into semi-integer defects connected by a disclination running through the droplet's bulk. This rearrangement significantly affected the internal shell surface, which shifted in the direction of lower disclination density. To quantify this effect, Fig. 5(c) plots the relative displacement of the internal droplet's center of mass  $\mathbf{r}_1^{\text{cm}}$  with respect to the external droplet's center of mass  $\mathbf{r}_2^{\text{cm}}$  (details are given in Appendix C 2). The internal droplet's asymmetry increased markedly with chirality and stabilized once the system entered the BP phase at  $\kappa \simeq 0.75$ . In this phase, the displaced inter-

nal core favored the formation of a network of double-twist cylinders in the region vacated by the core [Fig. 5(a)(iii)]. Upon further increasing  $\kappa$ , the center-of-mass displacement stabilized until we reversed the process by quasi-statically decreasing chirality. During this stage, the inner core was pushed rapidly toward the outer shell, as shown in Figs. 5(a)(iv) and 5(c). This effect arose because the decreasing chirality enlarged the double-twist cylinders, generating elastic pressure in disclination-rich regions that progressively pushed the inner core outward. This process also caused the elastic free energy to decrease more rapidly than during the chirality-increasing phase [Fig. 5(b)]. Below  $\kappa \simeq 0.4$ , the disclination network collapsed, leaving behind metastable spurious configurations [Fig. 5(a)(v)], which eventually transitioned to the 2B-2DL state at  $\kappa \simeq 0.3$ . However, unlike the thermalization experiments used to draw the phase diagram of Fig. 3, which began from a high-temperature isotropic state, the internal core in this case remained off-center, almost contacting the external surface [Fig. 5(a)(vi)]. In this configuration, the disclination lines had minimal extension, enabling a defect-free bulk configuration in the limit of vanishing chirality. The energy of this state matched that of the 4B configuration [Fig. 5(a)(i)], as shown in the inset of Fig. 5(b).

## VI. MULTICORE CHIRAL SHELLS

We now turn to liquid crystal droplets containing multiple isotropic inclusions. In this set of numerical experiments, the size of the inner droplets is kept constant, while we

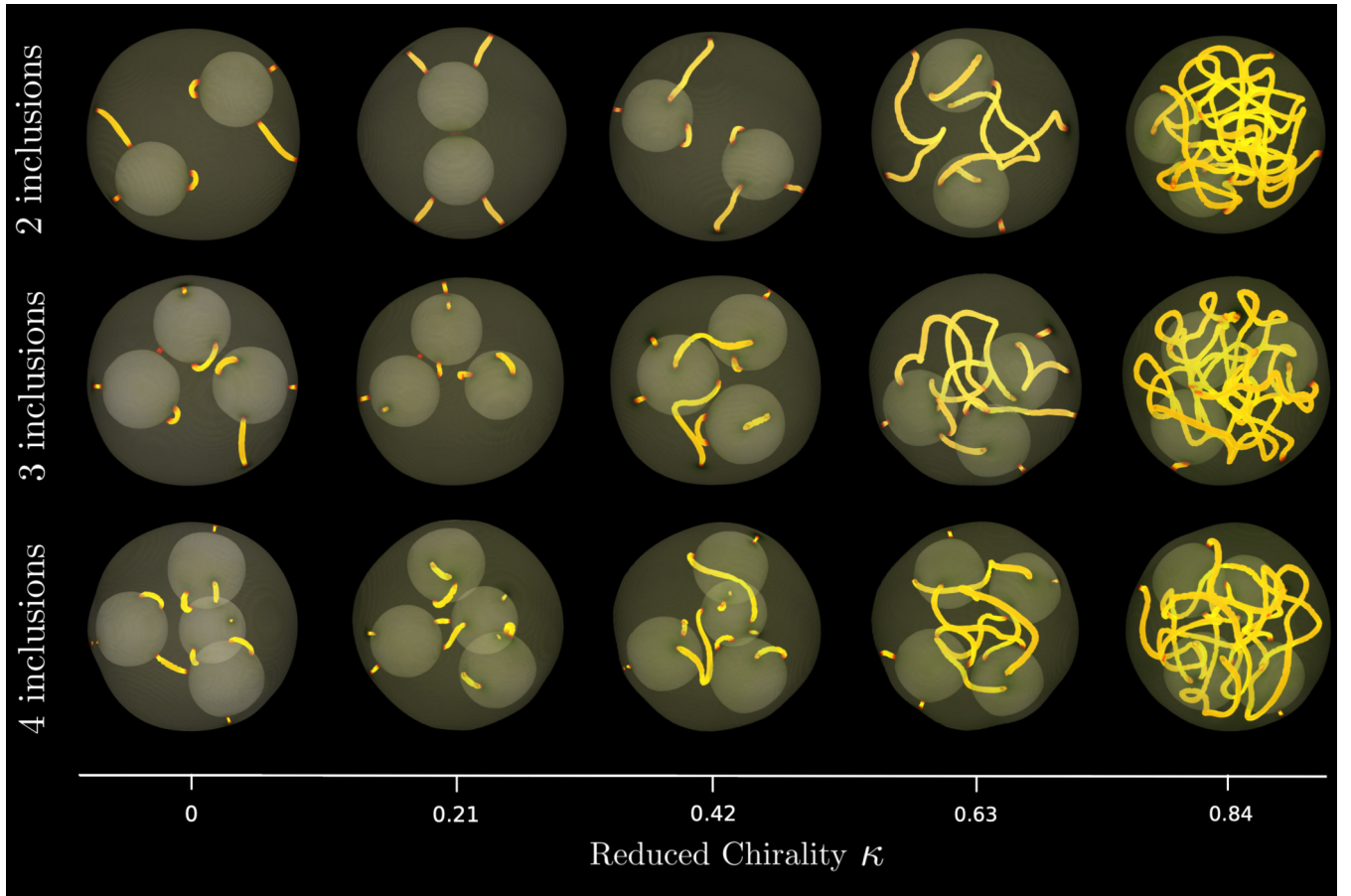


FIG. 6. Multiple inclusions. Gallery of defect configurations observed in multicore chiral shells. Disclination lines are highlighted in yellow. The size of the inner isotropic droplets is fixed at  $R_1 = 12$  ( $r = 0.68$ ).

systematically vary both the number of inclusions and the reduced chirality parameter,  $\kappa$ .

We begin by considering the achiral case ( $\kappa = 0$ ), shown in the first column of Fig. 6. For droplets with two inclusions, the defect configuration is relatively symmetric: The outer shell hosts four  $+1/2$  disclinations, each connected to a  $+1/2$  defect on the surface of an inclusion. To satisfy the topological constraint on each inclusion, a boojum forms on its surface. Increasing the number of inclusions to three introduces a more intricate network. One inclusion is connected to the outer shell by two disclination lines and carries a boojum, while the other two inclusions are linked to the outer shell through single disclination lines and are mutually connected by an additional disclination; these inclusions also exhibit two boojums each. When the number of inclusions reaches four, the system reorganizes into two pairs of droplets. Within each pair, the inclusions are connected by a disclination line terminating at  $+1/2$  defects on their surfaces, while each droplet also connects to the outer shell through a separate disclination and carries a boojum.

Introducing chirality modifies these configurations differently for cases with different number of inclusions. For moderate chirality ( $\kappa \lesssim 0.5$ ), the overall topology for two-core emulsions remains unchanged, with chirality primarily affecting the length, and twisting of the disclination lines. However, for higher number of inclusions (three- and

four-core emulsions), the system undergoes a topological reorganization. In these cases, two inclusions preferentially localize near the outer shell and are connected to both the outer shell and each other by disclination lines, while the remaining inclusion(s) remain isolated, each hosting two boojums (see also Movies S1 and S2 in the Supplemental Material [58]). This behavior suggests that chirality energetically favors configurations with fewer disclination lines, promoting defect-free twisting regions instead.

Upon further increasing chirality ( $\kappa \gtrsim 0.6$ ), the system transitions into a blue phase. Here, the inclusions migrate toward the outer shell, with a dynamics similar to that previously described in the hysteresis test of Sec. V, leaving behind a continuous liquid crystalline matrix that develops an intricate defect lattice characteristic of the blue phase. The segregation of the inner cores to the outer surface effectively decouples the topology of the inclusions from the bulk defect network, therefore allowing the LC to develop double-twist structures expected in bulk for large chirality.

## VII. DISCUSSION AND CONCLUSIONS

In this study, we numerically investigated the interplay between topological and geometrical constraints in thick CLC shells. Our numerical approach enables the simulation of the full nemato-hydrodynamic equations in a shell geome-

try of arbitrary thickness while accounting for the intrinsic dynamics of the shell boundary mediated by surface tension and anchoring couplings with the confined LC. For nematic LC shells without chirality, we demonstrate that increasing shell thickness drives a transition from a 4DL configuration, where the inner and outer shell surfaces are connected by four disclination lines, to a configuration in which the two shell surfaces are topologically isolated, each featuring two boojums (4B). This transition arises from a balance between elastic bulk and surface energies. Thin shells favor the 4DL arrangement due to the reduced surface energy of  $s = +1/2$  defects, whereas thicker shells minimize bulk elastic energy by adopting the 4B state, which eliminates disclination lines. Introducing chirality stabilizes configurations not observed in nonchiral nematic LCs. At low chirality, corresponding to a pitch larger than the shell thickness, the behavior remains analogous to nematic shells, with transitions dominated by shell thickness alone. At higher chirality, however, chiral frustration leads to helical distortions of the disclination lines and the emergence of skyrmionic structures. For intermediate chirality and thick shells, we observe a mixed state (2B-2DL), which combines boojums and disclination lines, balancing elastic energy contributions from bulk and surface. At high chirality, the system transitions to the BP state, characterized by a disordered network of double-twist cylinders, resembling the unconfined blue fog phase. Interestingly, the transition to the BP state is strongly influenced by shell thickness, with thinner shells delaying the onset of this phase. Our simulations also reveal phase coexistence between 4DL, 4B, and 2B-2DL states, suggesting a rich energy landscape with multiple local minima. This underscores the importance of the system's history, initial conditions, and external perturbations in determining the equilibrium configuration.

Our findings are qualitatively consistent with prior experiments on CLC shells. In Refs. [52,53], Darmon *et al.* reported five distinct defect morphologies, three of which—4B, 4DL, and 2B-2DL—also arise in our simulations. The remaining two experimental states involve (1) two disclination lines joining the inner and outer interfaces with local  $+1/2$  and  $+3/2$  profiles, and (2) a single defect on each surface (net surface charge  $+2$ ) connected by a Dirac string (the Frank-Pryce structure [4]). We do not observe these latter configurations numerically, plausibly because they are metastable (the first is rare experimentally) and/or require elastic anisotropy or anchoring asymmetry beyond the one-elastic-constant approximation adopted here. Despite this, the close agreement on the most frequently realized states supports the validity of our framework, which we use to build the full phase diagram and reconstruct the three-dimensional organization of disclination networks. Importantly, to address potential metastability induced by geometric constraints, we integrate the full hydrodynamic  $Q$ -tensor dynamics, explicitly coupling elasticity and flow to capture defect transport, reconnection, and annihilation, which guide the relaxation of the system together with shell deformations, as highlighted by the hysteresis test. Finally, We further generalize to multicore shells and uncover a geometry-controlled topological recombination driven by the additional inner interfaces. Taken together, these results provide a quantitative, predictive bridge between theory and experiment and reveal design principles for steering defect topology in CLC shells.

These findings form the basis for future investigations into the dynamic behavior of these systems under external stimuli, mixed boundary conditions, or in multicore emulsions. Additionally, it would be intriguing to explore how activity and self-sustained flows [55,59–64,66] influence the topological states observed at equilibrium.

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [65].

## APPENDIX A: NEMATO-HYDRODYNAMICS

### 1. Corotational derivative

The explicit expression of the corotational derivative  $S(\mathbf{W}, \mathbf{Q})$  appearing in Eq. (7) is given by

$$S(\mathbf{W}, \mathbf{Q}) = (\xi \mathbf{D} + \mathbf{\Omega})(\mathbf{Q} + \mathbf{I}/3) + (\mathbf{Q} + \mathbf{I}/3)(\xi \mathbf{D} - \mathbf{\Omega}) - 2\xi(\mathbf{Q} + \mathbf{I}/3)\text{Tr}(\mathbf{Q}\mathbf{W}), \quad (\text{A1})$$

Here,  $\mathbf{D} = (\mathbf{W} + \mathbf{W}^T)/2$  and  $\mathbf{\Omega} = (\mathbf{W} - \mathbf{W}^T)/2$  are the symmetric and antisymmetric part of the velocity gradient tensor  $W_{\alpha\beta} = \partial_\beta v_\alpha$ , respectively. The flow alignment parameter  $\xi$  determines the aspect ratio of the LC molecules and the dynamical response of the LC to an imposed shear flow. Here, we choose  $\xi = 0.7$  to consider flow-aligning rodlike molecules.

### 2. Stress tensor

The explicit expression of the interface contribution  $\sigma^\phi$  to the total stress is given by

$$\sigma_{\alpha\beta}^\phi = \sum_{i=1}^2 \left[ \left( f - \phi_i \frac{\delta \mathcal{F}}{\delta \phi_i} \right) \delta_{\alpha\beta} - \frac{\partial f}{\partial (\partial_\beta \phi_i)} \partial_\alpha \phi_i \right], \quad (\text{A2})$$

while the explicit expression of the LC contribution is

$$\begin{aligned} \sigma_{\alpha\beta}^{LC} = & -\xi H_{\alpha\gamma} \left( Q_{\gamma\beta} + \frac{1}{3} \delta_{\gamma\beta} \right) - \xi \left( Q_{\alpha\gamma} + \frac{1}{3} \delta_{\alpha\gamma} \right) H_{\gamma\beta} \\ & + 2\xi \left( Q_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} \right) Q_{\gamma\mu} H_{\gamma\mu} + Q_{\alpha\gamma} H_{\gamma\beta} - H_{\alpha\gamma} Q_{\gamma\beta} \\ & - \partial_\alpha Q_{\gamma\mu} \frac{\partial f}{\partial (\partial_\beta Q_{\gamma\mu})}. \end{aligned} \quad (\text{A3})$$

## APPENDIX B: NUMERICAL EXPERIMENTS

### 1. Simulation protocol

We integrate the dynamics of the hydrodynamic fields in Eqs. (6)–(8) in a cubic grid of size  $\mathcal{L} = 128$  with a predictor-corrector hybrid lattice Boltzmann approach [67,68] consisting of solving Eqs. (6) and (7) with a finite-difference algorithm implementing first-order upwind scheme and fourth-order accurate stencils for space derivatives, and the Navier-Stokes equation through a predictor-corrector LB scheme on a  $D3Q15$  lattice. For technical details on the lattice Boltzmann method implemented here, refer to Refs. [68,69].

For the quenching experiments from high-temperature configurations used to produce the phase diagram in Fig. 3, the system is initialized in the following manner. The concentration field  $\phi_1$  is set as a spherical droplet with radius  $R_1$ , where  $\phi_1 = \phi_0$  inside the droplet and  $\phi_1 = 0$  outside it. Similarly, the concentration field  $\phi_2$  is initialized as a spherical droplet with radius  $R_2 > R_1$ , where  $\phi_2 = 0$  inside and  $\phi_2 = \phi_0$  outside. This configuration produces a global concentration field  $\bar{\phi} = \phi_1 + \phi_2$  that is zero within a spherical shell between the internal radius  $R_1$  and the external radius  $R_2$ , and reaches the value  $\phi_0$  outside this shell and within the internal inclusion. The  $Q$ -tensor is initialized in an isotropic state outside the shell, while inside the shell it is initialized with a small amplitude and a random orientation. The velocity field is initialized to 0.

For the hysteresis test presented in Fig. 5, we started from a 4B configuration obtained in absence of chirality and reduced thickness  $r = 0.71$ , according to the previous protocol. Then, we have steadily increased the chirality at a rate of  $\Delta q_0 / \Delta t = 3.5 \times 10^{-6} [\text{L}]^{-1} / \text{iteration}$ , corresponding to a variation of the reduced chirality of  $\Delta \kappa / \Delta t = 2 \times 10^{-6} (\text{iteration})^{-1}$ .

### 2. Simulation parameters and mapping to physical units

In this study, we varied two control parameters: (1) the radius  $R_1$  of the internal droplet, which was varied between 0 and 29 while keeping the outer droplet radius  $R_2 = 38$  constant, corresponding to a variation of the normalized radius  $r$  from 1 to 0.24; and (2) the chiral power  $q_0$ , which was varied between 0 and 0.54, corresponding to a variation of the reduced chirality  $\kappa$  from 0 to 1.6. In particular,  $a = 0.091$ ,  $\phi_0 = 2.0$ ,  $k_\phi = 0.13$ ,  $\epsilon = 0.01$ ,  $M = 0.35$ ,  $A_0 = 0.13$ ,  $\chi_0 = 2.85$ ,  $\chi_s = -0.25$ ,  $L = 0.03$ ,  $W = 0.05$ ,  $\xi = 0.7$ ,  $\gamma = 1.0$ ,  $\rho = 2.0$ , and the viscosity  $\eta = 1.67$ . A mapping from lattice units to physical units can be obtained by fixing the relevant length, time, and force scales, respectively, given by  $l^* = 100 \text{ nm}$ ,  $t^* = 1 \mu\text{s}$ ,  $f^* = 250 \text{ pN}$  which are set to 1 in LU [50,51,70]. Some relevant parameters are reported in Table I. Following Ref. [71], for the case of 5CB liquid crystal with 5% CD1 chiral dopant at  $30^\circ\text{C}$ , we propose here an approximate estimation of this experimental setting in terms of our rescaled dimensionless units. Assuming for the elastic constant  $L \approx 7.5 \text{ pN}$  the average value of the three elastic contributions (splay, twist, and bend) and a coherence length  $\ell_c \approx 50 \text{ nm}$  (see also Ref. [72]), we can therefore estimate the value of the bulk constant  $A_0$  from the relation  $\ell_c = \sqrt{L/A_0}$ . This gives for the reduced chirality  $\kappa \approx 0.6$ .

TABLE I. Mapping between simulation and physical units. Length scale  $l^* = 100 \text{ nm}$ , timescale  $t^* = 1 \mu\text{s}$ , and force-scale  $f^* = 250 \text{ pN}$  are set to 1 in LB units.

| Model parameter                           | Simulation units | Physical units                 |
|-------------------------------------------|------------------|--------------------------------|
| Shear viscosity, $\eta$                   | 5/3              | 41.67 mPa s                    |
| Elastic constant, $L$                     | 0.03             | 7.5 pN                         |
| Shape factor, $\xi$                       | 0.7              | dimensionless                  |
| Diffusion constant, $D = Ma$              | 0.03185          | 318.5 $\mu\text{m}^2/\text{s}$ |
| Coherence length, $\ell_c = \sqrt{L/A_0}$ | 0.48             | 48 nm                          |
| Internal radius, $R_1$                    | 0–29             | 0–2.9 $\mu\text{m}$            |
| External radius, $R_2$                    | 38               | 3.8 $\mu\text{m}$              |
| Pitch, $p_0$                              | $\geq 11.6$      | $\geq 1.16 \mu\text{m}$        |

### 3. Free energy relaxation

After initializing the  $Q$ -tensor to an isotropic state outside the shell and to a small amplitude, randomly oriented field inside, we fix the control parameters (chirality and inner droplet radius) and evolve the system to steady state. Steady state is identified by the total free energy reaching a time-independent plateau, where the relative free-energy variation satisfies  $\Delta \mathcal{F} / \mathcal{F} < 10^{-3}$ , with  $\Delta \mathcal{F}$  evaluated over a lapse of  $\Delta t = 10^3$  iterations. This typically requires  $5 \times 10^5$  iterations. Figure 7 shows the time evolution of the elastic free energy and its twist and splay-bend components for four representative parameter sets, corresponding to the phases displayed in Fig. 3.

In the phase coexistence region, the system exhibits strong sensitivity to initial conditions, as multiple locally stable configurations can emerge for the same set of parameters. Consequently, distinct random initial states may evolve toward different metastable equilibria. In this region, we perform up to ten independent simulations for each parameter point, starting from different random initial conditions. Among these, the configuration attaining the lowest final free energy is reported in the phase diagram of Fig. 3 as the representative state.

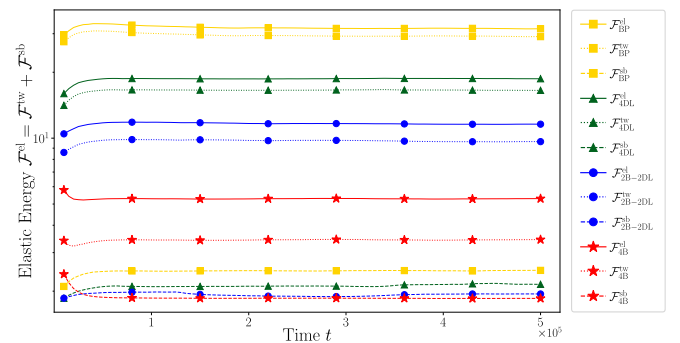


FIG. 7. Free energy evolution. Elastic free energy, decomposed into its twist and splay-bend components, is plotted as a function of time for the phases indicated in the legend: BP ( $\kappa = 1.25$ ), 4DL ( $\kappa = 0.61$ ), 2B – 2DL ( $\kappa = 0.44$ ), and 4B ( $\kappa = 0.19$ ). The inner isotropic droplet size is fixed at  $R_1 = 6$  ( $r = 0.84$ ).

## APPENDIX C: MEASUREMENTS

## 1. Disclination line identification

To classify the configurations among the states reported in the phase diagram of Fig. 3, we analyze the disclination lines as described below. We identify disclination lines and quantify their lengths using the disclination density tensor introduced by Schimming and Viñals [73]

$$\mathcal{D}_{ij} = \epsilon_{i\mu\nu} \epsilon_{jlk} \partial_l Q_{\mu\alpha} \partial_k Q_{\nu\alpha}, \quad (\text{C1})$$

where  $i, j, k, l, \alpha, \mu, \nu$  denote tensor indices, with summation implied over repeated indices. The Frobenius norm  $\|\mathcal{D}\|_F$  measures the local spatial variation of the  $Q$ -tensor and attains its maximum along the disclination cores, making it a natural measure for detecting disclination lines.

Near the core, the tensor  $\mathcal{D}(\mathbf{r})$  can be accurately approximated as rank one, reflecting a single dominant rotation axis and tangent direction. In the idealized core limit, the approximation is exact, allowing Eq. (C1) to be reformulated as

$$\mathcal{D}(\mathbf{r}) = s(\mathbf{r})(\boldsymbol{\Theta}(\mathbf{r}) \otimes \mathbf{T}(\mathbf{r})), \quad (\text{C2})$$

where  $\boldsymbol{\Theta}(\mathbf{r})$  and  $\mathbf{T}(\mathbf{r})$  are unit vectors representing the local rotation axis of the director field and the tangent to the disclination line, respectively. The scalar field  $s(\mathbf{r})$  quantifies the local magnitude of  $\mathcal{D}$  and is given by the Frobenius norm,  $s(\mathbf{r}) = \|\mathcal{D}(\mathbf{r})\|_F$ .

From Eq. (C2), the local rotation axis  $\boldsymbol{\Theta}$  and the disclination tangent  $\mathbf{T}$  are identified as the nondegenerate eigen-

vectors of  $\mathcal{D}\mathcal{D}^T$  and  $\mathcal{D}^T\mathcal{D}$ , respectively. Since  $\text{sgn}(\text{Tr}(\mathcal{D})) = \text{sgn}(\boldsymbol{\Theta} \cdot \mathbf{T})$ , fixing the direction of  $\mathbf{T}$  uniquely determines the orientation of  $\boldsymbol{\Theta}$ . It should be noted that the tensor  $\mathcal{D}$  is not strictly rank one except in the idealized core limit. However, the dominant eigenvectors of  $\mathcal{D}\mathcal{D}^T$  and  $\mathcal{D}^T\mathcal{D}$  still provide robust and reliable estimates of  $\boldsymbol{\Theta}$  and  $\mathbf{T}$  even away from the idealized core limit.

For the phases observed in our simulations, disclination cores are identified as regions where  $s(\mathbf{r}) \geq s_{\text{cut}}$ , with  $s_{\text{cut}} = 0.033$  for 4B,  $s_{\text{cut}} = 0.035$  for 2B – 2DL,  $s_{\text{cut}} = 0.038$  for 4DL, and  $s_{\text{cut}} = 0.025$  for the BP phase. For each phase, points exceeding the corresponding threshold are selected. The local tangent vector  $\mathbf{T}$ , obtained from the dominant eigenvector of  $\mathcal{D}^T\mathcal{D}$ , is then used to order the points along each line. After ordering, the resulting curves are smoothly interpolated to form continuous disclination lines, from which the total disclination lengths are then computed.

## 2. Analysis of the center of mass

The center of mass of each droplet has been computed as

$$\mathbf{r}_i^{\text{cm}} = \frac{\int d\mathbf{r} \, \mathbf{r} \, \vartheta_H(\phi_i - \frac{\phi_0}{2})}{\int d\mathbf{r} \, \vartheta_H(\phi_i - \frac{\phi_0}{2})}, \quad (\text{C3})$$

where  $\vartheta_H(x)$  is the Heaviside step function taking the value 0 when  $x < 0$ , 1/2 at  $x = 0$ , and 1 when  $x > 0$ .

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- [1] P. de Gennes and J. Prost, *The Physics of Liquid Crystals*, International Series of Monographs on Physics (Clarendon Press, Oxford, 1993).
  - [2] R. D. Williams, Two transitions in tangentially anchored nematic droplets, *J. Phys. A* **19**, 3211 (1986).
  - [3] O. D. Lavrentovich, Topological defects in dispersed words and worlds around liquid crystals, or liquid crystal drops, *Liq. Cryst.* **24**, 117 (1998).
  - [4] T. Lopez-Leon and A. Fernandez-Nieves, Drops and shells of liquid crystal, *Colloid Polym. Sci.* **289**, 345 (2011).
  - [5] P. J. Ackerman and I. I. Smalyukh, Diversity of knot solitons in liquid crystals manifested by linking of preimages in torons and hopfions, *Phys. Rev. X* **7**, 011006 (2017).
  - [6] G. Tóth, C. Denniston, and J. Yeomans, Hydrodynamics of topological defects in nematic liquid crystals, *Phys. Rev. Lett.* **88**, 105504 (2002).
  - [7] J. V. Selinger, Director deformations, geometric frustration, and modulated phases in liquid crystals, *Annu. Rev. Condens. Matter Phys.* **13**, 49 (2022).
  - [8] O. D. Lavrentovich, Liquid crystals, photonic crystals, metamaterials, and transformation optics., *Proc. Natl. Acad. Sci. USA* **108**, 5143 (2011).
  - [9] J. Money, J. G. Munguia-Fernández, S. Norouzi, M. Esmaili, J. A. Martínez-González, and M. Sadati, Photonic features of blue phase liquid crystals under curved confinement, *Chem. Commun.* **59**, 12231 (2023).
  - [10] L.-L. Ma, C.-Y. Li, J.-T. Pan, Y.-E. Ji, C. Jiang, R. Zheng, Z.-Y. Wang, Y. Wang, B.-X. Li, and Y.-Q. Lu, Self-assembled liquid crystal architectures for soft matter photonics, *Light Sci. Appl.* **11**, 270 (2022).
  - [11] M. Humar and I. Mušević, 3D microlasers from self-assembled cholesteric liquid-crystal microdroplets, *Opt. Express* **18**, 26995 (2010).
  - [12] H. Coles and S. Morris, Liquid-crystal lasers, *Nat. Photon.* **4**, 676 (2010).
  - [13] D. Foster, C. Kind, P. Ackerman, J.-S. B. Tai, M. Dennis, and I. Smalyukh, Two-dimensional skyrmion bags in liquid crystals and ferromagnets, *Nat. Phys.* **15**, 655 (2019).
  - [14] P. Ackerman, J. van de Lagemaat, and I. Smalyukh, Self-assembly and electrostriction of arrays and chains of hopfion particles in chiral liquid crystals, *Nat. Commun.* **6**, 6012 (2015).
  - [15] E. Bukusoglu, M. Bedolla Pantoja, P. C. Mushenheim, X. Wang, and N. L. Abbott, Design of responsive and active (soft) materials using liquid crystals, *Annu. Rev. Chem. Biomol. Eng.* **7**, 163 (2016).
  - [16] E. Kurt and E. Bukusoglu, Liquid crystal microcapillary-based sensors for affordable analytical applications, *Soft Matter* **18**, 4009 (2022).
  - [17] R. Kamien, The geometry of soft materials: A primer, *Rev. Mod. Phys.* **74**, 953 (2002).
  - [18] A. Fernández-Nieves, V. Vitelli, A. S. Utada, D. R. Link, M. Márquez, D. R. Nelson, and D. A. Weitz, Novel defect structures in nematic liquid crystal shells, *Phys. Rev. Lett.* **99**, 157801 (2007).
  - [19] G. Napoli and L. Vergori, Extrinsic curvature effects on nematic shells, *Phys. Rev. Lett.* **108**, 207803 (2012).

- [20] G. Napoli, O. V. Pylypovskiy, D. D. Sheka, and L. Vergori, Nematic shells: New insights in topology- and curvature-induced effects, *Soft Matter* **17**, 10322 (2021).
- [21] Y. Ishii, Y. Zhou, K. He, Y. Takanishi, J. Yamamoto, J. de Pablo, and T. Lopez-Leon, Structural transformations in tetravalent nematic shells induced by a magnetic field, *Soft Matter* **16**, 8169 (2020).
- [22] S. Čopar, Ž. Kos, T. Emeršič, and U. Tkalec, Microfluidic control over topological states in channel-confined nematic flows, *Nat. Commun.* **11**, 59 (2020).
- [23] J. Rojo-González, L. N. Carenza, A. de la Cotte, L. A. Hoffmann, L. Giomi, and A. Fernandez-Nieves, Defect-populated configurations in nematic solid tori and cylinders, *Phys. Rev. Res.* **6**, L012065 (2024).
- [24] H. Poincaré, Sur les courbes définies par les équations différentielles (III), *J. Math. Pure Appl.* **1**, 167 (1885).
- [25] H. Hopf, Vektorfelder in n-dimensionalen mannigfaltigkeiten, in *Collected Papers - Gesammelte Abhandlungen*, edited by H. Hopf and B. Eckmann (Springer, Berlin, 2001), pp. 87–112.
- [26] M. Urbanski, C. G. Reyes, J. Noh, A. Sharma, Y. Geng, V. S. R. Jampani, and J. P. F. Lagerwall, Liquid crystals in micron-scale droplets, shells and fibers, *J. Phys.: Condens. Matter* **29**, 133003 (2017).
- [27] N. D. Mermin, The topological theory of defects in ordered media, *Rev. Mod. Phys.* **51**, 591 (1979).
- [28] T. Lubensky and J. Prost, Orientational order and vesicle shape, *J. Phys. II France* **2**, 371 (1992).
- [29] T. Lopez-Leon, V. Koning, K. B. S. Devaiah, V. Vitelli, and A. Fernandez-Nieves, Frustrated nematic order in spherical geometries, *Nat. Phys.* **7**, 391 (2011).
- [30] M. V. Kurik and O. D. Lavrentovich, Topological defects in cholesteric liquid crystals, *JETP Lett.* **33**, 528 (1981).
- [31] J. Bezić and S. Žumer, Structures of the cholesteric liquid crystal droplets with parallel surface anchoring, *Liq. Cryst.* **11**, 593 (1992).
- [32] F. Xu and P. P. Crooker, Chiral nematic droplets with parallel surface anchoring, *Phys. Rev. E* **56**, 6853 (1997).
- [33] D. Seč, T. Porenta, M. Ravnik, and S. Žumer, Geometrical frustration of chiral ordering in cholesteric droplets, *Soft Matter* **8**, 11982 (2012).
- [34] T. Orlova, S. J. Aßhoff, T. Yamaguchi, N. Katsonis, and E. Brasselet, Creation and manipulation of topological states in chiral nematic microspheres, *Nat. Commun.* **6**, 7603 (2015).
- [35] Y. Zhou, E. Bokusoglu, J. Martínez-González, M. Rahimi, T. F. Roberts, R. Zhang, X. Wang, N. L. Abbott, and J. J. de Pablo, Structural transitions in cholesteric liquid crystal droplets, *ACS Nano* **10**, 6484 (2016).
- [36] L. Tran, M. Lavrentovich, G. Durey, A. Darmon, M. Haase, N. Li, D. Lee, K. Stebe, R. Kamien, and T. Lopez-Leon, Change in stripes for cholesteric shells via anchoring in moderation, *Phys. Rev. X* **7**, 041029 (2017).
- [37] V. Palacio-Betancur, J. C. Armas-Pérez, J. P. Hernández-Ortiz, and J. J. de Pablo, Curvature and confinement effects on chiral liquid crystal morphologies, *Soft Matter* **19**, 6066 (2023).
- [38] N. Johnson, L. C. Head, O. D. Lavrentovich, A. N. Morozov, G. Negro, E. Orlandini, C. A. Smith, G. M. Vasil, D. Marenduzzo, Clifford algebras and liquid crystalline fermions, *Phys. Rev. X* **15**, 041052 (2025).
- [39] C. Kocaman, O. Batir, and E. Bokusoglu, Optically responsive dry cholesteric liquid crystal marbles, *J. Colloid Interface Sci.* **671**, 374 (2024).
- [40] J. Pollard, G. Posnjak, S. Čopar, I. Mušević, and G. P. Alexander, Point defects, topological chirality, and singularity theory in cholesteric liquid-crystal droplets, *Phys. Rev. X* **9**, 021004 (2019).
- [41] D. C. Wright and N. D. Mermin, Crystalline liquids: The blue phases, *Rev. Mod. Phys.* **61**, 385 (1989).
- [42] G. P. Alexander and J. M. Yeomans, Stabilizing the blue phases, *Phys. Rev. E* **74**, 061706 (2006).
- [43] G. P. Alexander and D. Marenduzzo, Cubic blue phases in electric fields, *Europhys. Lett.* **81**, 66004 (2008).
- [44] O. Henrich, K. Stratford, D. Marenduzzo, and M. E. Cates, Ordering dynamics of blue phases entails kinetic stabilization of amorphous networks, *Proc. Natl. Acad. Sci. USA* **107**, 13212 (2010).
- [45] O. Henrich, K. Stratford, M. E. Cates, and D. Marenduzzo, Structure of blue phase III of cholesteric liquid crystals, *Phys. Rev. Lett.* **106**, 107801 (2011).
- [46] S. S. Gandhi and L.-C. Chien, Unraveling the mystery of the blue fog: Structure, properties, and applications of amorphous blue phase III, *Adv. Mater.* **29**, 1704296 (2017).
- [47] T. Paul and J. Saha, Origin and structure of liquid crystalline blue phase III, *Sci. Rep.* **10**, 16016 (2020).
- [48] J. Pišljarić, S. Ghosh, S. Turlapati, N. V. S. Rao, M. Škarabot, A. Mertelj, A. Petelin, A. Nych, M. Marinčič, A. Pusovnik, M. Ravnik, and I. Mušević, Blue phase III: Topological fluid of skyrmions, *Phys. Rev. X* **12**, 011003 (2022).
- [49] J. A. Martínez-González, Y. Zhou, M. Rahimi, E. Bokusoglu, N. L. Abbott, and J. J. de Pablo, Blue-phase liquid crystal droplets, *Proc. Natl. Acad. Sci. USA* **112**, 13195 (2015).
- [50] L. Carenza, G. Gonnella, D. Marenduzzo, G. Negro, and E. Orlandini, Cholesteric shells: Two-dimensional blue fog and finite quasicrystals, *Phys. Rev. Lett.* **128**, 027801 (2022).
- [51] G. Negro, L. N. Carenza, G. Gonnella, D. Marenduzzo, and E. Orlandini, Topological phases and curvature-driven pattern formation in cholesteric shells, *Soft Matter* **19**, 1987 (2023).
- [52] A. Darmon, M. Benzaquen, S. Čopar, O. Dauchot, and T. Lopez-Leon, Topological defects in cholesteric liquid crystal shells, *Soft Matter* **12**, 9280 (2016).
- [53] A. Darmon, M. Benzaquen, D. Seč, S. Čopar, O. Dauchot, and T. Lopez-Leon, Waltzing route toward double-helix formation in cholesteric shells, *Proc. Natl. Acad. Sci. USA* **113**, 9469 (2016).
- [54] C. R. Wand and M. A. Bates, Monte Carlo simulations of nematic and chiral nematic shells, *Phys. Rev. E* **91**, 012502 (2015).
- [55] G. Negro, L. C. Head, L. N. Carenza, T. N. Shendruk, D. Marenduzzo, G. Gonnella, and A. Tiribocchi, Topology controls flow patterns in active double emulsions, *Nat. Commun.* **16**, 1412 (2025).
- [56] G. Negro, L. N. Carenza, G. Gonnella, F. Mackay, A. Morozov, and D. Marenduzzo, Yield-stress transition in suspensions of deformable droplets, *Sci. Adv.* **9**, eadf8106 (2023).
- [57] N. D. Mermin, *Boojums All the Way Through: Communicating Science in a Prosaic Age* (Cambridge University Press, Cambridge, 1990).

- [58] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/1g3c-j9vp> for movies illustrating two of the configurations shown in Fig. 6.
- [59] L. Ruske and J. M. Yeomans, Morphology of active deformable 3D droplets, *Phys. Rev. X* **11**, 021001 (2021).
- [60] L. Carenza, G. Gonnella, D. Marenduzzo, and G. Negro, Rotation and propulsion in 3D active chiral droplets, *Proc. Natl. Acad. Sci. USA* **116**, 22065 (2019).
- [61] L. Carenza, G. Gonnella, D. Marenduzzo, and G. Negro, Chaotic and periodical dynamics of active chiral droplets, *Physica A* **559**, 125025 (2020).
- [62] L. C. Head, G. Negro, L. N. Carenza, N. Johnson, R. R. Keogh, G. Gonnella, A. Morozov, E. Orlandini, T. N. Shendruk, A. Tiribocchi, and D. Marenduzzo, Majorana quasiparticles and topological phases in 3D active nematics, *Proc. Natl. Acad. Sci. USA* **121**, e2405304121 (2024).
- [63] L. Metselaar, A. Doostmohammadi, and J. M. Yeomans, Topological states in chiral active matter: Dynamic blue phases and active half-skyrmions, *J. Chem. Phys.* **150**, 064909 (2019).
- [64] V. Jirón, M. Rajabi, H. Wang, and O. D. Lavrentovich, Dynamic control of active droplets using light-responsive chiral liquid crystal environment, *Commun. Phys.* **7**, 200 (2024).
- [65] A. Bulut, Y. Sariyar, G. Negro, and L. N. Carenza, Dataset for: Thick liquid crystalline cholesteric shells [Data set], Zenodo (2025), <https://doi.org/10.5281/zenodo.14611169>.
- [66] N. Kralj, M. Ravnik, and Ž. Kos, Chirality, anisotropic viscosity and elastic anisotropy in three-dimensional active nematic turbulence, *Commun. Phys.* **7**, 222 (2024).
- [67] S. Succi, *The Lattice Boltzmann Equation: For Complex States of Flowing Matter* (Oxford University Press, Oxford, 2018).
- [68] L. N. Carenza, G. Gonnella, A. Lamura, G. Negro, and A. Tiribocchi, Lattice Boltzmann methods and active fluids, *Eur. Phys. J. E* **42**, 81 (2019).
- [69] G. Negro, G. Gonnella, A. Lamura, S. Busuioc, and V. Sofonea, Growth regimes in three-dimensional phase separation of liquid-vapor systems, *Phys. Rev. E* **109**, 015305 (2024).
- [70] G. Negro, S. Busuioc, V. E. Ambruş, G. Gonnella, A. Lamura, and V. Sofonea, Comparison between isothermal collision-streaming and finite-difference lattice Boltzmann models, *Int. J. Mod. Phys. C* **30**, 1941005 (2019).
- [71] P. Popov, L. Honaker, M. Mirheydari, E. Mann, and A. Jáklí, Chiral nematic liquid crystal microlenses, *Sci. Rep.* **7**, 1603 (2017).
- [72] H. Stark, Physics of colloidal dispersions in nematic liquid crystals, *Phys. Rep.* **351**, 387 (2001).
- [73] C. Schimming and J. Viñals, Singularity identification for the characterization of topology, geometry, and motion of nematic disclination lines, *Soft Matter* **18**, 2234 (2022).