

Student understandings of discrete and continuous in quantum mechanics: A thematic analysis

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Students in a spins-first quantum mechanics course often have difficulties transitioning from spin systems to continuous systems like the infinite square well. Our study investigates how students understand discrete and continuous quantities in quantum mechanics and the connections the students make between them. We interviewed six students who were concurrently enrolled in a junior-level spins-first quantum mechanics course with a computational lab. During the interview, students organized a set of 20 cards containing various quantum mechanics concepts and representations, including snippets of Python code. Using a phenomenographic approach, we generated four themes: understandings of discrete, understandings of continuous, discreteness of representations, and discrete approximations of integrals. Participants often related discrete and continuous quantities in the context of code, suggesting that computational activities can support students in transitioning from discrete systems to continuous ones.

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I. INTRODUCTION

Upper-division quantum mechanics (QM) is a well-studied area of physics education research. For example, researchers have studied students' understanding of a variety of topics such as the math-physics connection in QM [1–3], energy measurement [4], time evolution [3,5], relative phase [6], probability [7], change of basis [8–10], and uncertainty principle [11]. There is also research on students' use of notation [12–14], representations [15,16], and their epistemologies and ontologies in QM [17–19].

Quantum mechanics has historically been taught by starting with the Schrödinger equation in the position representation for a particle in an infinite 1D square well [20], a “position-first” approach. Recently, a “spins-first” approach has become a common alternative [21,22]. Spins-first curricula introduce quantum mechanics in the context of Stern-Gerlach experiments with spin-1/2 particles.

Research comparing the efficacy of spins-first and position-first curricula is limited but has identified some differences in student learning. Generally, this research points to potential benefits of a spins-first curriculum. Sadaghiani *et al.* found that sophomore-level modern physics students taught with a spins-first course outperformed peers taught with a position-first course on the quantum mechanics concept assessment [23,24]. They

suggest that the comparatively simple mathematics of spin systems allows students to shift their focus from computing results to sensemaking. In a later study, Sadaghiani *et al.* compared spins-first students' and position-first students' ideas about expectation values [25]. They found that spins-first students often offered multiple methods for calculating expectation values and often wrote expressions in Dirac notation and summation notation. Position-first students often made errors in writing the order of the operator and state vectors in expectation value calculations, while no spins-first students made these errors.

Passante conducted a study to compare the resources spins-first and position-first students activate in the context of energy measurement [4]. Spins-first students were given an energy measurement question in the context of a spin-1/2 particle in a magnetic field, while position-first students were given a question in the context of an infinite square well. They found that the spin questions about the Hamiltonian and probabilities elicited more conceptual resources, while the position questions elicited more mathematical resources, but they were unable to conclude if the differences were due to instructional differences, question context, or question wording. Riihiluoma *et al.* conducted a comparative analysis on spins-first and position-first students' understanding of different quantum mechanics expressions. Using network analysis techniques, they found that spins-first students tend to strongly conceptually link bras and kets with generic vectors and quantum states, whereas position-first students conceptually link wave bras and kets with wave functions [26].

At Oregon State University, where our study was conducted, quantum mechanics is taught with a spins-first

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approach [15] using McIntyre’s textbook [21]. Instructors have noticed that the transition from spin systems to the infinite square well is difficult and that students do not leverage their understanding of discrete spin systems for computations in the context of the continuous infinite square well as much as the instructors expect. To understand this phenomenon better, we explore the following research questions:

- RQ1: How do these students understand “discrete” and “continuous” in the context of quantum mechanics?
- RQ2: What connections do these students make between their understandings of discrete and continuous objects in quantum mechanics?

We conducted semistructured interviews with six junior-level students concurrently enrolled in a spins-first quantum course and an accompanying computational lab course. In this paper, we present a breadth of understanding that participants had about the meanings of discrete and continuous in quantum mechanics. These understandings may not encompass all the ways students understand these topics, but we expect that descriptions of varied understandings identified in this study will be informative to instructors. We will describe our methodological choices, interview process, analysis, and results and will conclude with how the themes are related and their instructional implications.

A. Discrete and continuous systems

Our views of discrete and continuous are aligned with McIntyre’s description [21], the textbook we use in our quantum mechanics courses. Any state in a spin Hilbert space can be completely expressed as a superposition of a finite number of orthonormal basis states. The basis states are quantized and therefore discrete, meaning the eigenvalues associated with the basis states have permitted values with a defined spacing between them. We will refer to spin systems as discrete systems. For spin systems, the probability amplitude is defined as the inner product of two spin states in the same Hilbert space. This operation requires the summation of the element-wise products of the states and results in one scalar value, as shown in the example below.

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle, \quad (1)$$

$$|\psi_2\rangle = \frac{1}{2}|+\rangle + \frac{\sqrt{3}}{2}|-\rangle, \quad (2)$$

$$\begin{aligned} \langle\psi_2|\psi_1\rangle &= \frac{1}{2}\frac{1}{\sqrt{2}}\langle+|+\rangle + \frac{1}{2}\frac{1}{\sqrt{2}}\langle+|-\rangle + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\langle-|+\rangle \\ &\quad + \frac{1}{2}\frac{\sqrt{3}}{2}\langle-|-\rangle = \frac{1+\sqrt{3}}{2\sqrt{2}}. \end{aligned} \quad (3)$$

Our quantum mechanics course ends with the introduction of the infinite square well (or particle in a box) and the position basis. Any state in this particle-in-a-box Hilbert space can be completely expressed as a superposition of a countably infinite set of orthogonal basis states (i.e., energy eigenstates) or an uncountably infinite set of orthogonal basis states (i.e., position eigenstates). The eigenvalues of the energy eigenbasis have permitted values with a defined spacing between them and are therefore discrete. In the position representation, however, the state of a particle, or wave function, is not quantized and can take on any value within the box. We will refer to the infinite square well as a continuous system because it can be expressed in terms of a continuous basis (unlike spin systems). Even the discrete energy eigenstates of the infinite square well can be written in the position representation as functions. For infinite square well systems, inner products of two wave functions in the position representation require integration with respect to position, as shown below:

$$\langle\phi_n|\psi\rangle = \int_{-\infty}^{\infty} \phi_n^*(x)\psi(x)dx. \quad (4)$$

In both examples [Eqs. (3) and (4)], the inner product is a sum of products over each basis state. Wan *et al.* found that many students do not recognize the similar process by which probabilities and inner products are calculated regardless of the continuity or discreteness of observables [7].

II. METHODOLOGY

This section will address the rationale of using a phenomenographic approach [27], detail the reflexive thematic analysis we used to analyze data [28], and share details about concept images [29], the lens through which we viewed the data.

A. A phenomenographic approach

Our research employs a phenomenographic approach to explore our research questions. Phenomenography is a qualitative research approach developed in the 1970s by researchers at the University of Gothenburg [30,31] to explore the various ways that people understand and experience a phenomenon and is an established method in physics education research, e.g., Refs. [11,32–34].

Phenomenography adopts a relational ontology where there is no separation between an individual’s internal thinking and an external world. Rather, an individual’s thinking and the external world are related to each other through the individual’s awareness and experience of the world [31]. Phenomenography, as described by Marton [27], adopts a second-order perspective where researchers orient themselves to the descriptions and ideas

people have about a phenomenon as opposed to what the “essence” of the phenomenon is [27].

A phenomenography does not attempt to identify all the minute variations in understandings that people have of a phenomenon but rather seeks to find the few salient variations that comprise a collective experience [35]. These variations can be between how individuals understand the phenomenon but can also be within an individual’s understanding of the phenomenon. The correctness of these conceptions and variations of understanding are not relevant to phenomenography, as the relationship between the individual and the phenomenon exists regardless of whether the idea is valid or not. From an instructor’s perspective, though, the correctness or productiveness of students’ understanding of a phenomenon is relevant. We discuss the instructional implications of our findings in Sec. VI.

Phenomenographic studies typically use semistructured interviews to probe the participant’s understanding and relationship to a particular phenomenon [27,35,36]. The outcome of a phenomenographic study is a set of categories that describe the variation in participants’ understanding of the given phenomenon. We generated these categories by using a thematic analysis described in Sec. III and an outcome space described in Sec. V.

B. Concept image framework

The concept image framework was developed by Tall and Vinner [29] to address the ways that one understands a concept. They describe a concept image as the cognitive structure that accounts for all of the mental pictures, properties, impressions, experiences, and understandings on a given topic that an individual has. They explain that a concept image is something that develops over time and does not need to be coherent, i.e., parts of the concept image may not align or conflict. They further describe an *evoked* concept image, which is the portion of a concept image that is activated in response to a given stimulus. Different stimuli can evoke different portions of one’s concept image.

The focus of this research is not on presenting the participants’ individual concept images or contrasting them with a professional concept image. Rather, we used the concept image framework as a way of orienting ourselves to the data. The interview task (described in Sec. III) used multiple representations to cue participants into sharing verbal descriptions of their understanding of discreteness and continuity. The concept image framework naturally accounts for the coordination of multiple representations, which helped us to identify the meaningful descriptions offered by participants.

III. METHODS

In this section, we give context to the data we collected and share our data collection methods, as informed by our methodology.

A. Instructional context

Junior-level students at Oregon State University take a series of intensive, 5-week long, 7 contact hour per week courses called the *Paradigms* [37]. The quantum fundamentals course in the *Paradigms* series uses a spins-first approach to introduce quantum mechanics and transitions to the particle in a box during the last week of class, following McIntyre’s *Quantum Mechanics* [21]. It is the first quantum-focused course in Oregon State University’s physics curriculum and covers the postulates of quantum mechanics, quantum measurement, uncertainty, and time evolution.

Quantum fundamentals is paired with a computational lab where students solve quantum mechanics problems computationally. The computational lab introduces students to basic computational methods in Python and utilizes pair programming [38], a code writing technique that involves partners switching roles between typing code and directing what to be typed. This course covers the same topics as the quantum fundamentals course.

B. Broader research context

The study detailed in this paper is a part of a larger project on students’ transition from discrete observables to continuous observables. Each student co-enrolled in quantum fundamentals and the computational lab in Spring 2021 (approximately 30 students) was invited to participate in a research study that involved in-class observations and an interview at the end of the term. Six junior students agreed to participate in the study. These students were assigned pseudonyms Mary, Brandon, Charlie, Dave, Erica, and Fred. We did not collect demographic data because these factors were not salient to our analysis focusing on students’ conceptual understanding. Additionally, due to the small sample drawn from a single course, collecting and reporting demographic information may risk compromising participants’ identities.

The observation period allowed author Solorio to build rapport with each of the participants before conducting individual interviews. The rapport building allowed participants to become more comfortable expressing their understanding of physics concepts in the presence of an observer. It familiarized the interviewer with the language each participant used and provided course context for what participants said during the interview. By conducting the interviews after building rapport from the observation period, we were able to make informed interpretations of what participants said during the interviews. Because we would not have the same contextual nuance and rapport with other students, we did not solicit additional participants for interviews.

C. Card sorting task interview process

Interviews were conducted with the six participants and were semistructured, approximately 1 h long, and conducted over Zoom. The interviews were interactive in

nature, following the considerations offered by Solorio *et al.* [39]. Screen, audio, and webcam recordings were collected (for participants who had access to a webcam and were comfortable being on camera). The full interview protocol is provided in Appendix A.

The interview was structured as a card sorting task, an interactive research method where participants showed how they understand and organize concepts by organizing a set of cards [40]. This type of interview task allowed us to probe some of the broader connections students make between different systems, notations, and bases, which aligns well with our research questions. The 20 cards used for this interview are shown in Fig. 1. In the text, individual cards will be referred to by the symbolic representations shown on them, when possible. Cards that are difficult to display in text will be identified using the labels shown in Fig. 1.

We chose these 20 cards because they span the core concepts in the quantum fundamentals and computational lab courses (e.g., energy, states, probability, and inner products) for two systems (spin states and particle in a box) in a variety of representations (wave function, Dirac, matrix, and code). Each of the cards contained expressions and concepts that were commonly used throughout both courses and relevant to ideas of discreteness and continuity in quantum mechanics. Spanning course content allows for participants to make many potential connections between the cards and facilitates rich discussion of their understanding of the material. The richness of the activity was balanced with the cognitive load, resulting in limiting the number of cards provided to 20.

For the majority of the interview, participants organized 20 cards containing a variety of representations and notations of quantum mechanics content (see Fig. 1). To do the interview remotely, we used Optimal Workshop's web application Optimal Sort [41]. This web application allowed participants to move cards into groups, move cards between groups, name groups, and move groups. The interview followed a phased approach, with two open card sorts and then a fixed card sort [40]. During the free sorts,

participants were asked to group the cards without being given any criteria. For the closed sort, participants were asked to organize the cards into two groups (with the option to create more groups if participants wanted to). The different parts of the interview are outlined below:

- Description of cards: Participants looked at each card and gave a short description of how they understood it.
- Initial free sort: Participants organized the cards into groups in a way that made sense to them.
- Second free sort: If possible, participants organized the cards in another way that made sense to them.
- Discrete and continuous sort: Participants were asked to organize the cards into a discrete group, a continuous group, and any other categories they wanted to include.

Participants started the activity by providing a short description of each card to understand their interpretation of them and to familiarize participants with the set of cards. Following this, three phases of sorting the cards occurred. First, participants engaged in a free sort to further familiarize participants with the cards, capture any unprompted ideas about discreteness and continuity, and to elicit the organizational schemes participants used when not constrained by predefined categories. The purpose of the second free sort was to prompt participants to notice the variety of potential organization schemes they could use when discussing the cards. The discrete and continuous sort was used as a final task to elicit participants' ideas about discreteness and continuity in quantum mechanics.

D. Analysis

We value being transparent and detailed in our analysis approach to demonstrate its rigor. However, we recognize different readers may have various goals and priorities in reading this manuscript. For those primarily interested in the results of our analysis, see Sec. IV.

We adopted Braun and Clarke's reflexive thematic analysis approach to explore the data [28]. Thematic analysis allowed us to engage with the qualitative data

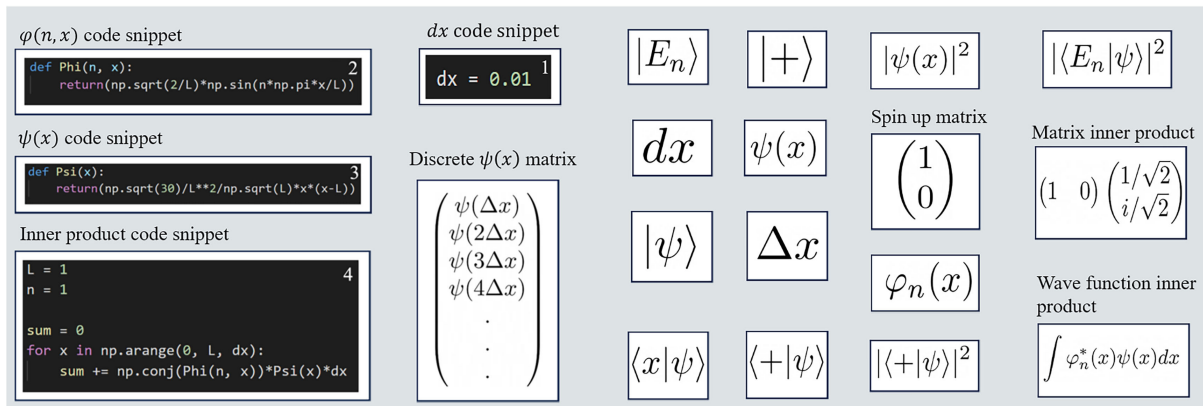


FIG. 1. Twenty cards participants were asked to sort during the interview task. The code snippets are labeled 1–4 in white in the top right corner for easier identification in images of participants' card sorts.

produced to conduct the phenomenography, i.e., finding the shared and varied experiences of a phenomenon between participants. Reflexive thematic analysis acknowledges and centers the role of the researcher. Rather than strive for interrater reliability, researchers using reflexive thematic analysis strive to be coherent, transparent, and aware of how their positionality and theoretical assumptions shape the themes we generate from the data [42]. Because reflexive thematic analysis forefronts the researchers' subjectivity and rejects positivist assumptions of objectively correct interpretations of data, the codes produced by one researcher may not align with codes generated by another researcher. The iterative, fluid process of coding makes meaning of the data to inform the formation of themes that capture those meanings. Consequently, the themes produced in our analysis may not be reproducible by other researchers. Our analysis, however, is still informative to quantum mechanics instructors and trustworthy to other researchers (further discussed in Sec. III E).

The analysis process was iterative between and within each step. In the first step, we familiarized ourselves with the data by watching each interview, noting the timestamps where a participant said or did something interesting (generally something displaying their understanding of discrete or continuous systems). We then transcribed portions of the interviews where participants said or did something related to their understanding of discrete or continuous systems. These relevant statements were made during each phase of the interview and were often unprompted during the initial free sort and second free sort and prompted during the discrete and continuous sort where participants were explicitly asked to explain their reasoning for placing a card into the discrete or continuous group.

These transcript excerpts were typically 15 to 60 s and were compiled into a spreadsheet. In the second step, initial codes were generated for the data. In reflexive thematic analysis, codes are researcher-generated labels that identify a feature of the data. To immerse ourselves in the data, we generated latent codes (which interpret participants' meanings) and semantic codes (which label explicit or surface meaning) for each datum [43]. Latent codes were developed through the concept image lens. Consider the following quote:

Brandon: I suppose the expectation value, we do get a scalar from that. I suppose that's discrete. Yeah, because it's just one point.

The semantic codes generated for this piece of data were "scalars are discrete" and "expectation values are discrete." This piece of data was given a latent code "discrete things are one point" (see Sec. IV A 1 for more details regarding the interpretation of this quote). These latent and semantic codes were discussed between the authors to come to some mutual interpretation.

The semantic codes were iterated throughout the analysis process. These semantic codes were then combined in the third phase of the analysis into loose, overarching themes. In the fourth step, the themes were discussed between authors, revisited, and iterated several times until clear. Subthemes, or variations within a particular theme, were also identified, discussed, and refined. In the fifth step, the themes were given a concise name and definition.

E. Establishing the trustworthiness of the analysis

We aim to provide faithful interpretations of the data we collected using methodological procedures to establish the trustworthiness of our analysis. We provide thick descriptions of the data used to generate our themes that detail the contexts, actions, and language used by participants in communicating their understanding throughout the interview [44]. Additionally, we establish the trustworthiness of our analysis using bracketing [35], reflexivity [42], and negative case analysis [45]. While reproducibility may be challenging because of the unique context in which participants learned quantum mechanics and coding, our analysis will provide valuable insight into the ways that students think about discrete and continuous quantum systems. We suspect the reasoning students use is not completely unique to our instructional context and therefore provide detailed descriptions of participants' responses to allow the reader to determine if the results may apply to their instructional context.

1. Bracketing

One way that phenomenographers practice interpretive awareness is through *bracketing* [35]. Bracketing is a process in which the researcher identifies their conceptual understanding of the phenomenon and then takes precautions to ensure their understanding does not influence their interpretation of the data. Our conceptual understanding of discreteness is given in the introduction. We incorporated the following practices suggested by Cope [46] while analyzing data:

- The authors discuss their understanding of discrete and continuous quantities in quantum mechanics.
- Themes and codes were generated from the data without the use of an *a priori* coding scheme.
- We considered how participants expressed their understandings of quantum mechanics and coding during the in-class observations to inform our interpretations of interview data.
- Authors Solorio and Gire discussed their understanding of interview transcript data to ensure accurate interpretation.

It is necessary to acknowledge that one can never fully bracket away their understanding of a phenomenon. This furthers the importance of the reflexive aspect of thematic analysis in this study.

2. Reflexivity

Reflexive thematic analysis involves intentional and thoughtful reflection to highlight the researcher's role in the analysis process [42]. In a reflexive thematic analysis, there is no expectation for codes and themes to be reproduced by another researcher [28]. Rather, we use reflexive thematic analysis to demonstrate the careful thought that went into generating the themes and to increase the trustworthiness of our analysis.

In the practice of reflexivity, we offer the positionality of the first two authors, who participated in the analysis. This background context is relevant as it may impact the understanding and identification of metaphors and language that participants used throughout the interviews and potentially influence our interpretation of the data. Author Solorio is a white, cis-gendered male. He is a first-generation college student with a Bachelor of Arts in physics from a small liberal arts school and a Ph.D. from a large, research-intensive public university. He did undergraduate research in computational cosmology, using C++, *Mathematica*, and Python. He had a spins-first approach for both his undergraduate-level and graduate-level quantum mechanics courses. He has observed the quantum fundamentals course twice and served as a graduate teaching assistant for it once. He has also observed the computational lab course twice.

Author Gire is a white, cis-gendered female. She has degrees in physics and astrophysics from large, research-intensive public universities. She has a little programming experience in C++, IDL, and Python. Her undergraduate and graduate courses in quantum mechanics took a position-first approach, but she has been teaching spins-first quantum mechanics since 2008 and has conducted research on students' understanding of quantum mechanics, their views of physics, representational fluency, and sensemaking practices. Both authors adopt the view of discrete and continuous quantum systems put forth in McIntyre's *Quantum Mechanics* [21], as discussed in Sec. I.

Additionally, our reflexivity is demonstrated by the following practices:

- Author Solorio sense-checked interpretations with Authors Gire and Roundy frequently throughout the analysis process.
- The authors had frequent conversations about the analysis process where Author Solorio provided reasoning for analysis decisions.
- Authors explored different possible interpretations of the data in in-depth conversations.
- The coding spreadsheet was periodically saved to view how the analysis changed over time.

3. Negative case analysis

A negative case analysis presents any pieces of data that contradict or do not support a theme within the body of data to illuminate the strengths and weaknesses of a theme [44].

As we established themes within the data, we consciously looked for any pieces of data that did not support those themes. For example, in Sec. IV D, we present a case where one participant's heuristic does not cleanly fit into the heuristics other participants used. By highlighting any present negative cases rather than omitting them, we provide a holistic perspective of the data and provide the reader more information to assess the limitations of the themes we present and the validity of our analysis [45].

F. Limitations

The six students who participated in this study were purposefully sampled, being chosen for their concurrent enrollment in the quantum fundamentals course and the computational lab course and their prior participation in observations of the computational lab course. This ensured a strong rapport with the interviewer and the participants' ability to well-articulate their ideas and understandings. Because this phenomenographic work was generated from a sample size of six students, there may be variations in understanding the phenomenon of interest that were not captured. Additionally, in the computational lab course, the participants of the study were always paired with another participant of the study. Participants frequently worked with each other, and it is possible that they developed similar understandings that other students in the course did not.

Participants sorted a total of 20 cards chosen to capture many of the representations and concepts that were introduced in the quantum fundamentals course while also not cognitively overloading participants. There were some representations (such as plots, Stern-Gerlach machines) and concepts (such as time evolution) that were not represented by the cards and may have influenced the thoughts participants shared during the interview, thus impacting the themes that came out of the analysis.

During the discrete and continuous sort, participants were asked to organize the cards into a discrete group, continuous group, and any other groups they chose to make. The way this sort was presented may have caused participants to think of discrete and continuous as being dichotomous descriptors, which could have influenced the way that participants chose to sort the cards. However, we note that Mary decided to make more than two groups during the discrete and continuous sort.

IV. THEMES

Four themes were generated from the interview data. These themes include variations in the ways that participants thought about and understood the theme. These were discussed between authors until there was agreement on the name, description, and data that supports the theme. The following subsections describe the theme and present an example of the theme. Each subsection also contains any negative cases relevant to the theme (we found only one

negative case in the data). Appendix B contains Figs. 5–10, which shows participants’ discrete and continuous sorts for additional context. See Supplemental Material [47] for the data from each participant’s initial, second, and discrete and continuous sort. Discussion of how these themes relate to our research questions can be found in Sec. V.

A. Understandings of discrete

Throughout the sorting task, participants shared their understanding of what it means for something to be discrete while sorting cards. We identified two main variations within this theme: (1) thinking of discrete quantities as single values, e.g., scalars (expressed by Brandon and Charlie) and (2) viewing discrete quantities as a set of particular points with no in-between values (expressed by each participant except Brandon). Charlie shifted between understandings over the course of the interview.

1. Single values

Charlie began the discrete and continuous sort by reviewing the cards in their “Bra Acting on Ket” group from the second free sort. This group contained the $\langle x|\psi\rangle$, matrix inner product, and $\langle +|\psi\rangle$ cards. Charlie determined those cards all belonged in the discrete category because they were “one value or one eigenstate.” They then changed the label of the group from Bra Acting on Ket to discrete.

Charlie used similar reasoning again when deciding to put the $|\langle +|\psi\rangle|^2$ card into the discrete group. They stated $|\langle +|\psi\rangle|^2$ is discrete because “it’s just asking for a simple value.” Brandon shared similar reasoning while sorting the Δx card, which they identified as an expectation value as opposed to a change in x , as other participants identified (to minimize the external impact on their thoughts and reasoning during the interview, they were not corrected). They stated the following:

Brandon: I suppose the expectation value, we do get a scalar from that. I suppose that’s discrete. Yeah, because it’s just one point.

Brandon’s understanding was that discrete describes objects that are one point (reasoning he used for several other cards throughout the interview). The examples from Charlie and Brandon developed the “single value” subtheme of what it means to be discrete, where something is discrete if it is a single point or a single value (e.g., a scalar). Notably, Brandon and Charlie did not sort any of the cards containing a function of position using this reasoning, even though a function outputs a single value for each input value.

2. Particular points

At the end of the discrete and continuous sort, Charlie was asked what they view to be the similarities and

differences between discrete and continuous systems in quantum mechanics:

Charlie: Discrete means like having particular values, but only particular values not anywhere in-between those particular values, whereas continuous is all values in-between some range.

Here they presented reasoning with more nuance. They stated that not just anything that outputs a single value or scalar is continuous. They stated that discrete means something has only particular values without anything in between. They contrast this with something that is continuous, which has an infinite number of possible values within a range.

Other participants used this kind of “particular points” reasoning while doing the discrete and continuous sort as well. When Erica was asked what similarities and differences there are between discrete and continuous cards, they explained,

Erica: Discrete would be like there’s only certain values possible. Like you can only get spin up or spin down, there’s no in-between. You can only get—like, energy comes in those packets. You can only get discrete amounts of energy each time.

After Erica answered the interviewer’s question, they decided to move the $|E_n\rangle$ and $|\langle E_n|\psi\rangle|^2$ cards in their “energy” group from the second free sort to the discrete group. Erica used spin and energy as analogies in their explanation, emphasizing that there are only particular values that are possible to measure. They noted that there are not any values in-between the particular spin or energy that can be measured. They referred to energy as coming in “packets,” perhaps referencing a concept they have encountered in another physics course or encountered in personal studies. At multiple points throughout the interview, Erica used their understanding of energy being discrete as a tool in deciding if something was discrete or continuous.

B. Understandings of continuous

Throughout the interviews, participants offered their understandings of what it means to be continuous in quantum mechanics. We found three variations in students’ understanding of continuous: (1) as referring to the continuity of the domain (expressed by each participant except for Brandon), (2) as referring to the smoothness or mathematical continuity of a wave function (expressed by Mary, Brandon, and Charlie), and (3) as referring to unboundedness, e.g., quantities extending to infinity (expressed by Charlie).

1. Continuity of domain

Fred started the discrete and continuous sort by moving around several cards from their previous sort

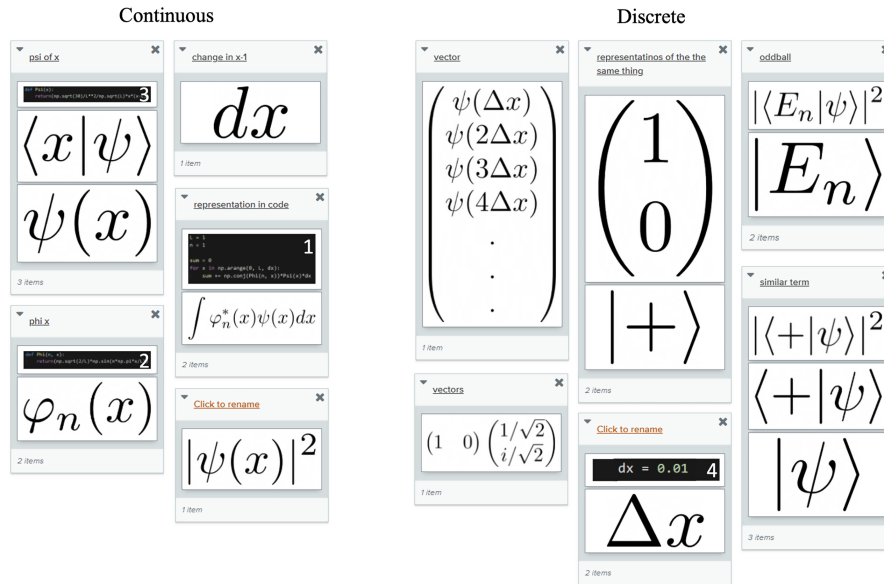


FIG. 2. Fred's discrete and continuous sort.

(see Supplemental Material [47] for a comparison of each participant's sorts for each phase of the interview). They arranged cards they thought of as continuous on the left and cards they thought of as discrete on the right, as shown in Fig 2. They explained how they chose to sort the cards as follows:

Fred: So, what I did was, for like the continuous, I just put what had some kind of function of x . Normally when you have that, you're going to be working with position which is a continuous thing, I think.

Fred associated position with continuity, which resulted in them placing cards that had a function of x in the continuous group. Almost all the cards they categorized as continuous were explicitly a function of x , except for the $\langle x|\psi\rangle$ and dx cards. The interviewer then asked why they chose to place the discrete $\psi(x)$ matrix card into the discrete category when it is a function of x . Fred explained,

Fred: So, what I remember working with that from class, it was kinda like this dx here [mousing over the dx code snippet card], it was always just like 0.01. So that's why I put it in [the discrete group]. I guess in practice it wouldn't be a continuous change in x .

In classifying the cards as discrete or continuous, they considered how continuous the position x was. This type of reasoning was present in interviews with other participants as well. Erica said,

Erica: For a continuous function, like all values are possible. Like you can go to any x position that you're looking at.

They note that continuous functions are defined for each position x . Charlie provided similar reasoning, stating,

Charlie: Discrete means like having particular values, but only particular values not anywhere in-between those particular values, whereas continuous is all values in-between some range.

These examples highlight that participants were attending to the continuity of the domain when deciding if a card should be classified as discrete or continuous. We call this kind of understanding "continuity of domain" because the salient feature that participants used to determine if a card was continuous was the continuity of position, which is commonly the domain of wave functions. We use the general term domain rather than specifying position because we believe this reasoning might similarly appear for the momentum basis. Each participant except for Brandon expressed this understanding at some point during the interview when classifying cards as discrete or continuous.

2. Continuity of function

Brandon offered some related but different reasoning during the discrete and continuous sort. They were very detailed in their identification of each card and discussed their choices in depth during the initial free sort. Consequently, there was not enough time during the interview to conduct a second free sort, so the discrete and continuous sort followed directly after the initial free sort.

Brandon started the discrete and continuous sort by asking the interviewer what they meant by discrete and continuous. The interviewer asked Brandon to share how they were interpreting those words, and they explained,

Brandon: I know discrete and continuous, that will relate to the functions—whether or not we see a continuous curve. You know, mathematically speaking, one which doesn't have points [makes a sharp angle using their two hands], doesn't suddenly drop [they extend their index finger out and quickly move their hand down], or rise [they quickly moved their index finger up], no asymptotes. You know, it just flows throughout space or time or whatever you're looking at. Continuously, that would be that. There's no issues, there are no holes.

Their explanation focused on the continuity of the wave function and its derivative, and they described the features that would make a function or its derivative discontinuous. When asked to clarify what they understood discrete to mean, they explained:

Brandon: I'm not entirely sure, but in contrast to continuous I would say it's a function which isn't continuous, but there are points at which it is, and you can kinda just be okay with points that aren't very continuous and just piece them together. Like piecewise functions are used in that way because there are asymptotes or holes or jumps, discontinuities and whatnot.

Brandon began to sort cards into a continuous group, specifically choosing to put the $\phi(x)$, $\phi(n, x)$ code snippet, and $\psi(x)$ code snippet cards into the continuous group because they were functions of x . Soon after, they placed the $\langle x|\psi\rangle$ into the continuous group because it was “just a different representation” of the $\psi(x)$ card. They then placed the $\int \phi_n^*(x)\psi(x)dx$ card into the continuous group because the integration “relates to the idea of it being a function.” We call this understanding of what it means to be continuous “continuity of function” because the salient feature used to sort cards was identifying if there was a continuous function (like a wave function) rather than a continuous domain (like position).

Mary and Charlie expressed a similar understanding at some point during their interviews. For example, Charlie shared,

Charlie: I think I would place this [$\psi(x)$ code snippet] in continuous because this wave function has to be continuous, at least that I think, has to be continuous inside of the box. So, it might not be continuous for all of space, but it is continuous inside of a box ... if you ask if it's continuous over all space, then no. If you ask if it's continuous inside of a box, I'd say yes.

Charlie was not concerned about the possible x values, recognizing that x can be outside of the box. They were,

however, concerned with the continuity of the *wave function*, stating that it must be continuous inside of the box.

3. Unboundedness

Charlie presented another understanding of continuous during their interview. They described something as being continuous if it has no upper bound or if it goes on to infinity. While this understanding was not shared by other participants, it was particularly salient to Charlie's understanding, and so we describe it here to provide insight into the landscape of ideas students hold on the meaning of continuous.

When deciding where to place the discrete $\psi(x)$ matrix card during the discrete and continuous sort, they said,

Charlie: So this would definitely be continuous because it goes on for infinity, this matrix.

Charlie reasoned that something that “goes on for infinity” is continuous. The next card they sorted was the $\phi_n(x)$ card. They reasoned,

Charlie: This $\phi_n(x)$ [card] is continuous because you can—it's just kind of a function of n , so you can just slap arbitrary n integer in there. This [$|E_n\rangle$ card] is continuous for the same reasons as this ψ_n [sic]. This [$|\langle E_n|\psi\rangle|^2$ card] kind of is continuous just because of this energy eigenstate [circling over the $\langle E_n|$ in the expression] being sort of continuous with this [circling around the $\langle E_n|\psi\rangle$ card] same idea.

Based on the description they provided, we interpret they meant “ ϕ_n ” rather than “ ψ_n .” The interviewer then asked them to expand on what they meant by “sort of continuous” and they responded,

Charlie: Well just because it's continuous meaning like you can keep plugging numbers in for n for as long as you want. I mean, at least theoretically I think. Because they represent eigenstates, I don't know if you can necessarily plug in forty million as your n number realistically or experimentally, but they analytically are continuous.

Charlie described continuous objects as having no upper bound or as going on to infinity. This understanding of continuous was presented only by Charlie.

C. “Discreteness” of representations

During the discrete and continuous sort portion of the interview, participants often used the representation on a card (e.g., bra-ket, matrix, function, and code) as a heuristic for determining if the card belonged in a discrete group or a continuous group. We found two variations: (1) notation-based heuristics, e.g., using matrix, Dirac, and function

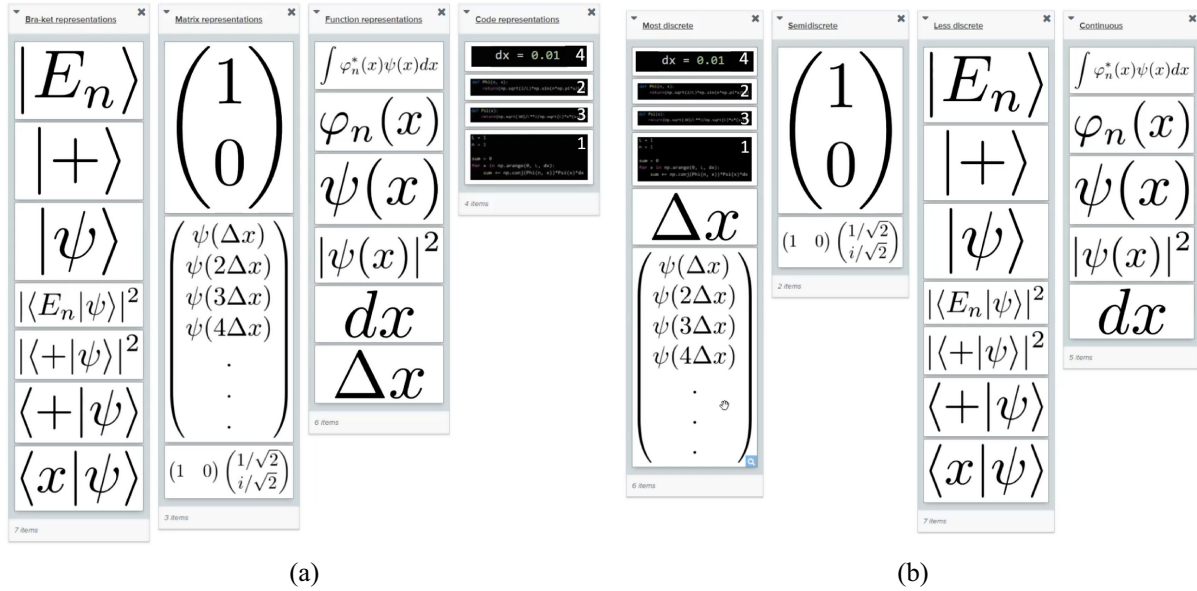


FIG. 3. A comparison of Mary’s second free sort and discrete and continuous sort. (a) Mary’s second free sort. (b) Mary’s discrete/continuous sort.

notation to determine if something is discrete or continuous (expressed by each participant) and (2) code-based heuristics, used to reason about code snippet cards being discrete or continuous (expressed by each participant). For purposes of clarity in our analysis, we distinguish representation (a concept expressed by an object, e.g., a quantum state) and notation (the symbolic convention used to convey that representation, e.g., $|\psi\rangle$), which may differ from participants’ use of the words.

1. Notation-based heuristics

During the second free sort, Mary decided to group the cards by what they called representation. They created four groups named “Bra-ket representations,” “Matrix representations,” “Function representations,” and “Code representations” as shown in Fig. 3(b). In explaining how the groups correspond to each other, they stated:

Mary: You definitely use this [the “Function representations” group] for an infinite square well, versus these [mousing over the “Bra-ket representations” and the “Matrix representations” groups] are best used probably for like spin systems.

Mary said the Function representations group was relevant to the infinite square well. In line with our definition presented in Sec. I, the infinite square well is a continuous system. They then associated Dirac and matrix notation with spin systems. Spin systems, according to our definition in Sec. I, are examples of discrete systems. Earlier in the interview, they mentioned that “Dirac notation is just like a more convenient way to represent

matrices.” Throughout the interview, Mary was unsure how or if a function could be represented by a ket.

When Mary was asked to do the discrete and continuous sort, they moved only a few cards around. They grouped all of the code snippet cards together and moved the discrete $\psi(x)$ matrix and Δx card into that group as well. They reasoned that those two cards were discrete because of their relation to computer code, which Mary viewed as discrete. They then renamed their groups “most discrete,” “semi-discrete,” “less discrete,” and “continuous” (see Fig. 3).

Mary was the only participant who chose to create more than two groups for the discrete and continuous sort (participants were encouraged to create more than just a discrete group and continuous group if they wanted to). The interviewer then asked Mary to explain what the semidiscrete group name meant. They explained that they were uncomfortable creating one group named discrete without being sure that all of the cards were truly discrete. Notably, the cards they associated with computation were in the most discrete category. This is in line with how they talked about code being discrete throughout the interview.

The less discrete group contained all of the cards that used bracket notation, including the $\langle x|\psi\rangle$ card that represents a wave function. All of the cards that had function notation, except for the discrete $\psi(x)$ matrix card which Mary associated with a computational activity, were present in the Continuous category. This further shows that Mary closely tied Dirac notation and matrix notation to discrete quantities and function notation to continuous quantities.

Each participant used the representation on a card to determine if it belonged in a discrete category or a continuous category. Generally, Dirac notation and matrix notation signaled a card as being discrete, while the

presence of a function of x in a card often signaled a card as being continuous. The extent to which a participant used representations to determine discreteness varied. Some participants exclusively classified cards with an x present as continuous and cards with Dirac or matrix notation as discrete. For example,

Brandon: I suppose this [$\varphi_n(x)$ card] is maybe less—because it’s a function, you could talk about it in a more continuous way. So yeah, maybe that would be better [in the continuous group] because it is a function of x rather than like a bracket notation or a matrix.

Brandon’s reasoning was that functions of x are continuous and contrasted the function representation with bracket and matrix notation. Other participants did not exclusively use representation to categorize cards. During the discrete and continuous sort, Erica said,

Erica: Energy is discrete. Energy eigenstates are like packets of energy, so I’ll throw all of the energies in [the discrete group].

They considered if the card referred to energy, which they had stated was discrete throughout the interview when organizing cards.

2. Code-based heuristics

Participants sorted the cards with Dirac notation, matrix notation, and function notation mostly in the same way, as previously described. There was more variety in the ways that participants sorted cards with code snippets, however. Three participants sorted the code snippet cards as a mix of continuous and discrete, two participants sorted each of the code snippet cards as continuous, and one participant sorted each of the code snippet cards as discrete. The ways that participants sorted the code snippet cards are summarized in Table I.

Dave sorted each of the cards as continuous. When reasoning about how to categorize the inner product code snippet, they stated,

Dave: I’m just thinking about the interchangeability between integration and summation. Because, you know, when

we think of integration, I guess, you know, that’s also interchangeable with summing a bunch of little things.

They noted that the inner product code snippet computed a sum of several “little things,” which they viewed as being the same as integration. For the other code snippet cards, Dave paid particular attention to the presence of an x . In sorting the $dx = 0.01$, $\psi(x)$, and $\varphi(n, x)$ code snippet cards, they reasoned,

Dave: Yeah, I guess it’s the same reasoning as before. I would want to put them there because of the x being involved with position.

Similar to the heuristic discussed earlier in this section, the presence of an x signaled that the code snippets were continuous to Dave. Brandon organized the code snippet cards in the same way, using the same kind of reasoning. Their reasoning for sorting the cards was based on what the code represented, the particle in a box in the position basis, which led them to categorize the code snippets as continuous.

Mary was the only participant who chose to sort all of the code snippets as discrete. When asked about their reasoning, they explained,

Mary: I don’t know if there’s a way in code to do things that aren’t technically discrete, but that’s because I don’t know a lot about code.

Mary considered the practical limitations of code when categorizing the code snippets. They were unsure of cases where computational processes are not discrete, citing that they did not have experience coding before the computational lab course. In this interview, they assumed that code required one to work with discrete quantities. This understanding of a code representation being fundamentally discrete resulted in Mary placing all of the code snippet cards into their most discrete category. Erica offered some similar reasoning, explaining,

Erica: The way this [dx] is being coded, you’re not really making a continuous function in actuality. It is trying to mimic a continuous thing, but it isn’t actually that.

They recognized that code can “mimic” or represent a continuous quantity through discretization.

TABLE I. The ways that participants organized the four code snippet cards during the discrete and continuous sort. For simplicity, Mary’s most discrete, semidiscrete, and least discrete categories have been labeled discrete.

Code snippet card	Mary	Brandon	Charlie	Dave	Erica	Fred
dx	Discrete	Continuous	Discrete	Continuous	Discrete	Discrete
$\psi(x)$	Discrete	Continuous	Continuous	Continuous	Continuous	Continuous
$\varphi(n, x)$	Discrete	Continuous	Continuous	Continuous	Discrete	Continuous
Inner product	Discrete	Continuous	Continuous	Continuous	Continuous	Continuous

To highlight a negative case, while sorting the code snippet cards, Charlie did not exactly follow the heuristics that other participants used (e.g., code being fundamentally discrete or code representing continuous systems). They recognized the discrete nature of code, but still placed the $\varphi(n, x)$ code snippet card and the inner product code snippet cards into the continuous category. This is in contrast to other participants who did not acknowledge this limitation and Mary who sorted all the cards into the discrete category because of the limitation. Charlie shared,

Charlie: These [code snippet cards] are kind of a little bit odd because anytime there's an n in there, it's the same concept as over here [moves the cursor over the $\varphi_n(x)$, $|E_n\rangle$, $|\langle E_n|\psi\rangle|^2$ cards in the continuous group] where it's continuous, but in computational you don't really do continuous for infinity [sic]. But I probably would still put them over here as continuous [moves the $\varphi(n, x)$ code snippet card to the continuous group].

They then chose to place the inner product code snippet card into the continuous group. The interviewer asked them to explain their reasoning and Charlie said,

Charlie: Cause [the inner product code snippet card] has the n in there, but—so you can change your n to whatever value you want up here in the code [mouses over the line $n=1$ in the inner product code snippet], but “probably” as meaning like because computation is more discrete in nature, you don't really have “continuous forever” values or else your computer would break, so [chuckles]. But that's not really a limit of the mathematics, it's a limit of hardware.

Charlie's understanding of what it means to be continuous included some ideas around being able to plug any value into a function. In particular, they stated that the wave function inner product card was continuous because of the “ n ” in the energy eigenfunction. They recognized that n is an integer, but they did not consider that in their explanation. In the quote above, they discussed the practical limitations of being able to plug any value into “ n ,” but still choose to sort the $\varphi(n, x)$ code snippet and the inner product code snippet cards as continuous. They cited the limitation as a consequence of hardware rather than mathematics, blending ideas about both the discrete nature of coding and what the code represents. Unlike other participants, Charlie's reasoning did not cleanly fit into the heuristics presented by other participants.

In summary, the participants who attended to the practical limitations of computers, noting that continuous things must be discretized in code, tended to organize code snippets as discrete. Other participants who attended to the system the code was representing tended to organize the code snippets as continuous. Three participants applied this

reasoning to some code snippets and not others, which resulted in them categorizing some cards as continuous and some cards as discrete.

D. Discrete approximation of integrals

This theme captures participants' reasoning about the connection between discrete and continuous systems through the lens of numerical integration in code, which primarily occurred when participants discussed the code snippet cards. We found two variations: (1) integrals as sums, e.g., understanding numerical integration through Riemann sums (expressed by each participant) and (2) the role of step size in numerical integration (expressed by Mary, Brandon, Charlie, and Dave). The discussion of code as an approximation varied between participants in length, detail, and the interview phase that it was brought up.

1. Integrals as sums

During the discrete and continuous sort, Brandon stated that the inner product code snippet and the $\int \phi_n^*(x)\psi(x)dx$ cards are the same thing in “different forms.” They went on to describe how the inner product code snippet card approximates the analytical inner product of a wave function and an energy eigenfunction as follows:

Brandon: So, [dx is] just a basic of integration. You just move over a space with really really small increments and you add up all of the different values ... if you talk about Riemann summation, you're doing that an insane amount of times. Like a whole lot. You're adding all that together and then you find some value from that. And the smaller this dx , the more steps you take, the more pieces you cut your region into, the more accurate your value is gonna be, but also it'll take longer to do that. It might be more difficult. If you're talking about a computer doing it, it'll take more processing power. Potentially making it take a lot longer and maybe the computer won't want to do it [chuckles] because it just won't have the ability or something.

Here, Brandon used what they know about Riemann summations to express how the integral is being approximated as a sum in the code snippet. They explicitly explain the role of dx in this process, stating that making dx smaller increases the number of terms in the sum and therefore gives a more accurate value of the integral. Brandon noted that in the dx code snippet card, the value for dx is something that the user chooses. They also pointed to the practical limitations of computation when they point to processing power as a limiting factor for the accuracy of the integration. Charlie also made a similar statement about hardware limitations during their interview.

2. Step size

Some participants specifically discussed the “chunking” that happens over the domain of an integrated function in code:

Mary: In code, you have to be really explicit and you have to define certain things, have exact values, and like running a for loop isn’t the same as an integral because an integral, you get so close to continuous that you basically are. Like your infinitesimals can be infinitesimally small, versus like these [code snippets] are—you’re chunking things.

Mary reasoned that the “infinitesimals” are not truly infinitesimal in computer code. They also noted that the infinitesimal size used in a for loop affects how accurate the numerical integration is. They referred to this discretization of the domain as chunking it.

Charlie and Brandon explicitly mentioned taking the limit as Δx goes to zero to get from a Riemann sum to integration. Dave explained that they see integration and summation as the same operation “in the background”:

Dave: If this integration represents a bunch of little sums, like when you’re doing an integral, that’s kind of what you’re doing in the background. You are finding some function that would represent that net change or whatever, or, if the answer from the integral is a function. But if it’s just a number, like a definite integral, yeah like you are working out a way to find that net change—that sum—without doing all those little sums painstakingly.

Participants coordinated ideas about continuity and computer code. They particularly relied on their knowledge

of Riemann sums to do this. Each of the participants explicitly referred to Riemann sums to express their understanding of the computational approximations made to evaluate integrals. Some participants included details about the role dx and the accuracy of the integration.

V. DISCUSSION

We have presented four themes that captured the various ways participants think about discreteness and continuity in quantum mechanics. Table II presents a summary of the themes and the variations within those themes. In this section, we will discuss how the data addresses the two research questions we set out to answer in this study.

A. Research question 1

Our first research question asks how these students understand discrete and continuous in the context of quantum mechanics. From the phenomenographic approach to the thematic analysis, we found variety in the ways that participants understand what it means for something to be continuous or discrete in quantum mechanics. The “understandings of discrete,” “understandings of continuous,” and “discreteness of representations” themes primarily inform this research question.

According to the participants’ descriptions, discrete objects have particular allowed values while continuous objects can take on any value within a range. Most participants understood discrete to refer to the property of a set of particular points without any points between them. In contrast to this understanding, some participants described discrete objects as being a single value. This understanding de-emphasized discrete as a quality that describes a set of points. Participants often understood continuous as referring to the continuity of a domain, such

TABLE II. This table provides a summary of the main themes, the variations within those themes, and descriptions of each variation.

Theme	Variations	Description
Understandings of discrete	Single values	Objects that return one single value are discrete
	Particular points	Discrete objects have particular values with no other possible values between them
Understandings of continuous	Continuity of domain	An object is continuous if the observable can take any value within a range
	Continuity of function	Continuous objects have a value for every possible position
	Unboundedness	Continuous objects have no upper bound or continue to infinity
Discreteness of representations	Notation-based heuristics	Objects in Dirac and matrix notation are discrete, while objects with a function of x are continuous
	Code-based heuristics	Code can be discrete or continuous depending on whether the discrete nature of computers or the system being represented is attended to
Discrete approximation of integrals	Integrals as sums	Integrals can be approximated as sums
	Step size affects accuracy	A smaller step size results in a more accurate numerical integration

as x in the position basis. Some participants, especially Brandon, understood continuous as referring to the continuity of a function, such as a wave function. Charlie was the only participant that understood continuous as relating to something being unbounded or “going on for infinity.” Participants attended to the allowed values an object has in determining if it is discrete or continuous.

As shown in the presentation of the discreteness of representations theme, Dirac and matrix notation strongly signaled that a card was discrete, while the presence of an x signaled continuous. The discreteness of the code representation varied between the participants, with Mary categorizing the code snippets as strictly discrete and Brandon and Dave categorizing the code snippets to be strictly continuous. The other participants categorized the code snippet cards as being discrete or continuous on a case-by-case basis. The representation on a card was not strictly the only criterion participants used to sort cards, but it was a heuristic many participants used in their reasoning for placing a card in a group. Wawro *et al.* found that students can and do reason about notations’ clarity, utility, and purposes [13]. From our findings, part of the reasoning evoked by notation may also include students’ conceptions about discreteness and continuity.

Marshman *et al.* found that students often struggle translating quantum states from Dirac notation to wave function notation [48]. Students who strongly correlate Dirac notation with discreteness, as we detailed, may encounter these issues. Riihiluoma *et al.* posit that the increased time that students work with Dirac notation in a spins-first course may strongly inform their conceptual schemata for those expressions [49]. Our findings suggest that discreteness may be another part of the conceptual schemata elicited by Dirac notation.

B. Research question 2

Our second research question asks what connections these students make between their understandings of discrete and continuous objects in quantum mechanics. From the analysis, we found that participants often used their understanding of what it means to be discrete to inform their understanding of what it means to be continuous and vice versa. This was present throughout the interviews for participants. One example, as given in Sec. IVA, was Dave’s comparison of continuous things being like a spectrum and discrete things being like belt notches. Discreteness and continuity were not just understood as individual things but in the context of each other. We also note that the particular points understanding of discreteness and the continuity of domain understanding of continuousness have significant overlap, as they are consistent with each other. Both of these lines of reasoning appeared to be productive for participants.

The computational context was a rich environment where participants connected ideas about discrete and

continuous systems. The code snippet cards often motivated participants to discuss the relationship between Riemann sums, numerical integration, and analytical integration. Numerical integrals are often calculated using a Riemann sum where a domain is broken into pieces of width Δx and the integral is approximated by multiplying the width by the value of the function (which can be determined with a variety of rules, such as the midpoint rule, trapezoidal rule, etc.). Each participant except for Charlie described the process of discretizing that occurs to do a numerical integration, which takes a continuous function and turns it into a discrete set of points spaced out by some length. Brandon, Fred, and Erica mentioned that the integration becomes more accurate with a smaller length between points. The computation context requires the coordination of multiple representations. Utilizing multiple representations in problems, as Marshman *et al.* suggest, may help students build nuance in their understanding of quantum states [48]. This may be particularly impactful in building connections between discrete and continuous objects for students that perceive discreteness or continuity within a given representational forms.

C. Outcome space

In phenomenography, the ways that categories of description, or themes, are logically connected to each other is called the outcome space [50]. Each of the themes described in Sec. IV is present in the outcome space illustrated in Fig. 4. This is not meant to be normative and does not fully represent the ways that students understand and connect their understandings of discrete and continuous. Rather, it is meant to visually organize the themes that were generated from our data.

In Fig. 4, there is a horizontal line that divides the discreteness of representations theme from the other themes. We consider the reasoning students used that comprises the discreteness of representations to be focused more on surface-level features present on the cards. The reasoning used by participants was focused on how something was being represented rather than what it was representing. In contrast, the understandings of discrete, understandings of continuous, and “discrete approximations of integrals” themes incorporated some deeper-level reasoning. These forms of reasoning that participants provided were more nuanced and coordinated multiple complex ideas. The arrows represent the various connections between the themes. We chose a double-headed arrow to symbolize the ways that these understandings inform each other. Participants’ understandings of discrete and continuous were connected in three ways: mediated through discreteness of representations, by comparing and contrasting their understandings of discrete and continuous, and mediated through discrete approximations of integrals.

The discreteness of representations theme was one way that participants coordinated their understandings of

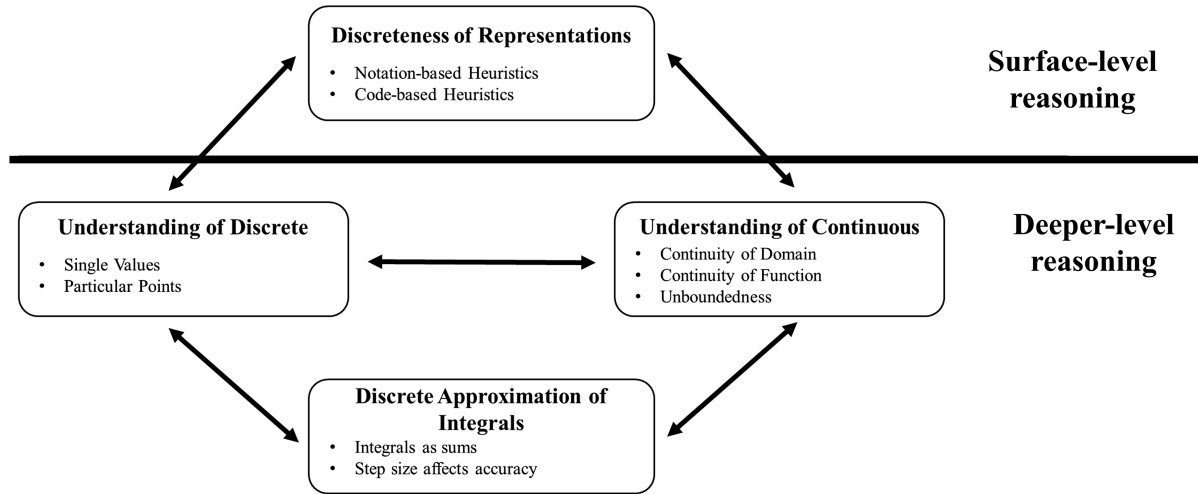


FIG. 4. Outcome space produced from the themes that were generated from the thematic analysis. The arrows represent the ways that students connect their understandings of discrete and continuous. The horizontal line divides the surface-level reasoning students used from the deeper-level reasoning.

discrete and continuous and represents some of the surface-level reasoning participants used. As shown in Sec. IV, participants closely associated Dirac and matrix notation with discrete systems, such as spin systems, whereas functions of x were closely associated with the infinite square well. Some participants used notation as a guide in categorizing some cards into discrete or continuous by noting the presence of a ket, a matrix, and x . The cards with code snippets, however, prompted some more nuanced thinking in participants and resulted in more variety in how those cards were sorted. For example, Charlie recognized the discrete nature of computation but sorted most of the code snippet cards as continuous because of the system the code was representing.

The middle arrows represent the direct relation of the understandings of discrete and understandings of continuous, where participants understood discrete by contrasting it with continuous and vice versa. In both cases, participants thought about the domain of a system to classify the cards they were grouping. In particular, the understanding that discrete objects are a set of particular points with a space between them and that continuous objects can take on any value within a range are ideas that were useful for participants. This reasoning was at a deeper level because participants used the underlying concepts to reason about whether something was discrete or continuous rather than focusing on surface-level features like the symbols used on the card. Some participants also used some dichotomous reasoning, e.g., this is not continuous so it must be discrete, to relate their understandings of discrete and continuous.

The discrete approximations of integrals theme was another deeper-level reasoning participants used to connect their understandings of discrete and continuous. Participants used their understanding of numerical integration to relate inner products in code (discrete sums) to analytic

inner products (integrals). Each participant explicitly referenced Riemann sums in their explanations of how an integral is computationally approximated as a sum. Participants coordinated their knowledge about mathematics and computation to produce explanations about how integration is carried out in code. Rather than keying into surface features like symbols, participants focused on the underlying concepts of inner products, using computation to connect them. With this reasoning, participants coordinated their physical, mathematical, and computational understandings to articulate how code can be used to approximate physical calculations. Some participants additionally noted the physical significance of the limiting process of decreasing the step size dx in code to better computationally approximate integration. This highlights the similarity between calculating and interpreting inner products for discrete basis sets (sum of products) and continuous basis sets (integration).

VI. CONCLUSIONS

This phenomenographic study investigated how students understand the concepts of discrete and continuous in a spins-first quantum mechanics course and the connections students make between those concepts. We found that students hold a variety of conceptions. Students in the study thought of discrete quantities as being single values or viewing them as sets of particular points with nothing in between. Students commonly understood continuous quantities through the continuity of domain (e.g., x) and the continuity of function [e.g., $\psi(x)$], with some students also invoking ideas of unboundedness. Students did not interpret discrete and continuous in isolation. Rather, their reasoning was mediated by comparing and contrasting those concepts and keying into the features of the domain of the card they were classifying (e.g., whether it involved

discrete values like spin states or continuous values like position in a wave function).

Participants used representation as a heuristic for determining if a card was discrete or continuous. Dirac and matrix notation often cued a classification of discreteness, while functions of x signaled continuity. Cards with code representations of a particle in a box prompted varied interpretations of their discreteness or continuity, depending on if students considered the discrete nature of computation or the continuous system the code represents. Computation emerged as a potential conceptual bridge, with students using ideas from Riemann sums and step size to connect continuous integrals to discrete approximations.

From this study, we found that participants' concept images of discrete and continuous quantum systems were shaped by their content knowledge and the representational contexts of quantum mechanics objects. Based on this research, we have several suggestions for instructors. First, leveraging the idea of the discretization of integrals can be helpful in discussing the similarities between discrete-basis inner products and continuous-basis inner products. As shown in the discrete approximations of integrals theme, participants often referred to Riemann sums when describing numerical integration. The transition from summation to integration via Riemann sums parallels going from discrete inner products to continuous ones for quantum systems. Consider the following equations for Riemann sums and integrals:

$$\sum_i f(x_i) \Delta x \rightarrow \int f(x) dx \quad (5)$$

and for inner products of discrete and continuous systems.

$$\sum_i \phi_i^* \psi_i \rightarrow \int \phi_n^*(x) \psi(x) dx. \quad (6)$$

The Riemann sum becomes an integral under the limit $\Delta x \rightarrow 0$. Similarly, when there is an uncountably infinite number of basis states, inner products must go from a sum to an integration. Notably, the Δx present in Riemann sums is implicit for discrete-basis inner products. Additionally, the integrated has different physical dimensions than the quantity being summed in the discrete case. In computation, it is natural to discuss integration as summation. Even in courses without a computational component, students' familiarity with Riemann sums can serve as a bridge between discrete and continuous systems. Using students' understanding of Riemann sums may motivate the need for a dx in a continuous-basis inner product even though there is not a Δx typically written for discrete-basis inner products. Instructors can implement scaffolded activities where the number of basis states for an inner product calculation increase to draw out parallels to between the transition from Riemann sums to integrals and discrete inner products to continuous inner products.

Students may also benefit from instruction that emphasizes that a generic quantum state in Dirac notation is not written in a defined basis. For example, $|\psi\rangle$, can be described in the energy basis or the position basis, but $|\psi\rangle$ itself is an abstract vector in a Hilbert space. Instructors can implement an activity to point out how a generic vector $\vec{v}$ can be described in multiple coordinate systems (Cartesian, rotated Cartesian, spherical, polar, etc.). The activity can draw a parallel to quantum states where a ket abstractly represents a vector that can be written in multiple bases by having students represent an infinite square well state in the energy basis (discrete) and position basis (continuous). Utilizing students' understanding of vectors in an activity that draws parallels to kets as quantum states will clarify that kets as representing an abstract quantum state not defined in a particular basis.

From the discreteness of representations section, participants often related Dirac and matrix representations to discrete systems. Because spins-first approaches introduce Dirac notation and matrix notation early on, students may build a strong association between those notations and discrete quantum systems, which is unproductive in the context of the infinite square well. Activities where students switch between Dirac and function notation for the infinite square well, such as through the use of a completeness relation, could demonstrate that notation is independent of the continuity of a system. Additionally, students may benefit from examples and problems for the infinite square well that stay in Dirac notation to further emphasize the utility of Dirac notation for continuous systems.

Finally, identifying that continuous references a property of the basis being used in describing a system may benefit students. In particular, we suggest that instructors explicitly distinguish between the terms discrete and "discontinuous." When Brandon was asked how they understand the term discrete in quantum mechanics, they described functions that are discontinuous (see the continuity of function variation of the understandings of continuous theme for additional context and explanation). They were perhaps thinking of a familiar context in mathematics courses where continuity of functions is often relevant. This may have led them to describe a part of their concept image that was not directly relevant to discrete systems, i.e., discontinuity.

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APPENDIX A: INTERVIEW PROTOCOL

The text is to be read aloud to participants during the interview. Bracketed phrases, e.g., [Show each card], represent actions the interviewer should take.

1. General introduction to the interview

Thank you for meeting with me today. You will see a prompt on your screen asking if you consent to be recorded. Let me know once you've consented to that or if you have any questions. This interview should be no longer than an hour. Your participation here is voluntary, so if you want us to stop at any point please let me know. If you would be comfortable and able, please turn on your webcam. Throughout the interview, I will be maintaining a neutral expression so that I don't influence what you say and do during this interview.

- Do you have any questions before we begin the interview task?

2. Introduction to the interview task

You will be working on a virtual card sorting task. I will send the link to an application that we will be using, but first I will talk about the task. There are 20 cards with a variety of quantum mechanics content written on them. I want you to organize these cards into categories that you decide on. There's no correct way to organize these cards, I am just interested in how you decide to them. The categories that you come up with may evolve over the interview. You can rename groups of cards, and you can move cards around after placing them into a group. While you are organizing these cards, I would like for you to think aloud. In other words, please verbalize what you are doing and thinking throughout the interview. I will occasionally ask you questions as you sort the cards. You will also have access to a Bingley whiteboard that you can use to write things down to supplement your explanations. The link to that board will be placed in chat. I will give you a demonstration of how to work the card sorting application momentarily.

- Do you have any questions about what I would like for you to do throughout the interview?

I will give you a quick demonstration of the things you can do in this card sorting software. [Show the following actions:]

- [Dragging multiple cards into the sorting area into separate columns.]
- [Naming one column "group 1" and another "group 2" and a third "group 3."]
- [Renaming "group 1" to "group 4."]
- [Moving group 4 below column 2, then above group 2, then between group 2 and group 3.]
- [Moving a card between groups.]
- [Deleting a group.]
- [Zooming into one of the code snippet images.]

This concludes the demonstration of what you can do in this sorting application.

- Do you have any questions about how to work the activity?
- I can either give you remote control of my computer for you to sort the cards, or you can navigate to the card sorting task and share your screen as you organize the cards. Which of these options do you prefer?

We will now begin the activity [Provide the link to the sorting activity or share screen and grant remote control. Provide the link to the virtual whiteboard.]

3. Interview tasks questions

- Task 1 (Description of cards): Before you start organizing the cards, I would like for you to look at all of the cards and tell me briefly what you think is on the card.
 - Task 2 (Initial free sort): Please organize the cards into groups in whatever way makes sense to you.
 - Task 3 (Second free sort): Please organize the cards into groups in a different way.
 - Task 4 (Discrete and continuous free sort): Please organize the cards into a discrete group, continuous group, and any other groups that you feel you need to make.
- [During the tasks, consider asking the following questions as appropriate:]
- Tell me about how you ended up organizing these cards.
 - Can you tell me about why you put the cards in each group together?
 - How are the cards between groups similar or different from each other?
 - How do your groups relate to each other?
 - Could you tell me about the names of these different groups and what they mean?
 - If you could create new cards to put in a pile, what would you create?

APPENDIX B: PARTICIPANTS' DISCRETE AND CONTINUOUS CARD SORTS

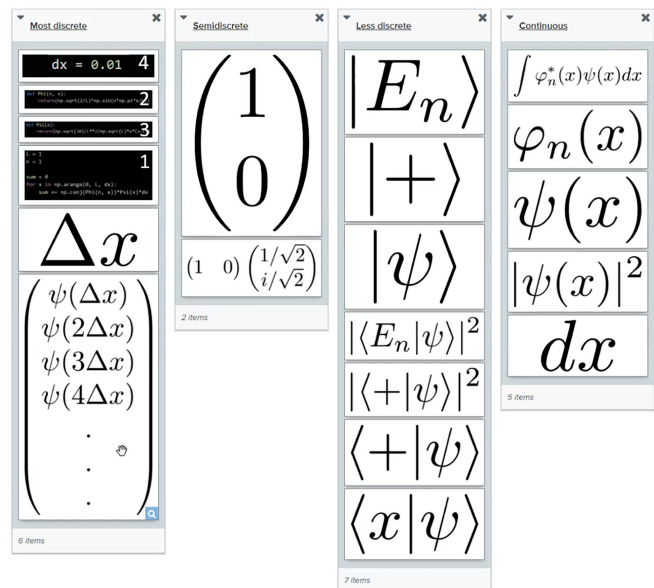


FIG. 5. Mary's discrete and continuous sort.

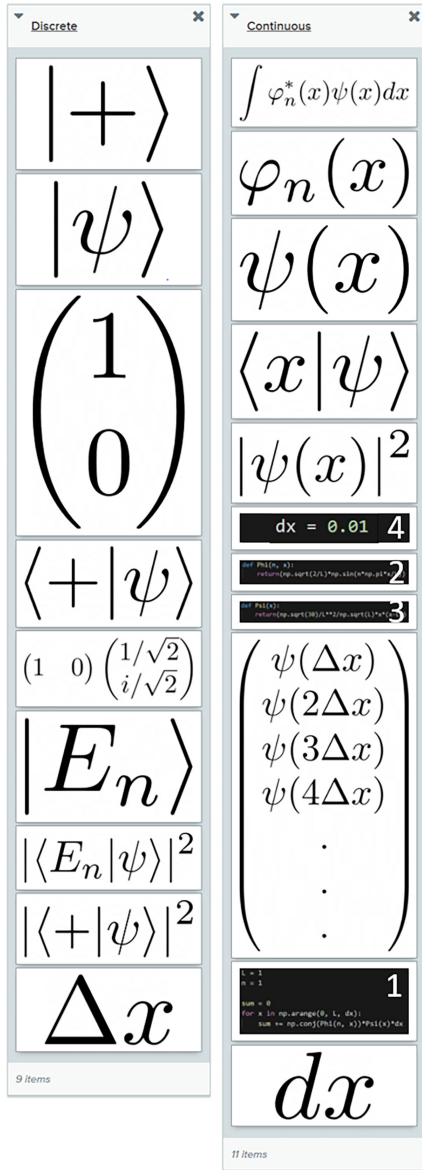


FIG. 6. Brandon's discrete and continuous sort.

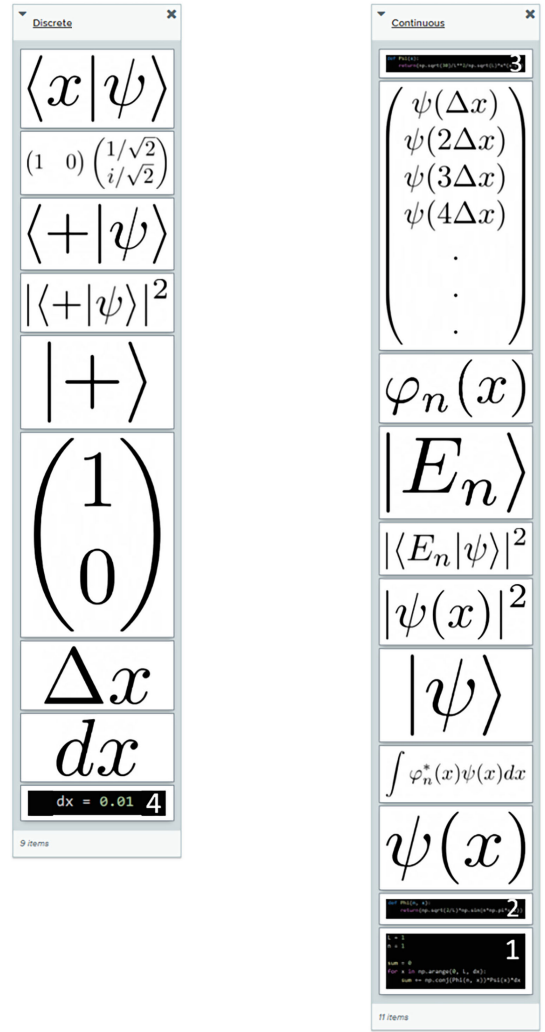


FIG. 7. Charlie's discrete and continuous sort.

Discrete

- $\begin{pmatrix} 1 & 0 \\ 0 & i/\sqrt{2} \end{pmatrix}$
- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\langle + | \psi \rangle$
- $|\psi\rangle$
- $|+\rangle$
- $|\langle E_n | \psi \rangle|^2$
- $|\langle + | \psi \rangle|^2$
- $|E_n\rangle$

8 items

Continuous

- $dx = 0.01$ 4
- dx
- Δx
- $\int \varphi_n^*(x) \psi(x) dx$
- 1
- $\langle x | \psi \rangle$
- $\psi(x)$
- $|\psi(x)|^2$
- 3
- $|\psi(x)|^2$
- 2
- $\varphi_n(x)$
- $\begin{pmatrix} \psi(\Delta x) \\ \psi(2\Delta x) \\ \psi(3\Delta x) \\ \psi(4\Delta x) \\ \vdots \end{pmatrix}$

12 items

FIG. 8. Dave's discrete and continuous sort.

Discrete

- $\begin{pmatrix} \psi(\Delta x) \\ \psi(2\Delta x) \\ \psi(3\Delta x) \\ \psi(4\Delta x) \\ \vdots \end{pmatrix}$
- Δx
- $|\langle + | \psi \rangle|^2$
- $|\langle E_n | \psi \rangle|^2$
- 2
- $|E_n\rangle$
- $\langle + | \psi \rangle$
- $\begin{pmatrix} 1 & 0 \\ 0 & i/\sqrt{2} \end{pmatrix}$
- $|\psi\rangle$
- $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $|+\rangle$

12 items

Continuous

- 3
- $\varphi_n(x)$
- 1
- $\langle x | \psi \rangle$
- $\int \varphi_n^*(x) \psi(x) dx$
- dx
- $\psi(x)$
- $|\psi(x)|^2$

8 items

FIG. 9. Erica's discrete and continuous sort.

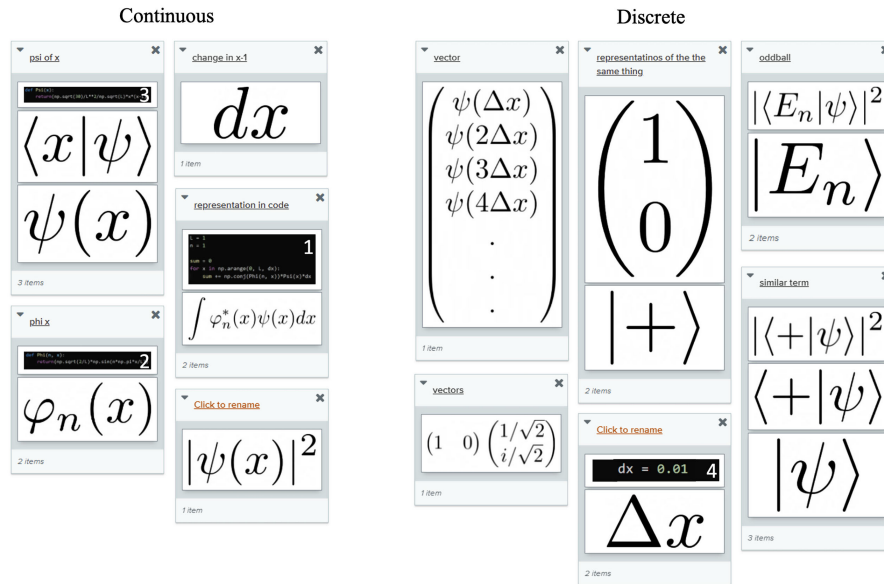


FIG. 10. Fred's discrete and continuous sort. The continuous and discrete labels were added for clarity.

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