When skill meets struggle: Mapping students' physmatic difficulties to physmatic skills in physics lessons

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The interplay of mathematics within physics—termed here as *physmatics*—presents unique cognitive challenges for students, requiring coordination between mathematical reasoning and physical sensemaking. While previous studies have documented common physmatic difficulties, capturing these challenges as they emerge in classroom instruction is rare. This study analyzed 273 teacher–student interactions from 23 video-recorded physics lessons taught across a diverse range of classrooms, aiming to identify and categorize physmatic difficulties as they naturally arose during instruction. Twenty-four codes of difficulties were identified through content analysis and grouped into five overarching clusters: (i) *questions about model components*, (ii) *questions about the model*, (iii) *difficulties applying mathematics in a physical context*, (iv) *interrepresentational fluency*, and (v) *unit-related difficulties*. A key finding of this study was the identification of systematic associations between the modeling skills teachers aimed to develop (i.e., mathematization, manipulation, interpretation, and validation) and the corresponding student difficulties. These findings underscore the centrality of modeling as a pedagogical framework for physics instruction and suggest that teachers who are aware of the links between skills and difficulties may be better prepared to anticipate student struggles, respond to them in real time, and even leverage them as part of their instructional practice.

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I. INTRODUCTION

The interplay between physics and mathematics—referred to in this paper as *physmatics* [1] requires the integration of conceptual and procedural knowledge from both disciplines. Students' difficulties in this respect (termed here as "physmatic difficulties") are often described as their struggle to connect mathematical reasoning with physical meaning [2]. Several theoretical models have been suggested to conceptualize these difficulties, including models that characterize modes of reasoning that blend mathematical and physical thinking, and the processes linking them [3] and frameworks that reinterpret and organize students' difficulties by mapping them onto algorithmic and conceptual cognitive frames [4]. These theoretical perspectives find support in empirical studies that demonstrate how such misalignments manifest in practice. For example, differences in the use of parameters and variables in the two disciplines can lead to confusion when interpreting a linear equation in the context of a physical phenomenon such as spring compression [5].

Recent research has provided valuable insights into physmatic difficulties among high school students—for instance, difficulties in moving between graphical, verbal, and mathematical representations [6–8], errors related to units or basic arithmetic [6,9], and formal use of formulas while ignoring their physical meaning [10]. These studies often employed structured instruments such as tests, interviews, or narrowly focused questionnaires but rarely, if at all, were designed to fully capture how these difficulties emerge and unfold during frontal classroom instruction.

Several studies have further indicated that students' physmatic difficulties often stem not from deficiencies in technical mathematical skills, but from a lack of understanding of the structural role mathematics plays within physics [11].

There is a challenge to identify the origin of such difficulties, whether they stem from a weak mathematical understanding, a conceptual misunderstanding in physics, or from a failure to comprehend how the two disciplines are interwoven in physics [12]. Some studies have shown that

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the physical context itself may complicate otherwise straightforward mathematical reasoning [13].

A study based on classroom observations revealed three major types of physmatic difficulties: arithmetic operations with physical context, overgeneralization of the zero-product property, and region of applicability of a formula [14,15].

In the current study, we investigate how physmatic difficulties emerge during classroom instruction and to what extent they relate to the physmatic skills that the teacher attempts to foster. In-depth analysis of videoed teacher-student discourse was employed to identify recurring patterns of difficulty and explore how they relate to the physmatic skills the teacher is trying to promote. The findings may contribute to a deeper understanding of how students experience and express physmatic difficulties during instruction, may help broaden the framework of physics teachers' physmatic pedagogical content knowledge (PCK) [16], and serve as a basis for developing tools to identify and address such difficulties more effectively.

II. THEORETICAL BACKGROUND

A. The role of mathematics within physics learning

The unique case of physics and mathematics showcases a longstanding and profound historical relationship. From the times of Archimedes, through Newton and Gauss, many pivotal figures contributed to both disciplines. Philosophical discussions about the distinctions and connections between physics and mathematics date back to Aristotle [1,17]. Building on this tradition, Hestenes [18] emphasized mathematical modeling as the foundation of physics education, and later scholars [2,19–21] expanded his framework into a set of essential skills for coordinating mathematics and physics (for a detailed account of these developments, see Levi *et al.* [15]). The process of constructing a mental model of a physical phenomenon often relies on specific skills for translating real-world situations into mathematical representations. We identified in the literature five key skills essential for physmatic modeling:

(A) Simplification is transforming a real-world phenomenon into a more idealized version to make it amenable to modeling [21,22]. This process may involve omitting certain variables (e.g., neglecting friction in free fall), reducing the dimensionality of the system (e.g., modeling motion in one dimension), or isolating a specific dependency by fixing one variable [21]. Taber [22] distinguishes between simplifications that preserve the logical structure of the physical concept and oversimplifications that can lead to misconceptions.

While simplification is an essential structural skill that initiates the modeling process [21], it is not always performed explicitly in classroom instruction. Teachers sometimes present a ready-made model without exposing the simplification decisions that underpin it [2,20], leading students to perceive scientific models as fixed truths rather than as the outcome of reasoned abstraction. Engaging students in the process of simplification, especially in open-ended experiments or explorations, can support their understanding of how physicists identify relevant features and reduce complexity in order to make sense of phenomena. In some cases, simplification occurs only after an initial empirical phase, when students first observe complex behavior and only later recognize that it can be divided into simpler components.

(B) Mathematization involves translating a simplified physical scenario into a formal mathematical language [19,21,23]. This process requires not only representing the physical system using variables, functions, and equations but also justifying the selection of mathematical structures based on the characteristics of the phenomenon. Karam *et al.* [19] emphasize that mathematization is a structural skill, grounded in abstraction, idealization, and awareness of the mathematical forms suitable for describing physical constraints. Redish and Tuminaro [23] refer to this as the *mapping* stage, in which the mathematical model that reflects key physical relationships is constructed.

Building on this view, White-Brahmia *et al.* characterize mathematization as providing the foundations for productive covariational reasoning, which is reasoning about how changes in one physical quantity are coordinated with changes in another [24,25]. Mathematization can also be approached differently in instruction: teachers may choose either a more abstract approach, where the mathematization is performed in a single conceptual leap, or a more gradual approach, involving progressive mathematization through various intermediate representations [21]. This choice influences the nature of students' learning. Uhden *et al.* [21] note that mathematization involves both structural reasoning and the use of technical mathematical operators, such as assigning variables, writing equations, or isolating terms.

(C) Manipulation refers to applying mathematical operations to the constructed model in order to derive results or make predictions [19–21]. This skill includes both symbolic procedures—such as algebraic transformations, substitutions, and solving equations—and deeper logical reasoning grounded in physical principles. Uhden *et al.* [21] distinguish between technical manipulations, which involve algorithmic application of mathematical tools, and structural manipulations, which involve deductive reasoning informed by the physics of the situation. Karam *et al.* [19] highlight that understanding how physical principles constrain or guide mathematical reasoning is crucial for meaningful manipulation.

Manipulation is often perceived by students as a purely technical step, frequently characterized by a *plug-and-chug* approach in which formulas are selected and applied mechanically without consideration of their physical meaning [21,26]. Teachers, therefore, play a key role in drawing

attention to the structural aspects of manipulation—such as the physical justification for algebraic decisions or the significance of choosing one path of derivation over another. By making structural reasoning visible, instruction can shift manipulation from a mechanical to an explanatory process, enabling students to understand how mathematical operations reflect and support physical insight.

(D) Interpretation refers to the process of attributing physical meaning to mathematical results within a given physical context [2,21]. This skill includes identifying limiting cases, generating predictions, reading equations through words or schemas, and recognizing analogies that reveal structural similarities between different physical situations [20]. Interpretation bridges mathematical formalism and physical intuition, allowing students to understand what equations tell us about the real world, beyond their symbolic manipulation.

From an instructional perspective, Interpretation plays a central role in helping students coordinate physical reasoning with mathematical representations. Rather than treating equations as purely formal objects, students can engage in *mixed processing*—blending intuitive ideas with symbolic operations [27]. Sherin's [28] notion of "symbolic forms" provides a framework for recognizing how students assign meaning to mathematical structures in physics. Interpretation, alongside mathematization, often recurs throughout lessons and serves as a key point where physical reasoning reenters the modeling process [19].

(E) Validation is assessing whether the mathematical results of the model are consistent with the observed behavior of the physical system [2,21]. This step concludes the modeling process by determining whether the assumptions, simplifications, and operations applied thus far lead to results that are meaningful in context. It provides a critical opportunity to reconsider or refine the model if inconsistencies arise. For many physicists, this is the most creative stage, as new insights often emerge when revising or abandoning models that fail to match reality [2].

From an instructional perspective, students often rely on authoritative answers or assume that mathematically correct results must also be physically valid. This tendency can be addressed by embedding opportunities for interpreting and evaluating throughout the modeling process—not only as a final check but also during earlier phases such as simplification and mathematization—thereby fostering more reflective engagement with the model and its limitations [2]. While not addressing validation directly, Lehavi *et al.* [29] showed that when students are not guided to question the empirical plausibility of mathematical results, they tend to use formulas mechanically, without evaluating whether the outcomes are physically meaningful. Inquiry-oriented instructional frameworks, such as the ISLE approach, articulate validation as a practice of examining assumptions and evaluating the coherence of model outcomes within a given context [30]. Within broader model-based instructional perspectives, validation is understood as an epistemic practice that involves comparing model predictions with empirical evidence and reconsidering the adequacy of the model and its assumptions [31–33].

These skills reflect the multifaceted nature of physmatics and are essential for meaningful integration of mathematics in physics learning and instruction. They are also aligned with Hestenes' [18] modeling framework, which outlines the main stages of constructing and using physical models. For further details, see our previous study [15].

While these skills provide a comprehensive framework for how students should ideally coordinate mathematics and physics, they also constitute potential sources of difficulty. In practice, students often encounter obstacles precisely at the points where these skills are required, leading to systematic patterns of physmatic difficulties that mirror the stages of modeling [2,21].

B. Physmatic difficulties

To frame the discussion, we first review general theoretical models that classify student difficulties at the math-physics interface and then outline specific categories of physmatic difficulties that have been identified in the literature.

1. General theoretical models of physmatic difficulties

Of the general theoretical models that address physmatic difficulties, the PCK-based model proposed by Lehavi *et al.* [16,29] offers a perspective grounded in teachers' pedagogical content knowledge. Building on the framework proposed by Magnusson *et al.* [34], as adapted by Etkina [35] to physics education, this model conceptualizes teachers' knowledge regarding the physics-mathematics interplay through five complementary dimensions: orientation toward teaching, referring to teachers' goals and beliefs regarding the teaching and learning of physmatic content; knowledge of curriculum structure, which concerns how physmatic content is organized and sequenced within instruction; knowledge of students, referring to teachers' understanding of students' prior knowledge and common difficulties in this domain; knowledge of assessment, which focuses on teachers' ability to design tasks that reveal students' physmatic understanding and difficulties; and knowledge of instructional strategies, involving the deliberate use of teaching approaches to foster physmatic thinking. A deeper interpretation of these dimensions is required to derive meaningful insights for teachers' PCK. Among other prominent models, Modir *et al.* [4] proposed a two-dimensional framework for classifying physmatic difficulties along the physics-mathematics and algorithmic-conceptual axes. Within this structure, students' reasoning can involve transition errors (difficulty shifting between quadrants), displacement errors (placing a problem in the wrong quadrant), or content errors (using inappropriate

strategies within the correct quadrant). For example [4] (p. 8), during a lesson on a particle in a box, undergraduate students were asked to identify which functions could represent an acceptable wave function. Although the task calls for examining the physical boundary conditions of the system, some students rejected functions simply because they did not match the standard sinusoidal form. Such reasoning reflects a displacement error, in which algorithmic mathematical reasoning is applied in place of conceptual physical reasoning.

A complementary perspective is offered by Gifford and Finkelstein [3], who presented a categorical framework for analyzing mathematical sensemaking in physics. Their model sheds light on the variety of reasoning patterns students employ when attempting to link mathematical expressions with physical meaning.

While such general models provide useful conceptual frameworks, empirical studies have revealed concrete difficulties observed in students' reasoning.

2. Specific physmatic difficulties reported in the literature

Previous research has identified various types of physmatic difficulties, particularly in relation to students' use of mathematics in physics contexts. Acevedo Nistal *et al.* [36] define representational fluency as students' ability both to work within a single representation and to translate between different representations. To differentiate between these two abilities, we refer to them as *intrarepresentational* and *interrepresentational fluency*. While this distinction between intrarepresentational and interrepresentational fluency is conceptually useful, empirical studies have primarily focused on students' difficulties in transitioning between representations. Accordingly, in the present study, interrepresentational difficulties are treated as a distinct category, whereas intrarepresentational difficulties are addressed within other physmatic difficulty clusters, depending on their conceptual characteristics. Subsequent empirical studies have documented students' difficulties with this latter, interrepresentational ability. For example, Planinic *et al.* [7] reported that even after instruction in kinematics, students encountered difficulties interpreting motion graphs, especially in assigning physical meaning to the slope and area under the curve. Similarly, Pratama *et al.* [37] and Prahani *et al.* [8] described difficulties in moving between representations in the contexts of impulse-momentum and heat, respectively. In another study of uniform motion and Newton's laws, Am and Istiyono [6] found that students demonstrated limited ability to shift across representations, particularly when translating physical descriptions into mathematical expressions. Bollen *et al.* [38] observed a gap between students' ability to construct graphs and their conceptual understanding of those graphs' physical meaning. In line with these findings, Niss [39] emphasized the challenges students face in identifying relevant mathematical components in physics problems and in moving flexibly between different representations. Similarly, Mumthas and Abdulla [9] reported that difficulties in applying mathematics to physics problems were perceived by teachers as the most significant challenge. In addition, Fehlinger *et al.* [40] identified two main strategies students use when interpreting graphs, one of which—focusing on the overall graph shape—was associated with more misunderstandings and lower performance, indicating a source of difficulty. These difficulties included selecting appropriate formulas, extracting information from diagrams, and constructing schematic representations grounded in physical reasoning.

Another commonly reported difficulty involves the inappropriate use of formulas without consideration of their physical meaning. Meli *et al.* [10], in a case study on mechanics, found that students often prioritized the application of formulas over conceptual understanding. Even high-achieving students tended to use mathematical expressions without regard for the conditions under which they apply. These findings underscore the critical need to reevaluate traditional teaching methods that separate mathematics from physics and support a more interdisciplinary approach to foster deeper conceptual understanding and address the documented difficulties in integrating the two domains. In addition, Badmus and Jita [41], emphasized that differences in the use of symbols between mathematics and physics can lead to confusion when students attempt to interpret and apply equations. Related difficulties in interpreting the meaning of signs, as well as difficulties with proportional relationships, have also been documented in the literature [24,25]. Such confusion may hinder students' ability to reason meaningfully with equations, particularly regarding the interpretation of symbols, parameters, and units.

Across studies, confusion surrounding the use and interpretation of dimensions (units) emerged as a common obstacle, often leading to inconsistencies or superficial reasoning. For example, Am and Istiyono [6] documented basic arithmetic errors, confusion regarding unit conversions, and instances in which students arrived at correct numerical results based on incorrect reasoning.

It is noteworthy that the empirical studies reviewed relied on structured research instruments such as tests, questionnaires, or interviews, and typically focused on narrowly defined topics within the physics curriculum. Despite these valuable contributions—both empirical and theoretical—none of these studies examined the dynamic nature of physmatic difficulties as they emerge during classroom teacher-student(s) interaction, nor did they connect the documented difficulties to the skills teachers aim to foster in their students. The present study addresses this gap by examining physmatic difficulties as they unfold in authentic classroom contexts.

III. RESEARCH QUESTIONS

1. What types of physmatic difficulties emerge in high school physics classrooms, how can they be

organized, and what is the relative frequency of each difficulty type during instruction?
2. In what way do the observed difficulties connect to the skills teachers foster in their students?

IV. METHODOLOGY

This study follows a mixed-methods approach. The qualitative part included code co-occurrence analysis (conducted using the ATLAS.ti program), which provided matrix-based tools to examine the overlap between coded segments [42], while in the quantitative analysis, we employed chi-square testing. The method allowed both in-depth interpretation of teaching-learning interactions and statistical examination of patterns across the data.

A. Data source: Lesson corpus and sampling rationale

We analyzed video excerpts from 23 physics lessons (19.25 h in total), covering a broad range of physics topics (see Table I). The lessons were taught by 14 different high school physics teachers who work across grades 8 to 12. This was to ensure that the difficulties found will not be specific topic or age oriented. The lessons analyzed in this study were regular classroom lessons taught by teachers who volunteered to participate. These teachers had not undergone any professional development specifically focused on physmatics. Lessons were included for analysis when they contained episodes in which students' difficulties involved both physical and mathematical components components of the same difficulty. The physmatic skills were identified based on the researchers' interpretation of the recorded lessons as a whole, drawing on the instructional actions observed throughout the lesson. The teachers themselves videoed the lessons, pointing the camera at the board due to the students' privacy considerations. Consequently, only the teacher is seen in the videos, while students are overheard. In line with established videodidactic practices [43–45], teachers retained ownership over the video recordings and determined which materials were shared for research purposes. A clear, nonhierarchical separation between teachers and researchers was maintained throughout the study. The study received the ethical approval of the Institutional Review Board. All lessons were viewed in full. Whenever a student's physmatic difficulty was identified, it was transcribed and subsequently analyzed in detail.

B. Data analysis

In accordance with Chi's [46] guidelines for content-based segmentation, which emphasize selecting meaningful units of analysis and considering contextual information when defining segment boundaries, we defined the unit of analysis as *physmatic event*—a segment of student–teacher interaction focused on the development of a specific physmatic skill and culminating in a related student difficulty. Each event was identified retrospectively, starting from the student's physmatic difficulty and tracing back through the interaction to identify the skill the teacher was fostering. We classified the skills based on the categorization we developed from the literature (see above and Table II). In order to identify the skill, we examined the interaction holistically, attending to the teacher's questions, instructions to students, and actions. For example, we examined whether the teacher was asking students to offer a mathematical model (mathematization), pointing them to parts of the model and asking for sensemaking (interpretation), or connecting the model and the phenomenon to examine whether they agree (validation). However, when the teacher's instruction clearly shifted to a different skill, a new event was defined, and subsequent difficulties were attributed to the newly introduced skill. The same type of difficulty may occur across different skills. As a result, the number of identified skill-difficulty Intersections exceeded the number of physmatic events.

TABLE I. Lesson distribution by physics domain.

Physics domain	Topics within domain	Number of lessons from which excerpts were taken (hours)	Number of physmatic events	Number of physmatic difficulties	Grades
Geometric optics	Reflection, refraction, lenses, prisms, and optical applications such as eyeglasses	5 (04:48:03)	37	63	10
Kinematics	constant velocity, constant acceleration, and free-falling bodies	5 (03:54:54)	28	54	10, 11
Dynamics	Forces, fluid pressure, work and energy, power, momentum, and circular motion	5 (01:44:10)	22	52	8, 11, 12
Electricity and magnetism	Electric charge, electric field, potential, Ohm's law, circuits, energy and power, and magnetism	6 (04:33:17)	38	75	12
Waves and quantum physics	Interference, Doppler effect, and wave-particle duality	2 (02:53:13)	21	29	12
Total		23 (19:14:16)	146	273	

TABLE II. Literature-based categorization of physmatic skills.

Category	Short description (for broader explanation, see the Theoretical background section above)
Simplification	Down-converting an observed phenomenon by omitting certain variables, reducing the dimensionality of the system or isolating a specific dependency by fixing one variable
Mathematization	Use of mathematical language to describe the physical (simplified) phenomenon
Manipulation	Collection of mathematical operations on the expression developed in the previous stage, Some operations follow directly from mathematical rules (e.g., isolating a variable and taking a derivative), while others rely on physical reasoning (e.g., cancelling a mass term based on assumptions about its nonzero value)
Interpretation	Giving meaning to mathematical results obtained or to concepts related to it in the context of the physical system in question
Validation	Evaluating whether the results of the model align with observations, thereby assessing its reliability

Our analysis followed Creswell's [47] spiral model of qualitative research and proceeded in four main stages. First, the first author watched the full recordings of all lessons and marked segments for transcription by a research assistant. Subsequently, a research assistant produced full transcripts of all segments, which were then imported into Atlas.ti. The transcripts were then thoroughly read to ensure immersion in the data. In cases where the transcription was not clear enough, we returned to the video and added information to the transcription.

Second, we coded the physmatic skills in a top-down manner, drawing on the established literature and based on the five core skills outlined in Table II, which we described in detail in the theoretical background. Each physmatic event was coded according to the specific skill the teacher seemed to intend to develop.

In parallel, we developed a coding scheme for the difficulties exhibited in each event (see Table III in Appendix A). We identified categories for physmatic difficulties inductively from the classroom discourse, following a bottom-up approach. Each physmatic event was therefore double-coded: both for the skill the teacher fostered and for the difficulty expressed by the student.

Third, we revisited the dataset in several iterative rounds to refine the definitions of the difficulty codes. These iterations followed the validity procedures detailed in the next subsection and resulted in an agreed-upon coding scheme. The difficulty codes were then organized into five clusters, which are presented in Table III.

Finally, we jointly conducted interpretation and synthesis by identifying cross-domain patterns using comparison matrices and co-occurrence analysis. We then examined these patterns statistically using a chi-square test to determine whether the appearance of student difficulties was randomly distributed or significantly associated with particular physmatic skills.

C. Development of the coding scheme

As mentioned, following the identification and segmentation of physmatic events, we analyzed the student difficulties that emerged within these events in order to develop a coding scheme. A total of 273 instances of student difficulties were identified across 146 physmatic events. Through a bottom-up analysis of these instances, we generated 24 distinct difficulty codes, each representing a recurrent type of student difficulty observed in the data.

Once the initial set of codes was established, we examined conceptual similarities and overlaps among them and compared our findings with the research literature to explore whether similar types of difficulties had been previously reported and how they were conceptualized. Although there was partial overlap, our codes did not fully correspond to existing frameworks, because in our analysis, each difficulty was required to involve both mathematical and physical components, whereas in prior studies, such dual involvement was not always required.

The clustering of codes was carried out through an iterative process that combined theoretical considerations with expert validation. Several rounds of refinement were conducted until we reached a set of five clusters, each grouping together codes with the highest possible degree of conceptual commonality. This clustered structure provided the basis for the subsequent quantitative analyses presented in the Results section.

D. Reliability and validity

Before conducting the external validation described below, we performed an expert validation to refine the coding scheme and the clustering of difficulties. In this stage, the codes were carefully reviewed to ensure that only difficulties that involved both a mathematical and a physical component were included in the final coding scheme. For example, a case in which students struggled to recall the mathematical notion of absolute value during a kinematics lesson was not included, as it reflected a purely mathematical difficulty without any physmatic interplay. Similarly, a segment in which students engaged only in a conceptual discussion of internal resistance within a battery, without any mathematical reasoning, was excluded as a purely physical difficulty. These difficulties were not included in the 273 difficulties.

To examine the validity of the coding, a peer validation process was subsequently conducted. Twelve analyzed physmatic events, each pairing a teacher skill with a single

student difficulty, were selected randomly from the lesson corpus and distributed among 58 validators, including high school physics teachers, mathematics teachers, and B.A. students in scientific disciplines. Each validator received a random set of 4 out of the 12 events. For each event, four possible coding categories of difficulties were presented, along with concise definitions. Validators were allowed to select more than one category per event. We employed a relatively large group of validators to capture diverse perspectives and to test the robustness of the coding scheme across different backgrounds. However, retraining the original group of validators was not feasible, so we extended the validation process to a research assistant, who received explicit and detailed instructions on the coding scheme and its underlying principles. The research assistant independently coded an additional subset of the data.

We compared the agreement between the coding of validators (including the research assistant) and the research team. Whenever validators categorized in accordance with the research team, even if only in one code, we counted it as agreement in coding, and the validated code is what was assigned to that segment. In cases where more than one difficulty code seemed plausible for a given event, we assigned the final code based on the most common interpretation. In all segments, the researchers' selected category was also chosen by at least some validators and was, in every case, the most frequently selected code.

Agreement rates between the researchers' coding and the validators' selections varied across the coded segments, with a mean of 80.34% and a standard deviation of 16.37%. In cases of disagreement (e.g., segments with lower agreement rates), final decisions were made through discussion among the three researchers who coauthored the paper. This expert-based validation helped resolve discrepancies and served to further support the consistency and theoretical grounding of the final coding scheme.

V. RESULTS

This section presents a quantitative description of the frequency of the different clusters of physmatic difficulties identified in the classroom data, followed by a Chi-square analysis examining their associations with the physmatic skills that teachers aimed to foster. The quantitative findings are complemented by selected classroom examples that illustrate the most common intersections between specific skills and specific clusters of difficulties.

We identified 146 physmatic events, which included 273 instances of student difficulties. These were coded into 24 codes and organized into 5 clusters, as described in the Methodology section.

The following section presents the five clusters of student difficulties identified in the data. Each cluster represents a distinct pattern of difficulties related to specific aspects of students' engagement with physical models and their mathematical representations. For the precise definitions of each code within the cluster system and detailed illustrative examples, see Table III.

Cluster 1: Questions about model components.

This is the largest cluster of all identified difficulties. Its distinct feature is that it includes difficulties in which students struggle with specific components of the physical model, but not with the model as a whole. By "model," we mean either the physical explanation addressed in the lesson or its mathematical representation as expressed in a formula.

The codes included in this cluster refer to various types of issues within a model: reference to an inappropriate physical quantity, lack of knowledge of the meaning of a physical quantity in a model, misunderstanding of the meaning of the sign in a formula, confusion between types of quantities, confusion between direct and inverse proportionality, omission of terms in a formula, considering a negligible quantity as non-negligible, and attributing universal validity to a standard value (e.g., assuming g is always 10 m/s^2).

Cluster 2: Difficulties applying mathematics in a physical context.

This cluster includes cases in which students encounter difficulties with mathematical operations or structures not in mathematics *per se*, but when these appear embedded within a physical context. These difficulties were coded only when the mathematical element was part of the physical model. For example, when a teacher taught a trigonometric procedure that was required for an optics lesson, students' mathematical struggles with that procedure were not categorized under this cluster because they were considered purely mathematical rather than physmatic.

The codes in this cluster refer to difficulties involving various aspects of mathematical reasoning within a physical context: misapplying arithmetic operators, connecting geometry to real-life descriptions, struggling with mathematical symbols, graph and diagram structure recognition (referring to the understanding of elements within a representation, not transitions between representations), and mixing two mathematical concepts.

Cluster 3: Questions about the model.

This cluster focuses on difficulties that concern the physical-mathematical model as a whole, rather than specific components of the model. In these cases, students use, interpret, or apply the model incorrectly at a global level—for example, by applying formulas or laws outside their valid scope or when the model conflicts with another model or with students' intuitive conceptions.

The codes in this cluster are: misidentifying the region of applicability of a formula, conflict between different models of physical phenomena, applying an incorrect or less appropriate law to the situation, overgeneralization of mathematical structure, and incorrect estimation from a formula or calculation.

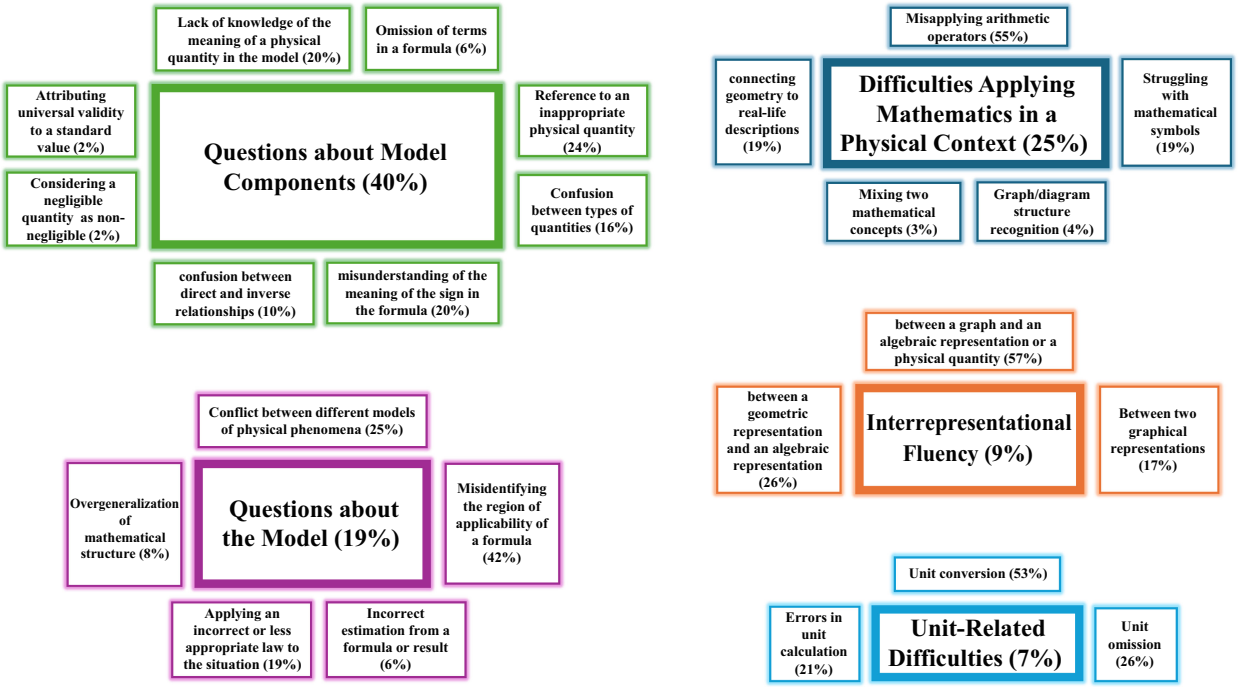


FIG. 1. Distribution of clusters across the dataset and their associated codes within teacher-student interactions.

Cluster 4: Interrepresentational fluency.

This cluster focuses on difficulties in translating between different mathematical representations of physical quantities or relationships (e.g., graphs, formulas, or geometric diagrams). Difficulties occurring within a single representation are addressed in other physmatic difficulty clusters, depending on their conceptual characteristics.

The codes in this cluster refer to transitions that may occur in either direction in each case: between a graph and an algebraic representation or a physical quantity; between a geometric representation and an algebraic representation; and between two graphical representations.

Cluster 5: Unit-related difficulties.

This cluster focuses on difficulties that occur at the interface between mathematical procedures and the physical meaning of the results, as reflected in students' use and interpretation of physical units.

The codes in this cluster are unit conversion, unit omission, and errors in unit calculation. Unit conversion refers to converting between two units of the same dimension (e.g., from meters to centimeters). Unit omission refers to cases in which units are omitted as part of the mathematical procedure rather than due to forgetting. Because this distinction can only be determined based on the interaction, cases were included in this category only when the teacher asked about the units and the student was unable to provide them immediately. Errors in unit calculation refer to assigning incorrect units to a physical quantity or failing to compute the appropriate resulting units.

Figure 1 shows the distribution of the five clusters across the dataset and provides an overview of their associated codes.

A. Relationship between physmatic skills and physmatic difficulties

To examine the relationship between the physmatic skills that teachers foster and the difficulties expressed by students, a chi-square test of independence was conducted. The analysis was based, as mentioned, on 146 physmatic events in which students expressed a total of 273 difficulties, resulting in 273 skill-difficulty intersections. Each intersection was defined as a classroom episode in which the teacher promoted a specific physmatic skill, followed by a student utterance reflecting a related difficulty. The physmatic skill of simplification was rarely identified in our dataset. Based on the literature [21], this is somewhat surprising, since this is a skill one would expect to emerge in many physics lessons. Moreover, in the few cases where it did appear, any student difficulties observed in the same episode were physical rather than physmatic. Accordingly, this category was excluded from the chi-square analysis. Therefore, the analysis included intersections between four skills—mathematization, manipulation, interpretation, and validation—and five clusters of difficulties: Questions about model components, difficulties applying mathematics in a physical context, questions about the model, interrepresentational fluency, and unit-related difficulties.

The results revealed a statistically significant association between physmatic skills and clusters of student

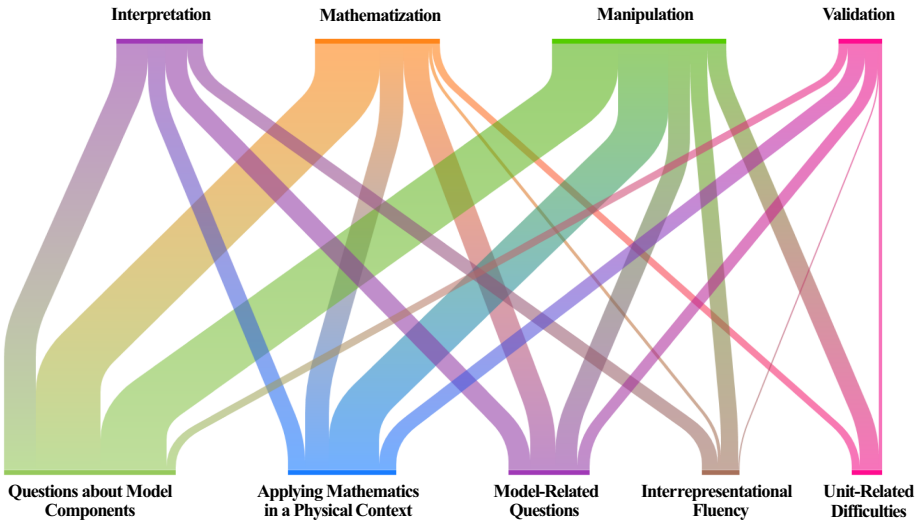


FIG. 2. Connections between physmatic skills promoted by teachers and clusters of students' expressed difficulties.

difficulties, $\chi^2(12) = 31.40$, $p = 0.0017$. Four cells (20%) had expected counts less than five. Figure 2 visualizes all skill-difficulty intersections across the dataset, illustrating the distribution of student difficulties within each physmatic skill.

To further interpret the findings, we examined adjusted standardized residuals (ARs), which identify cells where observed frequencies differ significantly from the expected frequencies under the null hypothesis of independence. AR values greater than |1.96| are considered statistically significant at the 5% level. The distribution of frequencies across skills and difficulty clusters is summarized in Table IV in Appendix B.

1. Positive associations

Several meaningful patterns emerged in the data.

Mathematization was positively associated with the category of difficulties of questions about model components (AR = 2.6). This suggests that during mathematization, students often encounter difficulties related to different components of the model as they work toward constructing a mathematical model that accurately describes the physical phenomenon.

Throughout all examples, square brackets indicate nonverbal actions or editorial clarifications.

The following example illustrates a code from the questions about model components cluster, specifically confusion between direct and inverse relationships.

During a grade 11 kinematics lesson, after the teacher explained the concept of uniform motion, she asked the students to produce a formula for constant speed.

Teacher: And now, how will I calculate the speed?
Student: Distance multiplied by time.
Teacher: How?
Student: Distance multiplied by time.
Teacher: Distance multiplied by time? Think again.
Student: Distance divided by time.
Teacher: Distance divided by time. [Writes on the board while saying again:] "Total distance divided by total time."[Writes on the board: $v=\Delta x/\Delta t$]

In this example, the teacher attempts to foster the skill of mathematization by guiding students to construct the formula for constant speed. The student misidentifies the type of mathematical relationship, treating an inverse (divisive) relationship as if it were direct (multiplicative), which leads to an incorrect mathematical model for velocity. The physical component involves understanding the relationship among the three quantities—displacement, time, and speed—and recognizing that speed is inversely related to time. The mathematical component lies in representing this relationship through an algebraic structure, in which the quantities are organized into a formula that reflects an inverse ratio. The difficulty reflects a confusion between inverse and direct relationships, resulting in a failure to construct the appropriate mathematical structure based on the underlying physical reasoning.

Another significant association emerged between manipulation and unit-related difficulties (AR = 2.1), implying challenges in handling or maintaining correct physical units during mathematical procedures. A frequent subcategory was unit conversion errors.

The following example illustrates a code from the unit-related difficulties cluster, specifically unit conversion.

During a grade 12 lesson on electric potential, the teacher guided students in performing a substitution using lengths given in centimeters:

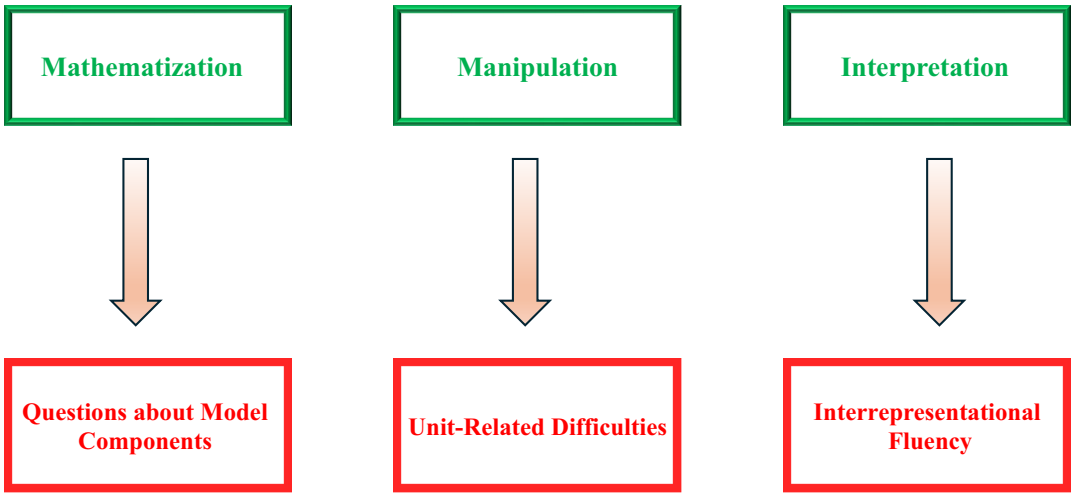


FIG. 3. Positive associations (AR > 1.96) between physmatic skills and difficulties during teacher-student interactions in classroom interactions.

Student: I substituted everything, and got 3.6.
Teacher: Times 10 to the power of what? 10 squared?
Student: Just 3.6.
Teacher [to all students]: Who can identify the mistake?
Student: Oh, right—I didn't convert from centimeters to meters.

In this example, the teacher promotes the skill of manipulation, as students were asked to isolate a variable from a formula for electric potential and then substitute values to compute the result. The student completes the substitution after isolating the variable but fails to convert the length from centimeters to meters. The mathematical component involves the algebraic manipulation of the formula—specifically isolating the desired variable and performing the substitution. The physical component lies in maintaining consistent units, ensuring that the quantities used in the calculation accurately reflect their real-world physical meaning. The difficulty arises when the student carries out the mathematical procedures correctly but misapplies the necessary unit conversion. This example highlights the importance of checking unit consistency during manipulation, as it is crucial for the next step of evaluating the validity of the result and identifying errors.

Another meaningful association emerged for interpretation, which was strongly linked to interrepresentational fluency (AR = 2.8). This indicates that when students were expected to interpret mathematical results, they had trouble moving between different representational forms. One observed subcategory was difficulty transitioning from a geometric representation to a formula or algebraic expression.

The following example illustrates an interrepresentational fluency difficulty, specifically involving a transition between a geometric representation and an algebraic representation.

During a grade 12 physics lesson on electric potential, after drawing equipotential lines for both a uniform field and a point charge, the teacher asked students to extract a more general conclusion:

Teacher: What general conclusion can we draw from these two cases? Something which is more general?
We already said where the potential is equal in a uniform field and where it is equal around a point charge.
I want us to try to draw a more general conclusion. I remind you: some of the …
Student: That the lines don't have to be straight—is that what you mean?

In this example, the teacher attempts to foster students' skill of Interpretation by encouraging them to draw a general conclusion from two geometric representations—equipotential lines in a uniform electric field and around a point charge—and to identify a physical principle that underlies both. Her goal was for students to recognize that equipotential lines are always perpendicular to electric field lines. Reaching this conclusion required integrating both mathematical and physical knowledge. The mathematical component involves inferring the geometric relationship between the field direction and the shape of the equipotential lines (as previously discussed and drawn). The physical component involves interpreting this geometric relationship in terms of electric quantities—understanding that field lines are always perpendicular to equipotential surfaces. The student's response ("that the lines don't have to be straight") reflects a focus on a superficial visual feature, without establishing the deeper connection between the geometric and conceptual understanding. This indicates a difficulty in

translating between different representations—from a geometric representation to a conceptual one.

To summarize these findings, Fig. 3 visually presents the statistically significant positive associations (AR > 1.96) between the physmatic skills promoted by teachers and the corresponding clusters of student difficulties observed during instruction.

2. Negative associations

Negative residuals revealed additional trends. Mathematization was negatively associated with interrepresentational fluency (AR = −2.3), manipulation with questions about the model (AR = −2.2), interpretation with unit-related difficulties (AR = −2.3), and validation with questions about model components (AR = −2.1), indicating that these combinations occurred less frequently than would be expected by chance. This pattern indicates that fewer difficulties are expected to occur when the teacher focuses on certain physmatic skills. For example, during mathematization, students are primarily engaged in constructing the model, so difficulties related to translating between representations are less likely to arise. During manipulation, students tend to focus on carrying out mathematical procedures, so difficulties related to the model as a whole are less likely to occur. Unit-related issues typically emerge earlier in the problem-solving process, making them less prominent during Interpretation. Finally, during validation, students attend to the model globally, which may reduce the occurrence of difficulties related to its individual components.

VI. DISCUSSION

This study mapped students' physmatic difficulties as they arose in a wide range of physics topics, classrooms, and teachers across the secondary school level. Using video-recorded lessons, we analyzed teacher-student interactions and identified specific events in which students expressed difficulties related to the physmatic aspects of physics learning. Applying content analysis, we coded these difficulties, grouped them into clusters, and linked them to particular physmatic skills that teachers promoted. By systematically coding the intersection between teaching and student responses, the study offers a grounded view of how physmatic challenges manifest during instruction. In this context, the physmatic skills used to frame the present analysis are grounded in the work of Uhden *et al.*, which presents a theoretical model of mathematical reasoning in physics [21] and characterizes the competencies involved in coordinating mathematical formalisms with physical meaning during the modeling process. While the framework proposed by Uhden *et al.* conceptualizes physmatic skills primarily as competencies to be fostered in students in order to support their development as future physicists, the present study extends this perspective by showing how the interplay between physmatic skills and students' expressed difficulties can also serve as a pedagogical lens for teachers. In this sense, our findings suggest that physmatic skills may function not only as learning goals for students, but also as analytic resources for teachers' interpretation of classroom interactions and for the development of their physmatic pedagogical content knowledge (PCK).

This study offers two main contributions. The first is theoretical: We mapped the connection between different types of physmatic difficulties and five core physmatic skills extracted from the literature. While previous research has documented recurring difficulties in physics learning, we found no prior work that systematically links these difficulties to the distinct skills required across the stages of the modeling process. The second contribution is methodological: We identified difficulties within authentic classroom contexts by analyzing video-recorded lessons. For each difficulty, we traced back to the instructional context in which it arose, focusing on the specific physmatic skill that the teacher seemed to promote in the moments preceding the student's response. Classroom interaction is the primary site where teachers can recognize and respond to student difficulties in real time—unlike in written assessments, where such difficulties surface only retrospectively. A teacher familiar with the typical connections between physmatic skills and student difficulties may be better equipped to address these challenges as they emerge.

The methodological contribution of this study, alongside identifying connections between physmatic skills and student difficulties, lies in making visible how these connections unfold during classroom interaction. While prior video-based research has primarily employed lesson recordings to support teachers' professional reflection, the analytic perspective developed here—focused on identifying physmatic skills and difficulties through systematic analysis of classroom interaction—may also inform such reflective use of video by teachers, for example by examining instructional strategies that appear productive or less effective in relation to students' physmatic reasoning, thereby pointing to possible directions for future research and video-based professional development [43–45].

In this study, we identified five core clusters of physmatic difficulties and examined their associations with five physmatic modeling skills described in the literature [18,21]. The cluster *questions about model components* was mainly associated with mathematization. This association highlights that mathematization is not a purely mathematical activity, but rather a process that brings out the roles of the physical components of the model—such as the meaning of quantities and their relationships—alongside the construction of appropriate mathematical structures. This might suggest that teachers or lesson developers should emphasize the physical components of the model in mathematization activities to reduce students' struggles.

The cluster unit-related difficulties was most frequent when manipulation processes, which involve performing

mathematical procedures on physical quantities, were promoted in the classroom. In these processes, students often concentrate on executing the algebraic steps and solving for the required variable, while paying less attention to conceptual considerations such as unit consistency. This may lead to situations in which the mathematical procedures are carried out correctly, but students lose the physical meaning of their actions in the process, which is expressed in overlooking units or incorrect unit handling. This finding is consistent with previous research showing that students often rely on the use of formulas as their primary strategy in physics, rather than applying dimensional analysis or other reasoning about units, even when such approaches are available and could support problem solving [7].

The cluster *interrepresentational fluency* was mainly associated with interpretation. In this case, the difficulty lies in recognizing that different representations—such as graphs, formulas, and diagrams—refer to the same underlying physical phenomenon. Previous studies [6–8,37,38] support the claim that interpretation requires a deep understanding of the physical concept—namely, the ability to analyze mathematical results in physical terms. To put it differently, in order to prevail in interpretation, one should master various representations. This might be the reason why learning this skill exposes this difficulty.

The clusters *difficulties applying mathematics in a physical context* and *questions about the model* were not strongly associated with any single physmatic skill but appeared across different modeling processes. Their broad distribution across skills highlights their fundamental role in the learning process and suggests that teachers should address them throughout instruction rather than in isolated phases. The fact that mathematical difficulties appear across most stages of the modeling process may suggest that the presence of a physical context itself places additional cognitive demands on students, making it harder for them to carry out even familiar mathematical procedures. This is in accordance with previous studies [15], or as Meli *et al.* [10] point out, students may make elementary arithmetic errors, such as multiplying fractions incorrectly, even when their mathematical performance is otherwise strong, due to the added complexity of applying mathematics within a physical context.

Based on the literature, which identifies simplification as an important component of mathematical modeling in physics [21], it is somewhat surprising that this skill was rarely identified in our dataset. Moreover, in the few cases in which simplification did appear, the student difficulties observed were conceptual physics difficulties, using the term as defined by Modir *et al.* [4], rather than physmatic difficulties.

The absence of physmatic difficulties associated with simplification does not imply that such difficulties do not exist; rather, it may be incidental that they did not emerge in the present dataset. However, the breadth and diversity of the data analyzed—including multiple physics topics, teachers, grade levels, and over 19 h of classroom instruction—may echo previous reports [2,20,24,25] pointing to a broader instructional tendency to present ready-made models, without explicitly engaging students in the simplification decisions that underpin them.

From this perspective, the present finding suggests that simplification may receive insufficient explicit attention in classroom instruction. This finding should signal physics teacher educators that simplification deserves greater attention in teacher professional development programs and should be addressed more explicitly within teachers' physmatic pedagogical content knowledge (PCK).

A. Supporting the development of physmatic pck of teachers

Beyond documenting the emergence of physmatic difficulties in teacher-student interactions, the findings of this study offer valuable insights for enriching physics teachers' PCK regarding the interplay between mathematics and physics. Drawing on the five dimensions identified by Lehavi *et al.* [16,29]—orientation to teaching, knowledge of curricular structure, understanding of students' conceptual difficulties, knowledge of assessment, and instructional strategies—we highlight how our results may inform each of these facets.

In terms of orientation to teaching, our findings suggest that teachers who are aware of the types of physmatic difficulties their students encounter, and of the connections between these difficulties and the modeling skills they aim to foster, may be better positioned to guide their instruction accordingly. Such awareness allows teachers to anticipate which difficulties are likely to arise at each stage of the modeling process (e.g., mathematization, interpretation, and validation) and to incorporate targeted support in advance [48]. This approach reflects an instructional orientation that emphasizes the development of modeling competencies—not merely solving problems. In this context, physmatics may be viewed as a distinct subdomain within physics education, with its own characteristic cognitive demands and pedagogical challenges [15]. Recognizing physmatics as such may encourage teachers to treat the coordination between mathematical and physical sensemaking as an explicit instructional goal, rather than an assumed or automatic process. A clear example of how a teacher can leverage the connection between physmatic skills and student difficulties to support instruction was observed in one of the classroom interactions analyzed in the study. In a lesson on total internal reflection in geometrical optics, the teacher aimed to develop the skill of validation and deliberately introduced the difficulty of "misidentifying the region of applicability of a formula"—a code within the *questions about the model* cluster, which in

our analysis appeared repeatedly in episodes involving the validation skill. During the lesson, students were asked to substitute extreme angle values into the sine function, which caused their calculators to return a "MATH ERROR" message. Rather than immediately offering an explanation, the teacher used the moment of confusion to create a sense of readiness for learning. He then explained that the error indicated the model was no longer valid—that is, refraction does not occur and the appropriate model is that of total reflection. In doing so, the teacher effectively integrated physical content with the development of a modeling skill, using a known difficulty to deepen students' understanding of formula validity and model applicability.

Regarding knowledge of curricular structure, the findings show that physmatic skills are relevant across a wide range of physics topics. Emphasizing these skills consistently throughout various topics may help students develop transferable strategies for engaging with physical phenomena while also reducing common difficulties, particularly those related to inappropriate mathematization, misinterpretation, or insufficient manipulation.

Beyond their distribution across topics, the findings also bear implications for teachers' knowledge of physmatic content, as they may point to the need to consider physmatic knowledge as a distinct content domain within physics teaching. Prior research has challenged the common view that difficulties in using mathematics in physics stem from insufficient mathematical knowledge, emphasizing instead that mathematics in physics differs qualitatively from mathematics learned in mathematics instruction and may require relearning within a physical context [2]. In this light, the present findings further support this perspective by showing that recurring difficulties are systematically linked to physics learning itself and to the modeling skills that physics teachers aim to foster. Awareness of physmatic skills that are uniquely developed within physics instruction may therefore support recognizing the interplay between mathematical and physical reasoning as an explicit instructional content, rather than an assumed prerequisite.

With respect to the dimension of knowledge of students, the taxonomy of 24 codes of student responses mapped into 5 overarching clusters provides a rich knowledge base for anticipating and interpreting students' challenges at the physics-mathematics interface. These categories align with the framework proposed by Modir *et al.* [4], which characterizes three types of errors—transition, displacement, and content—according to four modes of student thinking: conceptual physics, algorithmic physics, conceptual mathematics, and algorithmic mathematics. Our clusters of *unit-related difficulties* and *interrepresentational fluency* may reflect transition errors, as they involve moments when students are expected to shift between cognitive frameworks. For instance, when a student substitutes values into a formula without converting centimeters to meters, they apply algorithmic mathematical procedures without transitioning to the conceptual mathematical reasoning required to verify the coherence of units with the physical quantities. Similarly, difficulties in *interrepresentational fluency* also reflect transition errors, as they occur when students are required to shift between different cognitive frameworks—for example, from interpreting a graph mathematically to understanding its physical meaning—but fail to make this transition. Difficulties in *questions about the model* align with displacement errors, as they capture moments when students continue to rely on algorithmic mathematical thinking even when a shift to conceptual physical reasoning is needed. For example, a student who applies the formula for constant velocity on an accelerating body disregards the physical conditions required for the formula's validity. In contrast, difficulties in *applying mathematics in a physical context* may be related to content errors. For example, when a student incorrectly isolates a variable or makes a basic arithmetic mistake, such as in fraction multiplication, they are engaged in algorithmic mathematical thinking but apply it incorrectly.

In terms of knowledge of assessment, the findings highlight the existing gap in diagnostic approaches to physmatic difficulties. The systematic identification and mapping of physmatic difficulties provide a conceptual basis for the development of diagnostic questionnaires aimed at identifying specific physmatic difficulties in students' responses. This direction is consistent with prior studies demonstrating the potential of targeted diagnostic questionnaires for identifying physmatic difficulties in students' reasoning [14,15] and encourages further development of such assessment instruments in this domain.

Based on the findings of the present study, it is not possible to determine whether physmatic skills should be taught to students as decontextualized, general competencies independent of specific physics content. While our analysis highlights systematic connections between physmatic skills and students' difficulties, this does not imply that instruction should be reorganized around these skills as stand-alone curricular elements.

Rather, our findings may suggest (but do not prove) that, at the level of classroom instruction, it is more appropriate to follow the existing curricular sequence and to draw students' attention to physmatic aspects of problem solving when relevant difficulties arise within specific content areas. For example, when teaching velocity, students may be asked to construct a formula based on a physical situation (e.g., $\Delta x/\Delta t$), with explicit attention drawn to the process of mathematization. Later, in a different topic such as electric circuits, students may again be prompted to articulate a physical concept verbally and translate it into a mathematical expression (e.g., constructing a formula for current, $\Delta q/\Delta t$), making explicit that a similar physmatic

skill is involved and that certain difficulties (e.g., treating ratios as products) may recur across topics.

At the same time, we believe that the separation between physmatic skills and specific physics content may be more appropriate in the context of teachers' professional development. For in-service and preservice teacher education, professional development programs may focus explicitly on the relationships between physmatic skills and typical student difficulties, independent of particular domains such as mechanics, electricity, or waves, while remaining grounded in authentic classroom examples. Once these skills are named and characterized, as in the present study, teachers may then be supported in anticipating physmatic difficulties in the topics they are about to teach and in developing instructional strategies to address them. In this sense, the findings may inform multiple dimensions of teachers' PCK, including awareness of student difficulties, instructional decision making, and curricular planning.

Taken together, by linking specific physmatic skills with clusters of student difficulties, our findings bridge the key components of the model of Lehavi *et al.* [16]—namely, instructional goals (i.e., physmatic skills)—and the cognitive framework of Modir *et al.* [4], which classifies student difficulties by types, shedding light on the phys-math interplay.

VII. LIMITATIONS

This study has several limitations that should be considered when interpreting the findings. Although it included excerpts from 23 lessons taught by 14 different teachers and covering a range of physics topics, drawn from regular classroom instruction, certain features of the selected lessons—such as instructional style, interaction patterns, or the nature of the tasks—which were not used as selection criteria but emerged from the naturally occurring classroom contexts—may have influenced how physmatic difficulties manifested and how the connections between skills and difficulties were identified. For example, none of the lessons was in a laboratory setting. Second, the coding focused on difficulties that became visible during classroom interactions and may not capture all the challenges students actually experienced, particularly those that remained hidden or unspoken. Nevertheless, by focusing on classroom interactions, the study offers a valuable window into the emergence of physmatic difficulties and their connections to the instructional process.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available because they contain sensitive personal information. The data are available from the authors upon reasonable request.

APPENDIX A: CLUSTERS OF PHYSMATIC DIFFICULTIES

TABLE III. Clusters of physmatic difficulties with definitions and representative excerpts from teacher-student interactions. Excerpts were lightly edited for clarity (e.g., removal of slang and fillers). Double round brackets indicate nonverbal actions or editorial clarifications, and square brackets indicate added text for clarification. The teacher's skill preceding each difficulty is not shown here in order to focus on the difficulty itself.

Cluster 1: Questions about Model Components, 109, 40%			
Statements showing confusion relating elements within a physical model—such as quantities, formulas, relationships, or representations			
Code:	F	Definition:	Example:
Reference to an inappropriate physical quantity	27	Classroom interactions in which students refer to an inappropriate physical quantity in the model.	The teacher presents a question about Snell's law. Student: $c = 1$, right? Teacher: c isn't relevant here. I have n_1 and n_2, that's enough.

(Table continued)

TABLE III. *(Continued)*

Cluster 1: Questions about Model Components, 109, 40%			
Statements showing confusion relating elements within a physical model—such as quantities, formulas, relationships, or representations			
Code:	F	Definition:	Example:
Lack of knowledge of the meaning of a physical quantity in the model	21	Classroom interactions in which students do not know or misinterpret the meaning of a physical quantity in the model.	The teacher is using the constant acceleration formula ($v = v_0 + at$) Teacher: Excellent, very good. What is v_0 in this formula? Student: I don't know.
Misunderstanding of the meaning of the sign in the formula	22	Classroom interactions that indicate a misunderstanding of vector direction or incorrect sign assignment to physical quantities such as charge.	The teacher asks what happens to acceleration after the ball reaches 0 m/s. The positive direction is down. Student: So, acceleration becomes $-10\ \mathrm{m/s^2}$? Teacher: No, it stays $+10\ \mathrm{m/s^2}$.
Confusion between types of quantities	17	Classroom interactions that indicate confusion between different types of mathematical and physical quantities (e.g., variable vs parameter, vector vs scalar), reflected in difficulties recognizing familiar structures when represented with different symbols or notations.	The teacher asks if v, in uniform motion, is a variable or a specific value Students: It depends—could be one or more Teacher: No, it's one number—the constant speed of the object
Confusion between direct and inverse relationships	11	Classroom interactions that reflect confusion between direct and inverse relationships between physical quantities, for example assuming that increasing one quantity increases another when in fact it decreases.	The teacher asks how to calculate speed in uniform motion. Student: Distance multiplied by time. Teacher: Think again. Student: Distance divided by time. Teacher writes: $v = \frac{\Delta x}{\Delta t}$
Omission of terms in a formula	7	Classroom interactions that reflect the omission of terms when interpreting or applying a formula, for example ignoring a constant or a variable that should be included.	In this example, the teacher gives a problem on Coulomb's law: Student: So I got something different … I got 6.923. Oh, I know what my mistake was. Teacher: What was your mistake? Student: I forgot to include K in one of the parts.
Considering a negligible quantity as non-negligible	2	Classroom interactions that indicate students' consideration of a quantity that the model assumes to be negligible, reflecting a misunderstanding of the model's simplifying assumptions.	In this example, from a lesson on energy in electric circuits, the teacher and students are constructing an energy equation for the circuit. The teacher had previously explained that the kinetic energy of electrons can be neglected. Teacher: The energy supplied by the battery is dissipated in two ways: as internal energy and? Student: And as the kinetic energy of the electrons, right?

(Table continued)

TABLE III. *(Continued)*

Cluster 1: Questions about Model Components, 109, 40%			
Statements showing confusion relating elements within a physical model—such as quantities, formulas, relationships, or representations			
Code:	F	Definition:	Example:
Attributing universal validity to a standard value	2	Classroom interactions that reflect students' treatment of a standard or conventional value as a universal or absolute truth, rather than as a reference value used within a specific model or context.	In this example, the teacher and students are discussing a formula that fits a graph, and then examine the experimental estimate of g. Student: Wait, but what's the real g?
Cluster 2: Difficulties applying mathematics in a physical context, 69, 25%			
Difficulty applying mathematical knowledge in physical contexts, including operations, geometry, symbols, graph structure, and conceptual distinctions.			
Code:	F	Definition:	Example:
Misapplying arithmetic operators	38	Classroom interactions showing errors in basic mathematical operations or function application (e.g., missing sine, omitting a division sign).	The teacher gives a problem in which Snell's law is not applicable (e.g., a case of total internal reflection), but the student reports having obtained a result nonetheless. Teacher: Uh … what? How did you reach that conclusion? Student: I got a result that was more than … like, with a lot of digits after the decimal point.
Connecting geometry to real-life descriptions	13	Classroom interactions that reflect insufficient understanding or application of geometric concepts (e.g., plane, angle, perpendicular).	The teacher explains that rays and the normal to the surface lie in the same plane. Teacher: Clear? Student: Kind of … What does "same plane" mean?
Struggling with mathematical symbols	13	Classroom interactions that reflect unfamiliarity with mathematical symbols used to represent physical operations, quantities or their units (e.g., Δ for difference).	The teacher is deriving the velocity equation for uniform motion. Teacher: Now I'm saying—x_0 is the first x, the x from the beginning, when t was zero. Student: So its value is zero? Teacher: No, this is the initial position of the object.
Graph and diagram structure recognition	3	Classroom interactions that reflect difficulty identifying the type, structure, or components of a graph or diagram—such as axes, variables, or overall layout—regardless of its physical meaning.	The teacher discusses uniform motion in a repeated back-and-forth path, illustrating the direction with arrows. Teacher: Here the motion is uniform—the object keeps covering the same distance each time, but it changes direction. Here it moves this way, here the other way, and again … [draws arrows tangent to the direction of motion] Student: So it's not a position-time graph? Teacher: It's not a position-time graph—what is it? A motion diagram.

(Table continued)

TABLE III. *(Continued)*

Cluster 2: Difficulties applying mathematics in a physical context, 69, 25%			
Difficulty applying mathematical knowledge in physical contexts, including operations, geometry, symbols, graph structure, and conceptual distinctions.			
Code:	F	Definition:	Example:
Mixing two mathematical concepts	2	Classroom interactions that reflect confusion between two mathematical concepts—such as dimension vs length or angle vs ray.	The teacher explains that the wavelength 10^{-36} m is much smaller than a proton. Student: Zero dimensions? Teacher: I'm not talking about dimensions—just length in meters.
Cluster 3: Questions about the model, 52, 19%			
Misapplication or misinterpretation of the physical-mathematical model as a whole.			
Code:	F	Definition:	Example:
Misidentifying the region of applicability of a formula	22	Classroom interactions that reflect a lack of awareness of the physical conditions under which a formula is valid.	The teacher is teaching the phenomenon of total internal reflection. Student: It applies only when moving from a denser to a rarer medium, but not from a rarer to a denser one—or does it apply there too?
Conflict between different models of physical phenomena	12	Classroom interactions that reflect a conflict between the model introduced in instruction and another model held by the student—either a previously learned model or an intuitive model of reality.	In this example, the teacher asks about the dimensionality of light waves: Teacher: What about light waves? What type of wave is it, in terms of dimensions? Student: One-dimensional? Teacher: Can you explain? Student: Because … it just goes (points forward) straight through the air.
Applying an incorrect or less appropriate law to the situation	11	Classroom interactions that reflect the use of an incorrect or less appropriate physical law or principle to analyze a situation.	After teaching refraction through a prism, the teacher presents a situation in which a light ray enters the prism at zero angle, and then reaches the second surface at a small angle to the normal—a situation where the ray should refract rather than undergo total internal reflection: Student: There will be reflection here, right?
Overgeneralization of mathematical structure	4	Classroom interactions that reflect a misinterpretation of an algebraic structure, based on the assumption that a certain mathematical pattern always applies—even when this contradicts physical logic.	The teacher explains Ohm's law to the class. Student: If there is a circle with no resistor, so because $R = 0$, according to Ohm's law, $V = RI$, also $V = 0$.

(Table continued)

TABLE III. *(Continued)*

Cluster 3: Questions about the model, 52, 19%			
Misapplication or misinterpretation of the physical-mathematical model as a whole.			
Code:	F	Definition:	Example:
Incorrect estimation from a formula or result	3	Classroom interactions that reflect an incorrect interpretation or estimation of a numerical result obtained from a formula.	During a quantum mechanics lesson, the teacher calculates the velocity of an electron in a hydrogen atom: $v = \sqrt{\frac{ke^2}{mr}} = 1.6 \times 10^6$ m/s Student: We got the speed of light!
Cluster 4: Interrepresentational fluency, 24, 9%			
Statements showing difficulty in translating back and forth between different representations of physical quantities or relationships (e.g., graphs, formulas, schematic or geometric diagrams).			
Code:	F	Definition:	Example:
Between a graph and an algebraic representation or a physical quantity	14	Classroom interactions that reflect difficulty in transitioning between a graphical representation and a physical concept or formula or between a formula and a graphical representation.	The teacher is giving a lesson on constant acceleration and asks a student to draw a graph of velocity as a function of time. The student draws a horizontal straight line. Teacher: I see you drew the blue line, but okay … is that your choice? Is that correct? Is this what you want it to be?
Between a geometric representation and an algebraic representation	6	Classroom interactions that reflect difficulty translating between a geometric or schematic representation and a formula or Conceptual connection.	The teacher shows equipotential lines around a point charge and in a uniform field and asks for a general conclusion about their relationship to the field. Teacher: What general conclusion can we draw from both cases? Student: That the lines don't have to be straight—is that what you mean?
Between two graphical representations	4	Classroom interactions that reflect difficulty in translating between two graphical representations, such as from a position–time graph to a velocity–time graph.	The teacher presents a position-time graph showing constant velocity, and asks students to identify the matching velocity-time graph. Student D: So, it's A. [Graph A includes a rising velocity segment where a flat (zero-slope) position segment was expected]
Cluster 5: Unit-related difficulties, 19, 7%			
Statements reflecting difficulties in using, converting, or interpreting physical units—including wrong conversion, omissions, and errors in unit calculation.			
Code:	F	Definition:	Example:
Unit conversion	10	Classroom interactions that reflect either a missing or incorrect unit conversion—such as keeping values in centimeters instead of meters, or performing the conversion inaccurately.	While discussing how to compute diopters from a near point of 50 cm, the teacher asks how to convert the distance into meters. Teacher: I know that the distance is 50 cm. How do I convert that to meters? Student: 0.05. Teacher: 0.05? Student: 0.5.

(Table continued)

TABLE III. *(Continued)*

Unit omission	5	Classroom interactions in which students provide a numerical answer without specifying units, even when the physical quantity requires them. Cases in which students omitted units out of convenience but were able to state the correct units when asked were not included.	During a lesson on the Doppler effect, the teacher asks students to calculate the time between two ball detections. Student: [sends numerical answer in chat] Teacher: G, what are the units of what you wrote? The question asks for a time.
Errors in unit calculation	4	Classroom interactions that reflect difficulty assigning correct units to physical quantities, or lead to incorrect units for the required result. *(If no unit is given, see Unit omission.)*	During a circular motion problem, students compute the coefficient of friction. Student: So … what will be the units of this? Teacher: It's unitless—it has no units.

APPENDIX B: CROSS-TABULATION ANALYSIS OF DIFFICULTIES AND SKILLS

TABLE IV. Frequency of student difficulties following different physmatic skills, based on cross-tabulated analysis.

	Questions about model components	Difficulties applying mathematics in a physical context	Questions about the model	Interrepresentational fluency	Unit-related difficulties
Mathematization	41	15	16	2	5
Manipulation	42	32	14	11	12
Interpretation	20	11	14	10	0
Validation	6	11	8	1	2

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