How students can use coherence to reconstruct a partially forgotten equation

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Introductory physics instruction emphasizes fluency with routine problem-solving procedures. However, even when applying these procedures, students frequently encounter challenges. This paper investigates how students navigate such moments when answering qualitative electricity and magnetism problems in interviews. Students frequently noted they had partially forgotten a key equation on a problem involving *RC* circuits. We present focal cases that show how coherence-seeking approaches were employed to overcome this problem-solving challenge. In attempts to reconstruct these equations, participants identified and chained qualitative dependencies and sought coherence between qualitative and mathematical understanding of the physical system. These moments of forgetting and reconstructing equations are a useful site for studying broader physics learning goals. While prior work investigates the use of mathematical sensemaking by examining how students respond to explicit prompts, our cases illustrate how students can spontaneously use mathematical sensemaking strategies. We reflect on these cases to consider how such adaptive reasoning can be a target for instruction and assessment.

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I. INTRODUCTION

Physics problem solving research has revealed the ways physics students and experts can leverage their understanding of physical concepts to create, interpret, and check mathematical equations and solutions [1–15]. Strategies that go beyond just formal derivations have been categorized as *mathematical sensemaking in physics*—the different ways in which alignment between physical conceptual understanding and mathematical equations is leveraged in physics reasoning. Though an important part of disciplinary physics practice, the application of these strategies—such as dimensional analysis, checking functional dependencies, and limiting case analysis—is not widely evident in students' typical coursework. To assess these mathematical sensemaking skills, research commonly employs tasks that explicitly prompt students to create, interpret, or check mathematical expressions [9,10,16]. The trade-off of this approach is that it does not reveal how strategies are opportunistically applied and coordinated with other reasoning in the course of problem solving.

To investigate the opportunistic and unprompted use of mathematical sensemaking skills, this study draws on a series of problem-solving interviews where multiple students noted that they had (partially) forgotten a key equation on one particular problem (the four circuits problem). We illustrate the range of strategies that students turn to after noting that they could not precisely remember a key equation. In particular, we present focal cases that highlight the ways that students leverage coherence between their qualitative and mathematical understanding of a physical system to guide reconstruction of that equation. We explore how mathematical sensemaking relies on coherence between physical concepts and mathematical equations and is thus a subset of *coherence-seeking strategies*, which utilize the inherent connections and consistency of knowledge [17]. These cases illustrate students' unprompted use of mathematical sensemaking (and coherence seeking, more broadly) to overcome a problem-solving challenge. We reflect on these cases to consider how such reasoning can be a feasible target for instruction and assessment.

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II. THEORETICAL BACKGROUND

Connecting conceptual understanding to mathematical equations is core to physics and has been investigated throughout the history of physics education. Early physics education research on problem solving had a focus on (re)producing the accuracy and systematicity of expert solution methods [15]. Investigations of expert problem solving revealed that identification of the key physical concepts for a problem situation drives experts' use of mathematics [18]. That is, the physical concepts are pointers that indicate which particular mathematical equations one should use in problem solving. These investigations have led to the development and implementation of physics problem-solving frameworks that scaffold students' initial physical analysis of the problem situation to guide the subsequent selection and use of mathematical equations [19–26]. While this procedural problem-solving framework can increase students' fluency with routine procedures, this research does not explore *adaptive problem-solving expertise*—the ability to productively modify existing knowledge and procedures for application in new contexts [27]. Flexibility in reasoning makes one's problem-solving robust against deviations from these routine physics problem-solving procedures. Therefore, moments of forgetting an equation are a useful opportunistic research context for studying students' adaptive problem solving in physics.

What adaptive strategies could students use to reconstruct an equation that was forgotten and cannot be looked up? One could rederive them from first principles. For instance, if one forgets the equation $v^2 = v_0^2 + 2a\Delta x$, it could be rederived from the definitions of velocity and acceleration or by combining Newton's second law with the work-energy theorem. Similarly, the time constant for RC circuits, $\tau = RC$, can be rederived by analyzing an RC circuit using Kirchhoff's laws in combination with Ohm's law and the definition of capacitance, $Q = CV$.

While such formal derivations are valuable, they are not the only strategy one could use. There are many ways in which an understanding of physical concepts can be used to constrain the mathematical forms that can describe those physical systems. Sherin [2] showed how undergraduate physics students could create novel equations to describe physical systems using their knowledge of symbolic forms—conceptual schemata that are associated with particular equation symbol templates. By understanding physical quantities as being comprised of parts-of-a-whole or competing terms, students constructed equations to match those conceptual schemata without going through a first-principles derivation. For example, to describe the motion of a free-falling object that reaches terminal velocity due to air resistance, students wrote the equation $a(t) = -g + \frac{f(v)}{m}$ to indicate that the acceleration due to gravity is opposed by the acceleration due to a velocity-dependent resistance force $f(v)$. Here, the conceptual schema of opposing influences was represented by the symbol template $\square - \square$, where each $\square$ was filled in with a particular expression representing one of the two opposing influences. Investigations into symbolic forms have shown that this mathematical sensemaking strategy is applicable in a range of physics topics [28–30] and can be used to explain students' problem-solving insights [7].

Another way mathematical sensemaking can support reconstructing a (partially) forgotten equation is through the practice of checking a symbolic answer by matching its properties to the expected physical behavior. Research in this area has investigated student usage and uptake of approaches including dimensional analysis, checking functional dependencies, and limiting case analysis [11,31]. Dimensional analysis (or unit analysis) allows one to check that both sides of an equation or terms in a sum have the same dimensions (or units), a requirement for physical equations. Another method involves checking the physical plausibility of the functional dependencies within an equation—i.e., if X increases, would it make sense for Y to increase, decrease, or stay the same? One can also check that the equation produces the expected outcome for special or limiting cases. Research has investigated how and how often students evaluate their symbolic answers in courses that explicitly teach these mathematical sensemaking strategies [9,10,13].

Mathematical sensemaking strategies are a subset of the broader scientific reasoning practice of leveraging coherence [17]. For example, symbolic forms embody coherence by combining concepts with the symbolic structure of mathematical equations [2]. In addition, checks of symbolic solutions are explicit searches for alignment between mathematical expressions and expected physical behavior [9,10,13]. This paper further elaborates the connections between seeking coherence and mathematical sensemaking by illustrating how students use chaining. Chaining is the linking of different associations and relationships used to make inferences. In science, *chaining* is an important process, since many inferences and predictions cannot be made in one step; rather, intermediate conceptual and mathematical steps can connect the starting premises to the conclusions. In education research, chaining processes have been studied as a central part of scientific reasoning, such as mechanistic reasoning [32], mathematical sensemaking [12], and making qualitative inferences [33,34]. In the mathematical sensemaking context, chaining can provide the scaffolding necessary for mathematical and conceptual information to be linked. We will illustrate how chaining supports mathematical sensemaking and coherence-seeking practices for reasoning after forgetting an equation or dependence.

In sum, research around mathematical sensemaking in physics has investigated how physics students can (i) decide which equations to use through conceptual analysis,

(ii) create new equations from conceptual understanding, and (iii) check symbolic answers against expected physical behavior. This study builds on the existing research by investigating another activity where mathematical sensemaking is useful—reconstructing a partially forgotten equation and more generally, reasoning after forgetting an equation. Through this work, we further elaborate the mathematical sensemaking and coherence-seeking practices that physics students can employ.

III. METHODS

A. Data collection: Interviews

The data corpus is made up of semistructured, 1-h, one-on-one clinical interviews involving one interviewer (E. B.) and a participant from one of two institutions: a community college or a private four-year university. There were a total of 18 interviewees, nine from each school. Students were eligible to participate if they had completed or were currently enrolled in Introductory Physics II (electricity and magnetism). From the private four-year university, six students were undergraduates and three were graduate students; all were working toward degrees in physics or engineering. All of the private four-year university students had completed Introductory Physics II.

Participants were given qualitative, introductory-level electricity and magnetism questions during the interview. They were asked to reason aloud as they answered these questions. Participants were informed that this was not an evaluative activity but one focused on understanding how they thought about these questions.

While remaining more neutral, the interviewer supported this framing by asking follow-up questions that were mainly for clarification of previously stated ideas. After a student completed their thinking on a question, the interviewer sometimes also introduced new ideas to see how interviewees would think about them.

Participants were not asked to provide information on whether or how mathematical sensemaking strategies were taught in their physics courses. Therefore, we do not make any claims regarding how students learned or developed the reasoning strategies they used in the interview.

B. Case selection and analytic approach

A team of researchers (led by K. G.) produced content logs, attending to instances where physical reasoning, mathematical reasoning, and coherence between the two were evident. With this lens, we identified two problems from the interview protocol, both involving capacitors, which elicited rich reasoning across the data corpus. As we discussed these episodes and refined our interpretations, we noticed that multiple participants had difficulty recalling an equation during the four circuits problem (presented in the next section). "Forgetting an equation" was an interesting site to explore our analytic focus on mathematical sensemaking, because interviewees often employed a mixture of physical and mathematical reasoning to reconstruct or compensate for the forgotten equation.

Researchers then reviewed the four circuits problem in each of the 18 interviews to see how often interviewees explicitly sought to use an equation that they noted they could not exactly recall. When participants were identified as having "forgotten an equation," their interview segment was transcribed and analyzed in detail. Through an iterative process, these analyses became more focused on particular common themes—such as checking physical dependences, switching to a new line of reasoning, attempting to recall the equation, or displaying anxiety and uncertainty. After initial analysis, the focus on forgetting an equation expanded to include moments where students struggled to recall a variable dependence reflected within a relevant equation. Overall, 11 participants had 12 instances of either partially or totally forgetting an equation or variable dependence (one participant, Gabi, experienced two instances of "forgetting"). In the results, we present themes and illustrative examples from these cases. Preliminary analyses of three students' reasoning have previously been published [35,36].

While participants employed mathematical sensemaking in other areas of the interview, we limit our analysis to the four circuits problem. Focusing on this single problem allowed us to explore participants' unprompted use of mathematical sensemaking to overcome a difficulty due to the fortuitous fact that many students forgot a relevant equation or dependence on this problem.

During the analytic process, we remained aware that clinical interviews do not provide unbiased access to people's knowledge and problem-solving practices. We acknowledge that the interviewee's reasoning is situated in the context of the interview, emergent from their interactions with the physics tasks, interviewer, and framing of the interview activity [37–39]. Therefore, our analysis does not suggest that this is "typical reasoning" that students would employ outside of the interview. Nor do we claim that the interviewer's moves have no effect on what knowledge and approaches students bring to bear. Rather, our analysis describes the types of strategies students can use to navigate forgetting an equation or dependence without claiming that this is typical strategy use or that the participants of this study would spontaneously call upon these strategies in other contexts.

C. Focal question: The four circuits problem

The four circuits problem provides the following prompt, accompanied by the image shown in Fig. 1:

> The light bulbs, batteries, and wires are all the same in the four circuits. The capacitors in circuits A, B, and C are the same. The gap in the capacitor in circuit D is half of the other

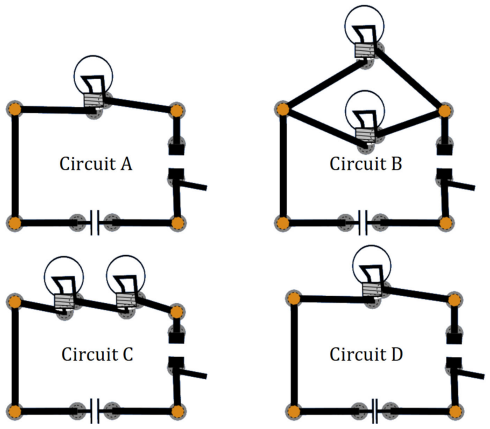


FIG. 1. Four circuits problem: Circuits A—D are each shown to have a switch, lightbulb(s), and a capacitor. Circuits B and C have two lightbulbs, configured differently, while circuits A and D only have one lightbulb. The gap in the capacitor in circuit D is half the width of the gap of the other capacitors.

> capacitors. All capacitors have individually been charged by connecting them to a 9 V battery separately from each other. They are almost fully charged, i.e., they have the maximal possible charge on either plate.
>
> You close the switches in all four circuits at the same time. Each capacitor takes at least 2 s to fully discharge. After 1 s, you open all the switches again.
>
> Which capacitor has the most amount of charge left on its plates? Which capacitor has the least amount of charge left on its plates? Explain your reasoning.

The correct answers are that circuit D has the most amount of charge left on its plates, and circuit B has the least amount of charge left on its plates. A symbolic method for answering this question is to compare the time constant for each RC circuit, $\tau = RC$, where R is the resistance in the circuit and C is the capacitance in the circuit. A larger time constant means that it will take longer for a capacitor charged to an initial voltage $V = 9$ V to discharge. If the resistance of one light bulb is R_0, then the resistances of circuits A, B, C, and D are R_0, $R_0/2$, $2R_0$, and R_0, respectively. If the capacitance in circuits A, B, and C is C_0, then the capacitance in circuit D is $2C_0$ since $C = \epsilon A/d$, where ϵ is a constant, A is the surface area of each capacitor plate, and d is the distance between the capacitor plates. Therefore, the time constants for circuits A, B, C, and D are R_0C_0, $R_0C_0/2$, $2R_0C_0$, and $2R_0C_0$, respectively. Since circuit B has the smallest time constant, it discharges the fastest and will have the least amount of charge remaining on its capacitor plates. Circuits C and D have the largest time constant, so they take the most time to discharge. Although circuits C and D will have the same remaining capacitor voltage, the capacitor in circuit D has more remaining charge Q since it has a larger capacitance and started with a greater initial amount of charge.

Alternatively, one could reason qualitatively about the physical processes of electric current and capacitor discharge to reach a similar conclusion. However, using the equation for the time constant $\tau = RC$ allows for precise and compact comparisons of a single quantity describing discharge rate. Therefore, recalling this time constant equation is advantageous in answering this qualitative question.

IV. RESULTS

A. Overview

As shown in Table I, of the 12 instances of forgetting an expression while attempting the four circuits problem, six involved partially remembering an aspect of the missing expression—students may have remembered two possible forms of an equation or dependence or may be unsure of whether a remembered equation or dependence is correct. In four of these instances (two instances for Gabi, one instance each for Tom and Rene), the interviewees attempted to reconstruct the missing equation or dependence using mathematical sensemaking. In the next subsections, we share illustrative cases of how mathematical sensemaking was used. The other two cases, where

TABLE I. Specifying the type of strategies used in the 12 cases when equations or variable dependencies were partially remembered or forgotten. The frequency of strategy usage and cases that will be presented in greater detail are noted.

Type of strategy used	Number of instances	Cases presented
Partially remembered equation or dependence		
Mathematical sensemaking to "recall" equation or dependence	4	Gabi (two cases), Tom, Rene
Spontaneous recall	1	
Selection without reasoning	1	
Totally forgotten equation or dependence		
Cohering multiple lines of reasoning	1	Blake
Conceptual arguments	2	Sam, Riley
Interviewer intervention	3	

participants partially remembered an equation or dependence, are not discussed in detail. In one of these two instances, the participant suddenly recalled the missing expression without further elaboration. In the other case, a student who recalled two possible expressions for equivalent resistance for series and parallel circuits ultimately decided one expression was for series and one expression was for parallel without further explanation.

There were six cases where students totally forgot an equation. These students noted that there was a relevant equation but did not discuss recalling any part of it. The cases of Blake, Sam, and Riley will be presented in greater detail. Blake attempted multiple lines of reasoning (conceptual arguments, reasoning qualitatively from equations, and analogical reasoning) and compared the results to determine the missing qualitative relationship. Two of these students, Sam and Riley, made progress by switching to purely conceptual and qualitative arguments. These students reasoned mechanistically to determine dependencies between relevant features of the four circuits problem. The other three cases, which will not be discussed further, were predominantly impacted by interviewer intervention to support or elicit particular reasoning pathways.

Next, we present more detailed cases of these students' mathematical sensemaking and conceptual arguments. As we did not ask participants to answer demographic questions, we use they/them pronouns when referencing individual participants.

B. Mathematical sensemaking to reconstruct equations or dependences

1. Example. Using qualitative dependencies to decide whether the time constant for RC circuits is RC or $1/RC$

One strategy participants used to reconstruct partially remembered equations was connecting qualitative dependencies to the form of a mathematical equation. In one case, Gabi (case no. 1) used this strategy to reconstruct the expression for the *RC* circuit time constant, $\tau = RC$. Gabi was interviewed during the summer before their junior year and was a physics major at the private four-year university. A preliminary analysis of Gabi's reasoning has been published previously [35].

After being presented with the four circuits problem, Gabi responded:

32 Gabi: Okay, so. (4 s pause) The, I think that there's like, in *RC* circuits, like, the time constant of how (5 s pause), but, like, the time constant of how long the capacitors take to discharge is—
33 Int.: Mhm.
34 G: *RC*?
35 I: Mhm.
36 G: It's either *RC* or 1 over *RC*, and let's see intuitively which one it should be. So, as the resistance increases it would take longer, wait, wait, time constant is the amount of time it takes to go to 1 over *e* of its original charge.
37 I: Okay.
38 G: Okay. So if it was a bigger resistance, it would make sense that it would take longer, I think. And then if it was more capacitance, pshh, I have a hard time getting an intuitive grasp on like what capacitance is like, like, so, capacitance is, is charge over, per voltage (writes "Q/V"), I believe.
39 I: Mhm.
40 G: So that's like amount of charge that the capacitor can hold, at the, or (4 s pause). So like a capacitor (places hands parallel to each other, separated by small distance) with larger capacitance (3 s pause), at the same, like, like, charge, oh here we go (points to "All capacitors have individually been charged by connecting them to a 9 V battery" text), charged by the same amount, has more charge on it.
41 I: Okay.
42 G: So it makes sense if it has more charge on it, it would take longer, maybe sh-, maybe longer to, uh, to uh, discharge. Sure, we'll say, okay, we'll say the time constant is *RC*.

One could mathematically derive the expression of the time constant by writing Kirchhoff's voltage law for an *RC* circuit and solving a differential equation. However, Gabi used an alternative approach: determining that the time constant is *RC*, and not $1/RC$, by talking through expected qualitative dependencies of how resistance and capacitance relate to the discharge time. They stated that a larger resistance would increase the discharge time. Although this statement was not explicitly justified with reasoning about a physical circuit model or relevant equations, Gabi noted that "it would make sense" (turn 38). While Gabi's stated qualitative dependence ("*more resistance → greater discharge time*") is sufficient to determine that *RC* is a reasonable expression for the time constant and $1/RC$ is not, Gabi then moved on to consider the dependence on capacitance.

For capacitance, they chained together qualitative dependencies to determine how *C* was related to discharge time. First, in turn 38, Gabi established that capacitance is charge per voltage (invoking the equation $C = Q/V$). Using this, they established in turn 40 that a larger capacitance means more charge on the capacitor plates when charged to the same voltage (9 V, given in the problem text). Finally, Gabi concluded that more charge means a greater discharge time (turn 42) and that therefore the time constant is *RC*. Again, Gabi stated that this conclusion "makes sense" (turn 42) without an explicit explanation of this dependence. To conclude that the time constant is *C* but not $1/C$, they

completed a chain of proportional dependencies: "*more capacitance → more charge → greater discharge time*."

Here, we make two points about Gabi's reasoning. First, the strategy of working out the form of the time constant RC with qualitative dependencies was supported by partially recalling its form and only considering possibilities that had distinct qualitative dependencies. If Gabi had also considered possibilities like $(RC)^2$, then qualitative dependencies of "*more resistance → greater discharge time*" would be insufficient to decide on the precise functional form. Second, although Gabi did not provide explicit explanations for qualitative dependencies that "made sense" to them, it is clear that they used these qualitative dependencies to determine what functional forms for the time constant are reasonable. If we were grading Gabi's reasoning for completeness, we might deduct points for the absence of these explanations. However, that Gabi glossed over these points to make quick progress suggests that this reasoning was not solely performative for the interviewer, where one may be more explicit in the explanations.

Gabi's reasoning that "*more capacitance → more charge → greater discharge time*" is an example of chaining. Specifically, to determine whether increased capacitance means a shorter or longer discharge time, Gabi used an intermediate inferential step relating to charge. This chaining enabled Gabi to identify the qualitative relationship that was then used in determining which mathematical expression was more reasonable. While it is not surprising that chaining is useful for answering a physics question [12,32–34], we present Gabi's reasoning to provide an example of how one can use chaining to reconstruct a partially forgotten equation.

In another example, Tom, a graduate student, similarly stated qualitative dependencies (without explanation) to decide whether the time constant should be RC or $1/RC$:

22 Tom: Um, and then, I'm trying to remember what the formula is uh—
23 Int.: For?
24 T: For, like, the time constant of an RC circuit. Um, it's either RC or 1 over RC. It's one of those two. Let's see if we can figure it out. If resistance, if the capac—, if we have a really, really big resistor and a really, really big capacitor, it should take a long time for it to do. And the same and the reverse is true, so it should be proportional to RC.

Like Gabi, Tom used qualitative reasoning to similarly decide that the time constant for RC circuits should be RC and not $1/RC$. They indirectly invoked the qualitative dependencies "*more resistance → greater discharge time*" and "*more capacitance → greater discharge time*." While Tom did not explain why these dependencies should be true, these qualitative dependencies, rather than explicit calculation, were used to determine which partially remembered functional form was correct.

2. Example. Chaining qualitative dependences to determine how capacitance depends on capacitor plate separation distance

To understand the effect of capacitor plate separation, Gabi (case no. 2) again used qualitative dependencies to decide between two possible forms of a functional relationship. After Gabi concluded that the time constant is RC, they moved on to evaluate the relative resistance and capacitance of each of the four circuits. Given that the capacitor in circuit D has a different plate separation distance than the others, they considered how this would affect the capacitance. Perhaps partially recalling the equation $C = \epsilon_0(A/d)$, Gabi worked to decide whether $C \sim A/d$ or $C \sim d/A$:

46 Gabi: Um, and the time constant for this one [circuit D], okay, so capacitance, uh also, is like area. It goes as either, like, area over distance or distance over area. If it has a larger area, it can hold more charge while being at the same voltage.
47 Int.: Okay.
48 G: And if it has a smaller distance, it can also hold more charge at the same area. Okay, we have some fixed, wait, what am I, okay, the voltage is fixed.
49 I: Okay.
50 G: The voltage is fixed, so the electric field times the length, so, okay, so if the voltage is fixed (holding hands parallel to each other), you shorten the length (decreases distance between hands), it means you have to have a bigger electric field.
51 I: Mhm.
52 G: Bigger electric field means more charge.
53 I: Mhm.
54 G: So a capacitor with more, with a narrower distance has a larger capacitance.
55 I: Okay.
56 G: So this [circuit D] has, uh, does it say, wait, does it say how much it's—
57 I: a half, yeah.
58 G: It's a half distance, so uh, it's, uh, so it [circuit D] has twice as much capacitance, so but, still one unit of resistance, so I believe it is T equals 2.

After stating the two possible dependencies for how capacitance depends on plate area and separation, Gabi immediately stated a qualitative dependence of "*larger area → more charge (while being at the same voltage).*" When connected with the previously stated idea that capacitance is the amount of charge held when charged

to a certain voltage (turn 40), their statement can be interpreted to mean "*larger plate area → more charge → greater capacitance*" (turn 46). As in the previous case, Gabi did not provide an explanation for this stated dependence and continued by considering the other variable, which in this case is distance.

In turn 48, Gabi started by stating a qualitative dependence, "*less plate distance → more charge*" at the same area (turn 48) but elaborated to consider voltage more explicitly, chaining together multiple qualitative dependencies. In considering the voltage, Gabi invoked "electric field times the length," implying the equation $V = Ed$, and then made a qualitative argument consistent with this equation: for a fixed voltage, "*less distance → bigger electric field*" (turn 50). They then stated the dependence "*bigger electric field → more charge*" (turn 52) and concluded that "*less plate distance → larger capacitance"* (turn 54), which is consistent with a chain of stated qualitative dependencies: for constant V, "*less plate distance → bigger electric field → more charge → larger capacitance*." The episode concluded with Gabi using the dependence $C \sim 1/d$ to determine that circuit D has twice as much capacitance as the other circuits (turn 58).

This episode provides another example of how equations can be invoked for qualitative reasoning rather than derivations through mathematical manipulations. As with $C = Q/V$ in the previous episode, the equation $V = Ed$ was utilized to determine a qualitative dependence that can be incorporated into a chain of three qualitative dependencies to conclude that $C \sim 1/d$. By contrast, these equations could be combined to eliminate V and find that $C = \frac{Q}{Ed}$, which is one step away from determining that $C = \epsilon_0(A/d)$ and that $C \sim 1/d$. The classification of Gabi's reasoning as mathematical sensemaking is because they reasoned about the intermediate qualitative dependencies embedded within equations rather than doing a mathematical derivation with those equations.

Not all interviewees who attempted to chain qualitative dependencies were successful. For example, Rene, a graduate student, also attempted to remember how the plate separation distance affects capacitance. When the interviewer prompted Rene to recall related ideas that they discussed in previous parts of the interview, Rene stated that they remembered capacitance being (directly) proportional to distance. When asked why they think it should be (directly) proportional, Rene attempted to construct a similar chain of qualitative dependencies as Gabi did:

34 Rene: Now, what does distance do? I don't remember.
35 Int.: What do you think? If you think about the, like, again, electrons and atoms and electric fields, and you talked before about the geometry of the plates.
36 R: Yeah, yeah, yeah. I definitely remember this, but, I remember it being proportional to d.
37 I: Because, why do you think it should be proportional?
38 R: Um (10 s pause).
39 I: Like the charges are sitting on the plates, right?
40 R: Yeah. (6 s pause) I'm trying to think of an intuitive reason for why it shoul- capacitance should be proportional to d. I guess one way we can try to [inaudible] is, Q is equal to CV (writes $Q{=}CV$). C is equal to Q over V (writes $C{=}Q/V$). [inaudible] proportional to d (writes $\sim d$). Can I make any statement that the charge should be proportional to distance? Well, that wouldn't make much sense. I mean, like, just the distance between them, how should that impact the amount of charge on it? Voltage, should that be inversely proportional to d? Maybe?

Even though this reasoning ultimately did not provide Rene with more certainty in a particular qualitative relationship, this case provides an additional example of how students chain qualitative dependencies to justify proposed functional forms. As with Gabi, Rene related qualitative dependencies as an "intuitive" (turn 40) way of approaching the task. The equation $C = Q/V$ was invoked again for qualitative reasoning. Here, Rene considered whether either proportionality, $Q \sim d$ or $V \sim 1/d$, made sense (turn 40) to justify $C \sim d$. However, unlike Gabi, they do not produce an explanation for either qualitative dependency. Gabi's use of $V = Ed$ allowed them to make intermediate steps toward determining the relation between capacitance and distance. Without such support for these qualitative relationships, Rene was unable to conclude that these relationships "make sense."

3. Example. Seeking coherence between physical and mathematical lines of reasoning

Blake, an undergraduate student at the private four-year university, connected multiple lines of reasoning to answer the four circuits problem. Blake's reasoning demonstrates how students can incorporate physical explanations to justify dependences, and that finding a physical reason to believe in a mathematical explanation can help students feel more certain in an answer. Here, we overview Blake's reasoning about how the capacitor plate distance affects capacitance. An extended analysis of Blake's reasoning has been published previously [36].

In considering how the capacitor plate distance affects capacitance, Blake noted that they forgot a relevant equation and then switched to reasoning about the physical motion of electrons using a line of reasoning we call "electrons jumping." Blake said that, "thinking logically,"

having the plates closer together makes it more likely for electrons to jump across and therefore makes capacitance smaller. Although this explanation is incorrect (i.e., it is the physical mechanism of capacitor breakdown, not of ideal capacitor behavior), it led to the dependence "*less distance → smaller capacitance*." Blake does not further explain the reasoning for this dependence, but it is consistent with the reasoning that less charge means smaller capacitance.

While Blake expressed doubt in this explanation, they proceeded with the question using this qualitative dependency. This doubt reappeared when Blake determined that, in using this dependence, there is no qualitative way to decide whether circuit B or D will have more charge remaining. While circuit B has a smaller equivalent resistance and therefore faster discharge rate, circuit D has less capacitance and less initial charge. In their words, "Well, it's like these two opposing variables … which leads me to believe, if this conclusion [circuit D having less capacitance] is correct at the beginning, then I wouldn't be able to distinguish … which one has more charge."

Subsequently, they considered a new approach, using equations to reconsider the qualitative relationship between distance and capacitance. In this second line of reasoning, "capacitor equations," Blake recalled two equations, $C = Q/V$ and $V = Ed$, and combined them to derive $C = \frac{Q}{Ed}$. Whereas Gabi chained the intermediate qualitative dependencies from each of the two equations, Blake's mathematical manipulation required the identification of just one qualitative dependence from the final equation: "*less distance → greater capacitance*."

At this point, Blake's mathematical approach with Capacitor Equations produced a conclusion that would make the four *RC* circuits question solvable, though it conflicted with the physical reasoning of Electrons Jumping. Following up on conclusion from Capacitor Equations, the interviewer asked, "Why would that be?" While Blake could have responded with an explanation of their equation-based reasoning with Capacitor Equations, they instead chose to answer why it would physically make sense that "*less distance → greater capacitance*":

35 Int.: Why would that be? (10 s pause) Just trying to, make sense of your reasoning.

36 Blake: Right, right, right, exactly. Um, and now I am trying to piece, like, because I do remember something about the ability for electrons to jump over, I mean that was one of the reasons, I mean, also piecing in dielectrics, the reason for improving a dielectric, adding a dielectric increases the capacitance. And what does a dielectric do? It decreases the, uh, the voltage difference between there. Um, and so having the plates closer together … wait, what? Having the plates closer together … increases the voltage difference. No wait, we just said the voltage was, um … electric field times distance … (4 s pause) Sorry, yeah yeah, having the plates closer together decreases the potential difference, if we're assuming constant electric field …

There are multiple indicators that Blake is seeking coherence between physical reasoning and their mathematical reasoning in Capacitor Equations to answer why their reasoning makes sense. Blake started by immediately reconsidering their physical reasoning in Electrons Jumping, perhaps in an attempt to "piece" together how physical lines of reasoning do or do not align with the Capacitor Equations conclusion. Blake then introduced an alternative physical line of reasoning, "dielectric analogy." The situation of adding a dielectric in between capacitor plates linked increasing the capacitance to decreasing the voltage. This provided a physical basis for supporting the qualitative dependence between capacitance and voltage, which was implicit in $C = Q/V$ when it was used in Capacitor Equations. Blake also explicitly considered the qualitative dependence "*less distance → smaller potential difference*" embedded within $V = Ed$ ("… the voltage was, um … electric field times distance …"). While this equation was present in Capacitor Equations, it was solely used in a mathematical derivation. Here, instead of a mathematical derivation combining two equations, Blake explicitly considered the qualitative dependences embedded within those two equations that, when chained together, would support the same conclusion of Capacitor Equations: "*less distance → less voltage→ more capacitance*."

At the end of this episode, Blake evaluated their three lines of reasoning. Blake concluded that their "first reasoning," in Electrons Jumping, was wrong and that "having the plates closer together decreases the voltage difference, which increases the capacitance," a conclusion which aligns with their derived equation in Capacitor Equations. However, it is not simply that Blake rejected their physical reasoning for a mathematical line of reasoning. Rather, Blake attended explicitly to an inconsistency between their physical and mathematical reasoning. To do so, they generated a new line of physical reasoning, Dielectric Analogy, that was consistent with their mathematical reasoning before concluding that it was correct. In this way, the process of seeking coherence between physical and mathematical reasoning was part of how Blake decided which conclusions were trustworthy.

C. Reasoning qualitatively about physical entities and processes

In Blake's episode, qualitative reasoning from a physical mechanism initiated their reasoning. We identified two additional participants, Sam and Riley, who switched to purely qualitative reasoning about the physical entities and

comparative processes in the circuits. These cases showed alternative ways students reasoned, after forgetting an equation, that did not rely on mathematical sensemaking.

Sam, a community college student, explicitly noted that "I forgot the formula again, but it's cool." Rather than trying to reconstruct the equation, they used purely qualitative reasoning tied to physical analogies. A preliminary analysis of Sam's reasoning has been published previously [35]. Sam started with a physical analogy of "electrons as people" to explain the potential physical behavior of electrons. This physical analogy, which is different than the ones Blake used, has been documented before in the analogical reasoning literature [40]. They stated, "I treat electrons kind of like people." Sam continued, "they kinda just make choices, you know? They kind of just pick their best-fit scenario." Sam utilized this analogy of electrons as people to reason about the physical process of current flow. They concluded that circuit B's parallel circuit would discharge more quickly than circuit A's single path because the current "has two different ways to go."

Sam proceeded with the problem by considering the other major difference between the four circuits—capacitor plate separation. In a previous part of the interview, they invoked the analogy that capacitance is the "ability to be like a battery impersonator" and that when a capacitor acts like a battery, it has to have an electric field between its plates. Considering the extreme case of when the distance between plates is very large, Sam reasoned that the electric field "won't even work," so increasing the plate distance would therefore decrease the capacitance. Here, on the four circuits problem, Sam concluded that circuit D will end with the greatest charge, because "the smaller distance increases capacitance" and D started with the greatest amount of charge. Although Sam does not explicitly make the connection between greater capacitance and more initial charge, this connection is implicit in their reasoning. Comparing circuit D to circuit A, Sam says D would definitely end with more charge, because those two capacitors started with a different amount of charge, are closed for the same amount of time, and identical in every other way.

Ultimately, Sam concluded that circuit B ends with the least amount of charge and circuit D ends with the greatest amount of charge of the four circuits. They did so using qualitative reasoning rooted in physical analogies. Although Sam made significant progress in reasoning qualitatively, there were a few areas in which their solution was incomplete. They did not explicitly compare the slower discharge rate of circuit C against the greater initial charge of circuit D, nor did they explicitly compare circuit C to circuits A and B.

In another example, Riley, an undergraduate student at the four-year university, similarly reasoned from the perspective of the electrons, articulating mechanisms involved in electron flow in the circuit to determine the relative discharge times between circuits. At the start of considering the four circuits problem, they explicitly noted that they could not recall the equation that relates capacitance to plate separation distance: "So there's definitely an equation I know that somewhere includes the distance. Do I remember it? No." Riley proceeded with invoking the same Electrons Jumping mechanism that Blake used: "I would think that if [the capacitor plates] are closer, it is easier for the charges to move from one side to the other … Um [4 s pause], yeah, because of what's happening is that the charges are jumping from one side to the other (motions hands from right to left)." Riley added, "So I think if they're closer together, it's easier to jump if I'm an electron (laughs)." From here, Riley connected electrons jumping across plates to the discharge of the circuit (a physically incorrect mechanism). Using this (incorrect) dependence of "*smaller plate distance → more charges jumping → greater discharge rate*," Riley concluded that circuit D would have the least charge remaining. In addition, Riley reasoned that more resistance means less current and less charges jumping, and that therefore circuit C, having the largest resistance, would have the most charge remaining.

Even though Riley drew on an incorrect physical mechanism to infer an incorrect qualitative dependence on how capacitor plate distance will affect discharge rate, their reasoning demonstrated an attempt to employ one's resources to support physical, qualitative reasoning. Without the equation, Riley shifted to connect a physical model of electrons jumping to chain together qualitative relationships regarding the amount of remaining charge, plate separation, resistance, and "charges jumping."

V. DISCUSSION

A. Coherence seeking: An adaptive strategy when forgetting an equation

The results provide another empirical illustration of how students seek coherence between mathematical expressions and other ideas, in this case applied to reconstruct forgotten equations. Though these skills are often categorized under the label of "mathematical sensemaking," they also speak to the practices of coherence seeking more broadly. Gabi and Tom considered which mathematical expressions cohered with the qualitative dependences between quantities (either directly stated as "making sense" or justified through chaining) in deciding which expression to trust. Though they were unsuccessful, Rene also used chaining to break down a proposed qualitative dependence in an attempt to justify the chained subdependences. This coherence between qualitative dependences and mathematical expressions has been previously studied in the context of answer checks [9–11] and creating equations from one's conceptual understanding [2]. The cases in our study illustrate another venue for this coherence: adapting to a forgotten equation.

Another type of coherence seeking is illustrated by Blake resolving incoherence between a physical line of reasoning (in Electrons Jumping) and a mathematical line of reasoning (in Capacitor Equations) by finding another physical idea (in Dielectric Analogy) that coheres with the mathematical reasoning. Although Dielectric Analogy does not directly relate to the physical system at hand, it provides a physical example that produces the same qualitative dependence as Capacitor Equations, making this mathematical conclusion perhaps seem more physically reasonable. This resolution between lines of reasoning resonates with previously illustrated cases of how students resolve conflicts between equation-based and conceptual approaches by finding conceptual resolutions that cohere with mathematical expressions [4].

These coherence-seeking strategies speak to routine and adaptive problem-solving expertise in physics education. While much of physics education focuses on developing students' fluency with the routine problem-solving procedures of the discipline, physicists adapt those routine skills in novel ways to extend our scientific understanding of the physical world [15]. Though the cases shown here may seem far from the innovative reasoning in physics research, they illustrate how students can innovate and adapt to a roadblock that can arise in executing even standard methods—forgetting a relevant equation. These cases indicate that students' problem-solving skills are not limited to brittle procedures that fail at the first sign of trouble. Here, coherence-seeking strategies like comparing equations or physical models with qualitative dependences are implemented as redundancy systems that can help students when their approach gets off track. Additionally, these strategies are consistent with the type of reasoning we would expect expert physicists to employ. We propose that future research should investigate possibilities for developing students' adaptive expertise by teaching mathematical sensemaking and coherence-seeking skills.

B. Reflections on retrieval practice

One relevant area of research that addresses student forgetting is retrieval practice—the deliberate practice of attempting to recall information from memory. Retrieval practice has been shown to improve learning and performance across multiple contexts [41,42]. Retrieval practice is also a growing area of interest in PER, with recent studies showing that retrieval practice can benefit factual recall and application of solution methods for physics content [43–45].

Our results illustrate how reconstruction of partially remembered equations, namely through chaining of qualitative dependences, is an alternative to direct retrieval of physics equations from memory. What are the implications of these results for using retrieval practice in physics instruction? If these students had stronger recall of the RC time constant, it would have certainly been more efficient, as students would not have needed to reconstruct $\tau = RC$ or $C \sim A/d$ to answer the four circuits problem. However, prescribing retrieval practice of the RC time constant expression alone may not foster the use of the desirable problem-solving skills, like the coherence seeking and mathematical sensemaking, illustrated in our findings.

That is not to say that retrieval practice could not be of service to the reconstructive processes presented here. For one, some students, like Gabi and Blake, did have good recall of other equations (e.g., $Q = CV$ and $V = Ed$), and they used these recalled equations to ground their justifications for qualitative dependences. While qualitative dependences need not be grounded solely from known equations, being able to recall equations from memory can provide a reliable starting point for students' mathematical sensemaking. It may be that retrieval practice involving more fundamental equations balances providing a reliable starting point while not overemphasizing procedural memory in physics learning. Furthermore, retrieval practice that emphasizes coherence between an equation and other ideas—including connections with other equations, physical models, or specific examples—may help students recognize coherence seeking as an approach that can be called upon in explanation, argumentation, and problem solving.

C. Connections to student epistemologies

The reason to investigate the ability to reconstruct forgotten equations is not that it is a key learning outcome. Students often have access to equation sheets on exams and do not need to memorize the exact equations, because physics education values knowing how to use the equations rather than being able to recall their exact form. However, the opportunistic focus on the act of forgetting and reconstructing an equation is a useful site for studying broader physics learning goals.

For example, this work connects to research on physics students' epistemologies. On the Maryland Physics Expectations Survey 2 (MPEX2) [46]—a survey measuring students' epistemological views in physics—one question asks students to rate the extent to which they agree or disagree with the following statement: "If I don't remember a particular equation needed for a problem in an exam, I can probably figure out an (ethical!) way to come up with it, given enough time." The "favorable" response, reflecting experts' own beliefs and their instructional goals for students, is to agree—to believe that physics knowledge can be reconstructed from one's knowledge rather than simply memorized (or forgotten). Some students in this study enacted these favorable epistemological views by working to reconstruct a (partially) forgotten equation. Furthermore, they demonstrated the coherence-seeking practices for connecting physical concepts, qualitative

dependences, and mathematical expressions that can be used to make progress when forgetting an equation. While there has been continued work investigating the development of physics students' epistemologies, expectations, and attitudes [47], there has been less research on the knowledge and strategies aligned with these beliefs. Students may only believe that they can reconstruct forgotten equations if they have strategies for doing so and experiences of doing it successfully.

D. The need for assessments targeting the integration of mathematical sensemaking in practice

The example of reconstructing equations also speaks to the challenge of detecting students' "integrated sensemaking practices" rather than discrete mathematical sensemaking skills. Like other interview studies, we identified the ways that students use mathematical sensemaking in the course of talking through their reasoning and problem-solving approaches. However, in written assessments in a course context, the unwritten mathematical sensemaking, that students may use in the course of their problem solving, goes undetected. Therefore, when aiming to assess students' mathematical sensemaking skills on written assessments, a common approach is to directly ask students to demonstrate those skills, either with general prompts like "check whether this expression makes sense or not" or specific prompts eliciting specific strategies (e.g., dimensional analysis, checking qualitative dependences). However, this does not capture whether students would spontaneously call upon these skills in the course of their problem solving. For this reason, it is not widely known whether current physics instruction fosters the spontaneous and opportunistic use of such mathematical sensemaking strategies. While one study has shown that targeted free-response exam questions can detect students' spontaneous use of mathematical sensemaking in service of coherent and efficient solutions [13], developing additional, similar assessments will be needed to facilitate mainstream adoption of mathematical sensemaking and coherence seeking as a physics learning goal.

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The data that support the findings of this article are not publicly available because they contain sensitive personal information. The data are available from the authors upon reasonable request.

[1] D. Hestenes, Toward a modeling theory of physics instruction, Am. J. Phys. **55**, 440 (1987).
[2] B. L. Sherin, How students understand physics equations, Cognit. Instr. **19**, 479 (2001).
[3] I. M. Greca and M. A. Moreira, Mental, physical, and mathematical models in the teaching and learning of physics, Sci. Educ. **86**, 106 (2002).
[4] B. Sherin, Common sense clarified: The role of intuitive knowledge in physics problem solving, J. Res. Sci. Teach. **43**, 535 (2006).
[5] E. Brewe, Modeling theory applied: Modeling instruction in introductory physics, Am. J. Phys. **76**, 1155 (2008).
[6] O. Uhden, R. Karam, M. Pietrocola, and G. Pospiech, Modelling mathematical reasoning in physics education, Sci. Educ. **21**, 485 (2012).
[7] E. Kuo, M. M. Hull, A. Gupta, and A. Elby, How students blend conceptual and formal mathematical reasoning in solving physics problems, Sci. Educ. **97**, 32 (2013).
[8] T. O. B. Odden and R. S. Russ, Sensemaking epistemic game: A model of student sensemaking processes in introductory physics, Phys. Rev. Phys. Educ. Res. **14**, 020122 (2018).
[9] K. T. Hahn, P. J. Emigh, M. Lenz, and E. Gire, Student sense-making on homework in a sophomore mechanics course, *presented at PER Conf. 2018, Cincinnati, OH*, pp. 160–163, 10.1119/perc.2017.pr.035.
[10] T. Sikorski, G. D. White, and J. Landay, Uptake of solution checks by undergraduate physics students, *presented at PER Conf. 2018, Cincinnati, OH*, pp. 368–371, 10.1119/perc.2017.pr.087.
[11] M. Eichenlaub and E. F. Redish, Blending physical knowledge with mathematical form in physics problem solving, in *Mathematics in Physics Education*, 1st ed. (Springer International Publishing, Switzerland, 2019), Vol. 7, pp. 127–151.
[12] J. D. Gifford and N. D. Finkelstein, Categorical framework for mathematical sense making in physics, Phys. Rev. Phys. Educ. Res. **16**, 020121 (2020).
[13] E. Kuo, M. M. Hull, A. Elby, and A. Gupta, Assessing mathematical sensemaking in physics through calculation-concept crossover, Phys. Rev. Phys. Educ. Res. **16**, 020109 (2020).
[14] F. F. Zhao and A. Schuchardt, Development of the sci-math sensemaking framework: Categorizing sensemaking of

mathematical equations in science, Int. J. STEM Educ. **8**, 10 (2010).

[15] E. Kuo Cognitive, *Epistemic, and Affective Theoretical Underpinnings of Research on Physics Learning Two Perspectives on Physics Problem Solving and Their Relation to Adaptive Expertise* (AIP Publishing, Melville, NY, 2023), pp. 1–26.

[16] M. Lenz, P. J. Emigh, and E. Gire, Surprise! Students don't do special-case analysis when unaware of it, *present at PER Conf. 2019, Washington, DC*, pp. 231–234, 10.1119/perc.2018.pr.Lenz.

[17] P. Thagard, Coherence, truth, and the development of scientific knowledge, Philos. Sci. **74**, 28 (2007).

[18] M. T. H. Chi, P. J. Feltovich, and R. Glaser, Categorization and representation of physics problems by experts and novices, Cognit. Sci. **5**, 121 (1981).

[19] F. Reif and J. I. Heller, Knowledge structure and problem solving in physics, Educ. Psychol. **17**, 102 (1982).

[20] A. Van Heuvelen, Overview, case study physics, Am. J. Phys. **59**, 898 (1991). 10

[21] P. Heller, R. Keith, and S. Anderson, Teaching problem solving through cooperative grouping. Part 1: Group versus individual problem solving, Am. J. Phys. **60**, 627 (1992).

[22] D. Huffman, Effect of explicit problem solving instruction on high school students' problem-solving performance and conceptual understanding of physics, J. Res. Sci. Teach. **34**, 551 (1997).

[23] E. Gaigher, J. M. Rogan, and M. W. H. Braun, Exploring the development of conceptual understanding through structured problem–solving in physics, Int. J. Sci. Educ. **29**, 1089 (2007).

[24] J. L. Docktor, N. E. Strand, J. P. Mestre, and B. H. Ross, Conceptual problem solving in high school physics, Phys. Rev. ST Phys. Educ. Res. **11**, 020106 (2015).

[25] R. J. Dufresne, W. J. Gerace, P. T. Hardiman, and J. P. Mestre, Constraining novices to perform expertlike problem analyses: Effects on schema acquisition, J. Learn. Sci. **2**, 307 (1992).

[26] J. H. Larkin and F. Reif, Understanding and teaching problem– solving in physics, Eur. J. Sci. Educ. **1**, 191 (1979).

[27] G. Hatano and K. Inagaki, Two courses of expertise, in *Child Development and Education in Japan* (W H Freeman & Co, New York, 1986), Vol. 17, pp. 262–272.

[28] Q. X. Ryan and B. P. Schermerhorn, Students' use of symbolic forms when constructing equations of boundary conditions, Phys. Rev. Phys. Educ. Res. **16**, 010122 (2020).

[29] W. D. Riihiluoma, Z. Topdemir, and J. R. Thompson, Applying a symbolic forms lens to probability expressions in upper-division quantum mechanics, *presented at PER Conf. 2022, Grand Rapids, MI*, pp. 383–388, 10.1119/perc.2022.pr.Riihiluoma.

[30] A. Kapodistrias and J. Airey, Rearranging equations to develop physics reasoning, Eur. J. Phys. **45**, 035701 (2024).

[31] E. Redish, Using math in physics: 5. Functional dependence, Phys. Teach. **60**, 18 (2022).

[32] R. S. Russ and P. Hutchison, It's okay to be wrong: Recognizing mechanistic reasoning during student inquiry, in *Proceedings of the International Conference of the Learning Sciences* (Indiana University, Bloomington, IN, 2006).

[33] B. A. Lindsey, M. R. Stetzer, J. C. Speirs, W. N. Ferm, and A. van Hulten, Investigating student ability to follow reasoning chains: The role of conceptual understanding, Phys. Rev. Phys. Educ. Res. **19**, 010128 (2023).

[34] J. C. Speirs, M. R. Stetzer, and B. A. Lindsey, Utilizing network analysis to explore student qualitative inferential reasoning chains, Phys. Rev. Phys. Educ. Res. **20**, 010147 (2024).

[35] K. Gifford, G. S. Ehrlich, E. Bumbacher, and E. Kuo, Seeking coherence and switching reasoning after forgetting an equation, *presented at PER Conf. 2021*, 10.1119/perc.2021.pr.Gifford.

[36] G. S. Ehrlich, K. Gifford, E. Kuo, and E. Bumbacher, Seeking physical/mathematical coherence by recruiting and reconciling reasoning: A case study in E & M, *presented at PER Conf. 2021*, 10.1119/perc.2021.pr.Ehrlich.

[37] A. A. Disessa, An interactional analysis of clinical interviewing, Cognit. Instr. **25**, 523 (2007).

[38] R. S. Russ, V. R. Lee, and B. L. Sherin, Framing in cognitive clinical interviews about intuitive science knowledge: Dynamic student understandings of the discourse interaction, Sci. Educ. **96**, 573 (2012).

[39] A. Gupta, A. Elby, and V. Sawtelle, Bridging knowledge analysis and interaction analysis through understanding the dynamics of knowledge in use, in *Knowledge and Interaction* (Routledge, New York, 2015), pp. 276–307.

[40] D. Gentner and D. Gentner, *Flowing Waters or Teeming Crowds: Mental Models of Electricity* (Erlbaum, Hillsdale, NJ, 1983), pp. 99–129.

[41] S. K. Carpenter, S. C. Pan, and A. C. Butler, The science of effective learning with spacing and retrieval practice, Nat. Rev. Psychol. **1**, 496 (2022).

[42] P. K. Agarwal, L. D. Nunes, and J. R. Blunt, Retrieval practice consistently benefits student learning: A systematic review of applied research in schools and classrooms, Educ. Psychol. Rev. **33**, 1409 (2021).

[43] V. Gjerde, B. Holst, and S. D. Kolstø, Retrieval practice of a hierarchical principle structure in university introductory physics: Making stronger students, Phys. Rev. Phys. Educ. Res. **16**, 013103 (2020).

[44] V. Gjerde, V. Havre Paulsen, B. Holst, and S. D. Kolstø, Problem solving in basic physics: Effective self-explanations based on four elements with support from retrieval practice, Phys. Rev. Phys. Educ. Res. **18**, 010136 (2022).

[45] T. Zu, J. Munsell, and N. S. Rebello, Comparing retrieval-based practice and peer instruction in physics learning, Phys. Rev. Phys. Educ. Res. **15**, 010105 (2019).

[46] T. L. McCaskey, Comparing and contrasting different methods For probing student epistemology and epistemological development in introductory physics, Ph.D. thesis, University of Maryland, 2009.

[47] A. Madsen, S. B. McKagan, and E. C. Sayre, How physics instruction impacts students' beliefs about learning physics: A meta-analysis of 24 studies, Phys. Rev. ST Phys. Educ. Res. **11**, 010115 (2015).