Relativistic electrodynamics with a universal length scale

Tiemo Pedergnana *
Department of Mechanical and Process Engineering, ETH Zürich, Leonhardstrasse 21, 8092 Zürich, Switzerland and Department of Mechanical Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA

Florian Kogelbauer
Department of Mechanical and Process Engineering, ETH Zürich, Leonhardstrasse 21, 8092 Zürich, Switzerland

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We derive the analogues of the Dirac and Pauli equations from a spatially fourth-order Klein-Gordon equation with a universal length scale. Starting from a singularly perturbed variant of Maxwell's equations, we deduce a 32-dimensional variant of the Dirac equation for spin-1/2 particles through an algebraic factorization procedure. We illustrate an experimental test of the theory from the split lines of the electron beam in a Stern-Gerlach experiment. This hyperfine splitting leads to four distinct eigenvalues of the spin operator, which can be grouped into two pairs centered around the classic values of $\pm\hbar/2$. The modified electrodynamic framework features an oriented, micropolar spacetime.

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I. INTRODUCTION

A. Motivation: the classic Klein-Gordon and Dirac equations

The Klein-Gordon equation was the first successful attempt to reconcile quantum mechanics with special relativity, providing a relativistic wave equation for spin-0 particles by promoting the classical energy-momentum relation $E^2 = c^2p^2 + m^2c^4$ to the operator identity:[1]

$$(\hat{E}^2 - c^2\hat{p}^2)\psi = m^2c^4\psi, \tag{1}$$

or, written out explicitly,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{m^2c^2}{\hbar^2}\psi, \tag{2}$$

where $\psi = \psi(x, y, z, t)$ is a scalar wave function, m is the particle mass, and c is the speed of light [1,2]. The Klein-Gordon equation laid the groundwork for quantum field theory by showing how particle creation and annihilation arise naturally when one interprets ψ as a field rather than a single-particle wave function. In modern physics, the Klein-Gordon equation underpins our understanding of mesons and other scalar fields [3], serves as the prototype for constructing more complex gauge and spinor theories, and illustrates fundamental concepts such as antiparticles, propagators, and the necessity of a field-operator formalism [4].

Although successful in the description of spin-0 particles, the second-order time derivatives and negative-energy solutions of the Klein-Gordon equation created interpretational difficulties, particularly in a single-particle context.

The Dirac equation [5,6] for relativistic spin-1/2 particles overcame these issues by formulating a first-order differential equation in both space and time, naturally incorporating spin through the introduction of four-component spinor wave functions and gamma matrices. Remarkably, the Dirac equation also predicted the existence of antiparticles, a profound insight that was later experimentally confirmed with the discovery of the positron [7]. In this way, the Dirac equation not only extended the relativistic quantum framework but also complemented the Klein-Gordon approach, both of which are now seen as essential components in the development of quantum field theory, where particles of different spin are described by corresponding field equations.

Indeed, the Dirac equation is derived by a factorization of the Klein-Gordon equation by looking for an operator $\mathcal{D}$ and a wave function Ψ satisfying

$$\begin{aligned}\mathcal{D}^2\Psi &= \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi \\ &= \frac{m^2c^2}{\hbar^2}\Psi,\end{aligned} \tag{3}$$

where $\hbar = h/(2\pi)$ is the reduced Planck constant. The Dirac equation is then given by the first-order system:

$$\mathcal{D}\Psi = \frac{mc}{\hbar}\Psi. \tag{4}$$

In the nonrelativistic limit of Eq. (4), Dirac famously recovered the Pauli equation for spin-1/2 particles [5,8]. The dimensions of $\mathcal{D}$ and Ψ along with their algebraic form are unknown *a priori*. Dirac's fundamental insight was to assume Ψ no longer as a scalar function, but much rather as a vectorial

*Contact author: tiemop@mit.edu

[1]In the following, the hat notation for operators is dropped for brevity.

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quantity. The operator $\mathcal{D}$ is then given by a 4×4 complex matrix differential acting on the four components of the vectorial wave function Ψ. The dimensionality of these quantities derives from algebraic constraints on the four Dirac matrices $\gamma_i, i = 0, 1, 2, 3$, which enter the picture through the following ansatz for $\mathcal{D}$:

$$\mathcal{D} = \gamma_0 \frac{i}{c}\frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z}. \tag{5}$$

By substituting this expression into Eq. (3), one deduces that the following anticommutation relations hold for the Dirac matrices [9]:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad \forall\, i, j = 0, \ldots, 3, \tag{6}$$

where curly brackets denote the anticommutator,

$$\{a, b\} = ab + ba. \tag{7}$$

B. A Klein-Gordon equation with a universal length scale and broken Lorentz symmetry

Special relativity considers the speed of light c as an invariant in all inertial frames moving in uniform translation with respect to each other. As a consequence, any speed can be expressed relative to c, while distance remains an absolute quantity. To remedy this asymmetry in phase space, an additional universal length scale ℓ_0 is introduced through the fourth-order equation proposed in Ref. [10]:

$$\left(-\ell_0^2\nabla^4 + \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{m^2c^2}{\hbar^2}\psi. \tag{8}$$

For vanishing right-hand side, this equation was previously considered by Mindlin and Tiersten [11] in the context of micropolar plate mechanics and by Pavlopoulos [10] as a nonlocal field theory with explicitly broken Lorentz symmetry. In particular, Pavlopoulos sought to tackle divergencies in present relativistic field theories (see also [12–15]) by replacing the d'Alembert operator:

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}, \tag{9}$$

appearing in the hyperbolic wave equation $\Box\psi = 0$, by the fourth-order operator:

$$-\ell_0^2\nabla^4 + \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{10}$$

The present work seeks to extend the analogy of Pavlopoulos by replacing $\Box$ in the Klein-Gordon equation (2), and exploring the phenomena predicted by the resulting, modified wave equation.

The operator (10) studied in Ref. [10] arises naturally from a variant of the vacuum Maxwell equations [16] under a singular, solenoidal (divergence-free) perturbation:

$$\nabla \cdot E = 0, \tag{11}$$

$$\nabla \cdot B = 0, \tag{12}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{13}$$

$$\nabla \times \left(B + \ell_0^2 \nabla \times (\nabla \times B)\right) = \frac{1}{c^2}\frac{\partial E}{\partial t}. \tag{14}$$

Indeed, taking the partial time derivative of Eq. (13), applying the curl to Eq. (14), and comparing the results yields

$$\nabla \times \nabla \times B + \ell_0^2 \nabla \times \nabla \times \nabla \times \nabla \times B = -\frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}. \tag{15}$$

By the solenoidal nature of B [see Eq. (12)], Eq. (15) is equivalent to

$$\left(-\ell_0^2\nabla^2\nabla^2 + \Box\right)B = 0. \tag{16}$$

An analogous procedure, in which the partial time derivative of Eq. (14) is taken and the curl is applied to Eq. (13), yields the same equation, but for E. No gauge conditions are required to arrive at these wave equations. In summary, the symmetries of Eq. (8) correspond to those of the modified electrodynamic framework given by Eqs. (11)–(14).

The introduction of the length scale ℓ_0 in the higher-order wave equation (8) breaks Lorentz symmetry. Indeed, the dispersion relation obtained in Ref. [10] from setting the left-hand side of Eq. (8) equal to zero is given by

$$\omega^2 = c^2k^2\left(1 + \ell_0^2k^2\right). \tag{17}$$

The negative sign before the fourth-order term in Eq. (8) ensures that the frequency ω is strictly positive. From Eq. (17), we see immediately that any linear symmetry transform has to act independently on space and time, because k and ω are of different leading order. Similarly, the two spatial differential operators of unequal order break dilation symmetry. Consequently, the classic linear action of the Lorentz group on Eq. (8) is reduced to Euclidean transformation of the spatial part.

C. Minimal coupling substitution

The minimal coupling substitution (see Ref. [17], p. 205 and Ref. [18], p. 409) is central to the derivation of the Pauli equation [8] from the Dirac equation [5,6]. The same is also true for the ℓ_0 analogues of these equations, which are the focus of this study. For a particle with charge q, enforcing minimal coupling involves replacing the Hamiltonian H and the impulse p as follows:

$$H \to H + q\phi, \quad p \to \underbrace{p - qA}_{=\pi}, \tag{18}$$

where ϕ is the scalar electromagnetic potential,

$$\pi = p - qA \tag{19}$$

is the kinetic impulse, and A is the magnetic vector potential. Under this substitution, the dynamics of a charged particle with (classical) Hamiltonian $H = p^2/(2m)$ reproduce the Lorentz force law. Conversely, the Lorentz force law is equivalent to the Maxwell-Faraday equation combined with Faraday's law (see Ref. [19], pp. 205–209). Since Eqs. (11)–(14) leave the Maxwell-Faraday equation [Eq. (13)] unchanged, the minimal coupling substitution (18) reproduces the conventional Lorentz force law also in this modified framework of electrodynamics. This fact is used here to justify the application of the minimal coupling substitution (18) in the context of the higher-order wave equation (8).

II. DERIVATION OF THE DIRAC EQUATION WITH A UNIVERSAL LENGTH SCALE: ALGEBRAIC FACTORIZATION

The factorization method discussed above is now applied to Eq. (8), which can be written out explicitly as

$$\begin{aligned}&\left(-\ell_0^2\nabla^4+\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi\\&=\left[-\ell_0^2\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+\frac{\partial^4}{\partial z^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+2\frac{\partial^4}{\partial x^2\partial z^2}\right.\right.\\&\quad\left.\left.+2\frac{\partial^4}{\partial y^2\partial z^2}\right)+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi\\&=\frac{m^2c^2}{\hbar^2}\psi.\end{aligned}\tag{20}$$

To factorize this equation in the sense that

$$\mathcal{D}_0^2\Psi=\left(-\ell_0^2\nabla^4+\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi,\tag{21}$$

in analogy to Eq. (5), the following ansatz is proposed here:

$$\begin{aligned}\mathcal{D}_0=&\Gamma_0\frac{i}{c}\frac{\partial}{\partial t}+\Gamma_1\frac{\partial}{\partial x}+\Gamma_2\frac{\partial}{\partial y}+\Gamma_3\frac{\partial}{\partial z}\\&+i\ell_0\left[\Gamma_4\frac{\partial^2}{\partial x^2}+\Gamma_5\frac{\partial^2}{\partial y^2}+\Gamma_6\frac{\partial^2}{\partial z^2}\right.\\&\left.+\sqrt{2}\left(\Gamma_7\frac{\partial}{\partial x}\frac{\partial}{\partial y}+\Gamma_8\frac{\partial}{\partial x}\frac{\partial}{\partial z}+\Gamma_9\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)\right].\end{aligned}\tag{22}$$

By comparison of Eqs. (21) and (22), one can readily deduce that the Γ matrices need to satisfy the anticommutation relations:

$$\{\Gamma_i,\Gamma_j\}=2\delta_{ij}\quad\forall\, i,j=0,\ldots,9.\tag{23}$$

These relations describe a Clifford algebra with ten elements, whose matrix representation has dimension $2^{10/2}=32$ (see Ref. [20]). An explicit construction of these matrices is due to Brauer and Weyl [21], and uses the Pauli matrices σ_i ($i=1,2,3$):

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\tag{24}$$

The specific form of the Brauer-Weyl representation used here is slightly adapted from the version given in the original reference to ensure that the particle-antiparticle duality of the standard Dirac equation is also maintained for its ℓ_0 analogue:

$$\Gamma_0=\sigma_3\otimes I_2\otimes I_2\otimes I_2\otimes I_2,\tag{25}$$

$$\Gamma_1=\sigma_2\otimes\sigma_3\otimes I_2\otimes I_2\otimes I_2,\tag{26}$$

$$\Gamma_2=\sigma_2\otimes\sigma_2\otimes\sigma_3\otimes I_2\otimes I_2,\tag{27}$$

$$\Gamma_3=\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_3\otimes I_2,\tag{28}$$

$$\Gamma_4=\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_3,\tag{29}$$

$$\Gamma_5=i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_{11}\Gamma_6\Gamma_7\Gamma_8\Gamma_9,\tag{30}$$

$$\Gamma_6=\sigma_2\otimes\sigma_1\otimes I_2\otimes I_2\otimes I_2,\tag{31}$$

$$\Gamma_7=\sigma_2\otimes\sigma_2\otimes\sigma_1\otimes I_2\otimes I_2,\tag{32}$$

$$\Gamma_8=\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_1\otimes I_2,\tag{33}$$

$$\Gamma_9=\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_1,\tag{34}$$

where I_2 is the 2×2 identity matrix. Additionally, the chiral matrix $\Gamma_{11}=\sigma_1\otimes I_2\otimes I_2\otimes I_2\otimes I_2$ is defined, which also satisfies the anticommutation relations (23).

The Γ matrices can be written as

$$\Gamma_0=\begin{pmatrix}I_{16}&0\\0&-I_{16}\end{pmatrix},\tag{35}$$

$$\Gamma_k=\begin{pmatrix}0&G_k\\-G_k&0\end{pmatrix},\tag{36}$$

$$\Gamma_{11}=\begin{pmatrix}0&I_{16}\\I_{16}&0\end{pmatrix},\tag{37}$$

for $k\in\{1,\ldots,9\}$, where G_k are 16×16 matrices given in the Supplemental Material [22], and I_{16} is the 16×16 identity matrix. Multiplication with Γ_0 changes the sign of the bottom half of the Γ matrices:

$$\Gamma_0\Gamma_k=\begin{pmatrix}0&G_k\\G_k&0\end{pmatrix},\quad k\in\{1,\ldots,9\},\tag{38}$$

$$\Gamma_0\Gamma_{11}=\begin{pmatrix}0&I_{16}\\-I_{16}&0\end{pmatrix}.\tag{39}$$

For the anticommutators of the G matrices, the following relation holds:

$$\{G_j,G_k\}=-2\delta_{jk},\tag{40}$$

for $j,k\in\{1,\ldots,9\}$.

The above steps lead to an explicit construction of the 32-dimensional analogue of the Dirac equation corresponding to the modified Klein-Gordon equation (8):

$$\mathcal{D}_0\Psi=\frac{mc}{\hbar}\Psi.\tag{41}$$

In contrast to Eq. (4), Ψ is now a 32-dimensional wave function and $\mathcal{D}_0$ a 32×32 complex matrix differential operator. After multiplying both sides with $c\hbar\Gamma_0$, Eq. (41) can be rearranged to yield

$$i\hbar\frac{\partial}{\partial t}\Psi=\underbrace{\left\{cp\cdot\left[\Gamma_0\left(\Gamma_s+i\frac{\ell_0}{\hbar}\mathcal{G}p\right)\right]+mc^2\Gamma_0\right\}}_{=H\Psi}\Psi,\tag{42}$$

where $p=-i\hbar\nabla$ is the impulse operator and

$$\mathcal{G}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\Gamma_4&\Gamma_7&\Gamma_8\\\Gamma_7&\sqrt{2}\Gamma_5&\Gamma_9\\\Gamma_8&\Gamma_9&\sqrt{2}\Gamma_6\end{pmatrix},\tag{43}$$

$$\Gamma_s=-i\begin{pmatrix}\Gamma_1\\\Gamma_2\\\Gamma_3\end{pmatrix}.\tag{44}$$

The right-hand side of Eq. (42) is identified as the Hamiltonian H acting on Ψ. After applying the minimal coupling substitution (18), Eq. (42) reads

$$i\hbar\frac{\partial}{\partial t}\Psi = \left\{c\pi\cdot\left[\Gamma_0\left(\Gamma_s + i\frac{\ell_0}{\hbar}\mathcal{G}\pi\right)\right] + q\phi + mc^2\Gamma_0\right\}\Psi, \tag{45}$$

where π is the kinetic impulse (19).

If all terms except the last inside the curly brackets are neglected, Eq. (45) reads

$$i\hbar\frac{\partial}{\partial t}\Psi = mc^2\Gamma_0\Psi. \tag{46}$$

The 32 eigenvalues of the energy operator on the left-hand side are $\pm mc^2$, each with 16-fold degeneracy, corresponding to free particle and antiparticle solutions, respectively.

We note that, while the classic Dirac equation is invariant under spatial rotations [6], it is not immediately obvious that the rotational symmetry translates to the ℓ_0 analogue (42). Indeed, as already pointed out in [10] and further elaborated in the discussion section of this paper, the full transformation group of (8) gives rise to a micropolar spacetime.

III. NONRELATIVISTIC LIMIT DERIVATION

A. Preliminaries

In this section, the nonrelativistic limit of Eq. (45) is derived. To aid in this derivation, the following auxiliary quantities are first defined:

$$V = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \tag{47}$$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}G_4 & G_7 & G_8 \\ G_7 & \sqrt{2}G_5 & G_9 \\ G_8 & G_9 & \sqrt{2}G_6 \end{pmatrix}. \tag{48}$$

From the above definitions, it directly follows that

$$M_{kl}V_m + V_mM_{kl} = 0. \tag{49}$$

Contraction between two tensors P and Q of equal order is defined as

$$P : Q = P_{ijkl\ldots}Q_{ijkl\ldots}. \tag{50}$$

The Kronecker product between two tensors $P_{ij\ldots}$ and $Q_{kl\ldots}$ of arbitrary order is defined as

$$[P\otimes Q]_{ij\ldots kl\ldots} = P_{ij\ldots}Q_{kl\ldots}. \tag{51}$$

If P and Q are vectors, then $P\otimes Q = PQ^T$.

The spin vector S appearing below is defined as follows:

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \frac{i\hbar}{4}\begin{pmatrix} G_2G_3 - G_3G_2 \\ G_3G_1 - G_1G_3 \\ G_1G_2 - G_2G_1 \end{pmatrix}. \tag{52}$$

Another quantity of interest is

$$T_3 = \frac{i\hbar}{4}(G_4G_7 - G_7G_4). \tag{53}$$

The eigenvalues of $S_3 - k^2\ell_0^2T_3$, where k is an arbitrary wave number, are given by

$$\lambda_{++} = \frac{\hbar}{2}(1 + k^2\ell_0^2), \tag{54}$$

$$\lambda_{+-} = \frac{\hbar}{2}(1 - k^2\ell_0^2), \tag{55}$$

$$\lambda_{-+} = -\frac{\hbar}{2}(1 - k^2\ell_0^2), \tag{56}$$

$$\lambda_{--} = -\frac{\hbar}{2}(1 + k^2\ell_0^2). \tag{57}$$

Each of these eigenvalues corresponds to four linearly independent eigenvectors spanning the four-dimensional sectors E_{++}, E_{+-}, E_{-+}, and E_{--}. The matrix $S_3 - k^2\ell_0^2T_3$ thus has full rank and its eigenvectors are given in the Supplemental Material [22].

B. Decoupling particle and antiparticle wave equations

To obtain the nonrelativistic limit of Eq. (45), it is first rewritten in component form by making the substitution

$$\Psi = e^{-imc^2t/\hbar}\begin{pmatrix} \xi \\ \chi \end{pmatrix}. \tag{58}$$

In these new variables, the left-hand side of Eq. (45) reads

$$i\hbar\frac{\partial}{\partial t}\Psi = e^{-imc^2t/\hbar}\left(mc^2 + i\hbar\frac{\partial}{\partial t}\right)\begin{pmatrix} \xi \\ \chi \end{pmatrix}, \tag{59}$$

while the right-hand side reads

$$\begin{aligned}
&\left\{c\pi\cdot\left[\Gamma_0\left(\Gamma_s + i\frac{\ell_0}{\hbar}\mathcal{G}\pi\right)\right] + q\phi + mc^2\Gamma_0\right\}\Psi \\
&= \left(c\pi\cdot\left\{\Gamma_0\left[-i\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} + i\frac{\ell_0}{\sqrt{2}\hbar}\begin{pmatrix}\sqrt{2}\Gamma_4 & \Gamma_7 & \Gamma_8\\ \Gamma_7 & \sqrt{2}\Gamma_5 & \Gamma_9\\ \Gamma_8 & \Gamma_9 & \sqrt{2}\Gamma_6\end{pmatrix}\pi\right]\right\} + q\phi + mc^2\Gamma_0\right)\Psi \\
&= \left\{-ic\pi\cdot\left[\begin{pmatrix}\Gamma_0\Gamma_1\\ \Gamma_0\Gamma_2\\ \Gamma_0\Gamma_3\end{pmatrix} - \frac{\ell_0}{\sqrt{2}\hbar}\begin{pmatrix}\sqrt{2}\Gamma_0\Gamma_4 & \Gamma_0\Gamma_7 & \Gamma_0\Gamma_8\\ \Gamma_0\Gamma_7 & \sqrt{2}\Gamma_0\Gamma_5 & \Gamma_0\Gamma_9\\ \Gamma_0\Gamma_8 & \Gamma_0\Gamma_9 & \sqrt{2}\Gamma_0\Gamma_6\end{pmatrix}\pi\right] + q\phi + mc^2\Gamma_0\right\}\Psi
\end{aligned}$$

$$= e^{-imc^2t/\hbar}\left[-ic\begin{pmatrix} 0 & \pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right) \\ \pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right) & 0\end{pmatrix} + q\phi + mc^2\begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16}\end{pmatrix}\right]\begin{pmatrix}\xi\\ \chi\end{pmatrix}. \tag{60}$$

In arriving at this expression, the block properties of the Γ matrices expressed in Eqs. (35)–(39) were exploited. By comparing Eqs. (59) and (60) and rearranging (recall that ξ and χ are 16-dimensional wave functions), one obtains

$$i\hbar\frac{\partial}{\partial t}\xi = -ic\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\chi + q\phi\xi, \tag{61}$$

$$i\hbar\frac{\partial}{\partial t}\chi = -ic\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\xi + q\phi\chi - 2mc^2\chi. \tag{62}$$

In the nonrelativistic limit $c\to\infty$, both the $\hbar$- and q-modulated terms in Eq. (62) are small compared to the other terms, leading to

$$\chi \approx -i\left[\frac{\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)}{2mc}\right]\xi. \tag{63}$$

Substituting this result back into Eq. (61) yields

$$i\hbar\frac{\partial\xi}{\partial t} = -\frac{\left[\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\right]\left[\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\right]}{2m}\xi + q\phi\xi. \tag{64}$$

The first term on the right-hand side of this 16-dimensional equation is now examined, noting that the entries of π commute with those of V and M:

$$\begin{aligned}&\left[\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\right]\left[\pi\cdot\left(V-\frac{\ell_0}{\hbar}M\pi\right)\right]\\ &= \pi_k\left(V_k-\frac{\ell_0}{\hbar}M_{kl}\pi_l\right)\pi_m\left(V_m-\frac{\ell_0}{\hbar}M_{mn}\pi_n\right)\\ &= \underbrace{\pi_k\pi_m V_k V_m}_{=\mathrm{I}} - \underbrace{\frac{\ell_0}{\hbar}\pi_k(M_{kl}\pi_l\pi_m V_m + V_k\pi_m M_{mn}\pi_n)}_{=\mathrm{II}}\\ &\quad + \underbrace{\left(\frac{\ell_0}{\hbar}\right)^2\pi_k\pi_l\pi_m\pi_n M_{kl}M_{mn}}_{=\mathrm{III}}.\end{aligned} \tag{65}$$

Using the above definitions, Eq. (64) can be restated as follows:

$$i\hbar\frac{\partial\xi}{\partial t} = -\frac{\mathrm{I}-\mathrm{II}+\mathrm{III}}{2m}\xi + q\phi\xi. \tag{66}$$

I

The different terms in Eq. (66) are now separately examined. Let us begin with I:

$$\begin{aligned}\mathrm{I} &= \pi_k\pi_m V_k V_m\\ &= \frac{\pi_k\pi_m}{2}[(V_kV_m - V_mV_k) + (V_kV_m + V_mV_k)]\\ &= \frac{1}{4}\{\pi_k,\pi_m\}(V_kV_m + V_mV_k)\\ &\quad + \frac{1}{4}[\pi_k,\pi_m](V_kV_m - V_mV_k),\end{aligned} \tag{67}$$

where square brackets denote the commutator,

$$[a,b] = ab - ba. \tag{68}$$

Using the definition of the kinetic momentum operator π, noting that $[p_k,p_m]=0$ and $[A_k,A_m]=0$, one can simplify the commutator in the above expression to

$$\begin{aligned}[\pi_k,\pi_m] &= (p_k - qA_k)(p_m - qA_m)\\ &\quad -(p_m - qA_m)(p_k - qA_k)\\ &= -q(p_kA_m + A_kp_m - p_mA_k - A_mp_k)\\ &= -q([p_k,A_m] - [p_m,A_k])\\ &= i\hbar q\left(\frac{\partial A_m}{\partial x_k} - \frac{\partial A_k}{\partial x_m}\right).\end{aligned} \tag{69}$$

The term in the brackets represents the entries of a skew-symmetric matrix whose vector representation is the magnetic field B:

$$\frac{\partial A_m}{\partial x_k} - \frac{\partial A_k}{\partial x_m} = \varepsilon_{kml}B_l, \tag{70}$$

where ε is the Levi-Civita symbol. Combining this result with Eq. (69) yields

$$[\pi_k,\pi_m] = i\hbar q\varepsilon_{kml}B_l. \tag{71}$$

To make an analogy to the classic Pauli equation, the skew-symmetric 16×16 matrices Σ_l and S_l, $l=1,2,3$, are here defined via [see also Eq. (52)]

$$(V_kV_m - V_mV_k) = 2i\varepsilon_{kml}\Sigma_l, \tag{72}$$

$$S = -\frac{\hbar}{2}\Sigma, \tag{73}$$

where $l=1,\ldots,3$. The vector Σ is now written out explicitly by multiplying both sides of Eq. (72) by ε_{kmj}:

$$\begin{aligned}\varepsilon_{kmj}(V_kV_m - V_mV_k) &= 2i\underbrace{\varepsilon_{kmj}\varepsilon_{kml}}_{=2\delta_{jl}}\Sigma_l,\\ \varepsilon_{kmj}(V_kV_m - V_mV_k) &= 4i\delta_{jl}\Sigma_l,\\ 2V_kV_m\varepsilon_{kmj} &= 4i\Sigma_j.\end{aligned} \tag{74}$$

$$\Rightarrow \Sigma_j = -\frac{i}{2}V_kV_m\varepsilon_{kmj}. \tag{75}$$

Therefore,

$$\begin{pmatrix}\Sigma_1\\ \Sigma_2\\ \Sigma_3\end{pmatrix} = -\frac{i}{2}\begin{pmatrix}G_2G_3 - G_3G_2\\ G_3G_1 - G_1G_3\\ G_1G_2 - G_2G_1\end{pmatrix}. \tag{76}$$

Finally, from Eq. (40), one can deduce that

$$(V_kV_m + V_mV_k) = -2\delta_{km}. \tag{77}$$

Combining Eqs. (67)–(77) yields

$$\begin{aligned}
\mathrm{I} &= \pi_k\pi_m V_k V_m \\
&= \frac{1}{4}\{\pi_k,\pi_m\}(V_kV_m + V_mV_k) \\
&\quad + \frac{1}{4}[\pi_k,\pi_m](V_kV_m - V_mV_k) \\
&= -\frac{1}{2}\{\pi_k,\pi_m\}\delta_{km} + \frac{i\hbar q}{4}\varepsilon_{kml}B_l(V_kV_m - V_mV_k) \\
&= -\frac{1}{2}\{\pi_k,\pi_m\}\delta_{km} - \frac{\hbar q}{2}\varepsilon_{kml}B_l\varepsilon_{kmj}\Sigma_j \\
&= -\pi_k\pi_k - \hbar q\delta_{lj}B_l\Sigma_j \\
&= -\pi^2 - \hbar qB\cdot\Sigma \\
&= -\pi^2 + 2qB\cdot S. \qquad (78)
\end{aligned}$$

By substituting this result back into Eq. (66), the classic Pauli equation [8] is recovered at zeroth order in ℓ_0 (albeit with 16-dimensional spin matrices contained in S):

$$i\hbar\frac{\partial}{\partial t}\xi = \left(\frac{\pi^2}{2m} - g\frac{q}{2m}B\cdot S + q\phi\right)\xi + O(\ell_0), \qquad (79)$$

where $g = 2$ is the gyromagnetic factor predicted by the Dirac equation [5,6]. Note that $S_k = (i\hbar/4)\varepsilon_{ijk}[G_i, G_j]$, $k = 1, 2, 3$, is a 3-vector of Hermitian 16×16 matrices with eightfold degenerate eigenvalues $\pm\hbar/2$, analogous to the vector of spin matrices appearing in the classic Pauli equation.

II

The ℓ_0-modulated term II on the right-hand side of Eq. (65) is now examined. Noting that all indices in this term are dummy indices and using Eq. (49), one can write

$$\begin{aligned}
\mathrm{II} &= \frac{\ell_0}{\hbar}\pi_k(M_{kl}\pi_l\pi_m V_m + V_k\pi_m M_{mn}\pi_n) \\
&= \frac{\ell_0}{\hbar}(\pi_k\pi_l\pi_m M_{kl}V_m + \pi_m\pi_k\pi_l V_m M_{kl}) \\
&= \frac{\ell_0}{\hbar}\left[\pi_k\pi_l\pi_m \underbrace{(M_{kl}V_m + V_mM_{kl})}_{=0} + [\pi_m,\pi_k\pi_l]V_mM_{kl}\right]. \qquad (80)
\end{aligned}$$

The commutator in this equation is resolved as follows:

$$\begin{aligned}
[\pi_m,\pi_k\pi_l] &= [\pi_m,\pi_k]\pi_l + \pi_k[\pi_m,\pi_l] \\
&= i\hbar q(\varepsilon_{mkj}B_j\pi_l + \varepsilon_{mlj}\pi_kB_j). \qquad (81)
\end{aligned}$$

Combining Eqs. (80) and (81) yields

$$\begin{aligned}
\mathrm{II} &= \frac{\ell_0}{\hbar}[\pi_m,\pi_k\pi_l]V_mM_{kl} \qquad (82) \\
&= iq\ell_0\big(\varepsilon_{mkj}B_j\pi_l + \varepsilon_{mlj}\pi_kB_j\big)V_mM_{kl} \\
&= iq\ell_0\big(\varepsilon_{mkj}B_j\pi_l + \varepsilon_{mkj}\pi_lB_j\big)V_mM_{kl} \\
&= iq\ell_0\varepsilon_{mkj}\{B_j,\pi_l\}V_mM_{kl}. \qquad (83)
\end{aligned}$$

Defining the second-order tensors $\mathcal{B}_{jl} = \{B_j,\pi_l\}$ and $\mathcal{V}_{jl} = \varepsilon_{mkj}V_mM_{kl}$, the above expression can be simplified as follows:

$$\mathrm{II} = iq\ell_0\mathcal{V}:\mathcal{B}(\pi). \qquad (84)$$

III

Next, the third term III on the right-hand side of Eq. (65) is considered:

$$\begin{aligned}
\mathrm{III} &= \left(\frac{\ell_0}{\hbar}\right)^2\pi_k\pi_l\pi_m\pi_nM_{kl}M_{mn} \\
&= \underbrace{\frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2\{\pi_k\pi_l,\pi_m\pi_n\}(M_{kl}M_{mn} + M_{mn}M_{kl})}_{=\mathrm{IIIa}} + \underbrace{\frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2[\pi_k\pi_l,\pi_m\pi_n](M_{kl}M_{mn} - M_{mn}M_{kl})}_{=\mathrm{IIIb}} \\
&= \mathrm{IIIa} + \mathrm{IIIb}. \qquad (85)
\end{aligned}$$

Substituting this expression into Eq. (66) yields

$$i\hbar\frac{\partial\xi}{\partial t} = -\frac{\mathrm{I} - \mathrm{II} + \mathrm{IIIa} + \mathrm{IIIb}}{2m}\xi + q\phi\xi. \qquad (86)$$

To simplify the symmetric term in Eq. (85), we define the tensors $\mathcal{W}_{klmn} = \{\pi_k\pi_l,\ \pi_m\pi_n\}$ and $\mathcal{A}_{klmn} = (M_{kl}M_{mn} + M_{mn}M_{kl})$. With these definitions, we have

$$\mathrm{IIIa} = \frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2\mathcal{A}:\mathcal{W}(\pi). \qquad (87)$$

The π commutator in Eq. (85) can generally be rewritten as follows:

$$\begin{aligned}
[\pi_k\pi_l,\pi_m\pi_n] &= \pi_k[\pi_l,\pi_m]\pi_n + [\pi_k,\pi_m]\pi_l\pi_n \\
&\quad + \pi_m\pi_k[\pi_l,\pi_n] + \pi_m[\pi_k,\pi_n]\pi_l. \qquad (88)
\end{aligned}$$

If this term is contracted with $M_{kl}M_{mn} - M_{mn}M_{kl}$, because the tensor M is symmetric, one can exchange k with l and m with n. This procedure leads to

$$\begin{aligned}
&[\pi_k\pi_l,\pi_m\pi_n](M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= (\pi_k[\pi_l,\pi_m]\pi_n + [\pi_k,\pi_m]\pi_l\pi_n + \pi_m\pi_k[\pi_l,\pi_n] \\
&\quad + \pi_m[\pi_k,\pi_n]\pi_l)(M_{kl}M_{mn} - M_{mn}M_{kl}) \qquad (89)
\end{aligned}$$

$$
\begin{aligned}
&= (\{\pi_k, [\pi_l, \pi_m]\}\pi_n + \pi_m\{\pi_k, [\pi_l, \pi_n]\}) \\
&\quad \times (M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= \{\pi_m, \{\pi_k, [\pi_l, \pi_n]\}\}(M_{kl}M_{mn} - M_{mn}M_{kl}). \qquad (90)
\end{aligned}
$$

In summary, one has

$$
\begin{aligned}
\mathrm{IIIb} &= \frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2 [\pi_k\pi_l, \pi_m\pi_n](M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= \frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2 \{\pi_m, \{\pi_k, [\pi_l, \pi_n]\}\}(M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= \frac{i\hbar q}{4}\left(\frac{\ell_0}{\hbar}\right)^2 \{\pi_m, \{\pi_k, \varepsilon_{lnj}B_j\}\}(M_{kl}M_{mn} - M_{mn}M_{kl}).
\end{aligned} \qquad (91)
$$

Defining the fourth-order tensors $\mathcal{F}_{klmn} = \{\pi_m, \{\pi_k, \varepsilon_{lnj}B_j\}\}$ and $\mathcal{M}_{klmn} = M_{kl}M_{mn} - M_{mn}M_{kl}$, the above expression can be written as

$$
\mathrm{IIIb} = \frac{i\hbar q}{4}\left(\frac{\ell_0}{\hbar}\right)^2 \mathcal{M} : \mathcal{F}(\pi). \qquad (92)
$$

After substituting Eqs. (78), (84), (87), and (92) into Eq. (86), the nonrelativistic limit of Eq. (45) is obtained:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\xi = \Bigg[&\underbrace{\frac{\pi^2}{2m} - g\frac{q}{2m}S\cdot B}_{-\mathrm{I}/(2m)} + q\phi + \underbrace{\frac{iq\ell_0}{2m}\mathcal{V} : \mathcal{B}(\pi)}_{\mathrm{II}/(2m)} \\
&- \underbrace{\frac{1}{8m}\left(\frac{\ell_0}{\hbar}\right)^2 \mathcal{A} : \mathcal{W}(\pi)}_{=\mathrm{IIIa}/(2m)} - \underbrace{\frac{i\hbar q}{8m}\left(\frac{\ell_0}{\hbar}\right)^2 \mathcal{M} : \mathcal{F}(\pi)}_{\mathrm{IIIb}/(2m)} \Bigg]\xi.
\end{aligned} \qquad (93)
$$

This equation is the general expression for the ℓ_0 analogue of the Pauli equation corresponding to the analogue Dirac equation (45).

IV. EXAMPLE

In the classic Stern-Gerlach experiment, a collimated beam of silver atoms, each possessing a single unpaired electron, is sent through an inhomogeneous magnetic field, which deflects the beam in positive or negative z direction, depending on the electrons' spins [23]. An illustration is given in Fig. 1 [24].

A theoretical model of a Stern-Gerlach experiment [25–30] is now considered in the context of Eq. (93), which is assumed to be valid in this example. Furthermore, a number of assumptions are made that allow for simplification of that equation:

(1) The electron beam passes through the origin $x = y = z = 0$ and remains at the middle line $x = 0$ throughout the process.

(2) A solenoidal magnetic field with a generic higher-order inhomogeneity with characteristic dimension $d = \sqrt{|\beta_1/\beta_3|}$ is assumed:

$$
B = \begin{pmatrix} -\beta_1 x - \frac{\beta_3 x^3}{3} \\ 0 \\ B_0 + \beta_1 z + \beta_3 x^2 z \end{pmatrix}. \qquad (94)
$$

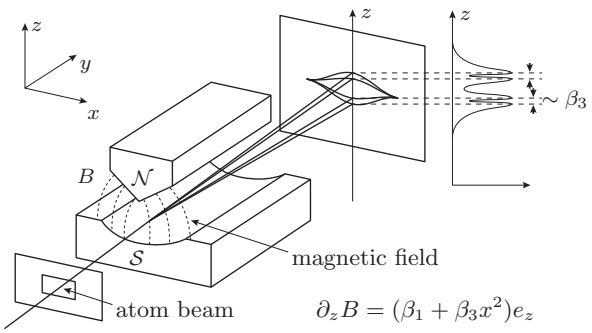


FIG. 1. The ℓ_0 analogue of the Pauli equation derived from the modified Klein-Gordon equation (8) predicts that in a Stern-Gerlach experiment, hyperfine splitting in the detected beam intensity occurs under a symmetric higher-order inhomogeneity in the magnetic field.

(3) The vector potential A corresponding to $B = \nabla \times A$ is given by

$$
A = \begin{pmatrix} 0 \\ B_0 x + \beta_1 xz + \beta_3 \frac{x^3}{3} z \\ 0 \end{pmatrix}. \qquad (95)
$$

The entries of A are identically zero at all times along the electron beam.

(4) All momentum terms, i.e., terms where π is left-multiplied with the wave function ξ, are neglected and only the terms involving products of the magnetic field B (or its derivatives) with ξ are considered.

(5) All precession effects and the scalar electromagnetic potential ϕ are neglected.

(6) To model the deflection of the electron beam by the magnetic field, the simplified treatment of Ref. [31], Sec. 9.1 is followed. For this approach, Eq. (93) is rewritten as

$$
i\hbar\frac{\partial}{\partial t}\xi = H\xi, \qquad (96)
$$

where H is the Hamiltonian operator. While passing through the magnetic field B, the wave function evolves like $\xi \sim e^{-iHt/\hbar}$, such that, after a time Δt spent inside the magnetic field, it will have acquired the impulse $p\xi = -\Delta t \nabla H \xi$. According to this model, the vertical deflection of the beam is governed by the z component of the dipole force operator $F = -\nabla H$.

The simplifying assumptions (4)–(6) serve to more clearly illustrate the main results and follow similar assumptions made in textbooks [31,32]. A more detailed analysis of spin precession [33,34] in the context of the higher-order electrodynamic framework considered here is left for future research.

The linear, β_1-modulated inhomogeneities in Eq. (94) are also found in classic textbook models [31,32], while an additional cubic term in the z component of the magnetic field with coefficient β_3 is considered here. Motivated by the symmetric geometry of the classic Stern-Gerlach experiment (see Fig. 1), the higher-order term is assumed to be symmetric in x. This nonlinearity does not change the prediction of the classic Pauli equation approach found in textbooks, but does come to bearing here due to the presence of higher-order derivatives

in Eq. (93). The terms in the x component in Eq. (94) were chosen to balance the terms in the z component when taking the divergence of B, ensuring a solenoidal magnetic field.

Due to assumptions (1)–(6), the terms II and IIIa in Eq. (93) do not contribute to the beam deflection in z direction, and the term I simplifies to just the magnetic dipole:

$$\mathrm{I} \to gq\, S \cdot B. \tag{97}$$

A crucial contribution comes from the term IIIb, which is now examined by simplifying Eq. (91). In particular, the following simplifications apply to the term $\{\pi_m, \{\pi_k, [\pi_l, \pi_n]\}\} = i\hbar q\{\pi_m, \{\pi_k, \varepsilon_{lnj}B_j\}\}$:

$$\{\pi_k, [\pi_l, \pi_n]\} \to \hbar^2 q\, \varepsilon_{lnj} B_{j,k}, \tag{98}$$

$$\{\pi_m, \{\pi_k, [\pi_l, \pi_n]\}\} \to -i\hbar^3 q\, \varepsilon_{lnj} B_{j,km}. \tag{99}$$

Now, using the simplified notation $[klmn] = M_{kl}M_{mn} - M_{mn}M_{kl}$, the expression IIIb can be reformulated as

$$\begin{aligned}
\mathrm{IIIb} &= \frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2 [\pi_k\pi_l, \pi_m\pi_n](M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= \frac{1}{4}\left(\frac{\ell_0}{\hbar}\right)^2 \{\pi_m, \{\pi_k, [\pi_l, \pi_n]\}\}(M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= -\left(\frac{\ell_0}{\hbar}\right)^2 \frac{i\hbar^3 q}{4} \varepsilon_{lnj} B_{j,km}(M_{kl}M_{mn} - M_{mn}M_{kl}) \\
&= -\left(\frac{\ell_0}{\hbar}\right)^2 \frac{i\hbar^3 q}{4} \varepsilon_{lnj} B_{j,km}[klmn] \\
&= -\left(\frac{\ell_0}{\hbar}\right)^2 \frac{i\sqrt{2}\hbar^3 q}{4} \Theta_j B_j \\
&= -\left(\frac{\ell_0}{\hbar}\right)^2 \frac{i\sqrt{2}\hbar^3 q}{4} \Theta \cdot B \\
&= -\sqrt{2}\ell_0^2 q\, \mathcal{T} \cdot B.
\end{aligned} \tag{100}$$

In Eq. (100), the following differential operators were defined:

$$\Theta_j = \frac{1}{\sqrt{2}} \varepsilon_{lnj}[klmn] \frac{\partial}{\partial k}\frac{\partial}{\partial m}, \tag{101}$$

$$\mathcal{T} = \frac{i\hbar}{4}\Theta. \tag{102}$$

The individual entries of the operator Θ are

$$\begin{aligned}
\begin{pmatrix}\Theta_1\\ \Theta_2\\ \Theta_3\end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix}[k2m3] - [k3m2]\\ [k3m1] - [k1m3]\\ [k1m2] - [k2m1]\end{pmatrix}\frac{\partial}{\partial k}\frac{\partial}{\partial m} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}
([1213] - [1312])\partial_x^2 + ([1223] - [1322])\partial_x\partial_y + ([1233] - [1332])\partial_x\partial_z \\
+([2213] - [2312])\partial_y\partial_x + ([2223] - [2322])\partial_y^2 + ([2233] - [2332])\partial_y\partial_z \\
+([3213] - [3312])\partial_z\partial_x + ([3223] - [3322])\partial_z\partial_y + ([3233] - [3332])\partial_z^2 \\
([1311] - [1113])\partial_x^2 + ([1321] - [1123])\partial_x\partial_y + ([1331] - [1133])\partial_x\partial_z \\
+([2311] - [2113])\partial_y\partial_x + ([2321] - [2123])\partial_y^2 + ([2331] - [2133])\partial_y\partial_z \\
+([3311] - [3113])\partial_z\partial_x + ([3321] - [3123])\partial_z\partial_y + ([3331] - [3133])\partial_z^2 \\
([1112] - [1211])\partial_x^2 + ([1122] - [1221])\partial_x\partial_y + ([1132] - [1231])\partial_x\partial_z \\
+([2112] - [2211])\partial_y\partial_x + ([2122] - [2221])\partial_y^2 + ([2132] - [2231])\partial_y\partial_z \\
+([3112] - [3211])\partial_z\partial_x + ([3122] - [3221])\partial_z\partial_y + ([3132] - [3231])\partial_z^2
\end{pmatrix},
\end{aligned} \tag{103}$$

where the shorthand notation $\partial_k = \partial/\partial_k$ was used, with $k, m \in \{1, 2, 3\}$ corresponding to (x, y, z).

The simplified ℓ_0 analogue of the Pauli equation (93) can now be written as

$$i\hbar\frac{\partial}{\partial t}\xi = \underbrace{-\left[g\frac{q}{2m}\left(S - \frac{\ell_0^2}{\sqrt{2}}\mathcal{T}\right)\cdot B\right]}_{H}\xi. \tag{104}$$

By the form of the magnetic field assumed in (2), the only terms in $\mathcal{T}$ that could contribute to the z component of ∇H along the beam are the second-order derivatives in x:

$$\mathcal{T} \to \frac{i\hbar}{4\sqrt{2}} \begin{pmatrix} [G_7, G_8]\partial_x^2 \\ \sqrt{2}[G_8, G_4]\partial_x^2 \\ \sqrt{2}[G_4, G_7]\partial_x^2 \end{pmatrix}. \tag{105}$$

As evident in Eq. (105), there is a preferred direction in this model, which is here assumed to be aligned with the x axis. The Hamiltonian H defined through Eq. (104) can be written out as follows:

$$\begin{aligned} H &= -\left[g\frac{q}{2m}\left(S - \frac{\ell_0^2}{\sqrt{2}}\mathcal{T} \right) \cdot B \right] \\ &= -\left[g\frac{q}{2m}\left(S - \frac{\ell_0^2}{\sqrt{2}}\mathcal{T} \right) \cdot \begin{pmatrix} -\beta_1 x - \frac{\beta_3 x^3}{3} \\ 0 \\ B_0 + \beta_1 z + \beta_3 x^2 z \end{pmatrix} \right] \\ &= -g\frac{q}{2m}\left[S_1\left(-\beta_1 x - \frac{\beta_3 x^3}{3} \right) + \frac{i\hbar\ell_0^2\beta_3 x}{4}[G_7, G_8] + S_3(B_0 + \beta_1 z + \beta_3 x^2 z) - \underbrace{\frac{i\hbar\ell_0^2\beta_3 z}{2\sqrt{2}}[G_4, G_7]}_{=\sqrt{2}\ell_0^2\beta_3 z T_3} \right]. \end{aligned} \tag{106}$$

The z component of the dipole force operator $F = -\nabla H$ has full rank and is given by

$$F_z = \mu\left[(\beta_1 + \beta_3 x^2) S_3 - \sqrt{2}\beta_3 \ell_0^2 T_3 \right], \tag{107}$$

where $\mu = gq/(2m)$ and $T_3 = (i\hbar/4)[G_4, G_7]$. At the location of the beam ($x = 0$), by Eqs. (54)–(57), the 16 eigenvalues of F_z are given by the following fourfold degenerate set (recall that $d = \sqrt{|\beta_1/\beta_3|}$):

$$\lambda_{++} = \frac{\mu\beta_1\hbar}{2}\left(1 + \sqrt{2}\frac{\ell_0^2}{d^2} \right), \tag{108}$$

$$\lambda_{+-} = \frac{\mu\beta_1\hbar}{2}\left(1 - \sqrt{2}\frac{\ell_0^2}{d^2} \right), \tag{109}$$

$$\lambda_{-+} = -\frac{\mu\beta_1\hbar}{2}\left(1 - \sqrt{2}\frac{\ell_0^2}{d^2} \right), \tag{110}$$

$$\lambda_{--} = -\frac{\mu\beta_1\hbar}{2}\left(1 + \sqrt{2}\frac{\ell_0^2}{d^2} \right). \tag{111}$$

The corresponding eigenvectors can be computed using the code provided in the Supplemental Material [22]. Depending on the alignment of the wave function ξ with the eigenvectors of F_z, the electrons will accumulate positive or negative momentum in z direction. Statistically, these positive and negative portions will each be evenly split around the mean values $(\lambda_{\pm+} + \lambda_{\pm-})/2$, which correspond to the classic textbook values and are independent of ℓ_0 and β_3. See Fig. 1 for an illustration of this result. Such hyperfine splitting can only be expected to be clearly observed if d is of the same order of magnitude as ℓ_0 and if the time Δt spent by the beam in the magnetic field is sufficiently long. Had one not assumed that the preferred direction evident from Eq. (105) is aligned with the x axis, a similar fine splitting as predicted by Eqs. (108)–(111) would generally result, albeit with slightly different numerical factors (depending on the specific alignment with the coordinate axes) before the ℓ_0^2-modulated terms.

V. DISCUSSION

This study has extended the analogy used in Ref. [10] of replacing the d'Alembert operator by a higher-order operator to the Klein-Gordon, Dirac, and Pauli equations. Thereby, a universal length scale ℓ_0 is introduced, which leads to a modified framework of relativistic electrodynamics, in which the Lorentz symmetry is replaced by a larger symmetry, which, so far, remains unknown. It was shown that, under standard assumptions, the ℓ_0 analogue of the Pauli equation derived in this manner predicts hyperfine splitting in the Stern-Gerlach experiment under a symmetric higher-order inhomogeneity in the magnetic field.

The models presented here could be used to describe phenomena involving the quantum number 16, which have been observed in nuclear science [35] and organometallic chemistry [36,37]. Note that the number 16 is not naturally distinguished in standard quantum mechanics (see Ref. [38] for an example of a nonstandard theory in which it is), but arises generically in the context of Eq. (104), which, under a symmetrically inhomogeneous magnetic field, predicts four different values of spin, which are more distinct for stronger inhomogeneities. This fourfold multiplicity of spin implies a secondary series of closed-shell quantum numbers (besides 2, 8, . . .) starting with the numbers 4 and 16.

The II term could also contribute to the hyperfine splitting reported here. This could occur, for example, if an asymmetric nonlinearity proportional to xz were included in the magnetic field (94). Interestingly, if ℓ_0 were taken as the classical electron radius as proposed in Ref. [10] and if $|\pi| \approx \hbar\lambda_C^{-1}$, where $\lambda_C = \hbar/(mc)$ is the reduced Compton wavelength, then the contribution of the II term would be of the order of the fine structure constant $\alpha = \ell_0/\lambda_C$. Similarly, the contribution of III would be of the order of α^2.

An interesting feature derived from Eq. (8) is the fact that space is oriented; i.e., there is always one preferred direction at each point [see, for example, Eq. (105)]. This feature relates to the last paragraph of

Ref. [10], wherein the need for extending special relativity into the realm of micropolar mechanics [39] was expressed. To date, the development of such a program remains an open task.

A. Application of the method to other models

Higher-order wave equations arise in a wide range of physical contexts from shallow water waves [40–44], nonlocal field theories [10,12,15,45–53], and doubly special relativity [54–61], to elasticity [62–68], micropolar media [11,39,69–75], and topological metamaterials [76–78]. Despite their ubiquity, the spin dynamics and symmetries of these models remain poorly understood, and the ansatz (5) is insufficient to capture the higher-order derivatives that appear in their governing equations.

Beyond its primary focus on electromagnetic phenomena, this work seeks to extend the algebraic factorization method underlying the derivation of the Dirac equation to higher-order equations and to make their spin dynamics accessible to systematic analysis. To avoid overshadowing the simplicity of this method, it is here applied to the specific example of Eq. (8), and a more general exposition is postponed to future research. Nevertheless, to exemplify the applicability of the method to another well-known example of a higher-order equation, in Appendix A, the first steps of the algebraic factorization procedure are reproduced for the equation of Bopp [12], given by $(\Box - \kappa_0^2)\Box\psi = 0$, where the reciprocal κ_0^{-1} is a universal length scale similar to ℓ_0 in Eq. (8).

VI. CONCLUSIONS

The concept of a universal length scale has been a fascination among physicists at least since Heisenberg's 1938 review article [79] and remains both a puzzling and promising research area until today [80]. This article has sought to make a contribution to this field by proposing a Dirac-like quantum electrodynamic theory based on a modified wave operator, wherein Lorentz symmetry is explicitly broken by the length scale ℓ_0. The full Lie group analysis of the fourth-order operator (10) is deferred to a later point due its complexity. This will also shed light on the micropolar spacetime induced by Eq. (8). Additionally, an alternative approach based on a simpler, more accessible transformation theory also involving a universal length scale could be pursued in the future.

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T.P. wrote the initial draft of the manuscript, including the algebraic factorization of the modified Klein-Gordon equation and the nonrelativistic limit derivation. F.K. led the study of the broken Lorentz symmetry for nonzero values of ℓ_0. Both authors contributed to the revision of the manuscript.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: FACTORIZATION OF BOPP'S EQUATION

The equation of Bopp is given by $(\Box - \kappa_0^2)\Box\psi = 0$, or written out as

$$\left(\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] - \kappa_0^2\right)\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi = 0. \tag{A1}$$

To factorize this equation in the sense that

$$\mathcal{D}_0^2\Psi = \left(\Box - \kappa_0^2\right)\Box\Psi, \tag{A2}$$

one could make the ansatz

$$\begin{aligned}\mathcal{D}_0 = i\kappa_0\left(\Gamma_0\frac{i}{c}\frac{\partial}{\partial t} + \Gamma_1\frac{\partial}{\partial x} + \Gamma_2\frac{\partial}{\partial y} + \Gamma_3\frac{\partial}{\partial z}\right) + \Gamma_4\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Gamma_5\frac{\partial^2}{\partial x^2} + \Gamma_6\frac{\partial^2}{\partial y^2} + \Gamma_7\frac{\partial^2}{\partial z^2}\\ + \frac{\sqrt{2}i}{c}\left(\Gamma_8\frac{\partial}{\partial t}\frac{\partial}{\partial x} + \Gamma_9\frac{\partial}{\partial t}\frac{\partial}{\partial y} + \Gamma_{10}\frac{\partial}{\partial t}\frac{\partial}{\partial z}\right) + \sqrt{2}\left(\Gamma_{11}\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \Gamma_{12}\frac{\partial}{\partial x}\frac{\partial}{\partial z} + \Gamma_{13}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right).\end{aligned} \tag{A3}$$

The dimension of these Γ matrices and the wave function Ψ would be $2^{14/2} = 128$.

[1] O. Klein, Quantentheorie und fünfdimensionale Relativitätstheorie, Z. Phys. **37**, 895 (1926).

[2] W. Gordon, Der Comptoneffekt nach der Schrödingerschen Theorie, Z. Phys. **40**, 117 (1926).

[3] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[4] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996), Vol. 2.

[5] P. A. M. Dirac, The quantum theory of the electron, Proc. R. Soc. A **117**, 610 (1928).

[6] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 1st ed. (Oxford University Press, Oxford, 1930).

[7] C. D. Anderson, The positive electron, Phys. Rev. **43**, 491 (1933).

[8] W. Pauli, Jr., Zur Quantenmechanik des magnetischen Elektrons, Z. Phys. **43**, 601 (1927).

[9] Depending on the chosen representation for the Dirac matrices, a signed diagonal matrix instead of the 4×4 unit matrix will be on the right-hand side of the following anticommutator relation.

[10] T. G. Pavlopoulos, Breakdown of Lorentz invariance, Phys. Rev. **159**, 1106 (1967).

[11] R. D. Mindlin and H. F. Tiersten, Effects of couple-stresses in linear elasticity, Arch. Ration. Mech. Anal. **11**, 415 (1962).

[12] F. Bopp, Eine lineare Theorie des Elektrons, Ann. Phys. **430**, 345 (1940).

[13] R. P. Feynman, A relativistic cut-off for classical electrodynamics, Phys. Rev. **74**, 939 (1948).

[14] R. P. Feynman, Relativistic cut-off for quantum electrodynamics, Phys. Rev. **74**, 1430 (1948).

[15] C.-R. Ji, A. T. Suzuki, J. H. O. Sales, and R. Thibes, Pauli-Villars regularization elucidated in Bopp-Podolsky's generalized electrodynamics, Eur. Phys. J. C **79**, 871 (2019).

[16] J. C. Maxwell, A dynamical theory of the electromagnetic field, Philos. Trans. R. Soc. London **155**, 459 (1865).

[17] W. Greiner, *Quantum Mechanics: An Introduction*, 4th ed. (Springer, Berlin, 2001).

[18] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms* (WILEYVCH Verlag GmbH & Co. KGaA, Weinheim, 1997).

[19] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 1st ed. (Pergamon Press, Oxford, 1960).

[20] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. (Cambridge University Press, Cambridge, 2001).

[21] R. Brauer and H. Weyl, Spinors in *n* dimensions, Am. J. Math. **57**, 425 (1935).

[22] See Supplemental Material at http://link.aps.org/supplemental/10.1103/1fj9-qccb for supplemental code.

[23] W. Gerlach and O. Stern, Der experimentelle Nachweis der Richtungsquantelung im Magnetfeld, Z. Phys. **9**, 349 (1922).

[24] Shown in Fig. 1 is an adaptation of the commemorative plaque of the Stern-Gerlach experiment at the Physikalischer Verein in Frankfurt.

[25] N. Rosen and C. Zener, Double Stern-Gerlach experiment and related collision phenomena, Phys. Rev. **40**, 502 (1932).

[26] J. A. Sidles, Folded Stern-Gerlach experiment as a means for detecting nuclear magnetic resonance in individual nuclei, Phys. Rev. Lett. **68**, 1124 (1992).

[27] H. Batelaan, T. J. Gay, and J. J. Schwendiman, Stern-Gerlach effect for electron beams, Phys. Rev. Lett. **79**, 4517 (1997).

[28] N. Hermanspahn *et al.*, Observation of the continuous Stern-Gerlach effect on an electron bound in an atomic ion, Phys. Rev. Lett. **84**, 427 (2000).

[29] Y. Li, C. Bruder, and C. P. Sun, Generalized Stern-Gerlach effect for chiral molecules, Phys. Rev. Lett. **99**, 130403 (2007).

[30] Y. Japha and R. Folman, Quantum uncertainty limit for Stern-Gerlach interferometry with massive objects, Phys. Rev. Lett. **130**, 113602 (2023).

[31] L. Ballentine, *Quantum Mechanics: A Modern Development* (World Scientific, Singapore, 1998).

[32] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd ed. (Cambridge University Press, Cambridge, 2018).

[33] J. Larmor, On the theory of the magnetic influence on spectra; and on the radiation from moving ions, Philos. Trans. R. Soc. London A **190**, 205 (1897).

[34] W. Thomas, The motion of the spinning electron, Nature (London) **117**, 514 (1926).

[35] A. Ozawa, T. Kobayashi, K. Y. T. Suzuki, and I. Tanihata, New magic number, $N = 16$, near the neutron drip line, Phys. Rev. Lett. **84**, 5493 (2000).

[36] C. A. Tolman, The 16 and 18 electron rule in organometallic chemistry and homogeneous catalysis, Chem. Soc. Rev. **1**, 337 (1972).

[37] N. P. E. Barry, R. J. Deeth, G. J. Clarkson, I. Prokes, and P. J. Sadler, Thermochromic organometallic complexes: Experimental and theoretical studies of 16- to 18-electron interconversions of adducts of arene Ru(II) carboranes with aromatic amine ligands, Dalton Trans. **42**, 2580 (2013).

[38] H. Gies, P. Heinzel, J. Laufkötter, and M. Picciau, Quantum field theories of relativistic Luttinger fermions, Phys. Rev. D **110**, 065001 (2024).

[39] E. Cosserat and F. Cosserat, *Théorie des corps déformables* (Hermann et Fils, Paris, 1909).

[40] D. J. Korteweg and G. de Vries, XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, London Edinburgh Dublin Philos. Mag. J. Sci. **39**, 422 (1895).

[41] N. J. Zabusky and M. D. Kruskal, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. **15**, 240 (1965).

[42] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. **27**, 1192 (1971).

[43] A. S. Fokas and M. J. Ablowitz, Linearization of the Korteweg-de Vries and Painlevé II equations, Phys. Rev. Lett. **47**, 1096 (1981).

[44] R. E. Goldstein and D. M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, Phys. Rev. Lett. **67**, 3203 (1991).

[45] A. Landé, Finite self-energies in radiation theory. Part I, Phys. Rev. **60**, 121 (1941).

[46] A. Landé and L. H. Thomas, Finite self-energies in radiation theory. Part II, Phys. Rev. **60**, 514 (1941).

[47] A. Landé and L. H. Thomas, Finite self-energies in radiation theory. Part III, Phys. Rev. **65**, 175 (1944).

[48] B. Podolsky, C. Kikuchi, and P. Schwed, A generalized electrodynamics part I: Mathematics, Phys. Rev. **62**, 68 (1942).

[49] B. Podolsky and P. Schwed, A review of generalized electrodynamics, Phys. Rev. **65**, 228 (1944).

[50] D. Montgomery, Generalized electrodynamics and the electron, Phys. Rev. **69**, 117 (1946).
[51] A. Green, Generalized electrodynamics and elementary particles, Phys. Rev. **72**, 628 (1947).
[52] B. Podolsky and P. Schwed, Review of generalized electrodynamics, Rev. Mod. Phys. **20**, 40 (1948).
[53] L. De Broglie, On a new conception of the interaction between charges and electromagnetic field, Phys. Rev. **76**, 862 (1949).
[54] G. Amelino-Camelia, Relativity: Special treatment, Nature (London) **418**, 34 (2002).
[55] G. Amelino-Camelia, Doubly special relativity: First results and key open problems, Int. J. Mod. Phys. D **11**, 1643 (2002).
[56] J. Magueijo and L. Smolin, Lorentz invariance with an invariant energy scale, Phys. Rev. Lett. **88**, 190403 (2002).
[57] S. Judes and M. Visser, Conservation laws in "doubly special relativity", Phys. Rev. D **68**, 045001 (2003).
[58] J. Kowalski-Glikman and L. Smolin, Triply special relativity, Phys. Rev. D **70**, 065020 (2004).
[59] S. Liberati, S. Sonego, and M. Visser, Interpreting doubly special relativity as a modified theory of measurement, Phys. Rev. D **71**, 045001 (2005).
[60] P. Jizba and F. Scardigli, The emergence of special and doubly special relativity, Phys. Rev. D **86**, 025029 (2012).
[61] J. J. Relancio and L. Santamaría-Sanz, Doubly special relativity as a non-local quantum field theory, Phys. Rev. D **110**, 086004 (2024).
[62] L. Euler, *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes* (Academiae Scientiarum Imperialis Petropolitanae, Saint Petersburg, 1744).
[63] A. E. H. Love, On the small free vibrations and deformations of elastic shells, Philos. Trans. R. Soc. A **179**, 491 (1888).
[64] S. P. Timoshenko, LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars, London Edinburgh Dublin Philos. Mag. J. Sci. **41**, 744 (1921).
[65] R. D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates, J. Appl. Mech. **18**, 31 (1951).
[66] J. N. Reddy, A simple higher-order theory for laminated composite plates, J. Appl. Mech. **51**, 745 (1979).
[67] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, 3rd ed., *Course of Theoretical Physics* (Butterworth-Heinemann, Oxford, 1986), Vol. 7.
[68] L. Pocivavsek, R. Dellsy, A. Kern, S. Johnson, B. Lin, K. Y. C. Lee, and E. Cerda, Stress and fold localization in thin elastic membranes, Science **320**, 912 (2008).
[69] R. A. Toupin, Elastic materials with couple-stresses, Arch. Ration. Mech. Anal. **11**, 385 (1962).
[70] C. Truesdell and W. Noll, The non-linear field theories of mechanics, in *Encyclopedia of Physics*, edited by S. Flügge (Springer, Berlin, 2004), Vol. III/3, Chap. Polar Elastic Materials, pp. 1–579.
[71] A. C. Eringen, *Microcontinuum Field Theories: I. Foundations and Solids* (Springer, New York, 1999).
[72] A. Merkel, V. Tournat, and V. E. Gusev, Experimental evidence of rotational elastic waves in granular phononic crystals, Phys. Rev. Lett. **107**, 225502 (2011).
[73] X. Xu, C. Wang, W. Shou, Z. Du, Y. Chen, B. Li, W. Matusik, N. Hussein, and G. Huang, Physical realization of elastic cloaking with a polar material, Phys. Rev. Lett. **124**, 114301 (2020).
[74] H. Nassar, X. Xu, C. Wang, and G. Huang, Polar metamaterials: A new outlook on resonance for cloaking applications, Phys. Rev. Lett. **124**, 084301 (2020).
[75] J. Kishine, A. S. Ovchinnikov, and A. A. Tereshchenko, Chirality-induced phonon dispersion in a noncentrosymmetric micropolar crystal, Phys. Rev. Lett. **125**, 245302 (2020).
[76] H. Fan, B. Xia, L. Tong, S. Zheng, and D. Yu, Elastic higher-order topological insulator with topologically protected corner states, Phys. Rev. Lett. **122**, 204301 (2019).
[77] M. I. N. Rosa, R. K. Pal, J. R. F. Arruda, and M. Ruzzene, Edge states and topological pumping in spatially modulated elastic lattices, Phys. Rev. Lett. **123**, 034301 (2019).
[78] P. A. Deymier and K. Runge, Revealing topological attributes of stiff plates by Dirac factorization of their 2D elastic wave equation, Appl. Phys. Lett. **120**, 081701 (2022).
[79] W. Heisenberg, Über die in der Theorie der Elementarteilchen auftretende universelle Länge, Ann. Phys. **424**, 20 (1938).
[80] S. Hossenfelder, Minimal length scale scenarios for quantum gravity, Living Rev. Relativ. **16**, 2 (2013).