

Interplay between mathematics and physics: Which aspects do instructors hope to convey in a master's program for high-school teachers?

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The interplay between mathematics and physics mutually affects both disciplines and is the subject of ongoing research. Here, we study the interplay in the empirical context of an MSc program for practicing high-school physics teachers putting strong emphasis on advanced physics courses taught by active physicists, together with pedagogical content courses taught by physics education researchers. Which aspects of the interplay between mathematics and physics do teacher instructors wish to convey to the teachers? We held semi-structured interviews with the physics instructors of the program, focusing on their goals for their courses. Thematic analysis of the interviews exposed several aspects of the interplay that instructors reported on integrating in their teaching, resulting in a characterization consisting of three families. The first strengthens established aspects in the literature: coherence, a tool for calculation, and mathematical derivation of new physics results (extraction). The second highlights new facets of existing aspects: the universality of geometry in physics, limits of approximations as invitations to explore, and simulations as virtual experiments (representations). The third comprises aspects that are less prominent in the physics education literature: concretization of abstract mathematics in physics contexts, and conflict as a mechanism for discovery (e.g., the Lorentz force's dependence on velocity conflicting with Galilean invariance). Our findings provide a framework bridging tertiary physics and teaching across all levels— aspects of the interplay between mathematics and physics that are manifest in advanced physics could inform curricular design for high school and beyond, and for teacher education.

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I. INTRODUCTION

The interplay between mathematics and physics is a mutual relationship. Mathematics has played a pivotal role in advancing physics, while physics has also contributed to central developments in mathematics [1]. Research on the teaching and learning of the interplay is multifaceted, as recently reviewed by Palmgren *et al.* [2]. Existing work in the field can be divided into three broad aspects. One, the learners: studying how they use, or could use, mathematics in physics. Two, the teachers: studying teachers' perceptions of learning goals, appropriate teaching methods, and their prevalence. Three, the professional development (PD) aspect: studying, on the one hand, changes in teachers' perceptions following participation in PD, and on the other hand, the underlying messages that PD leaders wish to convey to teachers. The present paper addresses the

last aspect in the context of a long-term teacher PD program putting a strong emphasis on advanced physics courses. In particular, we study how instructors in such courses frame the interplay between mathematics and physics. In the following, we will describe the existing literature on the three aspects and situate our work within it. But first, we will address the disciplinary aspect.

A. The interplay—the disciplinary aspect

The framework depicting the roles of mathematics in physics is still in the process of ongoing refinement. A seminal work in the field came from Pietrocola [3], who distinguished between a technical role and a structural role. The former includes rote calculations that do not involve understanding physics, such as the act of solving equations algorithmically. The latter describes a deeper connection including hierarchical organization of knowledge (such as Gauss' law being universal, as opposed to the particular Coulomb's law, see Kneubil and Robilotta [4]), and guiding reasoning (e.g., by checking limits). Karam [5] applied this framework in an empirical context, to analyze the teaching of an esteemed lecturer in a course on electromagnetism. He delved deeper into the two roles and refined the practices manifesting them. For example, justifying the mathematical objects chosen

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to represent physical quantities or identifying formal analogies between different phenomena are manifestations of the structural role of mathematics in physics.

Taking a historical perspective, Palmgren and Rasa [6] have recently provided a comprehensive framework for the roles that mathematics has played in physics research, referring also to their implications on physics instruction. In their characterization, the roles of mathematics in physics can be divided into four, slightly overlapping, categories. First, as a syntactic structure, organizing information and providing rules for reasoning and further development. Second, representation and semantics, allowing physicists to express and share information about physical phenomena. Third, reasoning and modifications, i.e., allowing logical deduction, theory testing and recognizing formal analogies. Fourth, extraction, solving a certain model or generating a new explanation or concept.

There are also empirical studies focusing on the disciplinary aspect. Recent work on the roles of mathematics in Science, Technology, Engineering and Mathematics (STEM) education as perceived by university instructors was published by Koerfer *et al.* [7]. They propose a hierarchy of roles echoing the work of Palmgren and Rasa, however, in the broader context of STEM education. Their characterization includes roles spanning from technical tool, through language for communication, to philosophical foundation.

B. The interplay—the learners’ aspect

A particular focus has been given to how students use mathematics in physics contexts, mostly in secondary or early undergraduate physics, e.g., using arithmetic, vectors, calculus, or evaluating their answers [8–11]. A common conclusion of such research is that success in mathematics classes does not guarantee success in physics class [12]. Other research has covered modeling of student thinking during problem solving, in progressively elaborate schemes and scenarios. For example, Redish and Smith [13] have provided a basic modeling cycle embodying the common thread in all work that followed: modeling the problem in mathematical terms, processing it to find a solution, interpreting the result and evaluating it in physical terms. Later, Uhden *et al.* [14] stressed the back-and-forth nature of the problem-solving process along the “degree of mathematization” axis, where each refinement of a model can be associated with a physical interpretation. Combining research on student difficulties with research on student reasoning, Wilcox *et al.* [15] proposed a framework consisting of Activation of the appropriate mathematical tool, Construction of a model, Execution of mathematical operations, and, finally, Reflection on the result (The ACER framework). Brahmia [16] proposed a model considering physics and mathematics expertise as complementary skills, stressing their gradual, intertwined development. In a similar vein, the blending of physics and mathematics

(i.e., using conceptual reasoning to facilitate manipulation of equations [17]) was illustrated, e.g., in the context of mechanics [18], the heat equation [19], and electrostatics [20]. Recently, a study by Reshef-Israeli and Kapon extended the reasoning process twofold: first, to layered problem-solving advancing progressively from ideal to “real,” and second, to collaborative work [21]. Another aspect of the interplay between mathematics and physics that has received attention is students’ epistemologies toward mathematics in physics [22–24]. Such research has found a correlation between epistemological views and students’ problem solving. In particular, students’ epistemological framing of a (problem solving) situation is tied to which resources [25] are active. For example, within the framing “mathematical consistency,” students seek similarity to another mathematical idea to validate their reasoning, rather than algorithmically following a calculation procedure.

C. The interplay—the teachers’ aspect

The aforementioned studies are student-centered, considering the interplay of mathematics and physics in learning processes happening in the learner’s mind in different physics education contexts. The breadth of literature indicates the complexity of the process and its evidently many facets. Another focus of interest is teachers’ attributes—their views, practices, and knowledge—regarding the interplay, and their role in shaping students’ learning processes.

Teachers’ attributes can be studied either from the perspective of actual classroom practice, or when they are influenced, either preservice or in PD programs for in-service teachers. From the perspective of actual practice, Yerushalmi *et al.* [26] interviewed physics faculty about the teaching of problem solving. Their interviewees believed, for example, that learning problem solving is a reflective process, yet they admitted that their courses did not explicitly strengthen reflectivity. In fact, they said that typical students should learn from looking at appropriate example solutions, but the examples they provided were not representative of expert (reflective) thought-processes. Another example probing schoolteachers’ views on the interplay is Pospiech *et al.* [27]. They suggested a knowledge structure for schoolteachers and tested it by interviewing highly experienced physics teachers. They found a gap between “the very general view that mathematics is important and... what role mathematics plays in the physical method” (p. 279). Only approximately half of the interviewed teachers mention elements of the structural role of mathematics in physics, described in Sec. I A. The researchers could not confirm that a specific view of the interplay correlated with specific teaching strategies. From the preservice perspective, Karam and Krey [28] devised a course for physics teachers in training focusing on historical aspects of the interplay and on multiple derivations

of prevalent physics equations along with their associated student difficulties. Following the course, some of their graduates exhibited a greater appreciation for derivations of physics equations as part of their “explanation repertoire,” an aspect of their pedagogical content knowledge (PCK) [29].

D. The interplay—professional development programs

As stated previously, teachers’ attributes regarding the interplay can be studied also in continued PD contexts, whether they focus on PCK or on content knowledge (CK). According to recent studies [30,31], both students and teachers say that a good physics teacher should have, among other characteristics, sound CK and PCK. Sometimes, CK can correlate negatively with PCK. For example, Thompson *et al.* [32] studied both out-of-field and in-field preservice teachers undergoing a graduate course for physics teachers. They found that the out-of-field teachers scored higher than the in-field teachers on knowledge of student ideas following the course. Indeed, PCK taxonomy, particularly in physics [33], includes knowledge of the curriculum, of student ideas, of instructional strategies, and of assessment methods. Other PCK models, e.g., in mathematics education [34], highlight also a “vertical” knowledge of content and curriculum, extending beyond the content that a teacher presently teaches. Here, one can expect a positive correlation between advanced CK and PCK, since teachers’ deeper understanding of the content structure permits them to hone their messages in teaching. For example, a teacher could present conservation of energy not only as a problem-solving technique, but as a fundamental pillar of physics.

McLoughlin and Sokolowska [35] surveyed existing strategies for physics teacher PD, such as lesson study [36], professional learning communities (PLCs) [37], research-practice partnerships [38] and practitioner inquiry [39]. Most of the above PD strategies are centered around teaching practices or content directly related to the school curriculum, led by physics education professionals. For example, Lippielo *et al.* organized a PLC focusing on the structural role of mathematics in physics. In particular, they suggested using the model of Uhden *et al.* [14] for activity design in the participants’ classes. In interviews after the PLC, the teachers appreciated both the theoretical aspect of the Uhden-Karam modeling cycle and the direct applicability of the proposed activities in their classroom teaching.

Other opportunities for PD could be academic programs or courses, introductory or advanced, where teachers also participate. Content courses, extending beyond the high-school curriculum, furnish a detailed portrayal of the interplay between mathematics and physics. In many places around the globe, and specifically in the Israeli context, such courses are part of the preservice education of high-school teachers, but their curricula are not specifically tailored to teachers, as their audience proceeds to

follow varied career paths [40]. By contrast, academic programs for in-service teachers focus mostly on pedagogy or PCK, for example, at CU Boulder [41]. We study graduate content courses *for teachers*—what are the messages that are suitable for such courses, in particular regarding the interplay between mathematics and physics? We approach the question by interviewing instructors who design and carry out these courses, as we discuss below.

E. The present work

This study intends to enrich our knowledge of aspects of the interplay between mathematics and physics as they are manifested in advanced physics courses for in-service teachers.

To that aim, we study an MSc program for science and mathematics teachers focusing on advanced disciplinary content. In our context, the course instructors are physicists who have been teaching similar courses to physicists and have had to adapt them to convey the messages they deemed important for teachers. We have conducted semi-structured interviews with instructors who have been instructing in the program for several years. Importantly, the instructors choose voluntarily to instruct teachers rather than, or in addition to, instructing physics graduate students. Such a choice indicates their commitment to advancing physics education, which they openly express. We performed thematic analysis of the interviews to identify aspects of the interplay between mathematics and physics that they emphasize in their instruction of teachers, guided by existing theoretical frameworks, mentioned in Sec. I A.

The study is situated in a longer research program. We wish to examine the affordances of advanced content knowledge in physics in high-school teaching. The present study is the first step, whose results will inform the next step: we will interview teachers who have graduated from the program on how they perceive the interplay, and how they manifest it in their teaching, i.e., how they integrate their advanced content knowledge into PCK.

The remainder of the paper is structured as follows: in Sec. II, we present our research approach. Section III provides details regarding the context in which we performed our research, the Rothschild-Weizmann Program for Excellence in Science and Mathematics Education. Section IV describes our methods in interviewing the instructors of the MSc program for teachers. Our main findings, including a characterization of the aspects of the interplay that we identified, are detailed in Sec. V. Discussion of the findings in light of previous work is in Sec. VI, followed by suggestions for application in teaching, in Sec. VII.

II. RESEARCH APPROACH

A context where instructors have had to formulate their thoughts over a long period of time, such as our program,

holds potential for deep insight into their views of what teachers should know regarding the interplay between mathematics and physics. Moreover, the intense interaction between instructors and teachers, year after year, serves to hone their views.

Consider instruction of quantum mechanics to high-school teachers. When deciding what to teach, the instructor may consider: (a) the knowledge structure to convey, and (b) teachers' existing knowledge and abilities. Regarding (a), the instructor asks themselves, what aspects of physics that are present in quantum mechanics do I wish to convey to teachers, even though much of the content is beyond the high-school curriculum (in Israel and elsewhere [42])? What are the essential concepts that could enrich teaching? Regarding (b), which adaptations should be made from a course for physicists? Many teachers have been teaching high school for several years, creating gaps in their knowledge, e.g., in the relation between the Hamiltonian's eigenvalues and energy. The instructor would have to bridge such gaps. Furthermore, in Israel, a shortage of physics teachers has resulted in many engineers (i.e., nonphysics majors) becoming physics teachers. See the next section for more information regarding the teachers enrolled in the program. We bring this description here to show that the instructors themselves have had to adapt their instruction to teachers, whether they are aware or not. Our purpose is to "reverse-engineer" the process and elicit the main elements that they had been planning to convey to teachers, with a particular focus on the interplay between mathematics and physics. Our interviewees, described in detail below, have all been involved in the program for several years.

III. THE ROTHSCHILD-WEIZMANN PROGRAM

The (nonthesis) MSc Rothschild-Weizmann Program for Excellence in Science and Mathematics Teaching (hereafter, the Program) at the Weizmann Institute of Science (WIS) marks the edge of the spectrum for in-service teacher PD in the local context. It spans 2 full days a week for the duration of 2 years, much longer than summer workshops or PLCs, that typically total 30–60 hours. It gives equal weight to advanced content courses, taught by active scientists, and pedagogical content courses, taught by science education researchers. For example, in the physics branch of the Program, physicists teach the content courses, while physics education researchers teach the pedagogical content courses.

The Program aims to enhance teachers' confidence and motivation by providing opportunities to advance their knowledge. The Program is led by the department of science teaching at WIS in partnership with the scientific departments, who devised courses tailored to the teacher audience. The participating teachers reduce their teaching load at school and receive a stipend from WIS. Many of the

Program's graduates later advance to leadership positions in their respective schools or in the ministry of education.

In the physics branch of the Program, approximately 15 teachers are enrolled at any time. They come from two main backgrounds: roughly a quarter have a physics BSc, and most others are second career teachers who were trained and have gained experience as engineers. The latter group obtained their degree either at research universities or at colleges for engineers, where they took introductory physics courses for engineers and advanced physics courses depending on their discipline. For example, mechanical engineers take a course on thermodynamics. A third group, teachers holding a BSc in physics education, has taken advanced courses in physics as well as (calculus-based) introductory courses, with their physics BSc counterparts. Most teachers have taken courses in calculus and linear algebra, with the most advanced level being that of the electrical engineering and physics graduates. Given that applicants are required to have at least 2 years' experience teaching, and that for many of the applicants teaching is a second career, 2–20 years have elapsed since they obtained their degrees.

The Program's emphasis on advanced content courses sets it apart from other (nonthesis) master's programs in science education in Israel. Facing such courses presents a significant challenge to teachers, given their backgrounds. Thus, their choice to apply to the Program, rather than the less demanding alternatives, indicates their motivation to enhance their knowledge of physics, which they also mention in their application interviews.

In terms of content, the teachers enrolled in the Program attend a preparatory workshop on "the mathematics of physics" before the first semester. During the Program, besides a full-week experimental project in an active laboratory, the teachers attend advanced courses in physics: classical mechanics, electrodynamics, quantum mechanics, statistical mechanics, and other smaller courses, e.g., applied optics, astrophysics, and particle and nuclear physics.

IV. METHODOLOGY

Our goal here is to expose instructors' considerations, formulating a characterization of the interplay between mathematics and physics that is deemed important for teachers by their instructors. We stress that our goal is *not* to study in detail the process that each instructor underwent while teaching in the Program, but to "gather" aspects of the interplay that they convey. Their long involvement with the Program makes their answers pertinent to teacher education. Thus, we intend to contribute both to the literature on the interplay between mathematics and physics and provide information to designers of teacher PD. Below, we detail the methods of data collection and analysis we employed.

A. Methods

We held semi-structured interviews with all ($N = 5$) instructors of the physics courses in the Program: Shalom, Omer, Ben, Andrew, and Hilda (all pseudonyms). We asked the instructors about their main goals for their courses, challenges that they faced, and processes of change that they had tried to elicit in the teachers. The questions are in Supplemental Material [43]. All interviews were fully transcribed. For data analysis, the first author suggested categories in addition to the top-down ones of structural versus technical aspects (Pietrocola [3] and Karam [5]). Inspired by the work of Palmgren and Rasa [6] and Koerfer *et al.* [7], we set out to identify *how* the different aspects of the interplay manifest, leading to bottom-up identification of codes from one interview. The research team discussed the proposed scheme, elucidating its definitions, and coding of all interviews ensued. A final codebook was agreed upon by the team. In addition, the first author and a fellow researcher (outside the research team) independently coded one interview. Initially, they agreed on 85% of the codes. After joint discussion, agreement improved to 92%.

As a former teaching assistant in the Program and a PhD graduate of the department of physics at WIS, the first author has a previous acquaintance with the instructors, establishing a common language with interviewees. Possible biases are balanced by the other members of the research team. In particular, the second author teaches pedagogical content courses in the Program and is its “homeroom teacher,” in direct contact with all teachers regarding Program-related issues.

B. Participants

Now that we have introduced the research methods, we may proceed to introduce the instructors and courses themselves. For the purpose of the current paper, we will focus on case studies of two instructors, Shalom and Omer, who have taught the main physics courses in the Program, for more than 5 years. They have had the longest interaction with the teachers, and their interviews have provided the richest data regarding the interplay between mathematics and physics. Where relevant, we will mention examples also from other instructors—in agreement or disagreement with the case study instructors.

Shalom is an emeritus professor, and one of the founders of the Program. He initially taught the courses on quantum mechanics, classical mechanics, solid state physics (a course that was later removed from the Program), and mathematics of physics. On occasion, he also taught a few classes in the statistical physics course. Today, he still teaches the quantum mechanics and classical mechanics courses. Not only was he instrumental in the foundation of the Program, but he has also had a significant effect on formulating its *raison d'être*: strengthening teachers’ knowledge in physics. To this day, he is the “art director,”

as he calls it, in charge of enlisting instructors from the physics department. He mostly selects instructors who are “pedagogically talented,” in his words. His main guidance to new instructors is “what you should [teach] is what will remain with them... what they will take to school... to excite students.” Concurrently, he has been teaching one of two main quantum mechanics courses for students at the physics department (graduate level) at WIS. He has recently won the physics faculty prize for excellence in teaching. In the early days of the program, he was aided by members of the department of science teaching at WIS. They observed his lessons and gave feedback on how to adapt teaching to the teacher audience.

The quantum mechanics course’s syllabus for the Program spans several topics. First, single-particle quantum mechanics: deriving the Schrödinger equations from de Broglie’s assumptions, to the energy levels of electrons in the hydrogen atom, with extensions for the periodic table. Second, quantization of fields, including the electromagnetic field, and third, light-matter interactions.

The classical mechanics course draws on Newtonian, Lagrangian, and Hamiltonian formulations. In particular, the syllabus emphasizes symmetries and conservation laws, small oscillations, the two-body problem, motion of a rigid body, and, time permitting, special relativity. Throughout the courses, the focus is less on involved mathematical technique but more on “organizing [teachers’] understanding.”

The mathematics of physics course, starting as a workshop 1 week before the first semester, refreshes teachers’ mathematical technique. Complex numbers, curvilinear coordinates, vector analysis, matrices, and (ordinary and partial) differential equations are covered. Importantly, they are taught *in a physics context*, e.g., matrices appear as the mathematical objects that embody rotations.

Omer is a researcher in nonequilibrium statistical physics. After an MSc in theoretical physics, he continued to a PhD in experimental optics. During his PhD, he was teaching high-school pupils toward their final exams, where he encountered “a collection of problematic things in what is taught.” At the same time, he was also a teaching assistant in the Program, under Shalom. After attaining his professorship at WIS, he started teaching graduate statistical physics to students at the physics department. A few years ago, he started teaching the same subject in the Program. Last year, he also taught the electrodynamics course.

The statistical physics course is computational modeling based. The teachers simulate the behavior of a two-dimensional gas of discs in a box using Newton’s equations. They make the connection between the microscopic equations and macroscopic variables, such as temperature and pressure. The electrodynamics course emphasizes the connection between electromagnetism and special relativity and designates Maxwell’s equations as the unifying framework for the subject.

The other instructors, who will complement the previous case studies, are the following. Ben is a theoretical astrophysicist teaching courses on astrophysics, one for graduate physics students and one for teachers in the Program. Andrew is a scientist in an experimental research group studying trapped ions who teaches the applied optics course. Hilda is a scientist in an experimental research group studying dark matter. She teaches the mathematics of physics course now, and the course on particle and nuclear physics.

We would like to stress that teaching in the Program is voluntary. All instructors in the Program teach by choice, with the aim of improving school teaching by “empowering” teachers and “enriching their knowledge,” as Andrew said. Beyond Omer, who had experience as a school-teacher, and Shalom, who has received guidance from the department of science teaching, none of the instructors have undergone dedicated PCK training tailored to the teacher population. Nevertheless, all instructors’ input was relevant for the refinement of categories owing to their long involvement with the teacher audience.

V. FINDINGS

Analysis of the interviews raised several aspects of the interplay between mathematics and physics that the instructors deem important for teachers. Table I contains the codes that resulted from our analysis. Some of them reproduce aspects that have already been discussed in the literature, while others expand the existing framework. The following subsections will describe each aspect. They are presented starting from aspects that have been described in detail in the literature (Secs. VA–VC), followed by those whose description we fortify (Secs. VD–VF), and finally those that are not salient in the literature (Secs. VG and VH).

A. Coherence

By far the most ubiquitous aspect of the interplay between mathematics and physics in our interviews was coherence—how mathematics connects seemingly different phenomena or theories. For example, Hamiltonian mechanics serving as the foundation for statistical and quantum mechanics, or Newtonian mechanics leading up to chaos and thermodynamics. As Omer said,

Hamiltonian mechanics or all that formalism, it may be conceptually very useful, later for statistical [mechanics], quantum [mechanics].

In fact if you understand mechanics, understand two-body collisions, you can teach temperature all the way... You talk about collisions in class, but in three dimensions and when... there

are many particles... it very quickly becomes chaotic.

Another example was Maxwell’s equations as a unifying framework for the apparently disparate electricity and magnetism:

I wanted also to reach Maxwell’s equations, so that they see how all of these separate things... [in school] you talk about... electrostatics in the beginning [of the year] and magnetic fields in the end, how that entire picture looks.

Yet another relation made apparent, by Shalom, was between waves and quantum mechanics:

I just noticed that... people don’t understand the relation between rays and waves. And when one understands... not solving equations, but drawing pictures and calculating phases along all these rays - one learns many many interesting things, and also naturally goes in the direction of Schrödinger, because that’s exactly how he did it.

Two other instructors also referred to coherence. Andrew spoke about realizing the greater context rather than particular topics:

If they teach geometrical optics at school...so they should know also beyond those drawings... and how it connects to other things, and what the limitations are.

Ben spoke about hydrostatic pressure being a key factor in the physics of stars as well as our own daily experience:

Later in the context of stars, then... like, a rather central topic in physics that you need to sort of deal with is pressure. Because you want to understand the hydrostatic structure of a star... but pressure is simply something that is very, you can sense it in class, measure it, understand its properties, and then also in the context of the star. It’s cool that **it’s also here, and also in the star.**

Finally, Shalom indicated his excitement and love of physics in the following way:

I call it the romance of physics... in the sense that you are baffled and wonder at **connections that you discover** and all sorts of deep insights.

B. Computational tool

As expected, the most rudimentary role of mathematics in physics made an appearance, albeit in fewer instances relative to other roles. This category summarizes the

TABLE I. The categories representing different aspects of the interplay between mathematics and physics, as derived from the interviews. The first three are known, the middle three are expansions of existing aspects, and the final two are not salient in the physics education research literature.

Category	Description	Example
Coherence	Mathematics connects seemingly different phenomena: rays and wave optics, (quantum) mechanics and waves, electricity and magnetism	“How did Schrödinger guess [his] equation? De Broglie said that it’s waves. So we look at equations we know. And there there is a refraction index... which is $E - V$, remember?... and this way we discovered the Schrödinger equation”
Calculational tool	Mathematics is used to technically solve a problem. Mathematical technique is addressed explicitly	“How you really sit and calculate this times this equals this...”
Extraction	Using mathematics to provide new explanations, predict new phenomena	“Gauss’ law... this is the place to talk about it... why electrostatics if we had lived in two dimensions would have been a catastrophe!”
Geometrical considerations	Implications of symmetries or geometry on physical quantities, or on possible predictions, and their universality	“I have two vectors, I want to define a product. I have nine options, right? What do I leave?... $xy - yx$ who cares?! So the correct way is because we, we have symmetries - rotations, right?”
Approximations	Awareness of limits of validity of different theories (e.g., in optics), and extension or reduction thereof	“Do you realize that... behind all these lines [rays], are waves?”
Representations	Mathematics allows passing between equations, plots, and other representations (e.g., Bloch sphere, visual simulations)	“I’m not one who likes the [formal] mathematics, for a physicist... Talk to me on the Bloch sphere”
Concretization	Derivations and operations (derivatives, integrals, matrix multiplication) are grounded in physical meaning	“We know what’s going on now, and we want to know what’s going to happen in a short time, right? We must have a derivative”
Conflict	New information conflicting with old information, or the same quantity appearing in two different contexts, requiring physics thought	“You teach electrostatics, and then jump to dc circuits. First you had the Coulomb force, meaning there’s acceleration, and suddenly the current [velocity] is linear in the voltage... what?!”

routine application of mathematical technique to reach an immediate goal, such as solving an equation, inverting a matrix, or calculating the divergence of a vector field. In particular, such calculations are done without thinking about the meaning of the different operations, and without any physics reasoning.

On the other hand, they are definitely not viewed as obsolete! Interestingly, the interviewees raised two arguments regarding the importance of sufficiently being in control of mathematical technique. Omer referred to Lagrangian mechanics as an important tool in solving extra-curricular problems that students may ask, drawing on his experience as a high-school teacher:

But when a student comes with a question that [they] can’t solve, then the solution will be either simple Newtonian mechanics, or Lagrangian mechanics. And it is a tool so that at least I will know the answer, and I know what’s going on, and I know how to guide them there... I can give such examples of a bead sliding on a rotating

hoop... In Lagrangian mechanics you find the answer very quickly and very easily.

Shalom referred to getting acquainted with such technicalities, without going into too much detail, as important for reducing the teachers’ cognitive load when they encounter them in their physics courses:

I very much tried not to go into too many technical details [in the mathematics of physics course]. On the other hand, do it such that the students aren’t afraid when [the instructors] write curl or divergence [in the later physics courses].

C. Extraction

Another aspect mentioned many times by the instructors is using mathematics to generate new knowledge. We take this category from Palmgren and Rasa, where it is defined as “a mathematical representation or its modifications [giving] rise to new information” ([6], p. 377). They discuss extraction both in problem solving, and in using

mathematics to uncover new structures or explanations in theoretical physics.

Shalom described extraction based on extending known principles to new scenarios, e.g., conservation of momentum indicating that not only particles, but also fields must have momentum:

What happens in relativistic mechanics? Actually, I don't act from here to there [action at a distance], I act from here to here to here to here [describing finite propagation velocity of disturbances]. Then if I believe in conservation of momentum, the momentum has to flow. Where does it flow? It flows to the field!

As we mention later in Sec. [VE](#), he also tells his students how realizing the limits of applicability of physics theories invites exploration beyond those limits, possibly leading to extraction.

Omer used numerical simulations to reveal the “hidden” microscopic physics laws behind macroscopic thermodynamics:

The thermodynamics that's relevant to daily life, by and large... stems from what they teach in mechanics... And you can do this with students... not analytically, but with simulations.

By solving Newton's equations for the microscopic dynamics of many particles in a box, he showed that the two-body collisions that are taught at school are the source of macroscopic thermodynamics, i.e., temperature and pressure.

He also taught how Maxwell's equations predicted electromagnetic waves:

Giving a sort of holistic picture of Maxwell's equations, that all of that together... showing them that **from there stems, you can explain what is light, and what electromagnetic waves are.**

In a similar vein, he used the universality of Gauss' law to study electrostatics in two dimensions “*Say, Gauss' law... what it means... why electrostatics if we had lived in two dimensions would have been a catastrophe!*”. Alluding to chaos appearing when many particles interact, he added “[*The teachers can say*] *look, something is happening here that is way, way beyond what we do in class...*”.

D. Geometrical considerations

A fundamental aspect of the interplay between mathematics and physics is the deep connection between symmetries and conservation laws, based on Noether's theorem. Furthermore, the universality of these and some

other relations exemplified below affords them a hierarchically higher status in physical reasoning.

In Sec. [VG](#), we will mention that Omer insists on the definition of the divergence and curl of a vector field in limiting geometric cases. Importantly, such definitions, rather than their technical Euclidean forms, are *universal*, i.e., apply in any dimension and geometry:

The goal is that they have a physical, intuitive, geometric picture, and that is why it's important for me that these definitions make contact with this picture.

Furthermore, the mathematical field of topology exemplified in Gauss' law, and its applicability in any dimension, make it an important focus for Omer:

I give you a point, how do you know whether it's inside or outside?... **it's something very, very deep.** It's more or less the deepest mathematical thing you see in high-school... so this is the place to open and talk about it and understand why it's so deep, **what would happen if we lived in a different [number of] dimensions...** which is nice!

Symmetries play perhaps an even deeper role in physics, since they constrain the mathematical objects that are permissible to describe phenomena: scalars, vectors, and higher-rank tensors, as Shalom remarked thinking about vector derivatives:

I have three components of a vector and I can differentiate them in three [directions]... I have nine derivatives. What do I call a derivative? What do I leave in the derivative? And then slowly, slowly I explain that **what is left are things that transform simply when I move from one frame [of reference] to another.** Because **that's the reason I invented the vector...** So the correct way is, **because we have symmetries** - rotations - right? And then I explained that in two dimensions it would be completely different!

Hilda echoed how symmetries (equivalent to conservation laws) constrain “the rules of the game” in a homework problem she assigned:

Saying if the interaction is such and such, given the conservation laws, whether it's possible or not possible.

In summary, we see that geometry is a common thread in physics, embodying both syntax (available mathematical objects) and semantics (universal laws applying in any dimension).

E. Approximation—simplification and invitation to explore

Approximations and awareness of their limits of applicability constitute another aspect of the interplay between mathematics and physics, in two ways. The first is reduction of a more comprehensive theory to one that applies only to a certain subset of the original domain of validity, usually implying simplification, e.g., ray optics from wave optics, as Shalom mentioned:

Lenses with many parallel lines entering, like, the lens and then leaving at angles and focusing... do you realize that behind... all these lines, are waves?

As Andrew said as well:

It's not that the geometrical optics is wrong... The wisdom is [knowing] when it is valid... As far as I'm concerned all of physics is like that.

The second manner in which approximation plays an important role is tied to another aspect, extraction (see Sec. VC). Realizing that physics laws are in fact approximations, limited in their applicability, invites extending theories beyond their currently known limits. Such extensions must reduce to the familiar physics laws in the appropriate limits. In particular, an example given by Shalom was that of Modified Newtonian Dynamics (MOND, [44]), where a nontrivial modification of Newton's laws for small accelerations explains measured galactic rotation curves:

Why is it ma , why not ma^2 ? Then I tell them the story [of MOND] because at small acceleration [dynamics is modified]... I'm just saying, here, one can **correct it**, and do all sorts of things and even explain phenomena.

F. Representations

The next aspect that we will discuss here is how mathematics allows physicists to represent physical phenomena in different forms. In particular, Omer values numerical simulations as visual experiments, giving a handle on information that is less accessible from the equations of motion:

First of all, seeing. **If I did the experiment, what would happen.** That's where the physics starts, in my opinion... In a video you get the intuition, really fast. In "kinetic energy becomes potential energy, back to kinetic energy" it sort of remains abstract.

Also, Hilda showed the teachers how to simulate the heat equation in order to gain access to the diffusion of

heat: *"It is something that pedagogically has value... when you go over the numerical procedure with them, it shows the processes that happen there."*

However, Omer warns that some representations can be misleading:

...in Gauss' law, when they draw on the white-board and count field lines, or at least that's what appears in the books, and of course the drawing is not valid... the drawing is two dimensional and... the reality is three dimensional and the density [of field lines] looks completely different.

He is referring to the fact that the density of field lines, an indication of the field strength, differs among dimensions. In two dimensions, the electric field decays as $1/r$, while in three dimensions, it decays as $1/r^2$.

Shalom advocates for drawing waves in teaching quantum mechanics, because a pictorial description is accessible to pupils:

[I want them to take it to school] Because in quantum [mechanics] it's waves, and you draw, and I'm sure that the waves will help them... and that thing that waves are usually quantized. Why are they quantized? Because they don't like sitting in a closed space like you want, only when it suits them, and that is their quantization.

He says that drawing waves in a closed domain (say, a potential well or a string) helps understand their quantization due to interference.

G. Concretization in physical context

Using mathematics in physics provides a tangible context in which to understand mathematical operations, e.g., derivatives and integrals can be viewed as rate of change and accumulating contributions, respectively [45,46]. In our interviews, other mathematical operations arose. The divergence and curl of a vector field were defined as the limits of flux divided by volume and circuit divided by surface, respectively, as Omer said:

Because first, that is the physical meaning of these things, and second, because electrodynamics is something you do with curved coordinates...

His comment on curved coordinates is related to his statement mentioned in Sec. VD, where the "geometric picture" allows a universal perspective.

Shalom, on the other hand, mentioned teaching matrices and determinants in the context of vector transformations:

When I teach them that a vector transforms... I tell them "here are x, y, z and here is x' "... Ah,

[so you need a] matrix... And then if I want to invert it, here, I know from x to x' ... so I already have a determinant...

Finally, Shalom mentioned the Fourier transform as providing an answer to the question “when is a wave a wave?” in a homework exercise. More precisely, it defines the frequency of a wave:

Here, this thing [drawing a truncated sinusoidal curve] - is this a wave or no? And what about this [drawing a longer truncated sinusoidal curve]? Is this a wave? And then I explain to them the difference, right? And the difference is in Fourier, right? So I say that eventually, a true wave is one that extends to infinity... as this thing [the truncated curve] becomes shorter, there are more frequencies.

Hilda summarized her approach to teaching the mathematics of physics course: “*This is how it [some mathematical operation] is done. And this is what it means in terms of physics.*”

Many of the quotes, especially the first, are indicative of how physicists blend physics and mathematics, a practice that has received attention in the context of problem solving by students [17,19].

H. Conflict as a discovery mechanism

The most unusual aspect that arose in the interviews was using conflicts indicated by the mathematics as vehicles of discovery. Theoretical results must cohere, and when they do not, one must ask: Why? For example, Shalom mentioned the dual role that mass has, once as inertial mass, and otherwise as a gravitational “charge”:

What is m ? Ah, more matter... but here there is confusion, right? Because matter usually says how much Earth pulls me... **gravitational mass** is not a wonder - it's a charge... **Inertial mass** is... I oppose when you push me? What's going on here? You know, it's something that is really not trivial.

Having the same quantity play two seemingly different roles invites inquiry into the origin of both. This deep connection can be seen as a precursor for the general theory of relativity, where Einstein realized that gravity is a fictitious force.

Similarly, Ohm's law is at odds with Coulomb's law, due to the voltage maintaining different relations to the velocity, as Omer put it:

You teach electrostatics, and then you jump to dc circuits... Even though you say that the voltage is related to the potential difference that you had

taught in electrostatics, but it is completely, completely, completely unclear... First you had the Coulomb force, meaning there's acceleration, and suddenly the current is linear in the voltage - what?

In Ohm's law, a potential difference is proportional to the current (velocity), while in Coulomb's law, it is proportional to acceleration. Coherence can be restored once drag forces are employed in the first case, leading to the terminal velocity that appears in the current. Drude's model is one method of incorporating scattering into the current [47].

Omer also mentioned a historically important example—the conflict between the Lorentz force and Galilean invariance:

For example the Lorentz force, that the force depends on velocity and in eleventh grade, the year before, we learned that the velocity is... I can choose any reference frame as long as it's inertial, the velocity is [changing, while the force should not]... so suddenly things don't add up.

It was the discrepancy between how the Lorentz force depended on velocity that eventually gave rise to special relativity, which restored coherence once more.

Finally, Shalom mentioned another fundamental example of a conflict related to classical trajectories in mechanics:

For example, take even the minimal action [principle]... how does a system know where to go?... Newton said... step by step. But then came... Hamilton and said that there are also these paths [the non-classical trajectories]... What is going on here? How do you suddenly globally know where you're going? And then suddenly came Feynman and said “no, no, no, it's not... every path has a probability!”

Here, once again the fact that the minimal action principles allows “scanning” through all possible trajectories, implying a system somehow knows how to “choose” the one corresponding to extremal action, was in stark contrast to Newton's differential formulation, which used initial conditions on derivatives to deterministically predict time evolution. Eventually, Feynman's path integral formulation gave a conceptually clear explanation for the classical trajectory contributing the most to constructive interference when the action was much greater than $\hbar$ [48].

VI. DISCUSSION

The interplay between mathematics and physics is a rich field of research that is becoming deeper and deeper [2].

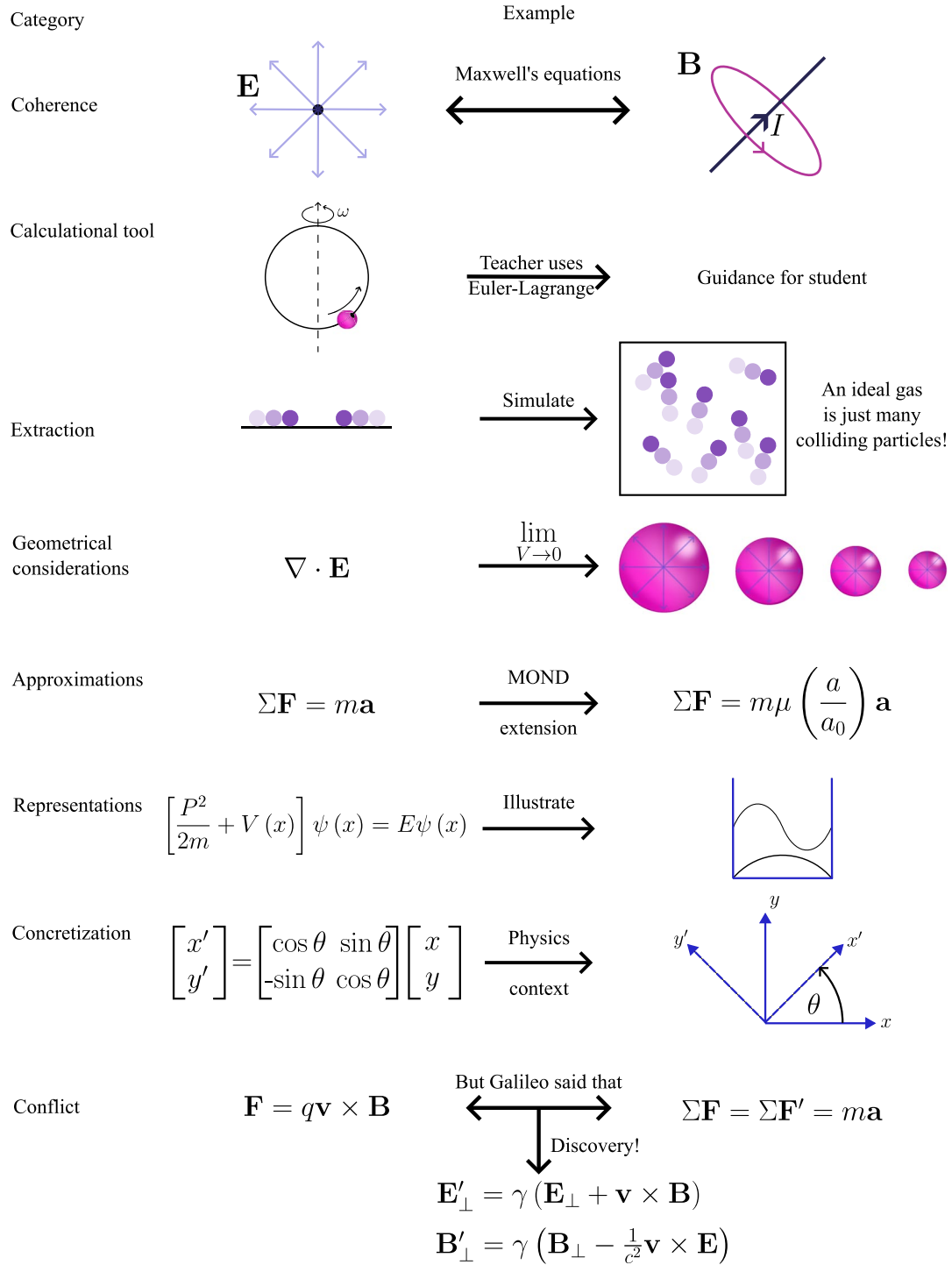


FIG. 1. Illustration of the aspects of the interplay between mathematics and physics, as exemplified in our findings.

In the present work, we were able to highlight aspects of the interplay, which were not salient in the literature and could be relevant for teaching, thanks to the unique context we studied, an MSc program for physics teachers with a strong emphasis on advanced disciplinary content. An illustration of the different aspects of the interplay is in Fig. 1. We will now relate our findings to previous work.

The first aspect, that mathematics provides a **coherent framework** for physics, has been discussed in the literature, e.g., by Palmgren and Rasa [6], in their roles “syntactic structure” and “representations and semantics.” Here, we reinforce their history-based findings with empirical evidence from teacher instructors, who, leveraging their knowledge as active research physicists,

emphasize the connections between different fields of physics. For example, they discuss how pressure is a variable playing a role both on Earth and in stars (mentioned by Ben), or how the mathematics of waves underlies both optics and quantum mechanics. Shalom expressed it poetically as “the romance of physics.”

The second aspect, mathematics seen as a **calculational tool**, is ubiquitous across STEM, as Koerfer *et al.* [7] have shown and is not unique to physics. In fact, most learners of physics see mathematics as limited mostly to this role [3,6]. Due to the plethora of structural aspects of the interplay between mathematics and physics, one may think that physicists view the technical role as “lesser.” However, as we have found here, the view of the Program’s teacher instructors is not so monolithic. In fact, one cannot help but remember the iconic “shut up and calculate!” that Mermin attached to the Copenhagen interpretation of quantum mechanics [49], indicating the important role of calculation in physics research. Omer, for example, drawing on his own experience, sees advanced formalism, such as Lagrangian mechanics, and its associated technique, as important for teachers to assist their students when they encounter extra-curricular problems. Shalom challenged the view advocated by Pietrocola [3] that teaching mathematics before physics perpetuates its technical role. He claims that early exposure to technicalities assists learners in paying attention to essential physics rather than technique later in their studies. In particular, Hilda, who now teaches the mathematics of physics course, agrees with Shalom that the goal of her course is to bring teachers “up to speed” on mathematical technique so that they can “focus on the physics, and not... how to diagonalize a matrix.” Indeed, the debate on the degree of integration between mathematics and physics in teaching is still ongoing [27]. Here, we highlight our interviewees’ suggestions for the potential of advanced technique (such as solving problems with Lagrangian mechanics) in helping teachers guide their more curious students.

Our third aspect is **extraction**. Using mathematics to explore beyond current knowledge is the driving engine behind discovery. If we did not have mathematics, we could not make statements beyond “The more A, the more B” [14]. In our interviews, extraction had several manifestations. First, applying a fundamental principle to a scenario that had seemed irrelevant to it, either applying conservation of momentum to fields as Shalom discussed, or Gauss’ law in two dimensions, as Omer mentioned. Second, solving seemingly simple equations for many particles, going from the single- or few-body limit to the many-body limit. In this context, using computer simulations revealed that the physics behind macroscopic phenomena such as temperature is no more mysterious than Newton’s equations, which was a key goal for Omer. Third, manipulating equations to make a prediction, such as deriving electromagnetic waves from Maxwell’s

equations as Omer had done. “Extraction” was identified by Palmgren and Rasa [6]. In addition, it is an amalgam of “a key to understanding” and “an agent for development” in Koerfer *et al.* [7].

The fourth aspect is the profound role that **geometrical considerations** play in physics. Since their role is so vital, we have decided to dedicate a separate category to their manifestations. For example, gauge symmetry is a beautiful notion at the basis of electromagnetic field theory, which served as the foundation of quantum chromodynamics [50]. Recognizing that much of physics can be derived using geometric reasoning, Hestenes [51] has suggested Clifford algebras as a unified language for mathematics and physics. The importance of similar ideas came across also in our findings, when both Shalom and Omer mentioned how a “geometric picture” of physics is useful for a physicist. On the one hand, it affords a universal perspective that can guide thought in any dimension, while on the other, it dictates the forms of physical laws due to symmetries. Therefore, geometric reasoning plays both a semantic and a syntactic role. In particular, the benefits of the geometric definition of divergence as the limit of flux over volume when the volume vanishes have been discussed in the context of an integrated mathematics/physics course [52]. Thinking using geometry could be included under Palmgren and Rasa’s role “reasoning and modifications,” however, we feel that it extends beyond “modification of representations.” We claim that geometric arguments stand at the extreme “semantic reasoning” that they mention; according to their definition, such reasoning “heavily relies on the insight the reasoner has on the representation and on how the representation connects and maps onto the physical background theory of the studied system.”

Asher Peres, who contributed much to the field of quantum information, wrote: “Physics is not an exact science. It is a science of **approximations**,” [53] which refers to our fifth aspect. Indeed, approximation methods constitute a significant part of mathematical modeling in physics [54]. Their role is not limited to simplification, though. An important aspect of approximations in physics is intimately tied to extraction. As Shalom made clear with the example of MOND [44], awareness of the approximations that physicists make is a hint to further examine cases where they break down. Interestingly, mention of approximation as an invitation to explore is scarce in the literature, though it pervades research; see, e.g., [55], where the researchers developed an approximation scheme extending beyond previous analytical approaches.

Fluency in different **representations**, e.g., equations, plots and diagrams, is a sign of expertise in physics [56]. For example, within the same representation, a *purposeful transformation* shifts the meaning of a representation based on disciplinary interests, making it “more useful for creating and communicating knowledge than its initial

form,” as defined by Kapodistrias and Airey in the context of manipulating the Friedmann equation [57]. In our interviews, computational modeling arose as an additional representation, which rather than being static, is dynamic. In fact, Ben, who teaches astrophysics, said that “*In physics, a lot of what we talk about are ideas that... are not very complicated, but they require some... imagination of space*”. Therefore, we suggest that computer simulations can serve as aids for those who are less inclined to “imagine,” as he put it. Furthermore, they allow visualization of physical processes and virtual experimentation that can promote understanding, as Omer pointed out. In their recent systematic review, Palmgren *et al.* [2] identified the role of mathematics as informing experimental planning and interpretation of experimental results. Virtual experiments are an important application of this role.

Physics **concretizing mathematical operations** that are abstract is another aspect. By explicitly connecting operations such as the divergence of a vector field to the flux through a closed envelope, learners can have a geometric mental model [58], which, according to Omer, is the proper physical meaning behind the divergence. Furthermore, the concept of wave frequency is normally introduced in relation to the time difference between two consecutive maxima, without explicitly referring to the length of the pulse. The Fourier transform puts the definition in a wider context, with pulses of arbitrary lengths. Cleverly, Shalom challenges the elementary definition in his homework exercise called “when is a wave a wave?” In the aspect of concretization, the interplay between mathematics and physics is indeed two-way as demonstrated by our interviewees’ examples: physics concretizes mathematical operations with, e.g., geometrical meaning, while mathematics extends a rudimentary, concrete definition to more elaborate situations. Note that the “service” that physics provides in this case to mathematics is not salient in the literature discussing the roles of mathematics in physics. In the mathematics education literature, by contrast, Towns *et al.* [59] studied the *mutual benefits* of meaning-making in physical chemistry using calculus, and of using physical chemistry contexts to make meaning of calculus concepts. A similar concept is the blending of mathematics and physics [17,60], which can be seen as an application of this aspect in the particular context of problem solving.

Our most unusual finding is **conflict as a discovery mechanism**. Sometimes, equations seem to contradict each other or basic principles. Such are the two following cases that bothered Omer, drawing on his experience as a high-school teacher. One, voltage in Ohm’s law is proportional to charge carriers’ velocity, whereas according to Coulomb’s law it indicates acceleration. The conflict is resolved and coherence restored by invoking drag forces in the first case. On a more fundamental level, relaxation to

equilibrium according to the second law of thermodynamics governs the first case [61]. Two, the mismatch between the Lorentz force’s dependence on velocity and Galilean invariance is a famous example of physicists’ pursuit of coherence leading eventually to a fundamental change in theoretical physics, the advent of special relativity. In other cases, different formalisms seem at odds, as Shalom discussed regarding Lagrangian and Newtonian mechanics. A system’s “choice” of Newtonian trajectory among all paths allowed in Hamilton’s least action formulation appeared to grant inanimate objects a consciousness. Feynman’s path integral formulation reconciled the apparent conflict with a conceptually clear explanation relying on quantum mechanics [48]. It is mathematics that pinpoints such discrepancies and motivates their resolution. In the characterization of Palmgren and Rasa, “extraction” is described as modifying a representation to glean previously inaccessible information. We, on the other hand, emphasize that conflict indicated by mathematical expressions or formalism is another mechanism for discovery. Perhaps the conflict can be viewed as a precursor to “extraction,” on the way to resolution.

Our interviews with teacher instructors in an MSC program, who are active research physicists, have yielded a plethora of aspects of the interplay between mathematics and physics that instructors wish to convey to teachers. Previous work has provided a broad characterization of the interplay on a conceptual level, with rather broad implications for teaching. We have taken Palmgren and Rasa’s invitation to heart: “This would also force us to rethink physics instruction, taking this entanglement [the interplay] into account and making it explicit in teaching” ([6], p. 379). We therefore propose our identified aspects as possible learning goals manifesting explicitly the connection between mathematics and physics in a far wider context than just technical.

VII. IMPLICATIONS FOR DEVELOPMENT

We appreciate the crucial role that teachers have in educating the next generation of physicists. Investing in their PD is therefore an important priority for physics education research. First and foremost, our findings indicate that advanced physics content could expose in-service teachers to “how a physicist thinks”, e.g., highlighting the importance of coherence. Research shows that advanced content knowledge could influence teaching. In chemistry teacher PD, for example, Tuvi-Arad and Blonder studied a workshop on molecular symmetry for high-school chemistry teachers. Exposing teachers to advanced content knowledge on symmetry helped them “approach the final exam in chemistry.” In addition, they themselves made the connections to the school curriculum (e.g., relating polarity and chirality to symmetry), potentially influencing their teaching [62]. In mathematics, Even and Mytlis studied teachers who had begun teaching

high-school mathematics after completing a nonacademic teacher education program. The teachers later enrolled in an academic program, and the interview-based research gave many examples of how their teaching had changed following the program. For example, one teacher reported that she now presents students with a small number of general formulas for antiderivatives, guiding them to use them in specific cases. Previously, she had given students long lists of formulas, which during her academic studies she noticed were examples of more general cases [63].

We highly recommend incorporating advanced content into PD programs, to portray the rich interplay between mathematics and physics. De Ataide and Greca have already said “If, these relationships [between mathematics and physics] are unclear for teachers, it is quite likely that students will also fail to grasp their true nature and will assume a naive attitude,” ([23], p. 1406), indicating the importance of teacher education in this context. What practical implications could our findings have for teaching in high school and beyond? In the following, we propose several activities inspired by our interviewees’ comments. Our suggestions require empirical testing in classroom teaching.

For instance, regarding approximations, in particular in optics, a high-school teacher can tell their students that ray optics is an approximation of wave optics and discuss when either one is valid, as both subjects are taught in high school.

Concretization of mathematical operations can be done by relating derivatives to rate of change [59], or integrals to accumulation of independent contributions [45,46]. An activity combining coherence, extraction, and conflict is the following. High-school students face a conundrum. In mechanics, they learn that energy is always conserved, yet friction somehow takes away energy from the point particles they model sliding down inclined planes. Where does that energy go? A solution would be invoking some form of the second law of thermodynamics, which can be explored in simulation without explicitly mentioning entropy: a single particle interacting with many particles in a box will lose (some of) its energy to “the environment,” leading to equipartition of energy [64]. As the “environment” grows, so does the energy loss.

Reasoning geometrically at undergraduate level can be done in the context of teaching Gauss’ law in the special case of a sphere, indicating how the electric field decays as the inverse squared distance from a point charge, $1/r^2$, while the area of a sphere increases as r^2 [4]. Proving Gauss’ law for arbitrary closed surfaces (see, e.g., Feynman [48]) could be an even more productive exercise. Furthermore, taking Gauss’ law as universal and then applying the same reasoning in two dimensions will yield a different dependence of the electric field on the distance from the point charge, which is relevant for vortex binding-unbinding transitions in two-dimensional superconductors [65].

Another possible example of extraction, together with representation, employed in high school by one of the Program’s graduates, could be simulating the two-body problem already in late high school. By changing the initial conditions, students will be able to explain comet trajectories, or discover open trajectories. Demonstrations of conflict leading to discovery could be those that were mentioned in the main text: the Lorentz force contradicting Galilean invariance, or Ohm’s law contradicting Coulomb’s law.

VIII. LIMITATIONS, CONCLUSIONS, AND OUTLOOK

Our study has several limitations. First, we interviewed only a few teacher instructors whose commitment to teaching yielded interesting findings. Therefore, our ability to draw general conclusions is limited. On the other hand, we have seen that our findings reproduce and extend previous characterizations of the interplay between mathematics and physics. Second, the first author, who graduated from the physics department at WIS, is well acquainted with some of the interviewees, naturally introducing biases. The involvement of the other members of the research team serves to balance these.

We have provided a characterization of several aspects of the interplay between mathematics and physics, both technical and structural, that teacher instructors deem important for teachers. The Program instructors’ disciplinary perspectives as active physicists are not necessarily articulated to teachers in other programs. In particular, one of our interviewees, a former high-school physics teacher, said that advanced formalism, with its associated technique, had helped him support his curious students at school. Advanced technique could potentially assist other teachers as well. Nevertheless, the discussion on integrating mathematics and physics in teaching remains unresolved. Another interviewee challenged the notion that learning mathematics before physics perpetuates its technical role in physics [3]. He argued that it allows learners to focus on the structural aspects when learning physics. For example, they could potentially notice aspects that we have found: how conflict leads to discovery, how approximations are invitations to explore, or the universal role of geometry in physics. In addition, we have offered several ideas on how to implement our findings in the high school or university classroom.

An important piece of the puzzle is missing in our work—the perspectives of the teachers who graduated from the Program. As mentioned in Sec. IE, we plan to interview graduates of the Program to see how they perceive different aspects of the interplay between mathematics and physics, and how they manifest them in their teaching.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available because they contain sensitive personal information. The data are available from the authors upon reasonable request.

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