

Experimental device-independent efficient certification of unsharp measurements

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Unsharp measurement is a unique type of measurement which provides a tradeoff between information gain and state disturbance. Such measurement is a crucial method in quantum theory and shows potential in various applications. We propose a device-independent scheme for certifying unsharp measurements with binary outcome in an efficient manner. Our scheme has low consumption of measurement resources and is robust against noise. In experiment, we demonstrate the certification of a typical kind of unsharp measurements. We achieve high fidelities above 0.9 when the strength of the unsharp measurement is less than 0.6 and get near unity fidelities in the low measurement strength range. Our scheme can be applied to enhance the feasibility and practicability of device-independent quantum information processing tasks.

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I. INTRODUCTION

Measurement theory plays an essential role in quantum mechanics [1]. The projective measurements make a quantum state irreversibly collapse into one of its eigenstates, thereby maximizing state disturbance. In contrast, positive-operator-valued measurements (POVMs) represent a more general form of measurements that extends beyond traditional projective measurements. Among POVMs, *unsharp measurements*, also referred to as weak measurements [2,3], can reveal partial information about a quantum state without completely collapsing it. These measurements provide a tradeoff between information gain and quantum state disturbance, displaying advantage over projection measurements in a variety of applications, such as quantum state tomography [4,5], unbounded randomness expansion [6,7], and entanglement recycling [8–10].

All these applications require an efficient and reliable method for certifying unsharp measurements. A canonical approach is to exploit quantum tomography technique, which allows one to reconstruct the density operator through an informationally complete set of measurements [11]. But such procedure requires a set of fully characterized inputs and

measurements, the number of which scales exponentially with the dimension of the system. An alternative technique is device-independent (DI) *self-testing*, where conclusions are drawn solely from observed statistics of measurement outcomes under the sole assumptions of no-signaling and the validity of quantum theory [12,13]. This method enables us to assess quantum information processing tasks with error-prone or untrusted devices [14,15]. The criteria of self-testing rely on the fact that certain quantum correlations—corresponding to extremal points of the quantum set—uniquely certify the underlying quantum state and measurements. Moreover, these quantum correlations can be certified by the maximum violation of some certain inequalities. Typically, the maximal violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [16] uniquely identifies two-qubit maximally entangled states [17]. Over the past decades, self-testing has received substantial attention. Self-testing scenarios for bipartite and multipartite entangled states have been presented in Refs. [18–22]. Recent advances have extended self-testing to quantum measurements, such as Pauli observables [23], entangled measurements [24,25], and quantum logical gates [26].

The extremal quantum correlation used for self-testing is usually achieved by projective measurements, and hence the self-testing of unsharp measurement is not straightforward [19]. Although some semi-DI self-testing schemes for unsharp measurements [27–29] have been proposed by utilizing quantum random access codes, they require an assumption on the system dimension. Also, a DI scenario was proposed [30] by self-testing multiple entangled states through different measurement devices. However, this strategy introduces redundancies in measurement resources and complicates experimental implementation. In this work, we address these

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limitations by proposing and experimentally demonstrating a fully DI scheme that efficiently certifies two-outcome unsharp measurements on two-dimensional Hilbert space while minimizing resource cost. Our scheme simplifies the measurement devices through a set of self-testing criteria for two-qubit entangled states, which requires only two measurement settings per party, significantly simplifying experimental implementation. Experimentally, we demonstrate the certification of a representative class of unsharp measurements and show robust noise tolerance across a broad range of measurement strengths. Our work presents an application of self-testing which could refine the practicability of quantum information processing tasks such as DI quantum key distribution and unbounded randomness expansion.

II. THEORETICAL BACKGROUND AND SCENARIO

Consider a quantum measurement $\bar{M}$ operating on a single-qubit system with binary outcomes $m \in \{+1, -1\}$. Such measurement can be represented in terms of two observables $\bar{M} = \sum_m m E_m$, where $E_m = \lambda_m (\mathbb{I} + \vec{n}_m \cdot \vec{\sigma})$ are POVMs with $\vec{n}_m$ ($|\vec{n}_m| \leq 1$) being the Bloch vector, $\lambda_m \in [0, 1]$ denoting the biased outcome probabilities, and $\vec{\sigma}$ representing the Pauli matrices [10,31]. The positivity and normalization of POVMs imply $\sum_m \lambda_m = 1$ and $\sum_m \lambda_m \vec{n}_m = 0$. Then the measurement $\bar{M}$ can be rewritten as [27,28]

$$\bar{M} = c\mathbb{I} + \vec{n} \cdot \vec{\sigma}, \quad (1)$$

where $c = 2\lambda_{+1} - 1$ denotes the outcome bias of the observable, and $\vec{n} = 2\lambda_{+1}\vec{n}_{+1}$ is the Bloch vector. The requirement $|\bar{M}| \leq 1$ is equivalent to the constraint $|c| \leq 1 - |\vec{n}|$. The measurement *strength* or *sharpness* [27,28] (also called its information gain [9,10]) is defined as $\eta = |\vec{n}|$. The cases of measurement strength $\eta = 0$ and 1 correspond to noninteractive and projective measurements, respectively, and an unsharp measurement is achieved with $\eta \in (0, 1)$.

Further, the POVM can be defined by Kraus operators K_m satisfying $E_m = K_m^\dagger K_m$, expressed as

$$\begin{aligned} K_{+1} &= \frac{1}{\sqrt{2}} (\sqrt{2\lambda_{+1} + \eta} \Pi_{+1|\vec{n}} + \sqrt{2\lambda_{+1} - \eta} \Pi_{-1|\vec{n}}), \\ K_{-1} &= \frac{1}{\sqrt{2}} (\sqrt{2\lambda_{-1} - \eta} \Pi_{+1|\vec{n}} + \sqrt{2\lambda_{-1} + \eta} \Pi_{-1|\vec{n}}), \end{aligned} \quad (2)$$

where the Bloch vector $\vec{n}$ is chosen as the direction of σ_z without loss of generality, and $\Pi_{\pm 1|\vec{n}}$ are the projection operators with eigenvalues ± 1 . Note that our scheme is valid for the unsharp measurements with the Kraus operators given in Eq. (2) under any unitary transformations.

Here we aim to verify the unsharp measurement in the form of Eq. (1) in a device-independent way. In a bipartite Bell-type scenario, to certify that a measurement described by M acts like a target measurement $\bar{M}$, it is sufficient to show that there exist isometry maps $\Lambda_{i/o}$ on the injection/output subspaces and subsystems respectively such that [26]

$$\begin{aligned} F(M, \bar{M}) &= \max_{\Lambda_B^i, \Lambda_B^o} F[(\mathbb{I} \otimes \Lambda_B^o \circ M \circ \Lambda_B^i) |\Psi^-\rangle \langle \Psi^-|, \\ &\quad (\mathbb{I} \otimes \bar{M}) |\Psi^-\rangle \langle \Psi^-|], \end{aligned} \quad (3)$$

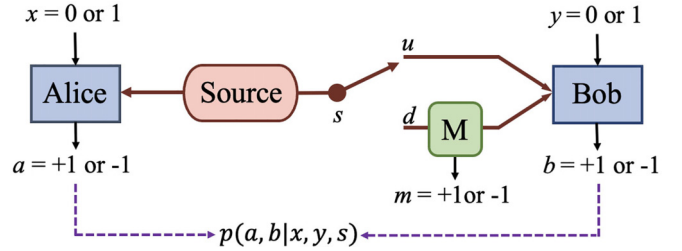


FIG. 1. The DI certification of unsharp measurements. The switch connects port u to characterize the source and the local measurements, and connects port d to certify the postmeasurement states after M . The statistics recorded in two paths are then used to certify the unsharp measurement M device-independently.

where $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ is the Uhlmann fidelity between two states ρ and σ . That is, a measurement can be fully described by its action on one half of a maximally entangled state $|\Psi^-\rangle$.

The DI certification scenario is shown in Fig. 1. The scenario involves two observers named Alice and Bob sharing an underlying state ρ_{AB} , which in the ideal case is the Bell state $\rho_{AB} = |\Psi^-\rangle \langle \Psi^-|$, with

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \quad (4)$$

Each party has access to a black box with two measurement inputs, denoted as A_x and B_y , $x, y \in \{0, 1\}$ for Alice and Bob, respectively. For each measurement input, dichotomic outcomes are obtained called a for Alice and b for Bob with $a, b = \pm 1$. Also, the unsharp measurement M to be self-tested is on Bob's side. A switch is controlled randomly by a stochastic variable s , determining whether to conduct M before the local measurements. The protocol with respect to the switch state is described in Table I.

The key ideal of our scheme is based on the result that different two-qubit entangled states can be self-tested via fixed measurement settings [18]. Taking into account the source and postmeasurement states under different m , three different entangled states are involved in our protocol, which can be expressed uniformly as

$$|\bar{\psi}_i\rangle = \cos \theta_i |01\rangle - \sin \theta_i |10\rangle \quad (5)$$

where $\theta_i \in (0, \frac{\pi}{4}]$ and the detailed form will be discussed later. Note that when $\theta = \pi/4$, the state is the ideal source in Eq. (4). It has been shown in Ref. [18] that the above states with different entangled degree can be self-tested by the generalized tilted-CHSH inequalities written as

$$\mathcal{B}_{[\alpha_i, \beta_i]} = \alpha_i A_0 + \beta_i (A_0 B_0 + A_0 B_1) + A_1 B_0 - A_1 B_1, \quad (6)$$

where $\beta_i \geq 1$. Its maximal classical and quantum bounds are $C_{[\alpha_i, \beta_i]} = \alpha_i + 2\beta_i$ and $Q_{[\alpha_i, \beta_i]} = \sqrt{(4 + \alpha_i^2)(1 + \beta_i^2)}$, respectively. The maximal quantum bounds $Q_{[\alpha_i, \beta_i]}$ are only achieved by the entangled states (5) with a fixed measurement setting

$$\begin{aligned} A_0 &= \sigma_z, \quad B_0 = \cos \mu \sigma_z + \sin \mu \sigma_x, \\ A_1 &= \sigma_x, \quad B_1 = \cos \mu \sigma_z - \sin \mu \sigma_x \end{aligned} \quad (7)$$

up to a local unitary, where α_i and β_i are determined by $\sin 2\theta_i = \sqrt{(4 - \alpha_i^2 \beta_i^2)/(4 + \alpha_i^2)}$ and $\sin(2\theta_i)/\beta_i = \tan \mu$.

TABLE I. Main protocol.

$s = u$	Self-testing of the source and local measurements 1. The source distributes the bipartite state ρ_{AB} to Alice and Bob. Two parties randomly choose measurement input x and y and observe output a and b, respectively. 2. Collect the joint statistics $p(a, b x, y, s = u)$. 3. The distributions $p(a, b x, y, s = u)$ maximally violate a biased Bell inequality that certifies the maximally entangled state and corresponding measurements.
$s = d$	Self-testing of the postmeasurement states 1. Distribute the source ρ_{AB} to two parties. The particle sent to Bob goes through M. 2. Alice and Bob randomly choose their inputs x and y to measure the received particle, and collect the probability distribution $p(a, b x, y, s = d)$ and the outcome m of M. 3. The distributions $p(a, b x, y, s = d)$ maximally violate the generalized tilted-CHSH inequality that certifies the partial entangled state dependent on the outcome m of M.
Criterion	The ideal self-testings of the source and states after M for two outcomes certify that M is equivalent to the target form $\bar{M}$.

Moreover, if the maximal quantum bound of Eq. (6) is observed, then there exist local isometry maps $\Lambda_A \otimes \Lambda_B$ such that

$$F((\Lambda_A \otimes \Lambda_B)\rho, |\bar{\psi}_i\rangle\langle\bar{\psi}_i|) = 1. \quad (8)$$

That is, the underlying state ρ is equivalent to the target form up to a local isometry.

In our scheme, the measurement setting is determined by $\tan \mu = \sqrt{4\lambda_{+1}^2 - \eta^2}/(2\lambda_{+1})$ for each target measurement M . According to the situations in Table I with switch s connecting to different paths, our scheme can be briefly stated as follows (see Appendix A for detail). When $s = u$, we aim to self-test the source and local measurements. Here we set $\alpha = 0$ such that the Bell inequality in Eq. (6) is simplified to the biased CHSH inequality where $\beta = 1/\tan \mu$, which belongs to the whole set of self-testing criteria for the Bell state [21]. The maximal quantum bound $Q_{[0,\beta]}^{s=u} = 2\sqrt{1 + \tan^2 \mu}/\tan \mu$ uniquely certifies the Bell states and corresponding local measurements defined in Eqs. (4) and (7) up to a local isometry.

When $s = d$, self-testing of the collapsed quantum states ρ_m conditioned on the outcome m of measurement M must be achieved, where

$$\rho_m = \frac{K_m \rho_{AB} K_m^\dagger}{\text{Tr}(\rho_{AB} K_m^\dagger K_m)} \quad (9)$$

and outcomes m correspond to distinct postmeasurement states of the source. By substituting the Kraus operators from Eq. (2), we obtain the ideal reference state $\bar{\rho}_m = |\bar{\psi}_m\rangle\langle\bar{\psi}_m|$, where $|\bar{\psi}_m\rangle$ follows Eq. (5) with $\theta_m = \cos^{-1}[\sqrt{(2\lambda_m - m\eta)/(\lambda_m)}/2]$. Taking θ_m into Eq. (6), the coefficients α and β can be written with respect to η and λ_m as $\alpha_m = 2\eta \tan \mu / \sqrt{(4\lambda_m^2 - \eta^2)(1 + \tan^2 \mu)}$ and $\beta_m = \sqrt{4\lambda_m^2 - \eta^2}/(2\lambda_m \tan \mu)$. With the same measurement setting for each m , if the maximal quantum violation $Q_{[\alpha,\beta]}^{m|s=d} = \sqrt{(4 + \alpha_m^2)(1 + \beta_m^2)}$ is observed, the corresponding entangled state $|\psi_m\rangle$ is uniquely determined up to a local isometry with the same measurement setting in Eq. (6). It is noteworthy that the projective measurement setting in Eq. (7) is only dependent on the unsharp measurement to be certified, that is, the measurement setting remains unchanged regardless of the state of the switch s .

Due to unavoidable imperfections in practical realization, the deviation of observation from the ideal values is inevitable during the self-test process. Combining the above two situations, the fidelity between the actual measurement M and the target form $\bar{M} = c\mathbb{I} + \eta\sigma_z$ can be bounded by fidelity of the states before and after M with triangle inequality [26,32]

$$F(M, \bar{M}) \geq \cos(\cos^{-1} F^i + \cos^{-1} F^o), \quad (10)$$

where $F(M, \bar{M})$ is defined as Eq. (3), F^i is the fidelity between the source and Bell state when $s = u$, and F^o is the fidelity of the source after M when $s = d$. It is noteworthy that F^o is defined as a weighted sum over measurement outcomes, that is $F^o \equiv \sum_{m=\pm 1} \sqrt{\lambda_m \bar{\lambda}_m} F_m^o$ where $\bar{\lambda}_m$ is the ideal value and λ_m is obtained from experiment. The maximal quantum values conditioned on each outcome m yield $F_{+1}^o = F_{-1}^o = 1$ and $\lambda_m = \bar{\lambda}_m$, consequently ensuring the overall value $F^o = 1$. Hence, if the maximal quantum bounds of the inequalities (6) determined by the quantum states (4) and (5) are simultaneously observed, then $F^i = F^o = 1$ which then guarantees $F(M, \bar{M}) = 1$. Thus we claim that the measurement M is equivalent to the target form $\bar{M}$ up to a local isometry.

To demonstrate the noise tolerance of our scheme, we employ the extraction channel method to derive tight fidelity bounds for each quantum state by optimizing relevant parameters and constructing appropriate Bell inequalities (see the Appendixes for details). This enables a comprehensive robustness analysis for self-testing generic unsharp measurements. In Fig. 2, we present contour plots illustrating the fidelity of two representative unsharp measurement implementations as functions of Bell inequality violation deviation at each protocol step. For computational convenience, we selected the representative case $\bar{M} = 1/2\mathbb{I} + 1/4\sigma_z$ with asymmetric outcome probabilities ($\bar{\lambda}_{+1} = 3/4$, $\bar{\lambda}_{-1} = 1/4$). The resulting robustness bounds are presented in Fig. 2(a). Moreover, we compare the fidelity of unsharp measurements $\bar{M} = \eta\sigma_z$ with the fidelity in previous work [30] for the same weak value $\eta = 0.34$. The results are shown in Fig. 2(b). The deviations are calculated as percentages of the maximal quantum violation value for each protocol Bell inequality. The robustness of our work only shows a small degradation compared to the work in Ref. [30]. This minor performance difference stems from two fundamental tradeoffs in our approach. On

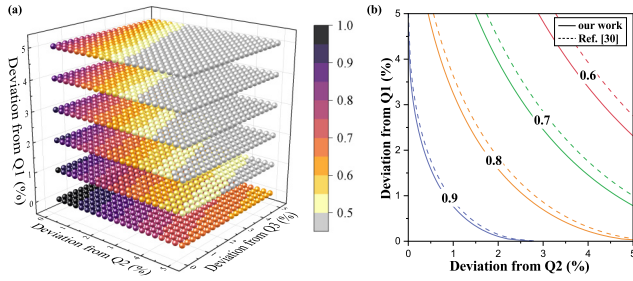


FIG. 2. Robustness analysis. The color gradient represents the fidelity value. (a) Robustness analysis of target measurement $\bar{M} = 1/2\mathbb{I} + 1/4\sigma_z$. (b) Comparison of robustness. Fidelity as a function of deviations from the maximal quantum bound when $\eta = 0.34$.

the one hand, a single measurement setup is used for both source and postmeasurement verification. On the other hand, biased-CHSH inequality shows reduced robustness (vs standard CHSH form) due to additional parameter sensitivity. This tradeoff enables certification of broader measurement classes while remaining experimentally feasible.

So far, we have provided the certification of unsharp measurement in a device-independent manner. To show the practicalness of our scheme, here we present an experimental demonstration and its robustness analysis based on a typical unsharp measurement.

III. EXPERIMENT

We demonstrate our protocol using the parametrization $\lambda_{\pm 1} = 1/2$ with the measurement direction $\vec{n}$ aligned along σ_z . This configuration yields the measurement operator $\bar{M} = \eta\sigma_z$ with $\eta \in [0, 1]$, which has unbiased outcomes. Furthermore, the corresponding angles $\theta_{\pm 1}$ satisfy that $\theta_{+1} = \pi/2 - \theta_{-1} = [\cos^{-1}(-\eta)]/2$, which means the reference states $|\psi_m\rangle$ in Eq. (5) have the same degree of entanglement and are equivalent up to the local unitary $-\sigma_x \otimes \sigma_x$. The experiment realization employs photonic polarization states, where the basis states $|0\rangle$ and $|1\rangle$ are defined as the horizontal and vertical polarization states, denoted as $|H\rangle$ and $|V\rangle$, respectively. As illustrated in Fig. 3, our experiment configuration can be divided into four components: source, the unsharp

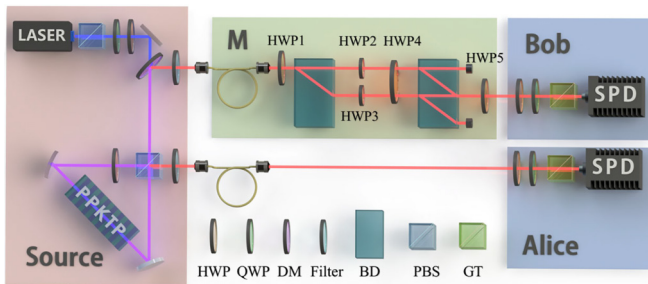


FIG. 3. Experimental setup. PPKTP, periodically poled potassium titanyl phosphate crystal; HWP, half-wave plate; QWP, quarter-wave plate; DM, dichroic mirror; BD, beam displacer; PBS, polarization beam splitter; GT, Glan-Taylor polarizer; SPD, single-photon detector.

measurement apparatus M , and the local measurement devices for Alice and Bob.

In the state preparation stage, we use a continuous-wave laser with a wavelength of 405 nm to pump a 20-mm-long periodically poled potassium titanyl phosphate (PPKTP) crystal in a phase-stable Sagnac interferometer [33]. Bidirectional pumping (clockwise and counterclockwise) generates frequency-degenerate photon pairs at 810 nm through type-II spontaneous parametric down-conversion in both propagation directions. Through proper control of the pump beam polarization, the target two-qubit maximally entangled state $|\Psi^-\rangle = (|HV\rangle - |VH\rangle)/\sqrt{2}$ can be generated. The maximum coincidence counting rate is 1200 Hz under a pump power of 4.5 mW. The interference visibilities with one photon projected to $|H\rangle$ and $(|H\rangle + |V\rangle)/\sqrt{2}$ states are measured to be 99.5 and 99.2%, respectively. Quantum state tomography yields an estimated state fidelity of 0.9929 (see details in Appendix C).

The entangled photon pairs are then distributed to Alice and Bob. On Alice's side, she performs local projective measurements A_x in Eq. (7), which can be realized by a combination of a half-wave plate (HWP), a quarter-wave plate (QWP), and a Glan-Taylor polarizer (GT).

On the other side the photons go through the switchable unsharp measurement apparatus before reaching Bob's local measurement devices. The unsharp measurement is implemented using a Mach-Zehnder interferometer [34] composed of five HWPs and two beam displacers which respectively transmit and displace horizontally (H) and vertically (V) polarized components. The interferometer is configured with HWP2 in the H path and HWP3 in the V path fixed at 0° and 45° , respectively. HWP4, shared by both paths, controls the measurement strength η by setting the optical axis angle as $45^\circ - (\cos^{-1} \eta)/4$. This wave plate additionally functions as a switch: when rotated to 22.5° , the measurement strength reduces to $\eta = 0$, and thus the photons are sent directly to Bob without unsharp measurement. The wave plates HWP1 and HWP5 are used to choose the measurement bases, and here they are respectively set at 0° and 45° to realize unsharp measurement in the σ_z basis. The middle output path of the interferometer corresponding to outcome $m = +1$ is routed to Bob's side, while the remaining ports are blocked. Note that the complementary measurement with $m = -1$ can be realized by simply setting the angle of HWP4 to $(\cos^{-1} \eta)/4$. We employ quantum process tomography (QPT) to reconstruct the matrix of our unsharp measurement device and achieve an average fidelity value above 0.97 (for details see Appendix A). After the unsharp measurement apparatus, photons are sent to Bob and the projective measurement B_y is implemented via a HWP and a QWP followed by a GT.

Finally, coincidence counts are obtained by single-photon detectors (SPDs) and a coincidence counter with a coincidence window of 500 ps. The joint probability distributions $p(a, b|x, y, s)$ for different measurement settings and outcomes are extracted from these coincidence counts within an integral time of 5 s.

IV. RESULTS

When $s = u$, we self-test the source and local measurements with the biased CHSH inequality given in Eq. (6)

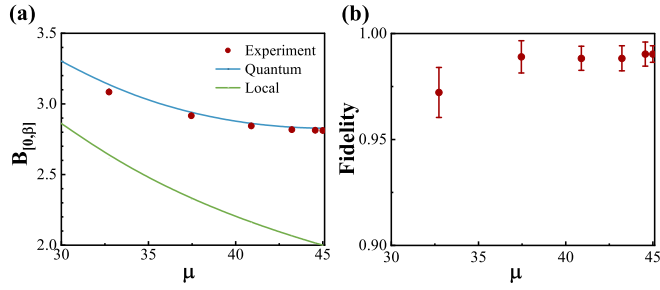


FIG. 4. Experimental results for $s = u$. (a) Observed value of Bell inequality $B_{[0, \beta]}$ with respect to measurement settings μ , where $\beta = 1/\tan \mu$. The blue and green curves represent the maximal quantum and local bounds, respectively, with red dots expressing the experimental results. (b) Fidelity with the Bell state subject to Bell violation with respect to μ .

with $\alpha = 0$ and $\beta = 1/\tan \mu$. In Fig. 4(a) we plot the experimental results of the violation value $B_{[0, \beta]}$ with respect to the parameter μ . All error bars are estimated by Poisson statistics of the experimental data. Our experimental results for the maximal quantum bound $Q_{[0, \beta]}$ are consistent with the theoretical values. Further, the fidelity between the source and Bell state is obtained by fixing the parameters v and t for each μ according to the definitions in the linear robust bound function given in Appendix B1 with results shown in Fig. 4(b). Note that for small values of μ , classical bounds approach quantum bounds, making the fidelity more vulnerable to noise. Particularly, when $\mu = 45^\circ$, the inequality covers the standard CHSH form. Under such situation, we achieve $B_{[0, 1]} = 2.813 \pm 0.008$ and a fidelity with 0.9903 which is higher compared with other inequalities.

When $s = d$, we certify the two partially entangled states $|\psi_m\rangle$ in Eq. (5). The observed violations for tilted-CHSH inequality $B_{[\alpha_m, \beta_m]}$ are calculated according to Eq. (6) with $\beta_m = 1$ and α_m determined by $\sin(2\theta_m) = \sqrt{(4 - \alpha_m^2)/(4 + \alpha_m^2)}$. The results are plotted in Figs. 5(a) and 5(b). Then fidelities of the states with the reference states are calculated via the method

in Appendix B. As shown in Figs. 5(c) and 5(d), the fidelities are above 0.95 for $\eta \in [0, 0.5]$ while they decrease for higher η . Since the tilted-CHSH inequality is susceptible to noise for weakly entangled states [35], the contamination of fidelity may come from the fact that a stronger measurement strength η will reduce the entanglement degree of the reference state.

Combining the fidelity of the source F^i and postmeasurement states F^o , we are able to estimate the lower bound of fidelity between the quantum instrument M and its target form $\bar{M}$ according to Eq. (10). The results are summarized in Fig. 5(e). It can be seen that the robustness works well for measurement strength ranging from 0 to 0.6, as they are above 0.9 and also in good agreement with the fidelities achieved by QPT. Considering the fact that most quantum tasks involving unsharp measurements require weak measurement strength, our scheme works well for practical applications.

V. CONCLUSION

We developed and experimentally validated a cost-effective, noise-tolerant certification protocol for general unsharp measurements that accommodates outcome distribution biases. The entangled states involved in our scheme can be self-tested by using only two binary measurements per party via the maximal violation of a family of Bell inequalities. Moreover, we presented a constructive method for extraction channel design specific to the generalized tilted-CHSH case, and a numerical optimization algorithm to derive the self-testing robustness bounds. These innovations collectively establish a robust framework for fidelity estimation of general unsharp measurements. The results confirm our scheme's superior noise resilience, while handling more general measurement classes. In experiment, we certified a series of unsharp measurement devices across a wide range of measurement strengths. We achieved high fidelities with the values above 0.9 over a large range of measurement strength, and especially in the weak strength range the fidelity values are approaching to 1.

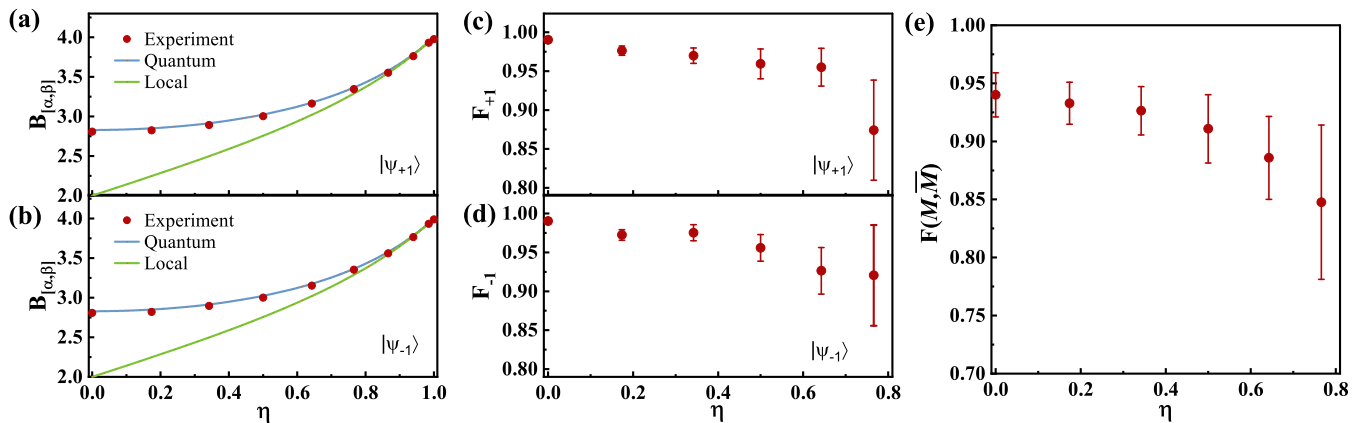


FIG. 5. Experimental results for $s = d$. (a), (b) Observed violation of generalized tilted-CHSH inequality $W_{[\alpha, \beta]}$ with respect to the outcome state (a) $|\psi_{+1}\rangle$ and (b) $|\psi_{-1}\rangle$. (c), (d) The fidelity bounds of the target state (c) $|\psi_{+1}\rangle$ and (d) $|\psi_{-1}\rangle$. (e) The lower bound of the fidelity with the target unsharp measurement against measurement strength η . The blue and green curves correspond to the maximal quantum and local bounds, respectively. The red dots represent the experimental values obtained.

In contrast to previous schemes for certifying unsharp measurements [27–30], our scheme certifies the unsharp measurements in a fully device-independent way through a single set of projective measurements. Hence, we demonstrate the feasibility of characterizing unsharp measurements through a fully untrusted device with low resource cost. Moreover, the method developed in our work can be extended to a variety of quantum information tasks due to its simplicity and robustness, such as entanglement recycling and the certification of high-dimensional nonprojective measurements.

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APPENDIX A: DEVICE-INDEPENDENT EFFICIENT CERTIFICATION OF UNSHARP MEASUREMENTS

Consider the canonical scenario involving two spatially separated parties, Alice and Bob. Each has access to a black box with inputs denoted respectively by x, y and outputs a, b . One could model these boxes with an underlying state $|\psi\rangle_{AB}$ and measurement projectors $\{M_x^a\}_{x,a}$ and $\{M_y^b\}_{y,b}$, which are commutative for different parties. After sufficiently many repetitions of the experiment one can estimate the joint conditional statistics, known as the behavior, $p(a, b|x, y) = \text{Tr}(M_x^a M_y^b |\psi\rangle\langle\psi|)$. Self-testing refers to a device-independent certification recipe where the nontrivial information on the state and the measurements is uniquely certified by the observed behavior $p(a, b|x, y)$, without assumptions on the underlying degrees of freedom. Importantly, Bell inequality is at the heart of self-testing. A Bell inequality consists of a function $\mathcal{B}$ of the probabilities $p(a, b|x, y)$ such that for sources producing separable states one has

$$\mathcal{B}(\{p(a, b|x, y)\}) \leq c. \quad (\text{A1})$$

The bound c is called the classical bound which holds for any separable systems. But such inequality can be violated by some entangled sources and achieves a maximal violation by a particular state and corresponding measurements up to local unitaries. Such situation can be understood geometrically in the vector space of probabilities shown in Fig. 6, where Bell inequalities are defined by linear hyperplanes [36,37]. This maximal quantum violation is an extremal point of a quantum set which allows one to determine the form of the state and measurement operators that produce the outcomes $p(a, b|x, y)$ [37–40]. Usually, self-testing can be defined formally in the following way [41,42].

Definition A1. The correlations $p(a, b|x, y)$ allow for self-testing if for every quantum behavior $(|\psi\rangle, \{M_x^a, M_y^b\})$ compatible with $p(a, b|x, y)$ there exists a local isometry

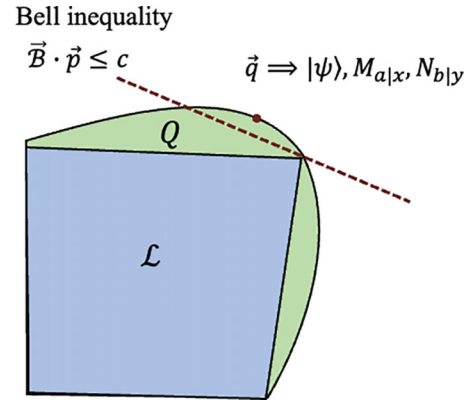


FIG. 6. Geometric representation of self-testing. The blue area $\mathcal{L}$ is the local set where the probability is obtainable with separable states. The green convex area $\mathcal{Q}$ represents the quantum correlation set. The dotted line represents a Bell inequality. The entangled quantum states are capable of producing probability vector $\vec{p}$ that lies outside of $\mathcal{L}$ and thus violates Bell inequalities in $\mathcal{Q}$. The extremal points of the quantum set obtained by the probability vector $\vec{q}$ maximally violate some Bell inequality, and admit up to a local isometry in terms of a particular entangled state and measurements.

$\Lambda = \Lambda_A \otimes \Lambda_B$ such that

$$\begin{aligned} \Lambda |\psi\rangle_{AB} |00\rangle_{A'B'} &= |\text{junk}\rangle_{AB} \otimes |\bar{\psi}\rangle_{A'B'}, \\ \Lambda (M_x^a |\psi\rangle_{AB} |00\rangle_{A'B'}) &= |\text{junk}\rangle_{AB} \otimes \bar{M}_x^a |\bar{\psi}\rangle_{A'B'}, \end{aligned} \quad (\text{A2})$$

where $|00\rangle_{A'B'}$ are the trusted auxiliary qubits attached by Alice and Bob locally into their systems and $(|\bar{\psi}\rangle, \{\bar{M}_x^a, \bar{M}_y^b\})$ are the target systems.

That is, the quantum correlations $p(a, b|x, y)$ certify uniquely the state and the measurements, up to a local isometry.

1. Self-testing of different entanglement resources via fixed measurement settings

In this section we review some previous results on the self-testing of the two-qubit state which are used as building blocks of our work.

It has been proved in Ref. [18] that a family of entangled two-qubit states $\{|\bar{\psi}_i\rangle\}_i$ in the form of

$$|\bar{\psi}_i\rangle = \cos \theta_i |01\rangle - \sin \theta_i |10\rangle \quad (\text{A3})$$

with $\theta_i \in (0, \frac{\pi}{4}]$ can be self-tested via a fixed measurement setting utilizing the generalized tilted-CHSH inequality by sum-of-squares decomposition technique. More precisely, conditioned on the input $i \in \{0, 1, 2, \dots, n\}$, the generalized tilted-CHSH Bell inequalities can be written as [43]

$$\mathcal{B}_{[\alpha_i, \beta_i]} = \alpha_i A_0 + \beta_i (A_0 B_0 + A_0 B_1) + A_1 B_0 - A_1 B_1, \quad (\text{A4})$$

where $\beta_i \geq 1$. The maximal classical and quantum bounds of the inequalities are $C_{[\alpha_i, \beta_i]} = \alpha_i + 2\beta_i$ and $Q_{[\alpha_i, \beta_i]} = \sqrt{(4 + \alpha_i^2)(1 + \beta_i^2)}$, respectively. The maximal quantum bound can be only achieved by the entangled state (A3) and

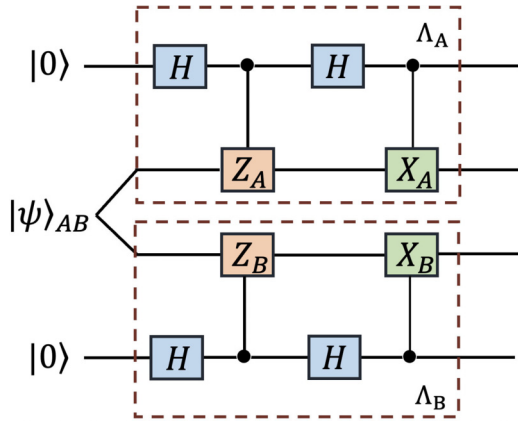


FIG. 7. The local isometry circuit $\Lambda_A \otimes \Lambda_B$. H is the standard Hadamard gate, and Z and X are constructed based on real performed measurements and controlled by the auxiliary qubits.

fixed measurements

$$\begin{aligned} A_0 &= \sigma_z, & B_0 &= \cos \mu \sigma_z + \sin \mu \sigma_x, \\ A_1 &= \sigma_x, & B_1 &= \cos \mu \sigma_z - \sin \mu \sigma_x \end{aligned} \quad (\text{A5})$$

where the angle $\mu \in (0, \frac{\pi}{4}]$, θ and μ satisfy $\sin 2\theta_i = \sqrt{(4 - \alpha_i^2 \beta_i^2)/(4 + \alpha_i^2)}$ and $\tan \mu = \sin(2\theta_i)/\beta_i$, respectively.

This result comes from the fact that, supposing the measurement settings are fixed by μ , for each state $|\psi_i\rangle$, a unique inequality $\mathcal{B}_{[\alpha_i, \beta_i]}$ can be constructed via setting the parameter α_i and β_i according to θ_i and μ , which is maximally violated by the underlying state $|\psi_i\rangle$ and fixed measurement in Eq. (A5). Thus different quantum states can be self-tested with a series of inequalities and a fixed measurement setting. Moreover, if the maximal quantum violation in Eq. (A4) is observed, then there exists an isometry map $\Lambda_A \otimes \Lambda_B$ that uniquely certifies the entangled pairs in Eq. (A3). The structure of the isometry map is shown in Fig. 7, which allows one to extract the information of the target state from the underlying system [18].

Specifically, $\mathcal{B}_{[\alpha, \beta]}$ has two special forms when $\alpha = 0$ and $\beta = 1$, which correspond to biased CHSH [44] and standard tilted-CHSH operators [42], respectively.

a. Biased CHSH inequality. If $\alpha = 0$, the Bell inequality in Eq. (A4) is simplified as a symmetrical biased CHSH inequality

$$\mathcal{B}_{[0, \beta]} = \beta(A_0 B_0 + A_0 B_1) + A_1 B_0 - A_1 B_1 \leq 2\beta, \quad (\text{A6})$$

where $\beta = 1/\tan \mu$. This inequality belongs to the whole set of self-testing criteria for the Bell state [21]. Its maximal quantum violation $\mathcal{Q}_{[0, \beta]} = 2\sqrt{1 + \beta^2}$ is able to self-test maximum entangled state $|\psi\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$ and measurements setting (A5). The inequality recovers to the standard CHSH inequality when $\mu = \pi/4$.

b. Standard tilted-CHSH inequality. If $\beta = 1$, the Bell inequality in Eq. (A4) turns out to be the standard form

$$\mathcal{B}_{[\alpha, 1]} = \alpha A_0 + A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 \leq 2 + \alpha, \quad (\text{A7})$$

with $\alpha = 2\sqrt{2\cos^2 \mu - 1}$. The maximal quantum violation of this inequality is given by $\mathcal{Q}_{[\alpha, 1]} = \sqrt{8 + 2\alpha^2} = 4\cos \mu$, which can be achieved with the measurement settings in Eq. (A5) and state in Eq. (A3) with θ satisfying $\sin 2\theta = \sqrt{(4 - \alpha^2)/(4 + \alpha^2)}$. Any pure two-qubit entangled state can be self-tested by the standard tilted-CHSH inequality with corresponding measurement basis [39, 45].

2. Demonstration for the device-independent certification of unsharp measurement

In this section, we explain our scheme in detail. We aim to self-test that an unknown measurement M behaves as the target unsharp measurement

$$\bar{M} = c\mathbb{I} + \vec{n} \cdot \vec{\sigma}, \quad (\text{A8})$$

which is defined in terms of two observables $\bar{M} = \sum_m m E_m$ with $m = \pm 1$. Here $E_m = \lambda_m(\mathbb{I} + \vec{n}_m \cdot \vec{\sigma})$ are POVMs with $\vec{n}_m$ ($|\vec{n}_m| \leq 1$) being the Bloch vector, $\lambda_m \in [0, 1]$ denoting the biased outcome probabilities, $\vec{\sigma}$ representing the Pauli matrices, $c = 2\lambda_{+1} - 1$ denoting the outcome bias of the observable, and $\vec{n} = 2\lambda_{+1}\vec{n}_{+1}$ denoting the Bloch vector. The strength or unsharp parameter of the measurement is defined as $\eta = |\vec{n}|$.

For the sake of convenience, we set $\vec{\sigma} = \{1, 0, 0\}$ in the rest of the statement without loss of generality. Then E_m can be expressed by Kraus operators K_m satisfying $E_m = K_m^\dagger K_m$, expressed as

$$\begin{aligned} K_{+1} &= \frac{1}{\sqrt{2}}(\sqrt{2\lambda_{+1} + \eta}|0\rangle\langle 0| + \sqrt{2\lambda_{+1} - \eta}|1\rangle\langle 1|), \\ K_{-1} &= \frac{1}{\sqrt{2}}(\sqrt{2\lambda_{-1} - \eta}|0\rangle\langle 0| + \sqrt{2\lambda_{-1} + \eta}|1\rangle\langle 1|). \end{aligned} \quad (\text{A9})$$

In a bipartite Bell-type scenario, the equivalence between the considered measurement and the target one is obtained by showing that there exist local injection and output isometrics Λ^i and Λ^o such that [26]

$$F(M, \bar{M}) = \max_{\Lambda_B^i, \Lambda_B^o} F[(\mathbb{I} \otimes \Lambda_B^o \circ M \circ \Lambda_B^i) |\Psi^-\rangle \langle \Psi^-|, (\mathbb{I} \otimes \bar{M}) |\Psi^-\rangle \langle \Psi^-|] \quad (\text{A10})$$

where $F(\rho, \sigma) = \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ is the Uhlmann fidelity between two states ρ and σ . The role of the isometries is to identify a subspace for the instrument's output and input between which the channel M acts as desired. In other words, a measurement can be fully described by its action on one half of a maximally entangled state $|\Psi^-\rangle$.

In our scheme, Alice and Bob share a source ρ_{AB} , which in the ideal case is the Bell state $\rho_{AB} = |\Psi^-\rangle \langle \Psi^-|$, with

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \quad (\text{A11})$$

The switch s randomly connects port u or d to characterize the source of the local measurements or the postmeasurement states after M . The reference projective measurement devices for Alice and Bob are set as in Eq. (A5). We aim to certify all instruments using a fixed measurement set, regardless of whether the particle passes through M or not. Accordingly, the measurement on Bob's side is associated with the parameters of M , with μ determined by $\tan \mu = \sqrt{4\lambda_{+1}^2 - \eta^2}/(2\lambda_{+1})$.

Now, we give a illustration of our scheme under different switch states. Starting with the situation $s = u$, the switch is connected to the upper path and the state is directly measured by Alice and Bob. To self-test the source and local measurements, we employ a family of biased CHSH inequalities by setting the coefficient $\alpha = 0$ in Eq. (A4), then the inequalities turn into a family of biased CHSH inequalities in Eq. (A6). If the maximal quantum bound $Q_{[\alpha, \beta]}^{s=u} = 2\sqrt{1 + \tan^2 \mu^2 / \tan^2 \mu}$ of the inequality is observed, then there exist local injection isometry maps $\Lambda_A \otimes \Lambda_B$ such that the source is equivalent with the Bell state up a local isometry, i.e.,

$$F^i \equiv F((\Lambda_A \otimes \Lambda_B) \rho_{AB}, |\Psi^-\rangle \langle \Psi^-|) = 1. \quad (\text{A12})$$

The quantum bound is uniquely achieved by Bell states and corresponding local measurements defined in Eqs. (A11) and (A5) up to local unitaries.

Then, we analyze the situation when the switch s is connected to d . The well-certified source and local measurements are used to self-test the unsharp measurement. Here we focus on the state after the measurement M . Generally, a projective measurement inevitably disturbs the system maximally. Hence, if the M performs projective measurement, the entanglement of the source is lost after M and there is no residual entanglement for Bob. Consequently, no inequality mentioned in this paper can be violated in this situation. In contrast, the observation of quantum violations indicates the fact that M must perform nonprojective measurements.

Measurement theory tells us that after going through M , the source state ρ_{AB} collapses to a state conditioned on the outcome m , and the state can be written as

$$\rho_m = \frac{K_m \rho_{AB} K_m^\dagger}{\text{Tr}(\rho_{AB} K_m^\dagger K_m)}. \quad (\text{A13})$$

By substituting the Kraus operators given in Eq. (A9), we get the reference states in the ideal case $\bar{\rho}_m = |\psi_m\rangle \langle \psi_m|$, with

$$|\bar{\psi}_m\rangle = \cos \theta_m |01\rangle - \sin \theta_m |10\rangle, \quad (\text{A14})$$

where $\theta_m = \cos^{-1}[\sqrt{(2\lambda_m - m\eta)/(\lambda_m)}/2]$.

As we have reviewed in Appendix A1 the two states in Eq. (A14) with different entanglement degrees can be self-tested by the maximal violation of the generalized tilted-CHSH inequality (A4) with the fixed measurements given in Eq. (A5). Here, we rewrite the inequality for each output m as

$$\mathcal{B}_{[\alpha, \beta]}^{m|s=d} := m\alpha_m A_0 + \beta_m A_0(B_0 + B_1) + A_1(B_0 - B_1), \quad (\text{A15})$$

where α and β are determined by $\alpha_m = 2\eta \tan \mu / \sqrt{(4\lambda_m^2 - \eta^2)(\tan^2 \mu + 1)}$ and $\beta_m = \sqrt{4\lambda_m^2 - \eta^2} / (2\lambda_m \tan \mu)$, respectively. The classical bound for this inequality is $C_{[\alpha, \beta]} = 2\beta_m + |m\alpha_m|$ and the maximal quantum violation is $Q_{[\alpha, \beta]}^{m|s=d} = \sqrt{(4 + \alpha_m^2)(1 + \beta_m^2)}$. If the maximal quantum violation of (A15) is observed, then there exist local output isometry maps $\Lambda_A \otimes \Lambda_B$ such that the postmeasurement state ρ_m is equivalent with target form $\bar{\rho}_m$ up a local isometry, i.e.,

$$F_m^o \equiv F((\Lambda_A \otimes \Lambda_B) \rho_m, \bar{\rho}_m) = 1. \quad (\text{A16})$$

We represent the fidelity in terms of two outcomes as $F^o = \sum_{m=\pm 1} \sqrt{\lambda_m \bar{\lambda}_m} F_m^o$. The maximal quantum bounds conditional on output m imply $F_{+1}^o = F_{-1}^o = 1$ ensuring $F^o = 1$.

Combining the above two situations, the fidelity between the actual measurement M and the target form $\bar{M} = c\mathbb{I} + \eta\sigma_z$ can be bounded by fidelity of the states before and after M with triangle inequality [26,30]

$$F(M, \bar{M}) \geq \cos(\cos^{-1} F^i + \cos^{-1} F^o), \quad (\text{A17})$$

where $F(M, \bar{M})$ is defined as Eq. (A10). Hence, if the maximal quantum bounds of the inequalities in Eqs. (A6) and (A15) are all observed, then $F^i = F^o = 1$ which guarantees $F(M, \bar{M}) = 1$. Thus we claim that the instrument M is certified to be $\bar{M} = c\mathbb{I} + \eta\sigma_z$ up to a local isometry.

To be more specific, for given values of $\lambda_{\pm 1}$ and η in the target unsharp measurement, we can deduce the following.

(1) For the output $m = +1$, the parameters of generalized tilted-CHSH inequality (A4) are $\alpha = 2\eta / \sqrt{8\lambda_{+1} - \eta^2}$ and $\beta = 1$ corresponding to the standard form. The maximally quantum violation is $Q_{[\alpha, \beta]}^{+1|s=d} = \sqrt{2(4 + \alpha^2)}$ which uniquely certifies the postmeasurement state

$$|\bar{\psi}_{+1}\rangle = \sqrt{\frac{2\lambda_{+1} - \eta}{4\lambda_{+1}}} |01\rangle - \sqrt{\frac{2\lambda_{+1} + \eta}{4\lambda_{+1}}} |10\rangle. \quad (\text{A18})$$

(2) For the output $m = -1$, the parameters of generalized tilted-CHSH inequality (A4) are $\alpha = 2\eta \sqrt{4\lambda_{+1}^2 - \eta^2} / \sqrt{(8\lambda_{+1}^2 - \eta^2)(4\lambda_{-1}^2 - \eta^2)}$ and $\beta = \lambda_{+1} \sqrt{4\lambda_{-1}^2 - \eta^2} / (\lambda_{-1} \sqrt{4\lambda_{+1}^2 - \eta^2})$. The maximally quantum violation is $Q_{[\alpha, \beta]}^{-1|s=d} = \sqrt{(4 + \alpha^2)(1 + \beta^2)}$ which uniquely certifies the postmeasurement state

$$|\bar{\psi}_{-1}\rangle = \sqrt{\frac{2\lambda_{-1} + \eta}{4\lambda_{-1}}} |01\rangle - \sqrt{\frac{2\lambda_{-1} - \eta}{4\lambda_{-1}}} |10\rangle. \quad (\text{A19})$$

Combining the relation $\sum \lambda_m = 1$, if the maximal quantum bounds of $\mathcal{B}_{[\alpha, \beta]}^{s=d, m}$ are all observed, then the parameter λ_m and the measurement strength η in the target measurement (A8) can be achieved.

APPENDIX B: ROBUSTNESS ANALYSIS

The robustness of the unsharp measurement is affected by the source and the postmeasurement states of the untrusted unsharp measurement. Our analysis proceeds in two stages: (i) developing the robustness bounds for entangled state self-testing using the generalized tilted-CHSH inequality and (ii) extending this formalism to unsharp measurement certification.

1. Robustness analysis of generalized tilted-CHSH inequality

To derive a tight bound on the fidelity of the measurement via experimental observed statistical probability $p(a, b|x, y, s)$, we utilize the method named self-testing from the operator inequality by the isometry map, as introduced in Ref. [40].

From the results of Ref. [40], the closeness of the unknown resource ρ_{AB} and the target state $\bar{\rho}$, i.e., the fidelity $F(\rho, \bar{\rho}) = \text{Tr} \sqrt{\sqrt{\rho} \bar{\rho} \sqrt{\rho}}$ between two states, can be obtained through local isometry maps Λ_A and Λ_B . For an arbitrary bipartite input state ρ_{AB} , the extractability of target $\bar{\rho}$ from

ρ_{AB} is defined as

$$\Xi(\rho_{AB} \rightarrow \bar{\rho}) := \max F((\Lambda_A \otimes \Lambda_B)\rho_{AB}, \bar{\rho}), \quad (\text{B1})$$

where the maximum is taken over all quantum channels of appropriate input and output registers. By the definition, $\Xi(\rho_{AB} \rightarrow \bar{\rho}) = 1$ if and only if there exist local channels $\Lambda_A \otimes \Lambda_B$ such that

$$\begin{aligned} F((\Lambda_A \otimes \Lambda_B)\rho_{AB}, \bar{\rho}) &= \langle (\Lambda_A \otimes \Lambda_B)\rho_{AB}, \bar{\rho} \rangle \\ &= \langle \rho_{AB}, (\Lambda_A^\dagger \otimes \Lambda_B^\dagger)\bar{\rho} \rangle \\ &= 1, \end{aligned} \quad (\text{B2})$$

where $\Lambda^\dagger$ refers to the dual channel of quantum channel Λ . It is equivalent to $(\Lambda_A \otimes \Lambda_B)\rho_{AB} = \bar{\rho}$.

To derive an operator inequality that yields a lower bound on the extractability, let Λ_A and Λ_B be local extraction channels that only depend on Alice's and Bob's local measurement operators respectively. Furthermore, define an operator K depending on local extraction channels as

$$K \equiv (\Lambda_A^\dagger \otimes \Lambda_B^\dagger)(\rho_{AB}). \quad (\text{B3})$$

Now assume that for some fixed real parameters s and t , the operator inequality

$$K \geq s\mathcal{B}_{[a,b]} + t\mathbb{I} \otimes \mathbb{I} \quad (\text{B4})$$

holds for all possible measurements. Then take the trace with the input state ρ_{AB} on both sides of (B4). Expanding the right-hand side yields

$$\text{Tr}(s\mathcal{B}_{[a,b]}\rho_{AB} + t\rho_{AB}) = sk + t \quad (\text{B5})$$

where $k = \text{Tr}(\mathcal{B}_{[a,b]}\rho_{AB})$ is the violation value of Bell inequality $\mathcal{B}_{[a,b]}$ for state ρ_{AB} . Then combining with Eq. (B2), we have

$$F((\Lambda_A \otimes \Lambda_B)\rho_{AB}, \bar{\rho}) \geq sk + t. \quad (\text{B6})$$

The fidelity is lower bounded by a linear function of the Bell violation. If the right-hand side reaches 1 when k equals the maximum Bell violation, then this is an ideal self-testing.

Notably, it has been proved that for $s = (4 + 5\sqrt{2})/16$ and $t = -(1 + 2\sqrt{2})/4$, the linear lower bound for the self-testing of the two-qubit maximally entangled state via the CHSH inequality is nearly optimal [40]. Moreover, in the cases of both the biased CHSH inequality Eq. (A6) and the standard tilted-CHSH inequality (A7), the parameters s and t admit exact analytical solutions [45,46]. For the biased CHSH inequality

$$\begin{aligned} s &= \frac{\cos \mu}{16 \sin \mu} [(6 - \sqrt{2} \cos \mu) \\ &\quad + \sqrt{(6 - \sqrt{2} \cos \mu)^2 - 34 \sin^2 \mu}], \\ t &= 1 - vQ_{[0,\beta]}, \end{aligned} \quad (\text{B7})$$

and for the standard tilted-CHSH inequality

$$\begin{aligned} s &= \frac{(\sqrt{8 + 2\alpha^2} + 2 + \alpha)(3\sqrt{8 + 2\alpha^2} - \sqrt{4 - \alpha^2} - \alpha\sqrt{2})}{4(2 - \alpha)^2\sqrt{8 + 2\alpha^2}}, \\ t &= 1 - s\sqrt{8 + 2\alpha^2}, \end{aligned} \quad (\text{B8})$$

respectively.

Now we aim to derive linear self-testing bounds of the form (B6) for the generalized tilted-CHSH inequality (A4), which certifies partially entangled states, where s and t are optimization parameters.

In the case of two binary observables per party, the observables can be written in a block-diagonal form with blocks of size at most 2×2 by Jordan's lemma [14]. Considering all the possibilities of the measurements for Alice and Bob, the two measurements in each party should follow arbitrary distribution in the Bloch sphere. For any two measurements, after defining the direction orthogonal to σ_z as the σ_x basis, each measurement per party can be characterized by one angle (a for Alice and b for Bob) in the σ_x - σ_z plane. So the measurements of Alice and Bob can be given by

$$\begin{aligned} A_x &:= \cos a \sigma_x + (-1)^x \sin a \sigma_z, \\ B_y &:= \cos b \sigma_x + (-1)^y \sin b \sigma_z \end{aligned} \quad (\text{B9})$$

for $x, y \in \{0, 1\}$. Since the measurements are distributed uniformly in the Bloch sphere, a and b take values between $[0, \pi/2]$ to cover all possible choices of observable. After unitary correction, each nontrivial block is fully characterized by the angle between the observables.

The generalized tilted-CHSH inequality (A4) can be expressed in block-diagonal form as

$$\begin{aligned} B(a, b) &= 2\beta(\cos a \cos b \sigma_x \otimes \sigma_x + \sin a \cos b \sigma_z \otimes \sigma_x) \\ &\quad + 2(\cos a \sin b \sigma_x \otimes \sigma_z - \sin a \sin b \sigma_z \otimes \sigma_z) \\ &\quad + \alpha(\cos a \sigma_x + \sin a \sigma_z) \otimes I, \end{aligned} \quad (\text{B10})$$

where the optimal quantum violation $Q = \sqrt{(4 + \alpha^2)(1 + \beta^2)}$ is achieved for $a = \pi/4$ and $b = \mu$. The optimal state with the density matrix is

$$\begin{aligned} \bar{\rho} &= \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} + \frac{1}{\sqrt{2}}(\sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_z) \right. \\ &\quad + \cos 2\theta \left(\frac{(\sigma_z + \sigma_x) \otimes \mathbb{I}}{2} + \mathbb{I} \otimes \sigma_z \right) \\ &\quad \left. + \frac{\sin 2\theta}{2} \left(\frac{1}{\sqrt{2}}(\sigma_x \otimes \sigma_x - \sigma_z \otimes \sigma_x) \right) - \sigma_y \otimes \sigma_y \right] \end{aligned} \quad (\text{B11})$$

which is equal to the state $|\psi\rangle = \cos \theta |01\rangle - \sin \theta |10\rangle$ up to a local unitary.

The extraction channels here for Alice and Bob are precisely used in Ref. [46] which only depend on Alice's and Bob's local measurement operators

$$[\Lambda_A(a)](\rho) := \frac{1 + g(a)}{2}\rho + \frac{1 - g(a)}{2}\Gamma(a)\rho\Gamma(a), \quad (\text{B12})$$

$$[\Lambda_B(b)](\rho) := \frac{1 + g(h_a(b))}{2}\rho + \frac{1 - g(h_a(b))}{2}\Gamma(b)\rho\Gamma(b), \quad (\text{B13})$$

where

$$\begin{aligned} g(x) &= (1 + \sqrt{2})(\sin x + \cos x - 1), \\ h(x) &= \begin{cases} \frac{\pi b}{4b^*} & \text{if } x \in [0, b^*], \\ \frac{\pi}{2} - \frac{\pi(\pi - 2b)}{4(\pi - 2b^*)} & \text{if } x \in (b^*, \pi/2], \end{cases} \end{aligned} \quad (\text{B14})$$

and

$$\Gamma(x) = \begin{cases} \sigma_x & \text{if } x \in [0, \pi/4], \\ \sigma_z & \text{if } x \in (\pi/4, \pi/2]. \end{cases} \quad (\text{B15})$$

We can check that for the optimal angles $a = \pi/4$ and $b = \mu$, the extraction channels (B12) and (B13) are identity (no dephasing) channels whereas $a, b = 0$ and $\pi/2$ correspond to full dephasing.

By combining the operator definition of K in Eq. (B3), the quantum channels (B12) and (B13), and the target state specification (B11), we derive the following:

$$\begin{aligned} K = & \frac{1}{4} \left[\mathbb{I} \otimes \mathbb{I} + \frac{1}{\sqrt{2}} [g(a)h(b)\sigma_z\sigma_z + h(b)\sigma_x \otimes \sigma_z] \right. \\ & + \cos 2\theta \left(\frac{[g(a)\sigma_z + \sigma_z] \otimes \mathbb{I}}{2} + h(b)\mathbb{I} \otimes \sigma_z \right) \\ & + \frac{\sin 2\theta}{2} \left(\frac{1}{\sqrt{2}} [\sigma_x \otimes \sigma_x - g(a)\sigma_z \otimes \sigma_x] \right) \\ & \left. - g(a)h(b)\sigma_y \otimes \sigma_y \right]. \end{aligned} \quad (\text{B16})$$

To determine optimal parameters (s, t) , we define the verification operator $T(a, b)$ expanding into the Pauli basis as

$$\begin{aligned} T(a, b) = & K(a, b) - s\mathcal{B}(a, b) - t\mathbb{I} \otimes \mathbb{I} \\ = & c_1\mathbb{I} \otimes \mathbb{I} + c_2\sigma_z \otimes \sigma_z + c_3\sigma_x \otimes \sigma_z \\ & + c_4\sigma_z \otimes \mathbb{I} + c_5\sigma_x \otimes \mathbb{I} + c_6\mathbb{I} \otimes \sigma_z \\ & + c_7\sigma_x \otimes \sigma_x + c_8\sigma_z \otimes \sigma_x + c_9\sigma_y \otimes \sigma_y. \end{aligned} \quad (\text{B17})$$

The coefficients c_i depend on the target state parameter θ , measurement angles (a, b) defined in Eq. (B9), and the optimization parameters (s, t, α, β) , with explicit forms

$$\begin{aligned} c_1 = & \left(\frac{1}{4} - t \right), \\ c_2 = & \frac{1}{4\sqrt{2}} g(a)h(b) + s2 \sin a \sin b, \\ c_3 = & \frac{1}{4\sqrt{2}} h(b) - 2s \cos a \sin b, \\ c_4 = & \frac{\cos 2\theta g(a)}{8} - s\alpha \sin a, \\ c_5 = & \frac{\cos 2\theta}{8} - s\alpha \cos a, \\ c_6 = & \frac{\cos 2\theta h(b)}{4}, \\ c_7 = & \frac{\sin 2\theta}{8\sqrt{2}} - 2s\beta \cos a \cos b, \\ c_8 = & -\frac{\sin 2\theta g(a)}{8} - 2s\beta \sin a \cos b, \\ c_9 = & -\frac{\sin 2\theta}{8} g(a)h(b). \end{aligned} \quad (\text{B18})$$

The operator inequality (B4) holds if $T(a, b) \geq 0$ for all measurement configurations (a, b) in the protocol. This positivity constraint ensures the validity of the derived fidelity bound across the entire parameter space.

We observe that the commutator $[T(a, b), \sigma_y \otimes \sigma_y] = 0$ holds for all parameters $a \in [0, \pi/4]$ and $b \in [b, \mu]$. This symmetry motivates the introduction of the projectors

$$P_x := \frac{1}{2}(\mathbb{I} \otimes \mathbb{I} + (-1)^x \sigma_y \otimes \sigma_y), x \in \{0, 1\}. \quad (\text{B19})$$

The positivity of $T(a, b)$ can be established by proving the positivity of each block $M_x = P_x T(a, b)$. Given that $\text{rank}(M_x) \leq 2$ the non-negativity conditions reduce to $\text{Tr}(M_x) \geq 0$ and $(\text{Tr} M_x)^2 - \text{Tr}(M_x^2) \geq 0$. Explicitly, these quantities evaluate to

$$\text{Tr}(M_x) = 2 \left[\left(\frac{1}{4} - t \right) - (-1)^x \frac{\sin 2\theta}{8} g(a)h(b) \right] \quad (\text{B20})$$

and

$$\text{Tr}(M_x^2) = 2 \sum c_i^2 + 4(-1)^x (-c_1 c_9 - c_2 c_7 + c_3 c_8) \quad (\text{B21})$$

and consequently yield

$$\begin{aligned} & (\text{Tr} M_x)^2 - \text{Tr}(M_x^2) \\ = & 2 \left(c_1^2 + c_9^2 - \sum_{i=2}^8 c_i^2 \right) + 4(-1)^x (c_2 c_7 - c_3 c_8). \end{aligned} \quad (\text{B22})$$

The functional dependence on multiple parameters (measurement angles a, b and inequality parameters α, β) makes analytical determination of s guaranteeing the positivity of Eqs. (B20) and (B22) intractable. We therefore implement a bisection-based numerical optimization scheme. The bisection algorithm proceeds as follows.

- (1) Initialize the search interval $s \in [0, 100]$.
- (2) At each iteration, (i) test the midpoint s_{mid} against all (a, b) combinations and boundary cases $x = [0, 1]$ and (ii) compute all derived quantities and validate conditions $\text{Tr}(M_x) \geq 0$ and $(\text{Tr} M_x)^2 - \text{Tr}(M_x^2) \geq 0$.
- (3) Update the search interval based on condition satisfaction.
- (4) Iterate until interval width $< \epsilon(10^{-6})$ (tolerance).

As a concrete demonstration, we numerically analyze the optimal parameter s as a function of the target state parameter $\theta \in (0, \pi/4]$ and the coefficient $\beta \in (0, 1]$ in the generalized tilted-CHSH inequality, as shown as in Fig. 8. The corresponding parameter t follows from the quantum bound $t = 1 - sQ_{[\alpha, \beta]}$. The existence of (s, t) ensuring the positive semidefiniteness of Eq. (B4) is guaranteed by construction. Combining this with the fidelity expression in Eq. (B2), we obtain the fundamental lower bound

$$F((\Lambda_A \otimes \Lambda_B)\rho_{AB}, \bar{\rho}_{AB}) \geq sk + t \quad (\text{B23})$$

where $k = \text{Tr}[\mathcal{B}[\alpha, \beta]\rho_{AB}]$ is the observed Bell violation. The bound represents a linear relationship with the violation value.

2. Robustness of the unsharp measurements

The robustness analysis in our work encompasses both the source $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and the unsharp measurements Eq. (A8).

For the initial stage of our protocol where the source remains unmeasured, the fidelity between the prepared state ρ_{AB} and the target Bell state $|\Psi^-\rangle$ admits the lower bound

$$F((\Lambda_A \otimes \Lambda_B)\rho_{AB}, |\Psi^-\rangle \langle \Psi^-|) \geq s^i k + t^i \equiv F^i, \quad (\text{B24})$$

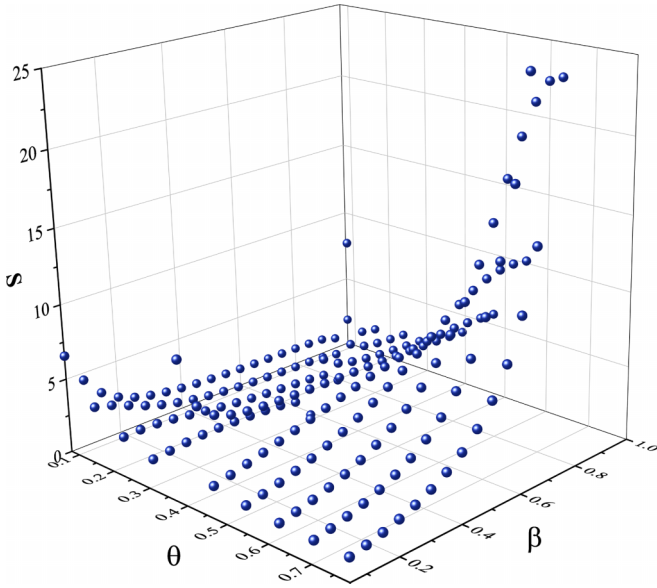


FIG. 8. The optimized value of parameter s with respect to β and θ .

where $k = \text{Tr}(\mathcal{B}_{[0,\beta]}\rho_{AB})$ is the observed violation of the biased CHSH inequality [defined in Eq. (A6)] and s^i and t^i are optimization parameters from Eq. (B7)

When $s = d$, the protocol employs the original measurement operator A_x and B_y defined in Eq. (B9), requiring no additional measurements. From the target state representations in Eqs. (A18) and (A19), we derive the coefficients of the generalized tilted-CHSH inequality (A15). This yields fidelity bounds for the postmeasurement states

$$F((\Lambda_A \otimes \Lambda_B)\rho_m, \bar{\rho}_m) \geq s_m^o k[\alpha_m, \beta_m] + t_m^o \equiv F_m^o, \quad (\text{B25})$$

where $k[\alpha_m, \beta_m] = \text{Tr}(\mathcal{B}_{[\alpha_m, \beta_m]}\rho_m)$ quantifies the observed violations of the standard tilted-CHSH inequality [Eq. (A7)] and its generalized form [Eq. (A4)]. The parameters (s_{+1}^o, t_{+1}^o) are specified in Eq. (B8), while (s_{-1}^o, t_{-1}^o) are determined via bisection optimization.

Formally, we introduce a register system where the classical outcomes $m = \pm 1$ are mapped to orthogonal quantum states $|r_m\rangle \in \{|+\rangle, |-\rangle\}$, respectively. Thus, the overall post-measurement state for two outcomes shared by Alice and Bob can be written in a compact form

$$\rho_{AB}^o = \lambda_{+1}\rho_{+1} \otimes |+\rangle\langle +| + \lambda_{-1}\rho_{-1} \otimes |-\rangle\langle -|, \quad (\text{B26})$$

where λ_m is the probability of the postmeasurement state ρ_m with respect to outcomes $m = \pm 1$. We define a weighted sum over measurement outcomes as $F^o = \sum_{m=\pm 1} \sqrt{\lambda_m \bar{\lambda}_m} F_m^o$.

Proposition B1. The strong concavity of the fidelity function and the orthogonality property of the register states enable us to derive fidelity bounds conditioned on specific measurement outcomes

$$F((\Lambda_A \otimes \Lambda_B)\rho_{AB}^o, \sum_{m=\pm 1} \bar{\lambda}_m \bar{\rho}_m \otimes |r_m\rangle\langle r_m|) \geq F^o. \quad (\text{B27})$$

Proof. The overall postmeasurement state for two outcomes after applying local quantum operations admits the

decomposition

$$\begin{aligned} \rho' &= (\Lambda_A \otimes \Lambda_B)\rho_{AB}^o \\ &= \lambda_{+1}\rho'_{+1} \otimes |+\rangle\langle +| + \lambda_{-1}\rho'_{-1} \otimes |-\rangle\langle -| \\ &= \sum_{m=\pm 1} \lambda_m \rho'_m \otimes |r_m\rangle\langle r_m| \end{aligned} \quad (\text{B28})$$

where $\rho'_m = (\Lambda_A \otimes \Lambda_B)\rho_m$ represents the processed subnormalized states.

The global postmeasurement target state is compactly expressed as

$$\sigma = \sum_{m=\pm 1} \bar{\lambda}_m \bar{\rho}_m \otimes |r_m\rangle\langle r_m|. \quad (\text{B29})$$

Using the strong concavity property of quantum fidelity, for any mixed state $\rho = \sum_i p_i \rho_i$ and $\sigma = \sum_i q_i \sigma_i$, the following inequality holds:

$$F\left(\sum_i p_i \rho_i, \sum_i q_i \sigma_i\right) \geq \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i). \quad (\text{B30})$$

Thus for our states, we can establish the lower bound

$$F(\rho', \sigma) \geq \sum_{m=\pm 1} \sqrt{\lambda_m \bar{\lambda}_m} F(\rho'_m \otimes |r_m\rangle\langle r_m|, \bar{\rho}_m \otimes |r_m\rangle\langle r_m|). \quad (\text{B31})$$

Because the pure states $|r_m\rangle$ are orthogonal, the fidelity decouples across the subspaces defined by m :

$$F(\rho'_m \otimes |r_m\rangle\langle r_m|, \bar{\rho}_m \otimes |r_m\rangle\langle r_m|) = F(\rho'_m, \bar{\rho}_m) = F_m^o. \quad (\text{B32})$$

This yields the final composite fidelity bound

$$F(\rho', \sigma) \geq \sum_{m=\pm 1} \sqrt{\lambda_m \bar{\lambda}_m} F_m^o = F^o. \quad (\text{B33})$$

We employ the triangle inequality property of the Uhlmann fidelity [26]

$$\cos^{-1}[F(\rho_1, \rho_3)] \leq \cos^{-1}[F(\rho_1, \rho_2)] + \cos^{-1}[F(\rho_2, \rho_3)], \quad (\text{B34})$$

where the relevant states in our framework are

$$\begin{aligned} \rho_1 &= (\Lambda_A \otimes \Lambda_B^o \circ M \circ \Lambda_B^i) |\Psi^-\rangle\langle \Psi^-|, \\ \rho_2 &= (\Lambda_A \otimes M \circ \Lambda_B^o) \rho_{AB}, \\ \rho_3 &= (\Lambda_A \otimes \bar{M}) |\Psi^-\rangle\langle \Psi^-|. \end{aligned} \quad (\text{B35})$$

From our definition of F^o , we have the direct relationship $F^o = F(\rho_2, \rho_3)$, while the input state fidelity is given by

$$F^i = F((\Lambda_A \otimes \Lambda_B^i) \rho_{AB}, |\Psi^-\rangle\langle \Psi^-|). \quad (\text{B36})$$

Using the nondecreasing property of fidelity under completely positive trace-preserving maps, we apply the transformation $(\mathbb{I} \otimes \Lambda_B^i \circ M \circ \Lambda_B^o)$ to F^i and obtain the fundamental

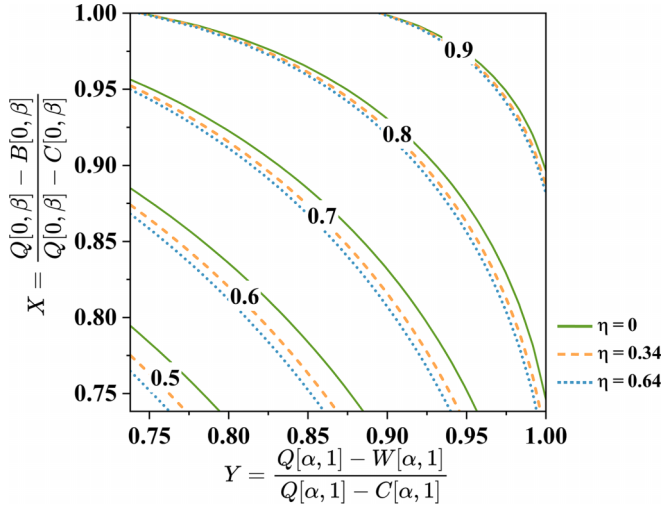


FIG. 9. Lower bounds on the fidelity ($M, \bar{M}$) with assumption that the second output state appears with equal probability (i.e. $\lambda_{\pm 1} = 0.5$). Axis X - Y represents the observation of inequality $\mathcal{B}_{[0, \beta]}$ and tilted-CHSH inequality $\mathcal{B}_{[\alpha, 1]}$ in two situations, respectively. The lower bound of the fidelity subject to the observations in two situations is marked on the curves.

inequality

$$F(\rho_1, \rho_2) \geq F^i, \quad (\text{B37})$$

which in turn implies $\cos^{-1}[F(\rho_1, \rho_2)] \leq \cos^{-1}(F^i)$. Further, notice that the measurement fidelity is identified by

$$F(M, \bar{M}) = F(\rho_1, \rho_3). \quad (\text{B38})$$

Finally, using the triangle inequality we can prove that the fidelity of the state before and after the measurement can be combined to bound the fidelity of the measurement itself, namely,

$$F(M, \bar{M}) \geq \cos[\cos^{-1}(F^i) + \cos^{-1}(F^o)]. \quad (\text{B39})$$

Whenever $Q_{[0, \beta]}^{s=u} = 2\sqrt{1 + \tan^2 \mu^2 / \tan \mu}$ and $Q_{[\alpha, \beta]}^{m|s=d} = \sqrt{(4 + \alpha_m^2)(1 + \beta_m^2)}$ are both achieved for the two output states, this bound guarantees that $F(M, \bar{M}) = 1$.

To evaluate the robustness of our protocol under realistic conditions, we plot $\bar{M} = \eta \sigma_z$ with $\eta = 0, 0.34$, and 0.64 as examples of the robustness analysis in Fig. 9. In this case, $\lambda_{+1} = \lambda_{-1} = 0.5$. The coordinate axis is normalized referring to different parameters for clarity.

Moreover, we analyze its performance when the quantum state is a Werner state $\rho_{\text{Werner}} = p\mathbb{I}/4 + (1-p)|\Psi^-\rangle\langle\Psi^-|$, where $\mathbb{I}$ represents the maximally mixed state (white noise) and p is the fraction of noise. Figure 10 compares the fidelity of our scheme with Ref. [30] in the main text across different noise levels by setting the target measurement $\bar{M} = \eta \sigma_z$ with $\eta = 0.34$. While our protocol demonstrates comparable noise resilience, its fidelity is marginally lower in high-noise regimes. This minor performance difference stems from the fundamental tradeoff in our approach.

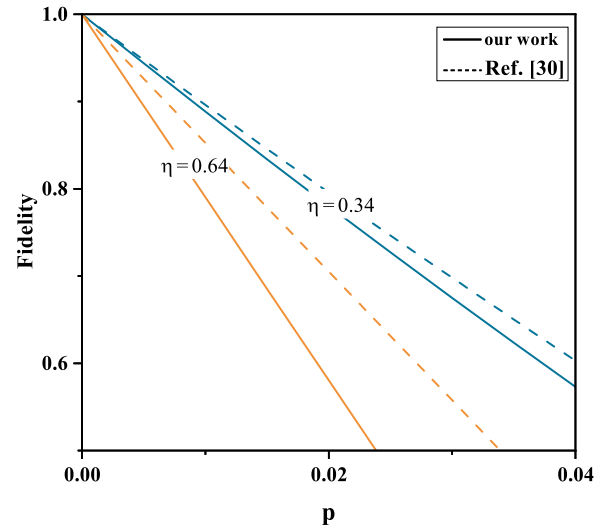


FIG. 10. Comparison of robustness. Fidelity as a function of the fraction of noise p . The color gradient represents the fidelity value.

These results highlight that our method maintains strong verification capability while offering greater applicability to practical experimental conditions.

APPENDIX C: QUANTUM TOMOGRAPHY OF THE STATE AND THE UNSHARP MEASUREMENTS

For the sake of reference, we perform quantum state tomography of the maximally entangled state [47] and QPT of the unsharp measurements [48]. The results are shown in Fig. 11.

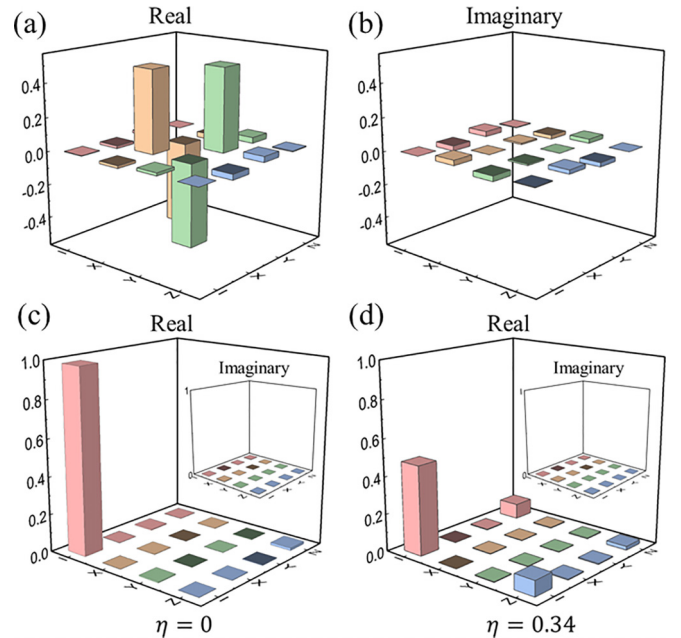


FIG. 11. (a), (b) Quantum state tomography of the maximally entangled state. (c), (d) Quantum process tomography of the unsharp measurement device with $\eta = 0$ and 0.34 .

The tomographic results for the maximally entangled state are depicted in Figs. 11(a) and 11(b). The estimated fidelity of the source state is 0.9929.

In Figs. 11(c) and 11(d), we present the density matrix of two typical measurements, i.e., the identity with unsharp parameter $\eta = 0$ and unsharp measurements with $\eta = 0.34$. It is worth noting that because of the property of unsharp measurement, the density matrix obtained is not necessarily

trace preserving. The fidelity between the observed density matrix and the target measurement is defined as

$$F(\chi, \chi_{\text{ideal}}) = \frac{\text{Tr}[\sqrt{\sqrt{\chi}\chi_{\text{ideal}}\sqrt{\chi}}]^2}{\text{Tr}[\chi]\text{Tr}[\chi_{\text{ideal}}]}. \quad (\text{C1})$$

The calculated process fidelities are 98.14 ± 0.51 and $97.44 \pm 0.31\%$, respectively, indicating a high-quality quantum process.

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