

Teachers' acceptance and understanding of a reduced Dirac formalism and the Bloch sphere

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Mathematical descriptions are the focus of quantum physics education research, as quantum physics is inherently described by its mathematical structures. However, there is a lack of research regarding the role of mathematical descriptions for understanding fundamental principles of quantum physics in teacher preparation and training. In order to investigate how preservice teachers and in-service teachers assess and understand the mathematical formalism in quantum physics, an explorative interview study was conducted. Therefore, we developed a teaching-learning sequence employing a reduced Dirac formalism in the context of single-photon polarization serving as an exemplary two-state system. In that respect, the focus was on two mathematical representations. The ket notation was used as an algebraic representation in connection with the Bloch sphere as a graphical representation of quantum states. On that basis, semistructured and material-based interviews with $N = 15$ preservice teachers and $N = 15$ in-service teachers were conducted. The data suggest that the reduced Dirac formalism is well understood and accepted by both preservice teachers and in-service teachers. However, some in-service teachers raise critical concerns about the abstract nature of the mathematical formalism, pointing to a perceived lack of experimental or phenomenological connection. The ket notation is valued not only for its communicative clarity and its role as a language but also for its structural role. By comparison, the Bloch sphere is met with more cautious acceptance—yet its use in visualizing experiments and measurement processes is positively assessed.

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I. INTRODUCTION

In recent years, there has been extensive research in quantum physics education, focusing on the development of teaching approaches and empirical research on implementing modern aspects of quantum physics such as quantum information sciences and technology in secondary education [1–3]. In this scope, a highly discussed approach concerns introducing quantum physics via exemplary two-state systems [4–8], which, from an educational point of view, appears promising for various reasons:

Integration of current technological developments: Due to the “second quantum revolution,” quantum technologies such as quantum communication or quantum computing receive a substantial degree of public attention [9]. Quantum physical two-state systems are particularly relevant for these technologies. Hence, teaching quantum

physics via two-state systems enables the integration of contemporary physical contexts, taking into account their technological and societal importance [10,11]. Including such contents can create awareness of the relevance of modern quantum technologies and enable interdisciplinary teaching, connecting physical, technological, mathematical, and computer science aspects [9,12]. In this scope, teaching-learning environments on quantum information science and technology such as quantum computation or quantum cryptography are highly discussed in recent research [7–9,12,13]. First empirical results suggest that teaching quantum technologies promotes both school and university students' interest in quantum physics [11,14,15].

Content-related educational aspects: Besides the opportunity of integrating current contexts, teaching quantum physics via two-state systems involves content-related potentials in a threefold way. First, integrating the physics of two-state systems enables the discussion of key concepts of quantum physics such as *superposition*, the quantum physical *measurement process*, *uncertainty*, or *entanglement* [4,9,16]. In this paper, we refer to these four key concepts as “fundamental principles of quantum physics.” Second, learning difficulties arising from the

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differentiation between classical physics and quantum physics can be reduced [9,11]. In particular, the discussion of properties of waves and particles can be left out, avoiding known learning difficulties related to wave-particle dualism [9,17]. Finally, the mathematical description of two-state systems is comparatively simple, as they are described within two-dimensional vector spaces and a discussion of position-space wave functions can be avoided. Therefore, teaching the physics of two-state systems seems suitable to explicitly highlight the role of mathematical formalism and its importance to quantum physics already in upper-level secondary school. In particular, it is possible to directly address the fundamental principles of quantum physics, as they inherently rely on mathematical structures [4,18,19].

So far, research on the role of the mathematical description of quantum physics for conceptual understanding has mostly focused on the university level and has been investigated only to a limited extent in the field of secondary education [1,2] as well as in teacher training. Based on the described situation, the presented study aims to investigate the role of a reduced mathematical formalism for a conceptual understanding of fundamental principles of quantum physics. Therefore, the research objectives of the study are:

- To develop a teaching-learning sequence on fundamental principles of quantum physics, introducing a reduced Dirac formalism, restricted to two-state systems without using linear operators.
- To investigate acceptance and conceptual understanding regarding the reduced Dirac formalism.

In the first step, the acceptance and conceptual understanding of preservice and in-service secondary school teachers we reinvestigated in an exploratory interview study. In this paper, results from this study are presented.

II. THEORETICAL BACKGROUND

Undoubtedly, mathematics plays an important role in physics. Here, an educational perspective on mathematics, both in physics in general and in quantum physics in particular, is discussed in this section.

A. Mathematics in physics: A general educational perspective

A systematic investigation and discussion of mathematics in physics education goes back to the early 2000s (for further references see [20]). Since then, numerous studies on educational aspects regarding the interplay between mathematics and physics as well as the role of mathematics in physics have been conducted [20,21].

Herewith, three different roles of mathematics in physics can be distinguished [20]:

- Mathematics can be used as a tool for conducting calculations, solving quantitative problems, and

focusing on algorithmic manipulations of formulas. This is referred to as the *technical role of mathematics in physics* [20,22].

- Mathematics can serve as a language that enables the description and modeling of physics and the representation of physical phenomena in a clear and concise way [23,24]. This role of *mathematics as a language* is distinct from a pure technical role, as mathematical expressions must be translated and have a meaning [20,25].
- Maybe the most important role in terms of conceptual understanding of physics is the *structural role of mathematics in physics*. Mathematics can serve as an instrument to describe structures of the world and form basic concepts of physics [22–24]. Furthermore, the structural role is characterized by building a connection between mathematical structures and physical reasoning [20].

As physical content can be communicated using different representations, focusing on mathematics in physics necessitates examining the role of mathematical representations in physics. In this paper, we follow the definition of representations as “notions or signs or symbols that stand for something in absence of that thing which typically is a phenomenon or an object in the external world but can be just in our imagination” [26] (p. 76). This definition emphasizes that representations can be externalized (e.g., drawings or formulas) or internalized as mental models [26,27]. Here, the term “representation” refers to external representations.

In educational contexts, multiple (mathematical) representations are usually used to describe physical contents. Since, in general, one representation does not contain all the information about a situation, multiple representations can provide *complementary information*. Furthermore, multiple representations help to *constrain* possible misinterpretations of a single one. Hence, they can help to *construct* a deeper understanding of a physical situation or concept [28,29].

Regarding mathematical representations, Geyer and Kuske-Janßen [26] distinguish between three different kinds. Mathematical representations such as terms, equations, or formulas are referred to as *algebraic* representations. Graphs or arrow representations of vectors in coordinate systems are called *graphical* representations. Finally, representations of numerical values such as tables with experimentally collected data are called *numerical* representations [26].

Especially in quantum physics education, graphical and algebraic representations of physical states play a crucial role [19] and are the main focus in this paper.

B. Mathematics in physics: A quantum education perspective

As quantum physics is inherently described by its mathematical structures, the structural role of mathematics

seems particularly important for teaching and learning quantum physics [19]. In combination with careful ascription of meaning to the mathematical structures, mathematical descriptions might enable students to develop coherent interpretations of both qualitative findings and quantum physical concepts and processes. Moreover, mathematics provides means for making (probability-based) predictions, for instance, concerning measurement outcomes [19]. Finally, mathematics may also support the formation of adequate mental models [19,30]. In this context, introducing quantum physics via two-state systems seems fruitful as they rely on only two-dimensional (complex) vector spaces, reducing the mathematical complexity. Hence, it seems plausible that introducing a corresponding reduced mathematical formalism may foster the conceptual understanding of key concepts of quantum physics already in upper-level secondary schools [18].

However, educational research on the role of mathematics in quantum physics has mainly focused on university level [1–3]. Here, many studies center on position-space quantum physics and aspects like wave functions and the Schrödinger equation [31–33]. Nevertheless, especially in recent years, there has been some research on approaches via two-state systems [34–39].

Donhauser *et al.* [2] affirm that, with respect to teaching quantum physics in school, the focus often lies on visualizations and on reducing the mathematical descriptions, implying that it is desirable to investigate school students' understanding of quantum physics regarding the mathematical formalism. Accordingly, there are some works addressing the potential of a mathematical description already in secondary school in recent years, in particular, with the focus on two-state systems [6,18,40–43]. Herewith, in some initial studies, the use of ket notation was investigated with respect to secondary school [40,41]. However, there is a lack of in-depth research on the role of a mathematical formalism for understanding quantum physical principles with respect to the school level. Furthermore, to our knowledge, the link between algebraic representations (e.g., ket notation) and graphical representations (e.g., Bloch sphere) of states has not yet been extensively focused on. However, as graphical representations can help bridge the gap between abstract algebraic representations and the physical world, investigating the interplay between these types of representations for learning quantum physics is desirable [44].

C. Teacher education in quantum physics

In the context of examining the role of mathematics in quantum physics with respect to school teaching, it is also essential to consider how teachers assess and understand mathematical descriptions. This is particularly important in light of the growing need for teacher training programs that incorporate modern approaches to quantum physics [45]. It can be argued that physics teachers should not be

mainly trained to employ mathematical formalisms for solving extended mathematical problems [46,47]. Rather, it is important that teachers get an overview-level knowledge of how mathematics is used in physics. Hence, an important focal point in teacher training should not be the handling of a mathematical formalism as it is done by theoretical physicists, but rather an understanding of mathematical structures in relation to their physical meaning [47]. In a study examining how physics preservice teachers in German teacher training degree programs assess their studies, the most frequently mentioned statement was that there is too much theoretical physics in the curriculum [46]. Specifically with respect to quantum physics, the results of a study described in Ref. [48] show that only 9% of the surveyed preservice teachers are interested in the mathematical formalism in quantum physics. Additionally, physics teachers are sometimes not able to clearly distinguish between concepts of classical physics and quantum physics, for example, in connecting physical reasoning with mathematical concepts [45]. Furthermore, studies show that preservice teachers do not feel confident in transferring the knowledge about quantum physics acquired during their studies to the content covered in school.

In summary, teachers seem to have difficulties in terms of the mathematical formalism in quantum physics in two respects [48]. On the one hand, physics teachers themselves have difficulties in conceptually understanding fundamental concepts of quantum physics and their mathematical descriptions. On the other hand, physics teachers consider their competence in transferring the content of their studies on quantum physics and theory to the classroom to be insufficient [48,49].

In a position paper, Micheli *et al.* [45] conclude that in view of the conceptual role of the mathematical formalism in quantum physics, physics teachers should understand this formalism even if they do not need it when teaching quantum physics in school. Furthermore, the authors argue that knowledge about the mathematical formalism is necessary for teachers as it shows how mathematical modeling and mathematical structures are inherent to quantum physics [45]. For this purpose, introducing teachers to the mathematical formalism by focusing on two-state systems seems a suitable way to provide related insights into the mathematical descriptions of quantum physics [45,50]. In addition, the investigation of teachers' assessment and understanding of the mathematical formalism in quantum physics can serve as a first step in exploring how the mathematical formalism could be used in upper-level secondary school.

III. RESEARCH QUESTIONS

In this study, the role of mathematical descriptions for understanding fundamental principles of quantum physics (*superposition, measurement process, uncertainty*, and

entanglement) is investigated. As suggested above, a reduced Dirac formalism [51] restricted to two-state systems is adopted (see Sec. IV). Furthermore, we emphasize the combination of the ket notation as an algebraic representation and the Bloch sphere as a graphical representation, following [4].

The target group consists of preservice and in-service physics teachers, as it seems reasonable to investigate secondary school teachers' acceptance and understanding of such content before implementation in school. Hence, we pose the following research questions:

RQ 1: How do secondary school preservice teachers and in-service teachers accept a reduced Dirac formalism for describing fundamental principles of quantum physics?

RQ 2: How do secondary school preservice teachers and in-service teachers understand the description of fundamental principles of quantum physics with the help of a reduced Dirac formalism?

RQ 1 is linked to an assessment of the reduced Dirac formalism. In this context, by *acceptance* we understand subjective attitudes and evaluations regarding the content, subjective feelings of understanding, and reflections on the extent to which the content is considered important or relevant, in accordance with [52,53]. Hence, acceptance is connected to whether the content seems plausible, logical, and coherent, as well as useful and meaningful to the participants.

To answer RQ 2, we operationalize *conceptual understanding* as outlined in the following. Conceptual understanding refers to the conceptual knowledge of physical concepts and their interrelations [54,55]. Furthermore, in terms of problem solving, conceptual understanding is not only linked to knowing what to do but also why to do it [56]. Hence, it is closely related to measurable learning objectives and conceptual knowledge [57]. Regarding RQ2 and in accordance with [54–57], conceptual understanding can be observed by correctly explaining and reasoning using the focused concepts. We therefore differentiate the following facets of understanding:

- Correct explanation of terms and concepts
- Correct description, justification, and explanation of facts
- Correct application of knowledge to new situations

IV. A TEACHING-LEARNING SEQUENCE ON INTRODUCING A REDUCED DIRAC FORMALISM

As a starting point, a teaching-learning sequence was developed to introduce fundamental principles of quantum physics employing a reduced Dirac formalism [51] in accordance with [4] (see also [58]). In the following, the use of single-photon polarization as an exemplary two-state system is motivated (Sec. IV A), and the reduced mathematical formalism is described (Sec. IV B).

Subsequently, an overview of the teaching-learning sequence is given (Sec. IV C), and remarks on paraphrasing and application tasks are made (Sec. IV D).

A. Single-photon polarization as an exemplary two-state system

For introducing fundamental principles of quantum physics, various physical realizations of two-state systems can be considered, e.g., which-path-encoded photonic states, single-photon polarization, double-well potential, or spin-1/2 objects [4,59,60].

In the proposed teaching-learning sequence, the focus is on single-photon polarization. From an educational point of view, the advantage of this exemplary two-state system is the possibility of an intuitive discussion of the fundamental principles as well as the introduction of the algebraic description of these principles using vectors. Especially the concept of basis states and their mathematical description using orthogonal vectors is easy to understand in this approach [18,42].

Furthermore, the concept of polarization of light is well known among teachers, as it usually is part of the school curriculum. It is therefore possible to refer to experiments with polarization filters without having to introduce and explain them in detail. Instead, the key concepts of quantum physics, which become visible in experiments with polarization filters, can be directly elaborated. Finally, experiments on anticorrelation at (polarizing) beam splitters and on entanglement can be closely aligned with evaluated teaching-learning concepts on quantum optics [17,61].

B. Reducing the mathematical formalism

In order to address the research questions, it was necessary to reduce the mathematical formalism of quantum physics suitable for school and teacher training. The aim was to reduce the mathematical formalism in such a way that it is only slightly above school level, in order to give teachers an insight into the mathematical descriptions of quantum physics while at the same time considering its transferability to schools. For our teaching-learning sequence, the considerations of Dür and Heusler [62,63] serve as a basis, as they focus on identifying the mathematical structures that underpin the modern understanding of quantum physics.

1. General considerations

The superposition principle, as a fundamental principle of quantum physics, is directly related to a vectorial description. Accordingly, in quantum physics, where the superposition of quantum states is discussed, the vectorial description of states should be the core of the mathematical formalism [62]. The next step is to discuss the properties of the quantum physical measurement process, its active

character, and the probabilistic nature of measurement results. The decisive factor here is that, after a measurement process, a quantum physical two-state system exists in one of two possible states, which in algebraic terms form a basis of the respective Hilbert space.

Therefore, the description of the measurement process of an observable can be reduced from the discussion of linear operators associated with the respective observable to the discussion of the corresponding eigenvectors, which correspond to the respective measurement basis. The discussion of the measurement process with respect to a certain measurement basis instead of considering the associated operators is particularly common in quantum informatics, for example, in quantum cryptography [8,62]. A further simplification is made in the description of possible results of the measurement process. Mathematically speaking, the results of a measurement process are the eigenvalues of the operator describing the respective observable, and the eigenstates correspond to the possible states after the measurement process. For simplification, literature sometimes refers to the eigenstates as possible measurement results instead of the associated eigenvalues [64]. These simplifications make it possible to discuss the measurement process without treating operators and eigenvalue equations. At the same time, the structural core of the measurement process is emphasized, namely, its active character and the realization of a possible basis state after the measurement process by resolving the superposition.

Not only for the discussion of the measurement process itself, but in particular for understanding the uncertainty principle, it is essential to understand that basis states of one basis are superposition states with respect to another basis [36,62], which is also centered in the teaching-learning sequence.

With entanglement as a fundamental principle of quantum physics, the mathematical description of systems of multiple quantum objects should also be discussed. In this scope, the structural role of mathematics in describing such systems is crucial. In particular, the possibility of a mathematical description of nonentangled systems, which can be considered independently of each other, as well as of entangled systems, where this is not the case, is discussed in the teaching-learning sequence. In this context, it is essential to consider the mathematical description of multiqubit systems using tensor products and to differentiate between product states and entangled states [63,64].

In summary, the reduced Dirac formalism focuses on the following aspects:

- The description of superposition as the addition of basis vectors
- The role of a measurement basis in relation to the measurement process and possible measurement results

- The complementarity of measurement processes with respect to different measurement bases
- The description of systems of multiple quantum objects using tensor products and the related distinction between entangled states and nonentangled states

As a result, in line with the structural role of mathematics in quantum physics, the focus is on the algebraic description of quantum physical states in two-dimensional Hilbert space with an emphasis on the concepts of basis and superposition while the discussion of linear operators is omitted.

2. Ket notation as an algebraic representation

Directly linked to the Dirac formalism is the algebraic representation of states using the ket notation (sometimes also referred to as “Bra-ket notation” or “Dirac notation”).

The ket notation is used as the focussed algebraic representation in the teaching-learning sequence due to its widespread use in university teaching [65] and its educational potential. The ket notation provides a compact and elegant possibility for writing down quantum physical states and presents a clear, universal notation that is also easy to transfer to more complex systems than two-state systems. Furthermore, it can serve as an anchor for linking different quantum physical concepts and phenomena as well as a support for problem solving [16,66–70]. Additionally, the ket notation not only allows the representation of superposition and the probabilities for the respective measurement result but can also directly represent a physical context, depending on the symbols used in ket notation [19,70].

3. Bloch sphere as a graphical representation

The use of multiple external representations plays an important role in physics and fulfills various functions [28]. Especially in school, visual and graphical representations are central for supporting the understanding of purely symbolic, algebraic representations. In particular, graphical representations offer the possibility to connect concrete physical systems and phenomena to abstract algebraic representations [19,44]. In the teaching-learning sequence, the focus is on the Bloch sphere, which offers a direct visualization of the mathematical structure of quantum physics as well as a geometric interpretation that provides additional perspectives on quantum physical concepts [64]. Particularly, the Bloch sphere enables a visualization of the (complex) states of a two-state system. An arbitrary superposition state of the polarization states $|H\rangle$ and $|V\rangle$ can be parametrized by two angles ϑ and φ :

$$|\psi\rangle = \cos\frac{\vartheta}{2}|V\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|V\rangle.$$

Hence, by using these two angles, a normalized polarization state can be visualized by a vector at the surface of a sphere with radius 1 (Bloch sphere).

Furthermore, the Bloch sphere allows an intuitive visualization of the measurement process and the uncertainty principle [62].

C. Overview of the teaching-learning sequence

Based on this discussion of the reduced mathematical formalism in connection with the research questions, the following aspects and principles were addressed in the teaching-learning sequence:

- Using single-photon polarization as an exemplary two-state system and discussing single-photon experiments (linear and circular polarization)
- Focus on fundamental principles of quantum physics
 - Superposition
 - Quantum physical measurement process
 - Uncertainty
 - Entanglement
- Employing a reduced Dirac formalism [51] using ket notation and the Bloch sphere as two mathematical representations

The fundamental objective for the design of the teaching-learning sequence is the discussion of the physical and mathematical content with respect to the research questions. The teaching-learning sequence is therefore primarily intended to form the basis for conducting an exploratory interview study to answer the research questions. The design of a learning environment with, e.g., self-study materials or a textbook was therefore explicitly not the aim of the design. Rather, the teaching-learning sequence was explicitly designed to be feasible within the interview setting of the presented study (see Sec. V). The teaching-learning sequence can indeed serve as a basis for developing materials for teacher training or learning materials for schools (see outlook in Sec. IX C), but this was not primarily the aim in the presented study.

The teaching-learning sequence consists of six individual units described in Fig. 1. Each unit comprises

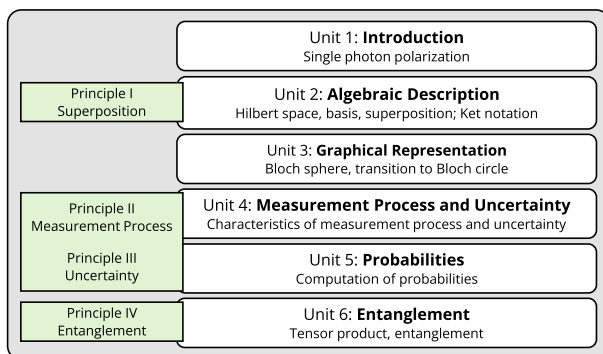


FIG. 1. Overview of the units and related fundamental principles of quantum physics.

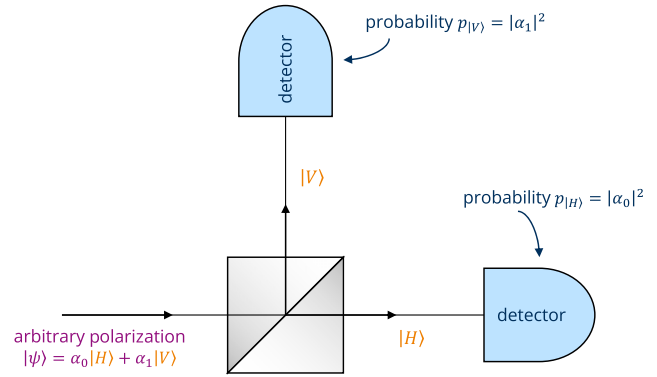


FIG. 2. Schematic representation of a polarizing beam splitter cube with subsequent detection and use of the ket notation as implemented in the teaching-learning sequence.

individual explanations with a duration of approximately 7 to 10 min. The introductory unit 1 focuses on single-photon polarization by discussing single-photon experiments with polarization filters and polarizing beam splitters. The experimental setups are shown in a schematic representation, and the results are discussed (see Fig. 2). In unit 2, the vectorial description of states is introduced. Key concepts and terms such as superposition and basis states are discussed. Furthermore, the ket notation of arbitrary states is introduced. After the algebraic description, the Bloch sphere as a graphical representation is presented in unit 3. By considering both linear and circular polarization, the Bloch circle for linearly polarized states is explained. In units 4 and 5, the measurement process as well as the

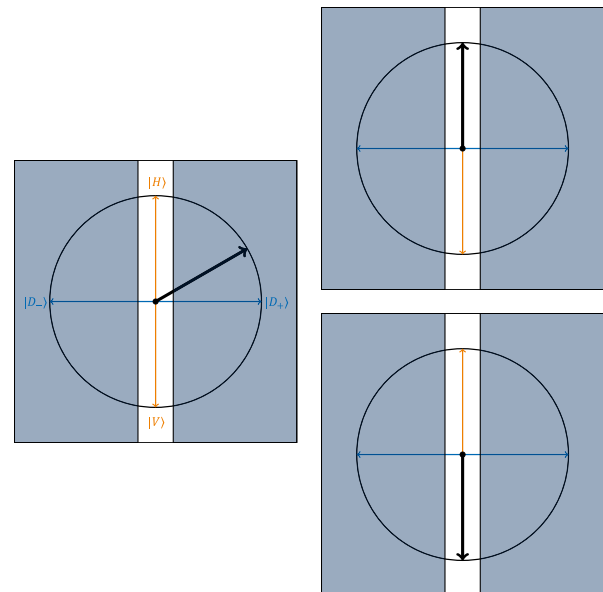


FIG. 3. The slit model of visualizing the quantum physical measurement process as suggested by [4]. The measurement apparatus and the measurement basis are represented by a slit. An arbitrary state is forced into the direction of the slit, resulting in one of the two possible measurement outcomes.

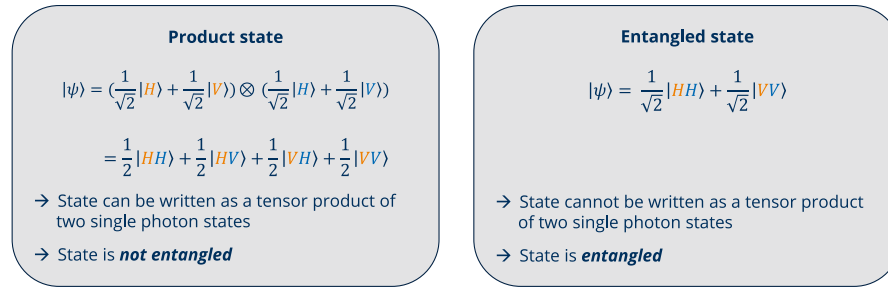


FIG. 4. Differentiation between product states and entangled states in ket notation as implemented in the teaching-learning sequence.

uncertainty principle are discussed. In these two units, both principles are addressed in a qualitative way with reference to the experiments in unit 1 and visualized using the Bloch sphere. For visualization of the measurement process, the slit model suggested by [4] is used (see Fig. 3). Furthermore, the computation of probabilities using squares of the absolute values of the amplitudes is introduced. Unit 6 addresses entanglement and its mathematical description using the tensor product. Here, the tensor product is introduced by discussing its central rules for computation. Especially the differentiation between entangled states and product states is taken into account (see Fig. 4). The content was designed in such a way that it can potentially be implemented in a slightly more simplified form in upper secondary school as well. The materials of the teaching-learning sequence are available in German upon request from the authors.

D. Tasks for paraphrasing and application

In addition to the individual explanations, tasks focusing on paraphrasing the contents as well as application tasks were designed. For each unit, two to three paraphrasing tasks and one to two application tasks were developed. Selected tasks are presented in Appendix C. Each task (T) is assigned an abbreviation indicating whether it is a paraphrasing task (P) or application task (A) and which unit it belongs to—e.g., the second application task in unit 4 has the abbreviation *A4T2*.

The tasks focus on paraphrasing the fundamental principles of quantum physics as well as on the explanation of experiments and mathematical descriptions discussed in the single units. The application tasks mainly focus on conceptual understanding with respect to the fundamental principles of quantum physics and its mathematical descriptions. Nevertheless, some tasks are also designed to address the handling of the mathematical formalism, e.g., the calculation of probabilities or the representation of states on the Bloch sphere.

V. METHODOLOGY

A. Study design

To answer the research questions and in view of the lack of literature regarding the focused target group (see

Sec. II B), an exploratory study design was considered appropriate, and a qualitative laboratory study was conducted [71] (see also [58]).

In detail, *Teaching Experiments* were conducted, often also referred to as “acceptance surveys” [11,53,72–75]. This form of material-based, semistructured individual interviews is particularly suitable for gaining insight into the perception of teaching concepts by learners as well as individual learning difficulties and learning processes [72,73].

At the start of the teaching experiments, interviewees were asked short introductory questions about their prior knowledge and ideas of quantum physics in order to ascertain the initial level. The main part of the teaching experiments consisted of five fixed phases, which were cyclically repeated for each unit of the teaching-learning sequence [11,53,73,74], as shown in Fig. 5:

1. Individual explanation

In an initial input phase, individual explanations were given to the participants by the interviewer.

2. Survey of acceptance

In the second phase, the acceptance of the explanations was investigated, as well as how participants assessed the discussed ideas.

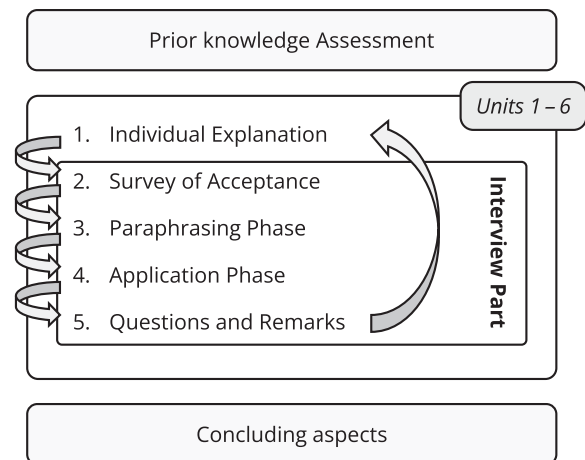


FIG. 5. Interview structure, following [11].

3. *Paraphrasing phase*

In the third phase, the participants were asked to paraphrase essential parts of the given explanations in their own words.

4. *Application phase*

In the fourth phase, the participants were asked to solve one or two application tasks.

5. *Questions and remarks*

In the fifth phase, the participants had the chance to ask further questions and to make further comments.

At the end of the teaching experiments, concluding aspects were discussed, and a reflection of the teaching-learning sequence in its entirety was given by the participants. In the following, the teaching experiments and their phases are referred to as “interviews” in a summarizing manner.

B. Sample

In total, $N = 15$ preservice physics teachers as well as $N = 15$ in-service physics teachers participated in this study.

All preservice teachers had already attended an introductory experimental course on wave optics and quantum physics. However, none of them had attended a course on quantum theory prior to the interviews. In particular, all of the participants were unfamiliar with the introduced Dirac formalism.

In contrast, the group of in-service teachers was more heterogeneous. While most of the teachers had less than 10 years of professional experience, with four of them even being in the phase of their in-service traineeship, the participant with the longest experience has been working as a teacher for 34 years.

As in Germany, teachers usually teach two different subjects in school, we focused on interviewing teachers with and without mathematics as their second subject (see

Fig. 6). In sum, $N = 7$ preservice teachers have mathematics as their second subject. The other second subjects are computer science ($N = 2$), English as a foreign language ($N = 2$), politics ($N = 3$), and art ($N = 1$). Among the in-service teachers, $N = 10$ interviewees have mathematics as their second subject. The other second subjects are English as a foreign language ($N = 1$), biology ($N = 1$), geography ($N = 2$), and computer science ($N = 1$).

The participants were recruited by advertising the interviews both personally and by mass mail.

C. Study implementation

The interviews were conducted in person at TU Dresden as well as in schools. Four interviews were conducted online via video call for organizational reasons.

The average duration of the interviews was 150 min for preservice teachers (SD = 10.9 min, Min = 130 min, and Max = 169 min) and 151 min for in-service teachers (SD = 20.2 min, Min = 111 min, and Max = 187 min).

The phases of the individual explanations had a duration of about 7–10 min on average per unit. Hence, approximately 60 min of each interview were allocated to the individual explanations. After unit 4, a short break was taken in each interview.

D. Data analysis

The interviews were audio-recorded and transcribed based on the transcription rules following [76,77], and the data were anonymized following [78]. The data were analyzed in two phases using different methods of qualitative content analysis, which are described below. For each phase of the analysis, a coding manual was developed, including category names and definitions, examples for coded sequences, and further coding rules and remarks, following [79].

1. *Evaluative qualitative content analysis*

In a first step, an evaluative qualitative content analysis following [79,80] was conducted. Hence, each response was rated with respect to acceptance, paraphrasing phase, and application phase on a deductively developed, three-point ordinal scale [11,53,73–75]. The goal was to obtain initial insights into the acceptance of the discussed content and an overview of the quality of the answers given in the paraphrasing and application phase. Therefore, for each phase, three different categories were differentiated as shown in Table I.

The coding manual for the evaluative qualitative content analysis consists of three coding schemes, one for each of the three phases. These coding schemes are shown in Appendix A (Tables VIII–X). Furthermore, exemplary

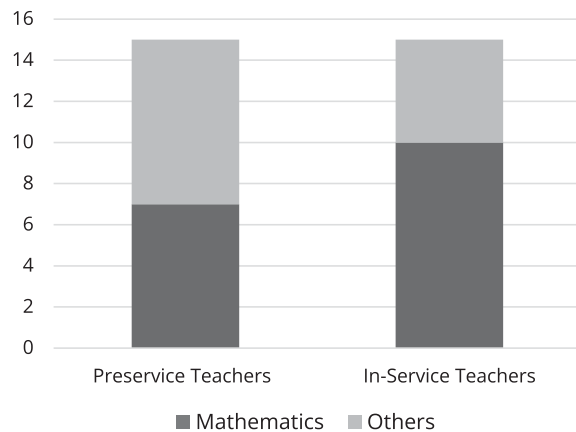


FIG. 6. Number of participating preservice teachers and in-service teachers, separated by their second subject.

TABLE I. For each of the main categories “acceptance,” “paraphrasing,” and “application,” three levels were differentiated. The best level of each category was coded with (+), the medium level with (0), and the lowest level with (−), respectively.

Acceptance	Paraphrasing	Application
Complete Acceptance (+)	Successful Paraphrasing (+)	Good Solution (+)
Limited Acceptance (0)	Satisfactory Paraphrasing (0)	Satisfactory Solution (0)
No Acceptance (−)	Poor Paraphrasing (−)	Poor Solution (−)

statements for each level of acceptance are shown in Appendix B (Table XI).

To ensure interrater reliability, seven interviews were drawn at random from all 30 interviews and were coded in the described way by a second independent rater. On this basis, linearly weighted interrater coefficients Cohen's Kappa κ_{Cohen} were determined, which can be considered substantial for each phase [81].

2. Analysis of assessment and learning difficulties

In the second step, a structuring qualitative content analysis was conducted following [79]. Using a deductive-inductive coding procedure, multiple aspects addressing research questions were analyzed:

- Analysis of the assessment and opinions regarding the discussed ideas (RQ 1)
- Analysis of specific learning difficulties and problems arising in relation to the explanations (RQ 2)
- Analysis of the assessment of the reduced Dirac formalism and the teaching-learning sequence as a whole in the overall reflection (RQ 1)

The resulting coding scheme consists of multiple main codes and subcodes. The data were analyzed in a deductive-inductive manner by multiple phases of going through the interview transcripts, and a first version of the coding manual has been developed during this phase [79].

In order to ensure the reliability of the coding manual, seven randomly chosen interviews were coded by an independent second coder using the first version of the manual, similar to the procedure described in Sec. V D 1. Following the procedure of consensus coding [79], unclear segments that were coded differently were discussed afterward, and a consensus was reached between the two coders. On this basis, the coding manual was adjusted accordingly, and the interviews were coded a second time using the adapted coding manual.

VI. FINDINGS

A. Evaluative qualitative content analysis

1. Findings of the survey of acceptance

Regarding preservice teachers (Table II), the explanations given in units 1, 2, 5, and 6 are largely assessed positively and highly accepted. (0) was only coded 2 to 3 times for each of the units. This demonstrates, in particular, that the introduction of single-photon polarization as well as the discussion of single-photon experiments involving polarization filters and polarizing beam splitters is generally highly accepted. Furthermore, the introduction of the formal mathematical description is highly accepted by preservice teachers. This applies to key aspects of the reduced Dirac formalism, such as the description of states in two-dimensional complex vector spaces and the use of ket notation. In addition, unit 6 on entanglement and the extension of the mathematical formalism to multiqubit systems and the tensor product are also accepted overall. While the evaluative qualitative content analysis of the survey of acceptance in unit 4 shows less positive results, two-thirds of the preservice teachers still show complete acceptance (+). Nevertheless, (0) is coded 4 times, and unit 4 is the only unit in which (−) was coded once. Regarding unit 3, preservice teachers show only mixed acceptance, with less than 50% of complete acceptance (7 times). Hence, the introduction of the Bloch sphere is only assessed as mediocre.

Regarding in-service teachers (Table III), the survey of acceptance for unit 1 and units 4 to 6 is quite similar in comparison to the preservice teachers. In those units, the majority shows complete acceptance, although the results are slightly lower overall. Hence, with regard to unit 1, we find that in-service teachers also highly accept the introduction via polarization of single photons. Unit 4 is the

TABLE II. Results of the evaluative qualitative content analysis of preservice teachers' survey of acceptance ($\kappa_{\text{acceptance}} = 0.69$).

	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6
Pre01	(+)	(0)	(+)	(0)	(0)	(+)
Pre02	(+)	(+)	(0)	(+)	(+)	(+)
Pre03	(+)	(+)	(+)	(0)	(+)	(+)
Pre04	(0)	(+)	(+)	(+)	(+)	(+)
Pre05	(+)	(+)	(0)	(0)	(+)	(0)
Pre06	(+)	(+)	(+)	(+)	(+)	(+)
Pre07	(+)	(+)	(+)	(+)	(+)	(+)
Pre08	(0)	(0)	(0)	(+)	(+)	(+)
Pre09	(+)	(0)	(0)	(0)	(0)	(+)
Pre10	(+)	(+)	(0)	(−)	(+)	(+)
Pre11	(+)	(+)	(+)	(+)	(+)	(0)
Pre12	(+)	(+)	(0)	(+)	(+)	(+)
Pre13	(+)	(+)	(+)	(+)	(+)	(0)
Pre14	(+)	(+)	(0)	(+)	(+)	(+)
Pre15	(+)	(+)	(+)	(+)	(+)	(+)

TABLE III. Results of the evaluative qualitative content analysis of in-service teachers' survey of acceptance ($\kappa_{\text{acceptance}} = 0.69$).

	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6
In01	(+)	(+)	(+)	(+)	(0)	(+)
In02	(+)	(+)	(0)	(+)	(+)	(+)
In03	(+)	(+)	(+)	(+)	(0)	(+)
In04	(+)	(0)	(+)	(+)	(0)	(0)
In05	(+)	(0)	(+)	(0)	(+)	(+)
In06	(+)	(+)	(+)	(+)	(0)	(0)
In07	(+)	(+)	(+)	(+)	(+)	(+)
In08	(0)	(+)	(+)	(+)	(+)	(+)
In09	(+)	(0)	(0)	(0)	(+)	(0)
In10	(+)	(+)	(0)	(+)	(+)	(+)
In11	(0)	(0)	(+)	(0)	(+)	(+)
In12	(+)	(+)	(0)	(+)	(+)	(0)
In13	(+)	(+)	(+)	(+)	(+)	(+)
In14	(0)	(0)	(0)	(-)	(+)	(+)
In15	(+)	(0)	(+)	(+)	(+)	(+)

only unit in which $(-)$ is coded once not only for preservice teachers but also for in-service teachers. The distribution of $(+)$ and (0) in unit 3 is also comparable to that of the preservice teachers, although slightly more positive. Nevertheless, it should be noted that, also in the structuring of qualitative content analysis, no strong differences between preservice and in-service teachers are noticeable. It is noteworthy that for the in-service teachers, unit 2 on the introduction of the reduced Dirac formalism shows the lowest acceptance rate, with only 60% of the conducted interviews coded with $(+)$. In fact, the formal mathematical description is evaluated less positively by in-service teachers, which is also evident in the analysis of assessment and learning difficulties (see Sec. VI B).

2. Findings of the evaluative qualitative content analysis of the paraphrasing phase

The results of the evaluative qualitative content analysis of the tasks of the paraphrasing phase show a rather positive picture for the preservice teachers (Table IV). In 10 of the 13 paraphrasing tasks, nearly three-quarters of the preservice teachers' solutions (73%) were coded with $(+)$. Only one paraphrasing task, P4T3 (explanation of the term "indeterminacy"), was categorized with $(+)$ for only nine preservice teachers, i.e., for less than two-thirds of this group of participants. The paraphrasing task solved best is P4T2 (explanation of how the measurement process can be visualized on the Bloch sphere) with 93% $(+)$. The worst solved paraphrasing tasks are P4T1 (explanation of the properties of the measurement process) with 67% $(+)$ and P4T3 (explanation of the term indeterminacy) with 60%, as well as P6T1 (explanation of the mathematical description of multiqubit systems) with 67% $(+)$. However, $(-)$ was also coded once or twice in 8 of the 13 paraphrasing tasks.

Overall, our results indicate that the paraphrasing tasks are better solved by the in-service teachers (Table V) than by the preservice teachers. The solutions of three tasks (P1T1, P3T2, and P5T1) were coded with $(+)$ for all in-service teachers. Furthermore, 11 of the 13 tasks were coded with $(+)$ for 73% or more in-service teachers. The solutions to tasks P4T3 (explanation of the concept of indeterminacy) and P6T1 (explanation of the mathematical description of multiqubit systems) were coded with $(+)$ for only nine in-service teachers and thus, for less than two-thirds. It can be seen that two of the three tasks, which are among the worst rated by the preservice teachers, also show the worst paraphrasing results among the in-service teachers. In particular, the concept of indeterminacy, also in differentiation from the measurement process, generally

TABLE IV. Results of the evaluative qualitative content analysis of preservice teachers' solutions of the paraphrasing phase ($\kappa_{\text{paraphrasing}} = 0.70$).

	Unit 1		Unit 2		Unit 3		Unit 4			Unit 5		Unit 6	
	P1T1	P1T2	P2T1	P2T2	P3T1	P3T2	P4T1	P4T2	P4T3	P5T1	P5T2	P6T1	P6T2
Pre01	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre02	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)
Pre03	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(-)	(+)	(+)
Pre04	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)
Pre05	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre06	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre07	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)	(+)
Pre08	(+)	(+)	(+)	(0)	(+)	(+)	(-)	(+)	(+)	(+)	(+)	(0)	(0)
Pre09	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre10	(+)	(+)	(-)	(0)	(0)	(0)	(-)	(0)	(-)	(-)	(+)	(0)	(+)
Pre11	(+)	(+)	(+)	(+)	(0)	(0)	(+)	(+)	(0)	(+)	(+)	(+)	(+)
Pre12	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(-)	(+)	(+)
Pre13	(-)	(0)	(0)	(-)	(0)	(-)	(0)	(+)	(+)	(+)	(0)	(0)	(0)
Pre14	(0)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(0)	(+)
Pre15	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)	(+)	(+)	(+)

TABLE V. Results of the evaluative qualitative content analysis of in-service teachers' solutions of the paraphrasing phase ($\kappa_{\text{paraphrasing}} = 0.70$).

	Unit 1		Unit 2		Unit 3			Unit 4		Unit 5		Unit 6	
	P1T1	P1T2	P2T1	P2T2	P3T1	P3T2	P4T1	P4T2	P4T3	P5T1	P5T2	P6T1	P6T2
In01	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In02	(0)	(+)	(0)	(-)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(0)	(0)
In03	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In04	(+)	(+)	(0)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In05	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)
In06	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In07	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)
In08	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(-)	(+)	(0)	(+)	(+)
In09	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In10	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)	(0)	(0)	(+)
In11	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)
In12	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(-)	(0)	(+)
In13	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(0)	(+)
In14	(0)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)	(+)	(+)	(+)
In15	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)	(+)	(0)	(0)

causes difficulties for the interviewees. This finding will be discussed in Sec. VI B.

It is noticeable that there are two interviewees among the preservice teachers performing considerably worse than the average, with only four (Pre10) or three (Pre13) paraphrasing tasks coded with (+). In particular, 7 of the 10 segments in which (-) was coded are related to these two preservice teachers. It should also be noted that there are no participants among the in-service teachers who are characterized by below-average performance in paraphrasing as clearly as those two preservice teachers. Six participants, three preservice teachers (Pre01, Pre05, and Pre06) and three in-service teachers (In01, In03, and In09), show successful paraphrasing (+) in all paraphrasing tasks.

3. Findings of the evaluative qualitative content analysis of the application phase

The results of the evaluative qualitative content analysis of the preservice teachers' application tasks (Table VI) show a pattern similar to that of the analysis of the paraphrasing tasks. In 8 of the 12 tasks, around three-quarters of the preservice teachers' answers (73%) or more are coded with (+). Task A2T2 (comparison of quantum physical and classical two-state system) shows 67% of the segments coded with (+), tasks A3T2 (drawing of states on the Bloch circle) and A5T2c (application of ket notation to a new example) show 60% of segments coded with (+). It is notable that only 20% of preservice teachers solved task A5T2ab (transferring knowledge to a three-state system) in a way that was coded with (+). This makes

TABLE VI. Results of the evaluative qualitative content analysis of preservice teachers' solutions of the application phase ($\kappa_{\text{application}} = 0.65$).

	Unit 1		Unit 2		Unit 3		Unit 4		Unit 5		Unit 6	
	A1T1	A2T1	A2T2	A3T1	A3T2	A4T1	A4T2	A5T1	A5T2ab	A5T2c	A6T1	A6T2
Pre01	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre02	(0)	(+)	(+)	(0)	(0)	(+)	(+)	(+)	(-)	(-)	(+)	(+)
Pre03	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(0)	(0)	(+)
Pre04	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)
Pre05	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(0)	(+)	(0)	(+)
Pre06	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
Pre07	(+)	(+)	(-)	(+)	(+)	(+)	(+)	(+)	(0)	(-)	(+)	(+)
Pre08	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)
Pre09	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(-)	(+)	(+)	(+)
Pre10	(+)	(0)	(-)	(0)	(0)	(0)	(-)	(+)	(0)	(0)	(+)	(+)
Pre11	(+)	(+)	(0)	(+)	(+)	(+)	(0)	(0)	(0)	(+)	(0)	(+)
Pre12	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)
Pre13	(0)	(+)	(0)	(0)	(+)	(+)	(-)	(0)	(0)	(0)	(0)	(+)
Pre14	(+)	(+)	(0)	(0)	(+)	(+)	(+)	(+)	(-)	(-)	(+)	(0)
Pre15	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)

TABLE VII. Results of the evaluative qualitative content analysis of in-service teachers' solutions of the application phase ($\kappa_{\text{application}} = 0.65$).

	Unit 1	Unit 2		Unit 3		Unit 4			Unit 5		Unit 6	
	A1T1	A2T1	A2T2	A3T1	A3T2	A4T1	A4T2	A5T1	A5T2ab	A5T2c	A6T1	A6T2
In01	(+)	(+)	(0)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)
In02	(+)	(0)	(0)	(+)	(+)	(0)	(+)	(+)	(0)	(0)	(0)	(+)
In03	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In04	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)
In05	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(0)	(+)	(0)	(+)
In06	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(+)
In07	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In08	(+)	(+)	(+)	(+)	(0)	(+)	(0)	(0)	(+)	(+)	(0)	(+)
In09	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)
In10	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)
In11	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)	(+)	(+)	(+)
In12	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(-)	(+)	(+)	(+)
In13	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(-)	(+)	(+)	(+)
In14	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(+)	(0)	(+)
In15	(+)	(+)	(0)	(+)	(0)	(0)	(0)	(+)	(0)	(0)	(0)	(0)

it by far the task with the lowest performance. This task requires a transfer of the knowledge regarding the measurement process and goes beyond a mere application of the content.

Just like in the evaluative qualitative content analysis of the paraphrasing tasks, it is obvious that, in general, the in-service teachers perform better in the application tasks (Table VII) in comparison to the preservice teachers. In particular, there are only two tasks in which less than 75% of respondents were coded with (+). In addition, there are two tasks, A1T1 (photons at the polarization filter) and A3T1 (reading off states at the Bloch circle), in which all in-service teachers' answers are coded with (+). A5T2ab also received the lowest score among the in-service teachers, with only 40% of segments coded with (+).

As in the analysis of the paraphrasing tasks, it can be seen that the students Pre10 and Pre13 perform the worst of all participants, although not as clearly as in the paraphrasing tasks. Furthermore, Pre01, In03, In07, and In09 are interviewees whose solutions were coded with (+) in all application tasks.

These findings give an overview of the acceptance of the discussed concepts and the performance in the paraphrasing and application tasks. However, a deeper analysis of occurring difficulties and possible reasons for the quality of the answers is discussed in the following sections.

B. Assessment and learning difficulties

To gain more detailed insights into teachers' perspectives, this section describes in detail how preservice and in-service teachers assess the discussed ideas, as well as the learning difficulties that arise in paraphrasing and application tasks.

1. Unit 1: Introduction

The content discussed in unit 1 is positively evaluated in general by both preservice and in-service teachers. The general structure and the explanations are assessed as logical, confirming the results of the evaluative qualitative content analysis. The introduction of single-photon polarization through simple experiments with polarizers and polarizing beam splitters is well received. Some participants also note that their preknowledge in the field of polarization of electromagnetic waves helps them understanding the explanations. For example, participant Pre09 states:

As I said, it was relatively easy to transfer the knowledge I already had. So it actually worked well. (Pre09, Pos. 55)

Nevertheless, some minor difficulties occur in terms of understanding the results of experiments with polarizers and polarizing beam splitters. For example, In10 and In11 state that polarizers only filter photons with a polarization deviating from the polarizer's setting and that polarizers do not actively change polarization. Participants Pre10 and In08 do not grasp the statistical character of the experiments and think that as soon as the polarization of the photons deviates a little from the filter setting, nothing is detected anymore. Some participants mention that they missed some explanation regarding "why" photons behave like they do when conducting the discussed experiments.

Another aspect is that the mentioned connection to preknowledge in terms of polarization of electromagnetic waves might lead to misconceptions that are subject to mechanistic ways of thinking. In the paraphrasing tasks, statements of three preservice and three in-service teachers

indicate a possible interpretation of single-photon polarization as “oscillating single photons”:

[...] What is observed when the second polarisation filter is set orthogonally to the first polarisation filter? Then, of course, we only detect photons that oscillate vertically. (In08, Pos. 41)

2. Unit 2: Algebraic description

Twelve preservice teachers and nine in-service teachers evaluate the content of unit 2—ket notation, basis, and superposition—as logical and understandable. However, some participants emphasize the discussed content to be more difficult in comparison to unit 1. The abstractness and use of the symbolic notation is seen as complicated by some participants:

For me, the problem lies a little in the mathematical descriptions. If you are not used to mathematical descriptions, it always takes a while to get used to the look, to put it mildly. So, to switch from the symbolic mathematical description to what is actually meant by it. I can cope with the mathematical descriptions. Whether I could reproduce them or calculate with them is not clear. But I can at least understand how you get there. (In12, Pos. 65)

Following on from this, In05 and In08 state that the link to the “real world” and physical phenomena is lost during unit 2. In addition, In05 also states that some mathematical concepts and tools seem complicated to him, as he is not a math teacher and certain mathematical aspects are new to him.

In general, the vectorial description is emphasized as positive for understanding polarization states, independently from the ket notation:

When you referred back to the fact that you can represent it as vectors, I think I had a visual idea in my head, also with the addition of these states. [...] It was this vector concept that was quite practical. (Pre05, Pos. 62–64)

Furthermore, five preservice teachers explicitly highlight the mathematical formalism to be “simple.” Five in-service teachers recognize aspects of the formalism and the ket notation from their former studies and state that their preknowledge of the formalism in particular and algebra in general was helpful for their understanding. Two in-service teachers emphasize the value of the ket notation, as it makes the description of different polarization states simple and understandable.

Some preservice and in-service teachers mention that they intuitively develop a mental image of polarization

states as real two-dimensional vectors, independent of the representations used in the individual explanation (ket notation). This may also be related to the mechanistic way of thinking discussed above.

In general, mainly two difficulties can be found. On the one hand, some participants show difficulties with the concepts of basis states and superposition, mainly in terms of the mathematical descriptions. In02 shows major difficulties in understanding the concept of superposition, especially because the concept of a basis is not clear from a mathematical perspective. In particular, In02 has problems in differentiating “two basis states of one basis” and “two bases.” Pre13 is unable to paraphrase both the concept of superposition as well as the concept of a basis. Additionally, Pre13 shows difficulties in paraphrasing the concept of superposition in the context of classical physics.

On the other hand, some participants show difficulties in understanding complex phases and circular polarization. This may be due to a lack of preknowledge of complex numbers and phases in general. In the case of Pre09, it becomes clear that the problem with complex numbers and circular polarization results from trying to create a pictorial image of circular polarization. Furthermore, Pre09 also shows the mental model of a polarization state being a real two-dimensional vector as discussed above.

To summarize, while most of the participants evaluate the discussed ideas as positive and logical, some criticize the abstractness of the content. The latter can be found a little more often within the sample of in-service teachers compared to preservice teachers.

Some interviewees explicitly assess the vectorial description as positive for their own understanding.

Overall, no single learning difficulty is shared by a large number of participants. In fact, specific difficulties occur only on an individual level, concerning one to three participants per category. Furthermore, difficulties often occur when basic mathematical or physical knowledge is lacking. For example, difficulties in the field of complex numbers or with mathematical concepts in general (e.g., basis of vector spaces or knowledge of vector algebra in general). Just like in unit 1, it can be seen that the participants independently make connections to wave optics that may lead to mechanistic ideas and hybrid models.

3. Unit 3: Graphical representation

Already the results of the evaluative qualitative content analysis of the survey of acceptance indicate that the assessment of the content in unit 3 is heterogeneous. However, our data show that this is due to the overall evaluation of the representation of polarization states on the Bloch sphere as abstract and complicated, rather than to specific learning difficulties:

I think it has also become a bit more challenging. Simply because you have to imagine it lying on a sphere. That is a bit more difficult to imagine. (Pre06, Pos. 75)

Two participants criticize that on the Bloch sphere, basis states, which are orthogonal in both their vectorial description and their polarization direction, are represented antiparallel on the Bloch sphere. Furthermore, participant In09 notes that one cannot represent vector addition on the Bloch sphere in the way it is represented in real vector space. Three interviewees explicitly question the sense of representing polarizing states on the Bloch sphere:

So what is currently unclear to me yet, is—We can now use this Bloch sphere or the Bloch circle to represent a polarisation somehow nicely, somehow on a circle, depending on how it is, with our spherical coordinates. What I need this for [...] is unclear to me now. Which is why I am still questioning the whole system a little bit. [...] But I think in order to be able to say, “yes, I have now understood or mastered it 1000 percent,” I still have to grasp the meaning behind. (In02, Pos. 169)

Nevertheless, In02 also states that the explanations are logical and understandable. These findings emphasize that there seems to be a difference between understanding the mathematical descriptions and accepting them in the sense of meaningfulness and usefulness. In contrast to the statements above, in total, 11 of the 30 participants state that the representation of polarization states on the Bloch sphere is logical and understandable. In06 evaluates the idea of representing two-dimensional and complex vectors on the Bloch sphere as “reasonable” and “logical.” Furthermore, some participants explicitly state that the transition to the Bloch circle seems logical when only considering linear polarization.

Main difficulties regarding the Bloch sphere occur with respect to the relation between the polarization angle and the angle on the Bloch sphere, as well as the representational change between ket notation and Bloch sphere. In application tasks A3T1 and A3T2, seven participants show difficulties with establishing the correct angular relationship between polarization angle and angle on the Bloch sphere.

Comparing preservice and in-service teachers, there are indications that in-service teachers show fewer difficulties with the representational change between ket notation and the Bloch sphere. The solutions of the paraphrasing and application tasks, as well as the results of the survey of acceptance, are more positive in comparison to preservice teachers. Nevertheless, some in-service teachers also evaluate the content as complicated and particularly

criticize that the connection between physics, the real world, and phenomenological aspects gets lost.

4. Unit 4: Measurement process and uncertainty

The majority of the participants evaluate the discussed ideas of unit 4 as understandable and traceable. In01 highlights the role of unit 4 as a connecting element between the experiments discussed in unit 1 and the mathematical descriptions of unit 2 and unit 3. Furthermore, In01 states that the characteristics of the quantum physical measurement process, especially in contrast to a classical measurement, are adequately worked out in unit 4. Overall, the participants’ assessments show that the main reason for limited acceptance is related to a lack of acceptance of the concept of uncertainty and the nature of the quantum physical measurement process itself rather than to the explanation. Some participants evaluate the uncertainty as not intuitive and difficult. A case worth mentioning is In14, who does not accept the characteristics of the quantum physical measurement process at all. In particular, the notion of considering a polarizing beam splitter combined with two detectors as a measurement apparatus is not accepted by In14:

When I see these images [...], then for me it is not a measurement process at that point, but an experimental setup. And the fact that an experimental setup can change a measurement result does not yet give rise to this crucial point that you can not determine it exactly. [...]. (In14, Pos. 145–154)

Besides aspects related to the principles of the measurement process and uncertainty themselves, the data show that the visualization of these concepts on the Bloch sphere and the slit model [4] is highly accepted. Eight preservice teachers and eight in-service teachers assess the visualization on the Bloch sphere highly positively:

I actually really liked the description of the Bloch circle because it was simply easier to understand with a visualisation. If you have a picture in front of you, you can see “Okay, I am jumping to there or there.” And you also have the angles that you can look at. [...] In the end, I saw that and it was intuitive, “yes, I would more likely jump to H than V.” So I really liked that. (Pre06, Pos. 110)

With respect to the visualization on the Bloch sphere, Pre01 and Pre03 point out that confusion may arise as the depiction of the slit along the two basis states might be mistaken for a model of a polarizing filter. Furthermore, In04 states that the slit model is highly abstract:

The Bloch sphere or the circle is already a very large abstraction. And then there is this

interpretation of the measuring process as simply an interpretation of the slit. [...] (In08, Pos. 177)

In04 highlights an important point, namely, that the slit on the Bloch sphere has no real physical meaning [4] and that a certain level of abstraction is required to comprehend this form of representation.

Overall, eight interviewees show difficulties when paraphrasing the concept of uncertainty. Especially, a mixing of the concept of nondeterministic measurement results and the concept of uncertainty can be found in the paraphrasing phase, which is the main reason for the paraphrasing tasks P4T2 and P4T3 to be coded as (0) or (−) (see Sec. VI A). Furthermore, the characteristics of the quantum physical measurement process cause difficulties, which can be seen in paraphrasing task P4T1 as well as in the application tasks of unit 4. The majority of these difficulties center on the participants' inability to fully relate the measurement process to the need for a specific measurement basis in order to describe it. This difficulty also becomes clear in the application tasks A4T1 and A4T2. Five participants cannot adequately paraphrase the term “nondeterminism.”

Comparing preservice teachers and in-service teachers, our data suggest that there are similar attitudes and difficulties regarding the content of unit 4. A special case is given by In14, who does not accept the quantum physical measurement and especially what we understand by a measurement apparatus at all.

5. Unit 5: Probabilities

Unit 5 is evaluated very positively, which can be deduced from the evaluative qualitative content analysis in Sec. VI A. Two preservice teachers explicitly evaluate the discussed content regarding the calculation of probabilities as “very easy.” The mathematical aspects discussed in unit 5, which refer to the calculation of probabilities of measurement results, are evaluated as logical and coherent. Overall, seven participants explicitly assess the calculations to be easy. Furthermore, it is stated that the contents are well connected to the previous units.

Four participants, two preservice teachers and two in-service teachers, show difficulties in paraphrasing the concept of a two-state system. Participant In08 states that a quantum object always is a two-state system. It is possible that this overgeneralization occurs since the whole teaching-learning sequence exclusively focuses on such systems. Participant Pre12 equates a two-state system with a superposition state and expresses that “a quantum object which is in a basis state is not in a two-state system.” Nevertheless, it can be seen that the term two-state system does not cause any major difficulties and seems to be understandable to most of the interviewees.

Fundamental difficulties can be found within the teachers' solutions of the application task A5T2, as can also be

seen in the results of the evaluative qualitative content analysis in Sec. VI A. Our data show that the participants have major difficulties in transferring their knowledge about the measurement process regarding two-state systems to a three-state system. In the solutions of task A5T2, two main difficulties can be found. First, some participants think that since a measurement process is always described by a two-dimensional basis, the system itself also has to be described by two basis states. Second, some participants interpret the state $|0\rangle$ as superposition of the states $|−1\rangle$ and $|+1\rangle$, reasoning that $0 = −1 + 1$.

No significant differences between preservice and in-service teachers are evident in this unit.

6. Unit 6: Entanglement

As can be seen in the results of the evaluative qualitative content analysis, unit 6 is positively evaluated overall. However, Pre05 and Pre12 assess unit 6 to be the most difficult one. Limited acceptance can be found mainly because of two reasons. First, some participants state that the mathematical formalism and the description of states of two quantum objects described by the tensor product are particularly difficult. Second, some participants state that the nature of entanglement seems unintuitive and illogical to them:

The process of entanglement simply sounds not logical. But it was not because of your explanations. (In04, Pos. 267)

Hence, just like in unit 4, it can be seen that not only the mathematical formalism but also the fundamental characteristics of quantum physics seem illogical for some preservice and in-service teachers. However, In09 states that the explanations regarding entanglement are in accordance with his preknowledge and plausible.

The mathematical description of states of two quantum objects and, in particular, the ket notation are perceived as helpful by many participants, which is explicitly mentioned by seven preservice teachers and two in-service teachers. The participants highlight that the ket notation helps them to understand the principle of entanglement. In particular, a crucial aspect is the simplicity of deciding whether a state characterizes entangled photons or not by taking a look at the correlations of possible measurement outcomes:

I only really understood the difference between entanglement and nonentangled states when you gave this specific example with the product state and the entangled state. I only realised it then. [...] So it is understandable and logical. Logical from a certain point onwards. (In01, Pos. 235)

Some participants considered the mathematical descriptions to be complex. In particular, it is noted that the

mathematical formalism entails a level of abstractness, which one must get used to. Similar to the introduction of the formalism in unit 2, it can be stated that extending the formalism to two quantum objects does not cause any difficulties or problems that arise for a large number of preservice teachers and in-service teachers, respectively. Despite the fact that some teachers consider the mathematical descriptions to be complex or abstract, there are only few specific difficulties and problems that arise.

Six participants have difficulties with the concept of Hilbert space and multiqubit Hilbert space. These difficulties occur mainly due to the fact that the concept of vector spaces itself causes difficulties. In02 shows a misconception by stating that only entangled states can be described in the multiqubit Hilbert space.

Overall, our data show no clear differences regarding the assessment and difficulties of the discussed content when comparing preservice teachers and in-service teachers.

7. Concluding aspects

After the discussion of the six individual units, the participants were given the opportunity to look back on the whole interview in a concluding overall reflection. This phase allows deeper insights into which aspects of the individual explanations seem particularly relevant for the preservice teachers and in-service teachers. Furthermore, it gives the teachers the opportunity to reflect on the teaching-learning sequence as a whole, describe possible changes they experienced throughout the interview, share their thoughts from a practitioner point of view, or highlight interconnections or conspicuousness they find important between the units.

In retrospect, all preservice teachers and almost all in-service teachers evaluate the teaching-learning sequence explicitly as comprehensible, well-structured, and logical in general. In13 states that the discussed contents should be understandable for teachers in general and notes that the content is well suited for teacher training workshops. Some in-service teachers mentioned they enjoyed the sequence and expressed appreciation for having learned new aspects of quantum physics.

Interestingly, our data show that all three roles of mathematics in physics, as described in Sec. II A, are mentioned by the participants. Hence, they assess the mathematical formalism as an important technical tool, a language, and a structuring element. Many interviewees explicitly highlight the structural role of the formalism, and different perspectives can be found in this regard. Some teachers mention that the algebraic structures in general, and in particular the description of states as elements of two-dimensional vector spaces, are helpful for understanding the concepts of quantum physics. Consequently, they point out that the formalism helps them to understand the physical phenomena, make

statements about results of measurement processes, and express relations precisely:

I think that it is very essential for understanding, because the pictorial idea of superposition states is so abstract that you can not actually achieve it. So we need something, we need a model that gives us the opportunity to interpret and understand this abstraction and to make clear statements from it. And here, of course, the Ket notation is by far the most interesting aspect for me. (In13, Pos. 262)

Furthermore, the structural role of the mathematical formalism is reflected within a strong connection to the ket notation as an algebraic representation of states. Especially when it comes to explaining and understanding entanglement and distinguishing between entangled and nonentangled states, ket notation is evaluated as very helpful, since correlations of measurement results can be predicted easily. Pre04 emphasizes that the mathematical descriptions become increasingly important throughout the interview, especially when it comes to explaining entanglement:

And then at some point at the end, when there was no other option, I switched to a mathematical description. And that became clear to me relatively quickly at the end, even if I did not have the clarity of purpose of the mathematical description at the beginning [...]. Because otherwise I can not conceptualise entanglements in any semantic way. Instead, I need the mathematical description to recognise the difference. (Pre04, Pos. 269)

Regarding the role of mathematics as a language in physics, the participants evaluate the ket notation positively as a convention. Pre03 mentions that ket notation is helpful to shorten the description of situations:

So I think the Ket notation is quite useful, because it really shortens a lot. And which, unlike in some other situations in physics, the designation by the letters is also quite useful. Because it always gives you an indication of what kind of polarisation or something like that is involved. (Pre03, Pos. 400)

Furthermore, seven preservice teachers and four in-service teachers emphasize that the ket notation is helpful for the calculation of probabilities in relation to the measurement process, as the link between probabilities and amplitudes is simple.

Similar to the assessments in unit 3 and unit 4, the Bloch sphere is seen as more controversial. In total, eight

participants assess the Bloch sphere to be helpful for their understanding. Participants Pre01 and In06 highlight that the Bloch sphere is especially useful for visualizing states with complex phases.

I thought the Bloch sphere was kind of cool. Because you could draw conclusions afterwards, so to speak. In higher-dimensional structures with circularly polarised photons, where I would have lacked the imagination to draw the right conclusions if I had not had this mathematical description. (In06, Pos. 331)

Regarding the visualization of the measurement process and uncertainty on the Bloch sphere, two different opinions can be found. Some participants evaluate the Bloch sphere as clearly helpful for visualizing these concepts. In contrast, some teachers note that the Bloch sphere does not seem necessary to them, as it does not add value to Dirac formalism and the ket notation:

I think you have basically left out the mathematics in the measurement process by looking at this slit. In general, I am a bit bothered by this Bloch sphere and whether it is really necessary to visualise the whole thing in a beautiful and attractive way. I mean, it is a visualisation, yes, but I am unsure whether the Bloch sphere really creates understanding. [...] At least you are visualising something. [...] But that is actually what I find most objectionable. (In03, Pos. 236)

Another aspect concerns the comparison between pre-service teachers and in-service teachers. Our data suggest that the assessment of the ket notation as an unfamiliar representation that you first have to get used to is made more explicit by in-service teachers than by preservice teachers. Furthermore, in the discussion of concluding aspects, noticeably more preservice teachers than in-service teachers assess the visualization of the measurement process and uncertainty on the Bloch sphere as positive and helpful. In addition, more preservice teachers assess the visualization of states on the Bloch sphere as helpful for their understanding in general.

VII. DISCUSSION

In the following section, the research questions are answered based on the two phases of data analysis described above.

RQ 1: How do secondary school preservice teachers and in-service teachers accept a reduced Dirac formalism for describing fundamental principles of quantum physics?

First, the introduction into quantum physical two-state systems via single-photon polarization as well as the

discussion of fundamental principles of quantum physics based on this exemplary two-state system is positively evaluated and accepted overall. Furthermore, the reduced Dirac formalism is generally assessed as understandable.

The ket notation is considered understandable and is positively evaluated in its function as a language and as a short and compact notation of states. It is particularly highlighted as a support for understanding entanglement and multiqubit states.

The acceptance of the Bloch sphere is mixed. The main reason can be found in the perceived abstractness of the graphical representation of polarization states on the Bloch sphere. However, our data show a positive change in acceptance throughout the interviews when employing the Bloch sphere to visualize the quantum physical measurement process and uncertainty.

Some participants show limited acceptance regarding the measurement process, uncertainty, and entanglement as fundamental principles of quantum physics, as they appear unintuitive, independently of the mathematical descriptions.

The data do not show clear differences between pre-service teachers' and in-service teachers' assessments of the content across the units. However, in-service teachers appear to be more critical of the abstractness of the mathematical formalism than preservice teachers. This also becomes evident in the discussion of concluding aspects in the interviews. This stronger criticism of the mathematical formalism and the lack of connection to physical phenomena among in-service teachers might have multiple reasons. The participating preservice teachers are still university students. Thus, they may be accustomed to handling mathematical formal descriptions of physics (e.g., in classical mechanics and electrodynamics) as part of their studies. Furthermore, it seems plausible that preservice teachers are more accustomed to being confronted with new mathematical-physical topics than in-service teachers, who often lack the time or opportunity to familiarize themselves with entirely new subjects in the course of their everyday work [82]. In contrast, it may be more difficult for in-service teachers to deal with a new physical topic, including its abstract mathematical descriptions. This is further reflected in the fact that some in-service teachers report limited recall of their university courses. However, it is also possible that some preservice teachers are not yet confident enough to critically question new content. In this context, they may not yet have fully developed the ability to relate the presented ideas to one another and to decide whether they make sense or not. Another possible reason is that the criticism of the mathematical formalism might reflect the in-service teachers' teaching experiences at school. Our data indicate that some in-service teachers directly link the discussed content to a possible transfer into their own teaching. However, as the results were obtained through an

exploratory qualitative data analysis, only interpretations and assumptions can be made.

RQ 2: How do secondary school preservice teachers and in-service teachers understand a reduced Dirac formalism for describing fundamental principles of quantum physics and which learning difficulties occur?

In general, the contents of the teaching-learning sequence are understood to a rather good extent by the participating preservice and in-service teachers. Some problems occur regarding the superposition principle and its mathematical description due to the lack of mathematical knowledge of complex numbers or vector algebra.

Besides minor difficulties related to complex numbers in general, difficulties with complex phases and how to visualize them on the Bloch sphere also appear. Especially, the transition between the algebraic representation of states in ket notation and the graphical representation on the Bloch sphere appears to be difficult for teachers. These difficulties regarding the relation between the angle of polarization $\vartheta/2$ and the angle on the Bloch sphere ϑ , the differentiation between Hilbert space and real space (see Sec. IV B 3), as well as difficulties with overall phases are in line with the results of other studies discussed in Ref. [64].

Regarding the application tasks, in particular task A5T2, which involves the transfer of the discussed ideas to a three-state system, causes difficulties as the participants are often not able to transfer their knowledge to this new system. On the one hand, we think that this question might be too difficult within the setting of the interviews in general, as the time for participants to process their knowledge was limited. On the other hand, the difficulty of transferring knowledge regarding a two-state system to other systems is also described in a former study by Weissman *et al.* [8].

Overall, the data show no clear differences in conceptual understanding between preservice teachers and in-service teachers. The distribution of successful, satisfactory, or poor solutions of the paraphrasing and application tasks suggests that in-service teachers tend to solve these tasks slightly better (see Sec. VI A). However, due to the small sample size and the qualitative data analysis, no clear statements can be made beyond the specific sample. Furthermore, the analysis of learning difficulties (see Sec. VI B) shows no clear differences between preservice teachers' and in-service teachers' understanding.

Furthermore, this study deliberately included preservice and in-service teachers with and without mathematics as their second subject, as it seems plausible that there are differences in the acceptance and understanding of these groups (see Sec. V B). However, no general differences between these groups could be found in the data. For example, Pre13, whose second subject is mathematics, shows many difficulties in the paraphrasing and application tasks, whereas In01, whose second subject is English

as a foreign language, shows only minor difficulties as well as an overall high acceptance (see Sec. VI A).

VIII. LIMITATIONS

Since the presented results were gained in the scope of a qualitative interview study with a small sample size, generalizations to other samples may be limited. As the study was conducted within a laboratory setting, our results cannot be directly transferred into practical teaching environments in teacher training programs. However, we consider learning difficulties observed in the teaching experiments to be likely to also occur in real-world settings.

As participation was voluntary, a positively biased sample has to be assumed, as probably mostly engaged participants interested in the topic took part in the study.

Furthermore, regarding the assessment of the presented content in interview studies, our data might be subject to socially desirable bias [81]. This assumption is supported, as, in the evaluative qualitative content analysis, some participants show a high level of acceptance in the survey of acceptance but only satisfactory solutions in the paraphrasing and transfer tasks. The latter may also be related to an illusion of explanatory depth [83], meaning that participants found the discussed ideas plausible without an in-depth comprehension of the underlying concepts [73].

Finally, since the conducted teaching experiments rely on the contents of the teaching-learning sequence and specific paraphrasing and application tasks, only limited generalizations to other teaching-learning sequences or teaching approaches in the context of quantum physics can be made [73].

IX. CONCLUSION

In this paper, we sketched a teaching-learning sequence on fundamental principles of quantum physics (*superposition, measurement process, uncertainty, and entanglement*) and the mathematical formalism. Furthermore, we presented the results of an exploratory interview study in which Teaching Experiments on mathematical descriptions of fundamental principles of quantum physics were conducted in order to assess both preservice and in-service teachers' acceptance and understanding of the explanations given.

A. Implications for research

Regarding the investigation of the role of a reduced Dirac formalism for the understanding of fundamental principles of quantum physics, some open questions for further research can be identified. Since only a few specific application tasks could be implemented in the teaching experiments, an examination of teachers' performance in more complex problem-solving tasks should be investigated in the future. Furthermore, as the teaching

experiments did not allow for investigating how teachers can handle the formalism in a technical way (e.g., performing calculations of probabilities), an examination regarding procedural knowledge would be desirable.

A key challenge in terms of the graphical representation of states on the Bloch sphere is that orthogonal basis states are depicted in antiparallel orientation on the Bloch sphere. Along with this, the differentiation and the relationship between vectors in the complex Hilbert space and the real space seem to be difficult. Therefore, it would be interesting to investigate if similar problems occur in the same manner when discussing the Bloch sphere representation within the context of other two-state systems, such as path-coded two-state systems [41] or electron spin [11]. These exemplary two-state systems have in common that the orthogonality of the basis states is not as obviously connected to physical properties as it is in the case of photon polarization.

Also, the use of the representation of states on the Bloch sphere within the context of quantum computing seems to be fruitful. Our results indicate that the Bloch sphere is evaluated more positively when starting to use it as a tool for visualizing the measurement process and uncertainty, and working with it goes beyond a mere representation of states. Hence, it seems plausible that further applications of the Bloch sphere, such as visualizing manipulations of qubits (e.g., quantum gates), might increase acceptance and its role for a conceptual understanding of quantum physics [12,44].

Furthermore, it would be desirable to investigate the influence of the reduced Dirac formalism presented in this paper on internal representations and mental models of preservice teachers and in-service teachers. This seems to be interesting, as we observed teachers' reasoning and statements suggesting an interference of classical and quantum mental models, especially within unit 1 of our teaching-learning sequence. Hence, a further investigation of the influence of the mathematical formalism on developing adequate quantum physical mental models seems to be desirable. First studies based on teaching-learning environments employing two-state systems indicate positive influences of the use of a mathematical formalism for the transitions from gestalt to functional thinking [30,84].

B. Implications for teacher training

As discussed in Sec. IV C, the teaching-learning sequence was primarily intended to form the basis for conducting the described interview study. However, building on the results of our study, the teaching-learning sequence can also serve as a basis for teacher training programs and workshops.

With respect to teacher training, the structural role of mathematics in quantum physics should be highlighted, as our data indicate that teachers assess the mathematical formalism to be helpful for their understanding in many cases. In this regard, it seems to be important to highlight

the connections between physical properties and mathematical structures. This is important not only for teachers' conceptual understanding of quantum physics [45], but—as our data suggest—also for the acceptance of the formalism itself. However, we think that an introduction into the mathematical formalism using single-photon polarization is suitable for teacher education. In addition, applications of the mathematical formalism with respect to two-state systems, e.g., in quantum information sciences [12], should be made clear to teachers.

An important aspect seems to be the consideration of teachers' preknowledge on mathematical tools and concepts such as vector algebra or complex numbers. Our data suggest that the assessment of the reduced Dirac formalism often depends on the extent of teachers' preknowledge on these aspects.

Regarding the two mathematical representations, our results suggest that the ket notation, in its role as a mathematical language [20], contributes to teachers' conceptual understanding, especially in the case of entanglement. With respect to the Bloch sphere, our data show a wide spectrum of opinions. Hence, when using visualizations on the Bloch sphere in teacher training, one should be aware that the Bloch sphere might not facilitate understanding for all teachers. However, our data suggest that the pictorial representations of experimental setups seem to fulfill a similar function for visualizing measurement processes as the Bloch sphere and thus should be taken into consideration as well.

C. Outlook

The data collected in this study provide a rich basis for future analyses. In particular, the analysis of resources used in the paraphrasing and application tasks, as well as reasoning structures, allows conclusions to be drawn about which aspects of mathematical formalism are used in solving these tasks. In this scope, our data enable the analysis of mathematical sense making [85] by preservice teachers and in-service teachers in the context of the teaching-learning sequence. In this regard, in the future, our data will be further analyzed. Furthermore, it is also desirable to investigate which opportunities and concerns teachers have regarding the use of a mathematical formalism in the classroom.

The results regarding the importance of the structural role of the mathematical formalism as well as the mainly positive assessment of the ket notation as an algebraic representation of states make it reasonable to extend the research to secondary school students. Especially, it should be investigated whether the same aspects of the formalism that are beneficial for teachers' understanding of fundamental principles of quantum physics can also be identified in the context of secondary school students. In particular, this concerns the structural role of mathematics, the use of ket notation as a language, and the role of

graphical representations on the Bloch sphere. Finally, it is essential to explore whether or not new learning difficulties and misconceptions in the understanding of the mathematical formalism for describing fundamental principles of quantum physics among secondary school students can be uncovered.

To address these aspects, a further study is in preparation in which we address similar research questions within the scope of the upper secondary level. Therefore, the teaching-learning sequence is being adapted, the formalism is slightly more reduced, and teaching experiments with upper-level secondary school students will be conducted.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CODING SCHEMES FOR THE EVALUATIVE QUALITATIVE CONTENT ANALYSIS

In the Tables [VIII](#), [IX](#), [X](#), the coding schemes for the evaluative qualitative content analysis of the survey of acceptance are shown.

TABLE VIII. Coding scheme for the evaluative qualitative content analysis of the survey of acceptance (see also [\[11,17\]](#)).

Level	Description	Coding rules and remarks
Complete acceptance (+)	The individual explanation is accepted without any restrictions. If applicable, the interviewee provides a good summary of the content on his or her own.	In the case of questions from the interviewee, this category is coded if the question indicates that the individual explanation itself has been understood and accepted and the question shows a transfer of the content.
Limited acceptance (0)	The individual explanation is accepted in some parts. For example, there is criticism of several individual aspects of the information offered, while there is still an overall acceptance of the content. Individual aspects may be explicitly considered too difficult. If applicable, the interviewee provides a satisfactory summary of the content on his or her own.	It is also regarded as criticism of individual aspects if the interviewee evaluates that content is missing, which is considered necessary for a coherent explanation, as long as such explanations are not part of subsequent unit. In the case of questions from the interviewee, this category is coded if the question indicates that minor aspects were considered illogical or not conclusive.
No acceptance (−)	The individual explanation is largely not accepted or not accepted at all. The content is considered not logical, not conclusive, or not meaningful. In particular, the content is explicitly considered too difficult or incomprehensible.	In the case of questions from the interviewee, this category is coded if extensive questions are asked indicating that central aspects were considered illogical or not conclusive.

TABLE IX. Coding scheme for the evaluative qualitative content analysis of the paraphrasing phase (see also [\[11,17\]](#)).

Level	Description	Coding rules and remarks
Successful paraphrasing (+)	The question is answered completely and correctly.	Not all technical terms need to be used correctly. This category is also coded if aspects are missing but completed on request.
Satisfactory paraphrasing (0)	The question is partially answered correctly. Either minor aspects are incorrect and are only answered with minor assistance or minor aspects are missing and are only answered correctly with minor assistance.	Differentiation from (+) in the case of queries by the interviewer: this category is coded if aspects are missing and are only reproduced with minor assistance. Differentiation from (−) in the case of queries by the interviewer: this category is coded if assistance is given, but no fundamental conceptual difficulties are identified.
Poor paraphrasing (−)	The question cannot be answered, or the answer is largely incorrect and cannot be corrected without mistakes, even with assistance.	Differentiation from (0) in the case of queries by the interviewer: this category is coded if extensive assistance is given and if major conceptual difficulties are identified.

TABLE X. Coding scheme for the evaluative qualitative content analysis of the application phase (see also [11,17]).

Level	Description	Coding rules and remarks
Good solution (+)	The task is solved completely and correctly.	Not all technical terms have to be used correctly. In case of written solutions, the mathematical notation does not have to be perfect. This category is also coded if aspects are missing but completed on request.
Satisfactory solution (0)	The task is partially solved correctly or is solved correctly with assistance.	This category is also coded if improper use of the mathematical formalism indicates problems in conceptual understanding. Differentiation from (+) in the case of queries by the interviewer: this category is coded if aspects of the solutions are missing and are only added with assistance. Differentiation from (−) in the case of queries by the interviewer: this category is coded if assistance is given, but no fundamental conceptual difficulties are identified.
Poor solution (−)	The task cannot be solved, or the solution is largely incorrect and cannot be corrected, even with assistance.	Differentiation from (0) in the case of queries by the interviewer: this category is coded if extensive assistance is given and if major conceptual difficulties are identified.

APPENDIX B: EXAMPLES FOR THE THREE LEVELS OF ACCEPTANCE

In Table XI, examples of participants' statements coded with (+), (0), and (−) in the evaluative qualitative content analysis are shown.

TABLE XI. Examples for coded answers for each of the three levels of acceptance (see Sec. V D 1).

Level	Example
Complete acceptance (+)	<i>I have to say this time, I actually found it very understandable again. So far, I don't have anything where I would say that it didn't make sense to me. I also found the mathematical descriptions useful, especially that it was well visualised again. So especially this slit, which is easy to draw, and then these arrows. I think you can imagine everything much better that way. (Pre02, pos. 205–206)</i>
Limited acceptance (0)	<i>[...] So in principle, that was good. What is perhaps not quite clear to me at the moment/[...] We now can represent a polarisation somehow nicely with this Bloch sphere [...] with our spherical coordinates. What do I need this for? Why do I somehow want to represent my quantum, which I have now somehow polarised [...] That's unclear to me now, that's why I still have a question mark behind the whole system [of the Bloch sphere] [...]. But from a purely factual point of view, step by step, the small individual steps, I found it easy to understand, in any case. But I think that in order to be able to say "Yes, I have now understood or mastered it 1000 percent," you still have to grasp the meaning behind it. [...]. (In02, pos. 158–162)</i> ⇒ The explanations are assessed as comprehensible and understandable. However, the meaning is questioned in detail.
No acceptance (−)	<i>Well, I think it was a bit too fast for me. I think I'll be a bit confused right away with the questions. But again, I found the mathematical descriptions [the Bloch sphere] better than the explanations at the beginning. [...] Well, I didn't quite understand it with the slit either. Does it now mean that measurements can really only be taken horizontally and vertically in this case? Or is it also in the area of the slot, a bit different from horizontal and vertical? [...] I was a bit confused with the slit, because then it looks like there could also be deviations from horizontal vertical inside this slit. (Stu10, pos. 208–221)</i> ⇒ The explanations are not assessed as comprehensible and understandable overall. Furthermore, the core of the explanations (slit model for visualizing the measurement process) is not fully understood. Thus, the explanations seem not to be plausible or coherent to the participants.

APPENDIX C: EXAMPLES FOR PARAPHRASING AND APPLICATION TASKS

The Tables XII and XIII show selected examples of paraphrasing and application tasks discussed in the study.

TABLE XII. Selected examples of paraphrasing tasks. The abbreviations refer to the respective tasks of the teaching experiment.

Abbreviation	Task
P1T1	Describe what can be detected after a single photon is sent through a polarization filter. What can be observed if the polarization filter is set orthogonal to the polarization of the photon?
P2T1	Explain what is meant by the “state of a quantum object.” Explain how the state of a quantum object can be described mathematically.
P3T2	Explain under which circumstances the Bloch circle can be used instead of the Bloch sphere.
P4T1	Describe how the quantum physical measurement process is characterized.
P5T2	Explain what is meant by a “two-state system.”

TABLE XIII. Selected examples of application tasks. The abbreviations refer to the respective tasks of the teaching experiment.

Abbreviation	Task
A1T1	A photon with arbitrary polarization is sent to a polarization filter rotated by 30° relative to the horizontal axis. Explain how a second polarization filter must be adjusted so that the photon can never be detected.
A2T1	Explain why the state $ \psi\rangle = \frac{2}{\sqrt{5}} D_+\rangle + \frac{1}{\sqrt{5}} D_-\rangle$ is a valid polarization state.
A3T2	Draw the following state in the Bloch circle: $ \psi\rangle = \cos\frac{120^\circ}{2} H\rangle + \sin\frac{120^\circ}{2} V\rangle = \frac{1}{2} H\rangle + \frac{\sqrt{3}}{2} V\rangle$
A4T1	Justify whether the following statements are true or false: (a) If a photon is in a basis state given by the measuring apparatus, the measurement result cannot be predicted and is subject to chance. (b) The polarization of a photon is measured with respect to the basis $(H\rangle, V\rangle)$. A horizontally or vertically polarized photon is detected with a probability of 50% in each case. For this reason, the photon was in one of the two basis states $H\rangle$ or $V\rangle$ prior to the measurement process. (c) Photons are considered that are known to be either in the $H\rangle$ or $V\rangle$ state. The polarization with respect to the basis $(D_+\rangle, D_-\rangle)$ is measured. It can be stated with certainty that the state of each photon is changed after the measurement process.
A5T2ab	An excited helium atom has a total spin of 1. This means that, after measuring a component of the total spin, the atom can be found in one of three distinct states. These states are denoted by $ -1\rangle, 0\rangle, 1\rangle$. (a) How many basis states are needed to describe all spin states of the atom? (b) Is the given system a two-state system? Justify your answer.
A6T2	A state is given by $ \psi\rangle = \frac{1}{2} P_0A_0G_0K_0\rangle B_0W_0\rangle + \frac{1}{2} P_1A_1G_1K_1\rangle B_0W_0\rangle + \frac{1}{2} P_0A_0G_0K_0\rangle B_1W_1\rangle + \frac{1}{2} P_1A_1G_1K_1\rangle B_1W_1\rangle$ Justify which systems are entangled and which are not.

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