

Gaze behavior as a key to revealing strategies for identifying indirectly proportional graphs in thermodynamic and mathematical context

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The ability to accurately interpret mathematical functions is crucial for understanding complex physical and mathematical concepts. Differentiating isothermal from adiabatic change of state in p - V diagrams respectively graphs of indirectly proportional functions from graphs of power functions in x - y diagrams presents a common challenge for students. This study examines the gaze behavior of 80 10th-grade students to identify the processing strategies they use when distinguishing these graphs of functions in mathematical and thermodynamic contexts. The cluster analysis of the gaze data revealed two main strategies: focusing on the value pairs, often resulting in a higher number of correct responses, and focusing on the graph itself, often resulting in misunderstandings. Interviews indicated that students who relied on graph-focused strategies often followed intuitive and less effective approaches. These findings underscore the need for adaptive support measures to address individual learning challenges and improve students' understanding of graphs of indirectly proportional functions. Beyond the specific focus on indirectly proportional and power functions, the methodology and results of this study offer a transferable framework for identifying learning difficulties across various function types and contexts, complementing existing research on students' understanding of linear functions.

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I. INTRODUCTION

In modern educational research, the analysis of learning processes is a central focus, particularly in the development of individualized teaching methods. A key aspect of this research is understanding how learners interact with different forms of representation to process complex information [1–3]. One of the most important forms of representation in STEM is the graph of a function. In the following sections, we will refer to this as simply a graph. These graphs illustrate functional relationships by showing how changes in one variable affect another, which is essential for understanding physical principles [4]. In this article, we will focus on graphs of thermodynamic state changes, specifically the indirectly proportional functions (isothermal state change) and the power functions (adiabatic state change). Especially indirectly proportional functions,

which describe situations where one quantity increases while another decreases, with the product of the two quantities remaining constant, play a fundamental role in physics and everyday life. In thermodynamics, for instance, they are crucial for describing phenomena such as the inverse relationship between pressure and volume [5]. In everyday life everyone knows that when traveling at a constant speed, it takes half the time to cover a distance if you double the speed.

Interpreting graphs often presents challenges [6]. Specifically, learners frequently struggle to distinguish indirectly proportional functions from other power functions in thermodynamic context [7]. Additionally, studies indicate that learners have difficulty applying mathematical knowledge to physical contexts, suggesting challenges in transferring knowledge between these contexts [8]. Teachers are often unaware of the strategies or thought processes their students use when solving problems and tend to overestimate their abilities [9]. This expert blindness illustrates how difficult it is for experts to understand the perspective of beginners [10].

Eye-tracking analysis allows for the identification of specific strategies and potential difficulties learners may have in interpreting functions [11–13]. The comparison of

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processing strategies across different disciplines, such as mathematics and thermodynamics, offers the opportunity to scientifically analyze differences in information processing shaped by the specific demands of each field. This also allows for the systematic investigation of transfer difficulties, that is, challenges in effectively applying mathematical knowledge and skills in other contexts [8]. Analyzing eye-tracking data in these contexts allows for the identification of both strategies and context-specific differences. These insights are valuable for developing subject-specific teaching methods, as they can help improve the learning and understanding of complex concepts in various fields, such as mathematics and thermodynamics. This study focuses on classifying and analyzing processing strategies for distinguishing between indirectly proportional functions and power functions. In the thermodynamics context, isothermal and adiabatic changes of state are examined.

II. THEORETICAL AND EMPIRICAL BACKGROUND

A. Indirectly proportional graphs in thermodynamic context

In thermodynamics, the indirect proportionality between pressure p and volume V occurs during isothermal changes of state when the temperature T of an ideal gas is kept constant. An ideal gas is a theoretical model composed of atoms and molecules with a specific radius, moving with statistically distributed velocities [14]. The elastic collisions between these particles obey the principles of conservation of energy and momentum [14]. For a given amount of ideal gas in the system, pressure p , volume V , and absolute temperature T are interconnected through the functional notation of state [5]:

$$f(p, V, T) = 0. \quad (1)$$

It is determined by the thermodynamic properties of a gas under very low, constant pressure [5]. The equation of state for an ideal gas with a mass m and a molecular weight M is given by

$$pV = \frac{m}{M}RT, \quad (2)$$

where $n = \frac{m}{M}$ represents the amount of substance and R is the universal gas constant [5]. The state variables pressure p , volume V , and absolute temperature T are interconnected by the equation of state, which describes the current state of the gas.

In thermodynamics, the state change of a closed system is often examined as it evolves from an initial to a final state, passing through a series of intermediate states [5]. This change of state can be graphically represented in a p - V diagram as a curve that illustrates the transition [5]. A state change is considered reversible if the system remains in

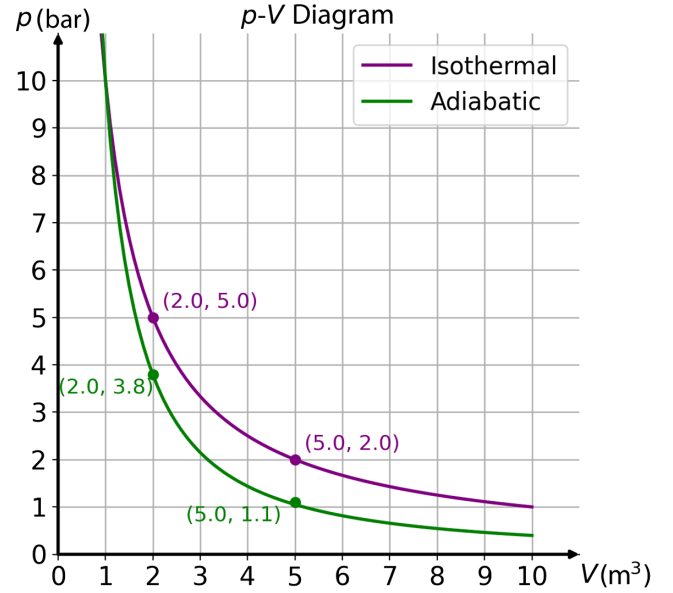


FIG. 1. Visualization of isothermal and adiabatic changes of state in a pressure-volume diagram (p - V diagram) with highlighted specific states.

thermodynamic equilibrium throughout the entire process and the successive states differ only by infinitesimally small amounts [5]. For a process to be reversible, it must be quasistatic, meaning it must occur so slowly that the system remains very close to equilibrium at all times.

In a reversible isothermal process, where the temperature T is kept constant, Eq. (2) becomes:

$$pV = \text{const.} \quad (3)$$

According to the equation of state, this means that the product of pressure and volume stays constant [5]. In a p - V diagram, isothermal changes of state are represented by hyperbolas, with the pressure and volume axes serving as asymptotes [5]. For instance, doubling the pressure results in a halving of the volume. In Fig. 1, the isothermal state change is depicted as the purple curve.

In contrast to isothermal changes of state, where heat is exchanged with the surroundings, adiabatic changes of state involve no heat exchange. A state change in a thermodynamic system is considered adiabatic if it occurs reversibly and the system is thermally insulated, so that no heat exchange with the surroundings can take place during the entire process [5]. Under these conditions, the following relationship from Eq. (2) applies [5]:

$$pV^\kappa = \text{const.} \quad (4)$$

In Fig. 1, an adiabatic change of state is also depicted in green, which qualitatively resembles the isothermal change of state [5]. However, because $\kappa > 1$, the adiabatic curve is steeper than the isothermal change of state [5]. The similar curve shapes of isothermal and adiabatic changes of state

TABLE I. Classification of strategies for proportional problems according to Fisher [21].

Strategy		Explanation
Correct strategies		
1	Proportional formula	Correct solution by equating two ratios or setting up an equation that expresses the equality of two products, including mention of the indirectly proportional relationship.
2	Proportional reasoning	Correctly solving the problem using a proportional strategy that does not rely on the proportional formula.
3	Algebra	Solution by setting up an algebraic equation without directly applying the proportional formula.
4	Other correct	A correct strategy that does not fit into categories 1–3.
Incorrect strategies		
5	No answer	No response or reaction to the given task.
6	Intuitive	Answer based on guesses or intuitive feelings, without logical reasoning.
7	Additive	Focus is mistakenly on additive differences between quantities, disregarding underlying multiplicative relationships.
8	Proportional attempt	Recognizes that a proportional relationship exists but cannot correctly express it.
9	Other incorrect	An incorrect strategy that does not fit into the categories 6–8.

make it difficult to distinguish between them, especially without closely examining the exact values on the graph.

B. Challenges in identifying isothermal change of state

The understanding of isothermal and adiabatic change of state in p - V diagrams poses a challenge for learners, as the visual differences are not immediately apparent [15]. Adiabatic changes of state are often confused with isothermal ones due to the difficulty of recognizing their specific characteristics in the diagram [7]. In contrast, isobaric and isochoric changes of state are easier to interpret because of their clear representation [7]. De Bock *et al.* [6] describe that difficulties primarily arise because learners misinterpret situations where a larger x implies a smaller y , as is the case with indirectly proportional functions. This misinterpretation stems from the tendency to transfer linear characteristics to nonlinear relationships [6]. Learners often face challenges transferring mathematical knowledge to physics contexts, revealing gaps in their ability to connect these disciplines [8,16,17]. Knowledge activation is heavily influenced by the specific context, with perceptions of the situation playing a key role [18].

In a study by Fehlinger *et al.* [19], the visual attention of learners was examined while identifying graphs of indirectly proportional functions and power functions in both thermodynamic and mathematical contexts. The findings revealed that learners had difficulty distinguishing between these graphs in both contexts [19]. Additionally, their patterns of visual attention differed: in the mathematical context, their focus was more directed toward the graphs themselves, whereas in the thermodynamic context, their attention was primarily on the axes of the diagram [19].

While this study highlights important context-related differences in visual processing, it leaves open which specific strategies learners employ during graph interpretation and

how these strategies are reflected in their gaze behavior. The present study addresses this gap by systematically analyzing the relationship between learners' interpretation strategies and their corresponding eye movement patterns across both contexts. This extension enables the identification of learners' interpretation strategies through their gaze behavior. These insights offer a valuable foundation for the design of adaptive learning environments. In the long term, such environments could employ gaze-based diagnostics to deliver targeted prompts that support strategy refinement and, consequently, promote more effective learning.

C. Processing strategies of indirectly proportional graphs

Ashcraft [20] defines processing strategies as cognitive methods for systematically solving tasks. Here, they will be referred to as strategies. Fisher [21] identified nine strategies, both correct and incorrect, for solving directly and indirectly proportional problems. A comprehensive explanation of these strategies can be found in Table I.

D. Investigation of strategies

Eye tracking and retrospective think-aloud (RTA) methods are effective for revealing these strategies [13,22]. Visual attention—assessed through eye tracking—offers insights into how learners focus and process information, as eye movements reflect the brain's cognitive interactions [23,24]. The eye-mind hypothesis [25] suggests that a learner's gaze directly aligns with current cognitive processing. Key eye-tracking metrics include fixation count, transition count, and saccade direction, all of which are closely tied to cognitive processes [26,27].

Fixation count indicates which parts of a task draw the most focus, revealing cognitive priorities. Transition count tracks shifts between focus areas, highlighting how participants integrate information. Saccade direction, or the angle

of eye movement, shows if participants systematically scan or focus on specific regions. These metrics collectively outline how learners organize visual attention and approach tasks.

Combining eye tracking with RTA enhances gaze data analysis, allowing learners to recall and explain their focus patterns [28,29]. In cued RTA, users are shown a replay of their interactions to aid memory recall [29]. The use of a video cue that visualizes eye movements (gaze video replay) has proven particularly effective in obtaining precise and detailed feedback [30]. This allows for a detailed insight into a learner's strategies. Our study extends the work of Fisher [21] by focusing on indirectly proportional graphs and integrating physical contexts. The addition of gaze data refines the analysis of visual processing and provides a richer basis for identifying interpretation strategies—an important step toward adaptive instructional support, as discussed in the following section.

III. RESEARCH QUESTIONS AND HYPOTHESES

Although previous research has advanced our understanding of how students interact with graphs, little is known about how learners process indirectly proportional functions across different contexts.

In particular, existing studies rarely examine the context-specific strategies learners apply when interpreting such graphs, especially when comparing thermodynamic and mathematical representations. This study addresses this gap by investigating gaze patterns and strategies in both contexts. The following research questions and hypotheses guide the investigation:

Question 1: What strategies do learners use to identify graphs representing indirect proportionality in mathematical and thermodynamic contexts?

Hypothesis 1: In line with Fisher [21], learners are expected to apply different strategies depending on the context in which the graphs are presented.

Beyond identifying these strategies, it is also essential to characterize and compare them in terms of gaze behavior, both across and within contexts. Therefore, two additional questions are posed:

Question 2.1: How do the gaze data associated with different strategies vary between the two contexts?

Hypothesis 2.1: As shown in other studies, different contexts are expected to show distinct gaze patterns reflecting context-specific cognitive processing [26,27].

Question 2.2: How does the gaze behavior for the same strategy differ between the two contexts?

Hypothesis 2.2: Building on Fehlinger *et al.* [19], we hypothesize that the same strategies may manifest differently depending on the context. Specifically, thermodynamic tasks may include more transitions between the diagram axes than mathematical tasks.

IV. METHODOLOGY

A. Participants

A total of 80 10th-grade students from secondary schools in Germany participated in this study (45 female, 34 male, 1 not reported). The average age was 15.47 years ($SD = 1.66$); one participant did not report their age. A total of 68 participants reported German as their native language, 10 reported a different native language, and 2 did not provide this information.

According to the curriculum and the information provided by their teachers, all participants were familiar with the particle model and the basic concepts of pressure, volume, temperature, internal energy, and thermodynamic changes of state. They also had the necessary mathematical knowledge of indirectly proportional functions.

B. Procedure

The study was conducted in three sequential phases—a repetition phase (phase 1), a training phase with eye tracking (phase 2), and a validation phase with eye tracking (phase 3)—involving all participants (see Sec. IV A). In phases 2 and 3, participants completed graph-related tasks followed by interviews (see Fig. 2). The primary focus of the strategy analysis lies in phases 1 and 3.

Phase 2 served to introduce the concept of gaze paths and to prepare participants for the subsequent reflection on their individual problem-solving approaches. To this end, participants first completed a graph identification task—based on the study design by Fehlinger *et al.* [19]—during which their gaze behavior was recorded without explicit instructions to reveal their strategy. The aim of this initial eye-tracking phase was to capture natural, unprompted processing behavior.

In the subsequent validation (phase 3), participants worked on comparable tasks but were explicitly instructed to make their gaze behavior reflect their underlying problem-solving strategy as clearly as possible (see Sec. IV C 2). Following this, participants reflected on their approach in an interview using their recorded gaze paths as a reference (see Sec. IV C 5).

A comparison of phases 2 and 3 revealed no significant differences in the observed strategies and only minimal variations in gaze behavior. The gaze patterns remained consistent; only their absolute frequency increased in phase 3. When viewed in relation to the total task duration, no differences were found. These findings suggest that the explicit instruction to make strategies visible did not distort natural strategy use and can thus be considered a valid method for eliciting individual strategies.

In phase 1, the necessary prior knowledge was reactivated using a handout (see Sec. IV C 1). Participants worked through the handout independently. They were not

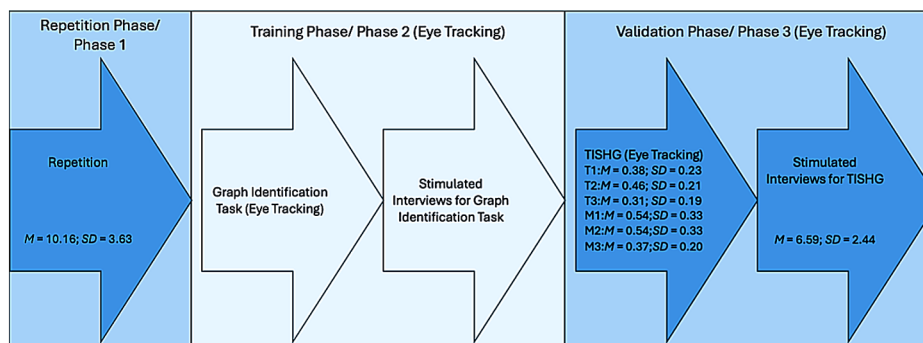


FIG. 2. Procedure for investigating strategies. Mean (M) and standard deviation (SD) of time spent in phases 1 and 3. Time was noted only in the depiction for the phases included in the analysis. The representation of phase 2 serves solely to illustrate the method and clarify the procedure.

allowed to take notes or ask questions. Fifteen minutes were allocated for this task, although students were allowed to proceed earlier if they finished the handout ahead of time.

Following a nine-point calibration of the eye tracker, participants completed phase 2. In this phase, they worked on graph identification tasks while their gaze behavior was recorded.

A subsequent interview used visualizations of the recorded gaze paths to support participants in reflecting on their task performance. Although this phase is not the primary focus of the present study, it served an important methodological function by introducing participants to the process of gaze recording and visualization.

This familiarity with the concept of gaze paths was a necessary prerequisite for phase 3 of the study, in which participants were asked to identify graphs and describe their approach based on their gaze behavior. This test is referred to as the Test for illustrating strategies in handling graphs (TISHG) (see Sec. IV C 2).

After a second calibration of the eye tracker, participants completed the TISHG, during which their gaze behavior was recorded once again. Responses were submitted via mouse click; no feedback was provided, and participants could not return to previous items. The test was intentionally untimed to reduce time pressure and allow for the most natural use of strategy possible.

During this phase, the researcher observed participants' gaze behavior in real time on a secondary screen. Notable gaze features—such as repeated fixations or unusual transitions—were documented and used to select relevant prompts from the interview catalog (see Sec. IV C 5).

In the final interview, participants viewed a playback of their individual gaze paths using visualizations generated in Tobii Pro Lab. These visualizations depicted fixations as circles and saccades as connecting lines, supporting memory recall and helping participants explain their approach.

The interviews served not only as a reflection tool but also as a means of validating the gaze data: participants verbalized their thought processes, enabling a more accurate interpretation of their visual attention. This phase was also untimed, allowing participants ample opportunity to articulate their strategies and reasoning in detail and without pressure. The complete study design, including the average time spent on each section, is illustrated in Fig. 2.

C. Materials

The following materials pertain exclusively to phases 1 and 3, as only these phases are relevant to the analytical objectives.

1. Repetition (phase 1)

A handout (see Supplemental Material Sec. A [31]) was developed to reactivate the necessary prior knowledge for interpreting graphs in both contexts. It includes definitions of thermodynamic changes of state and their representation in a p - V diagram. Additionally, worked examples illustrate how the characteristics derived from the definitions can be used to identify each specific state change within a p - V diagram.

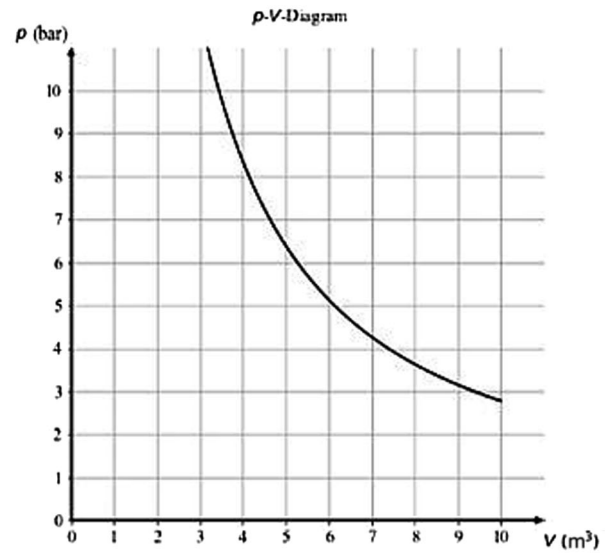
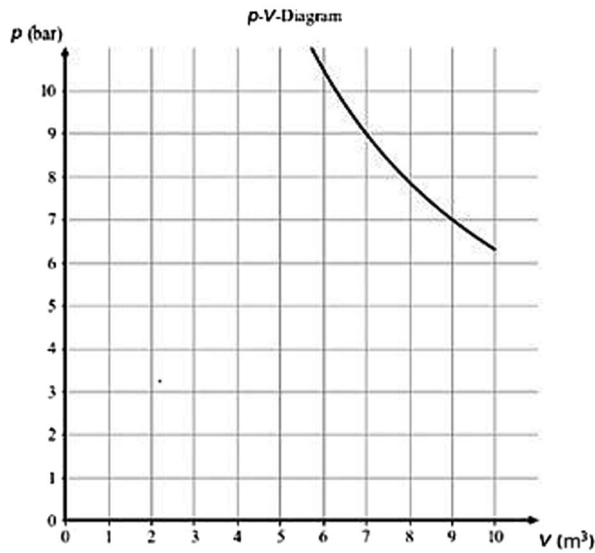
No explicit strategies were introduced. This handout ensures a consistent knowledge base across participants, allowing for the targeted investigation of strategies independent of potential prior knowledge differences.

2. Test for illustrating strategies in handling graphs (TISHG) with eye tracking (phase 3)

The test asks participants to identify indirectly proportional graphs in both mathematical and thermodynamic contexts. In the thermodynamic context, isothermal changes of state are represented in a p - V diagram.

Participants were also asked to illustrate their strategy based on their gaze behavior. Participants were given the following instruction:

Klicke auf die isotherme Zustandsänderung und verdeutliche deine Vorgehensweise durch dein Blickverhalten!



Klicke auf den indirekt proportionalen Funktionsgraphen und verdeutliche deine Vorgehensweise durch dein Blickverhalten!

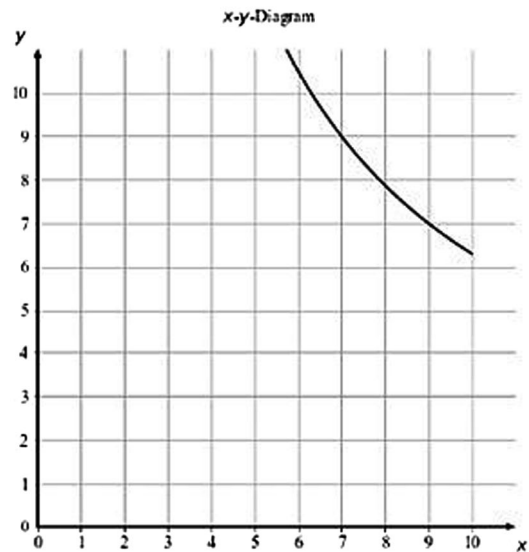
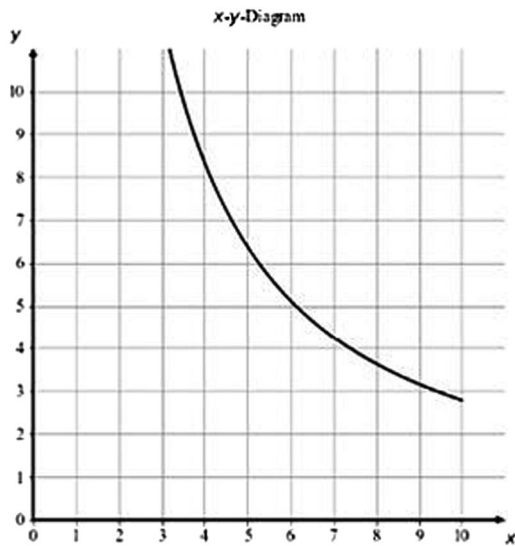


FIG. 3. Similar item pairs for identifying graphs of indirect proportional functions in thermodynamic (top) and mathematical (bottom) contexts. Thermodynamic task: click on the isothermal change of state and demonstrate your process through your eye movements! Mathematical task: click on the graph of the indirectly proportional function and demonstrate your process through your eye movements!.

Click on the ... and clarify your procedure through your eye behavior! (see for example Fig. 3)¹

The answer options included both the indirectly proportional graph and a power function graph that represents an adiabatic change of state in thermodynamics. The test consisted of six items, with three items focused on

thermodynamic contexts and three on mathematical contexts (see Table II).

3. Item design (phase 3)

To investigate potential differences in strategies between the thermodynamic and mathematical contexts, the corresponding items were designed to be surface similar.

This means that the graphical trajectories and underlying mathematical relationships were identical across both contexts.

¹Originally presented in German, the task has been translated into English here.

TABLE II. Analysis of scales for isothermal change of state and indirectly proportional function. “*” indicates that the values are significant at $p < 0.05$.

Scale	Cronbach’s α	Item	Difficulty index	Point-biserial correlation
Isothermal change of state	0.75	T1	0.525	0.817*
		T2	0.738	0.647*
		T3	0.625	0.695*
Indirectly proportional function	0.75	M1	0.538	0.737*
		M2	0.638	0.741*
		M3	0.675	0.707*

Accordingly, the same answer options were provided, although their order was occasionally varied. One difference is that the graphs in the mathematical context were presented in an x - y diagram, while the thermodynamic state changes were depicted in a p - V diagram.

An example of a surface-similar item can be seen in Fig. 3. The order of the tasks was arranged so that no surface-similar items immediately followed one another. The items alternated between the thermodynamic and mathematical contexts. The test was independently developed for this study, and the tasks can be found in Supplemental Material Sec. B [31].

4. Apparatus and preparation of measurement (phases 2 and 3)

The items were presented digitally on a 24-inch screen with a resolution of 1920×1080 pixels. Eye movements were recorded using the Tobii Pro Fusion stationary eye tracker, which sampled at 120 Hz with an average accuracy of 0.40° .

The participants maintained a consistent distance of approximately 60 cm from the screen by adjusting their seating position before each session and were instructed to adhere to this distance as closely as possible. Before each task, a nine-point calibration ensured accurate measurement of eye movements, starting with participants focusing on a central cross displayed on the screen.

5. Stimulated interview (phase 3)

Stimulated recall interviews were conducted using the retrospective think-aloud (RTA) method, in which the visualization of participants’ recorded gaze paths served as a prompt. These gaze replays acted as cues to support learners in recalling and verbalizing their thinking during the graph identification tasks.

This approach complements the task design in TISHG and thus provides a more comprehensive understanding of the strategies used.

While the gaze behavior of the participants was monitored in real time during task completion, the researcher concurrently note observations—for example, repeated fixations on specific values.

These observations were then used to ask targeted questions during the RTA interviews, based on the predeveloped interview guide. Each RTA session began with an open-ended prompt inviting participants to justify their decision in favor of a particular diagram. This provided insight into their individual strategies and served to validate the gaze data:

Describe the specific features in the selected diagram X that made it clear to you that it represents the graph of a function or a thermodynamic change of state.²

In the second step, participants were asked to formulate their approach in a way that would be understandable to peers:

Explain your steps in the process so that a classmate can understand your approach.

Participants were encouraged to reflect on their gaze path and describe their thought process for each step. These questions were posed in both the thermodynamic and mathematical contexts.

If changes in selection or approach were observed during task completion within these contexts, additional follow-up questions were posed to clarify these decisions:

Explain why you changed your strategy for this task.

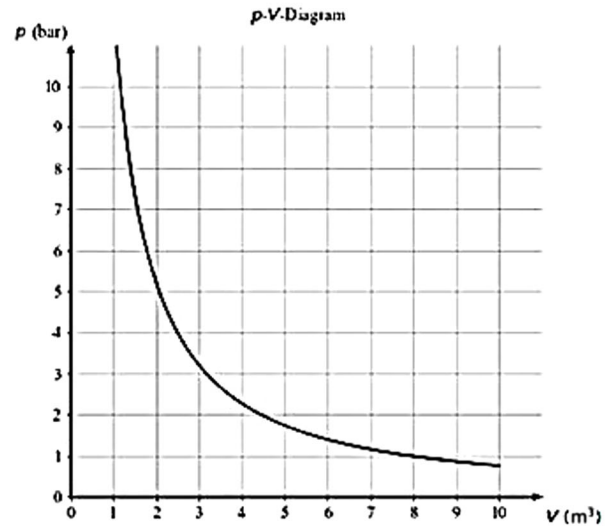
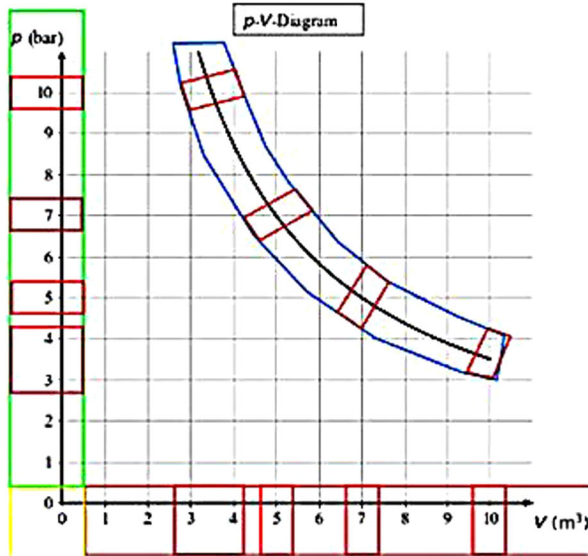
The complete interview guide can be found in Supplemental Material Sec. C [31].

D. Data analysis

To address the first research question, participants’ gaze data and interview data were triangulated to identify interpretation strategies. For the research questions 2.1 and 2.2, a detailed analysis of the gaze data was conducted, and the response frequency within the different strategies was highlighted.

²The questions were initially formulated in German and are presented here in their English translation.

Klicke auf die isotherme Zustandsänderung und verdeutliche deine Vorgehensweise durch dein Blickverhalten!



Klicke auf den indirekt proportionalen Funktionsgraphen und verdeutliche deine Vorgehensweise durch dein Blickverhalten!

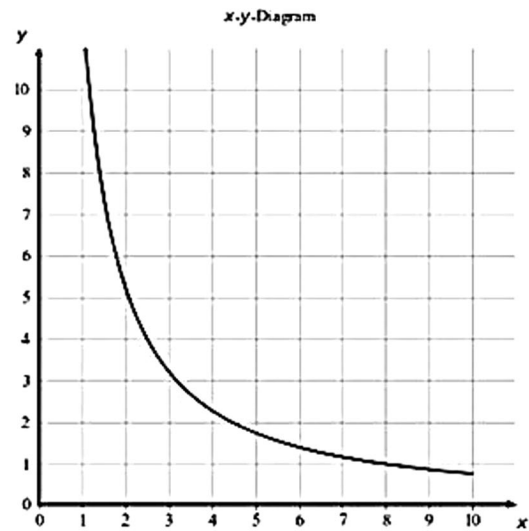
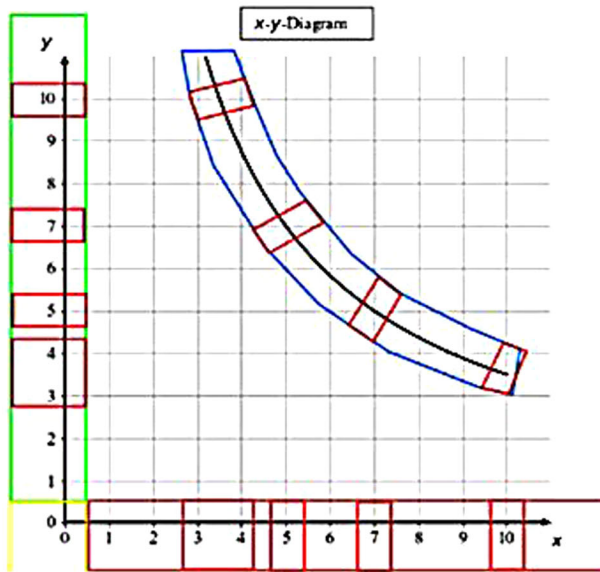


FIG. 4. The same exemplary item pair as in Fig. 2, with highlighted areas of interest (AOIs). Thermodynamic task: click on the isothermal change of state and demonstrate your process through your eye movements! Mathematical task: click on the graph of the indirectly proportional function and demonstrate your process through your eye movements!

1. Preparation of gaze data

To analyze strategies in identifying indirectly proportional graphs, areas of interest (AOI) were defined within the Cartesian coordinate systems and p - V diagrams.

These AOIs included the ordinate, abscissa, origin, axis labels, and the graph itself. For a more detailed analysis, additional AOIs were defined to cover specific points on the graph as well as arguments or function values along the axes (see Fig. 4). In some cases, AOIs overlapped.

Gaze data within these overlapping areas were fully assigned to each of the relevant AOIs, without splitting the eye-tracking metrics proportionally. This allowed each AOI to be analyzed independently.

The selection of AOIs was guided both by theoretical considerations and heatmaps generated from participants' cumulative fixations (see Supplemental Material Sec. D [31]). For surface-similar item pairs, AOIs were defined consistently across contexts to ensure comparability (see Supplemental Material Sec. E [31]).

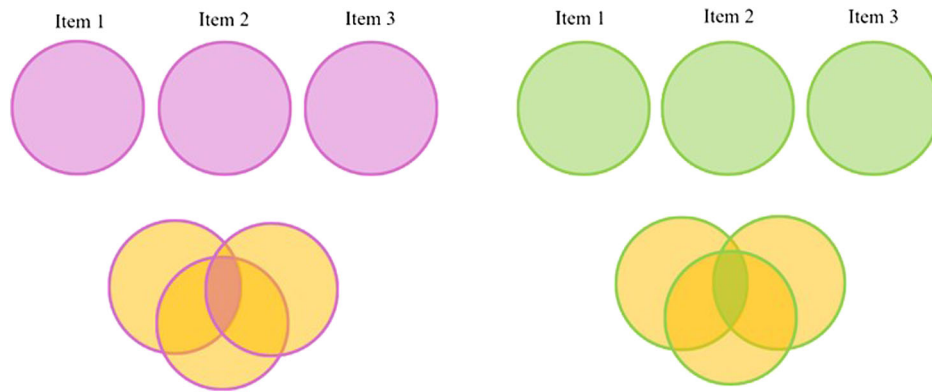


FIG. 5. Exemplary representation of the intersection comparison of the same strategies across all three items of a context to identify a single strategy for that context. Pink and green areas represent the existing strategies applied consistently. Orange areas, which are not part of the intersection, indicate an additional strategy.

Gaze data were captured and processed in Tobii Pro Lab. Eye movements were classified using the default “Identification by Velocity Threshold” algorithm, applying a velocity threshold of $30^\circ/\text{s}$ [32].

2. Categorization of strategies

Initially, *overarching strategies* were formed based on *gaze data* to gain a preliminary understanding of participants’ approaches. Subsequently, *interview data* were used to refine these categories into *subordinate strategies*.

To analyze the *overarching strategies* of the participants in both contexts, *gaze data* on the *number of transitions*, the *number of fixations*, and the *number of saccades in 5° intervals* were utilized. The analysis was based on defined AOIs.

To ensure comparability among participants, the relative frequencies of these metrics were calculated. Based on these normalized gaze data, a cluster analysis was conducted for each of the three items in both contexts. Ward’s method for hierarchical clustering with Euclidean distance and z score normalization was applied [33]. Ward’s method is an agglomerative clustering approach that minimizes the total within-cluster variance at each step. It is particularly suited for uncovering structure in complex behavioral data by forming compact, well-separated clusters. Since the strategies were analyzed across contexts rather than on an item-by-item basis, an intersection comparison was conducted. Participants were only assigned to a strategy if they consistently applied the same cluster-defined strategy across all three items within a given context (see Fig. 5 for an example involving three items). This resulted in the identification of the so-called main strategies. The naming of these strategies was based on a qualitative analysis of the respective gaze paths. Cases outside the intersection were labeled as *changing strategy*.

The *interviews* aim to describe the strategies within the clusters in detail and to form *subordinate strategies*. Only those strategies that were clearly identified through the intersection comparison were further detailed in the interviews. To achieve this, the interviews were first transcribed, and the subsequent analysis followed the principles of content-structuring qualitative content analysis as outlined by Kuckartz [34].

Deductive subcategories were derived from the literature previously described by Fisher [21] and then aligned with the overarching strategies from the cluster analysis to establish a consistent connection between the qualitative data and the clusters identified through the gaze data.

Relevant statements from the interviews were assigned to these subcategories, followed by a second round of coding.

To ensure coding reliability, an independent coding was conducted by a second person. The results were compared with the original coding to identify agreements and discrepancies. This approach enables a comprehensive examination of the strategies, enhances the validity of the findings, and validates the gaze data.

3. Analyze of response frequency

Response frequency was operationalized as the number of correct responses per item. The responses were coded as 0 (incorrect) or 1 (correct). Then, the number of correct answers for the three items in each context was added, resulting in a score between 0 and 3. This score was then calculated for each strategy.

4. Analysis of overarching strategies

The assigned overarching strategy categories based on gaze paths were examined in more detail by analyzing saccade directions and transitions within the areas of interest (AOIs) for each strategy in the respective contexts.

A kernel density estimation based on the von Mises distribution was calculated for the saccade directions within the AOIs of all items—separately for each strategy and context. This method allows for comparing the saccade directions associated with different processing strategies.

The results were visualized using polar diagrams. Polar diagrams are well suited for visualizing circular data like saccade directions, as they clearly show directional information across the full 360° range and highlight strategy-specific gaze patterns. To analyze these directions, kernel density estimation was used as a flexible, non-parametric method for estimating probability density. Since saccade directions are circular, the von Mises distribution—commonly seen as the circular equivalent of the normal distribution—was applied. This approach allows for a smooth and interpretable visualization of gaze patterns across strategies.

Additionally, the transitions within the AOIs of all elements in a context were analyzed. Analyzing transitions is important because they reflect shifts in visual attention between relevant elements. This reveals how information is connected and processed—key to understanding underlying strategies. The relative frequencies of transitions were calculated across all participants and displayed in a matrix.

This provides an aggregated overview of the transitions, highlighting general patterns and differences between strategies. The most common transitions that occurred across all strategies in both contexts were identified and examined in detail:

For *Question 2.1*, a permutation test was conducted to determine whether significant differences exist in these transitions between the strategies. The test used the variance of the means of the relative frequencies as the test statistic, as this measure captures overall differences in the average frequency of transitions across the groups.

This approach is especially suitable for comparing an undetermined number of groups [35], with the exact number of strategies (n) determined by the outcomes of the cluster analysis. The permutation test was conducted separately for each of the most frequently selected transitions across the strategies (i.e., n groups) in each context. To control false positives due to multiple testing, the Benjamini-Hochberg procedure was applied [36].

An additional analysis examined the relationship between the relative frequencies of specific transitions and a summed test score using Spearman correlation [37]. The p values were adjusted using the Benjamini-Hochberg procedure [36].

For *Question 2.2*, we conducted a permutation test to examine differences in transitions within the same strategies across the two contexts (two groups) [38]. The relative frequencies of each participant's transitions were considered. The absolute difference of means was used as

the test statistic. The resulting p values were adjusted using the Benjamini-Hochberg procedure [36].

Since some participants appeared in both groups, the data exhibited dependency. To account for this, a paired permutation test was applied for these participants, as their results are inherently related. For participants who differed between the groups, a standard permutation test assuming independence was conducted. This dual approach ensured that the dependency structure in the data was appropriately addressed, allowing for a more accurate assessment of the differences in transitions between the contexts.

In cases of significant differences, we identified which strategy group exhibited the highest relative frequency of each transition.

V. RESULTS

A. Strategies

Two main strategies were identified through a cluster analysis of the gaze data (see Fig. S in Supplemental Material Sec. F [31]). Additionally, the intersection comparison revealed that some participants switched between the two main strategies across all three items (changing strategy). This results in three overarching strategies.

The analysis of the interview data provided a more nuanced understanding of these main strategies, allowing for the identification of subordinate strategies (see Table III).

1. Overarching strategies

The focus on value pairs involves focusing on a specific point on the graph and its corresponding values on the axes (see Fig. 6). This strategy is applied in 36.25% of cases in the thermodynamic context and 18.75% of cases in the

TABLE III. Overview of overarching strategies, common strategies, and absolute frequencies in the thermodynamic H_T and mathematical H_M contexts. The values refer to the 80 participants. The bolded values indicate the absolute frequencies of the overarching strategies.

Overarching strategies	Subordinate strategies H_T H_M	
Focus on the value pairs		29 15
	Algebra	17 13
	Proportional reasoning	1 1
	Intuitive	8 0
	Other correct	3 1
Focus on the graph of function		21 28
	Intuitive	20 25
	Proportional attempt	0 1
	Other correct	1 2
Changing strategies		30 37

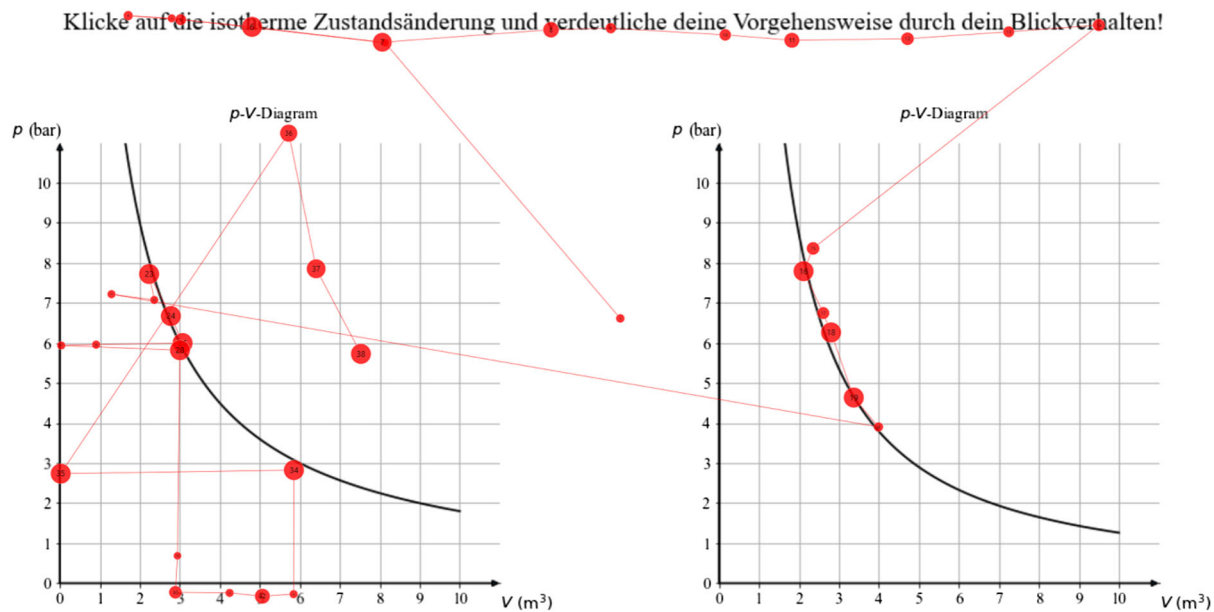


FIG. 6. Exemplary gaze path for the strategy focused on the value pairs, utilizing the subcategory of algebra (participant: DA, SE, MA, 05). Task: click on the isothermal change of state and demonstrate your process through your eye movements!

mathematical context. The focus on graph, on the other hand, centers primarily on the graph itself (see Figs. 7 and 8). It is used in 26.25% of cases in the thermodynamic context and 35% in the mathematical context.

Across the three tasks per context, participants did not consistently use the same strategies. This means that their strategy shifted between focusing on the graph of function and the value pairs. A pairwise comparison revealed that 37.5% of participants in the thermodynamic context and 46.25% in the mathematical context changed their strategies while solving the items.

In addition, the interview data provide in-depth insights into the strategies. This allows for further refinement of the strategies using the categories of Fisher [21]. As with the strategies, the same subcategories of the strategies appear in both contexts, although with different frequencies (see Table III). Notably, the most frequently occurring subcategory is identical in both contexts.

2. Subordinate strategies

The focus on value pairs was primarily based on algebraic structures. In this strategy, at least two points were considered, comparing their respective arguments and function values. For the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, a relationship was established where $x_1 = y_2$ and $x_2 = y_1$. The following are exemplary quotes from students who applied this strategy while solving the items.

So, I was able to clearly identify points again, and I chose 3 and 6, as well as 6 and 3. From that, I understood that it was essentially an isothermal

process. Isothermal is then indirectly proportional.³(Participant: DA, SE, MA, 05)⁴

The visual pattern for explaining the processing can be seen in Fig. 6. The following quote comes from solving a different item but follows the same strategy as the first one and generalizes it.

The x value from the first point must be the y value from the second point and the y value from the first point must be the x value from the second point. (Participant: FE, JU, MA, 07)

The strategy focused on the graph of function was primarily on examining its overall structure. The interviews reveal that learners predominantly concentrated intuitively on the shape and position of the graph, without providing any mathematical justification.

This strategy is categorized according to Fisher's framework as an intuitive strategy, which is considered flawed Fisher [21]. The approach is exemplified in the following section through quotes.

As mentioned before, it starts earlier than the graph on the left. It's also a larger graph, and it follows a more curved path. Or, it just behaves differently than the one on the left. (Participant: DA, MA, MA, 05)

³The participants' statements were translated from German into English to improve clarity.

⁴The ID numbers of the participants are in brackets.

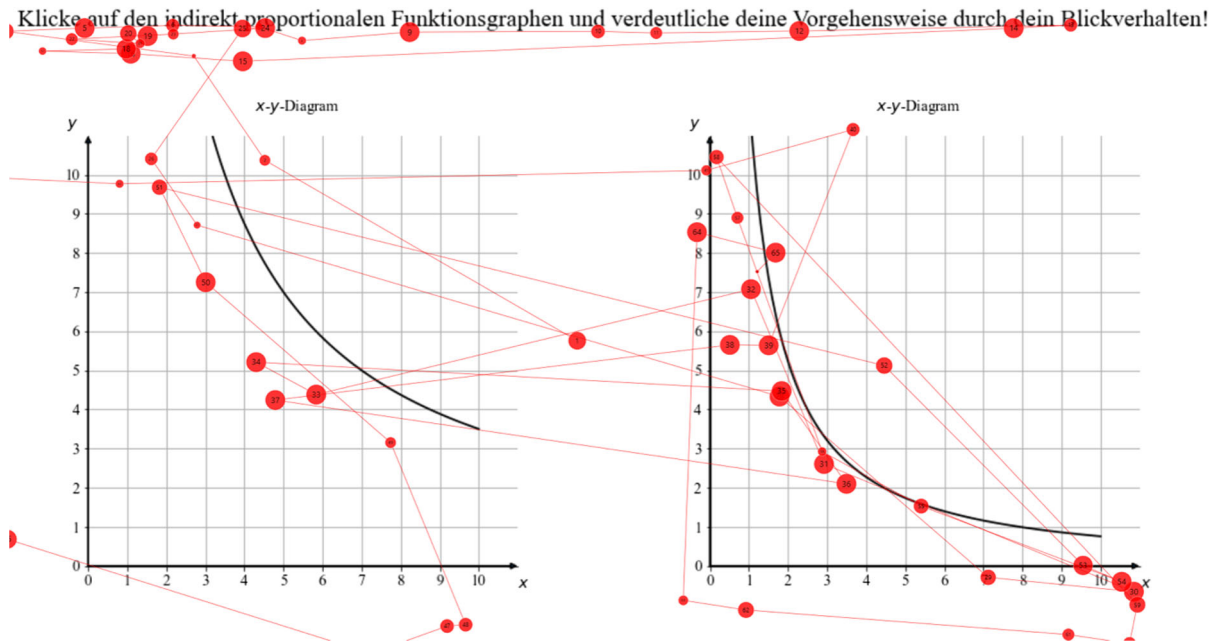


FIG. 7. Exemplary gaze path for the strategy focused on the graph of function, as illustrated in example 1 of an intuitive strategy (participant: DA, MA, MA, 05). Task: click on the graph of the indirectly proportional function and demonstrate your process through your eye movements!

The focus is on the earlier starting point, the size, the curvature, and the overall behavior of the graph compared to the other (see Fig. 7).

Another learner considers the midpoint of the graph to be particularly important. This strategy is explained in the following quote and is illustrated in Fig. 8.

... I looked at the midpoints and the endpoint of the graph. It became clear to me that the midpoint of the first graph has more of a bend compared to the second graph, and the first graph is more aligned with the x axis than the second one. (Participant: JA, JU, LA, 05)

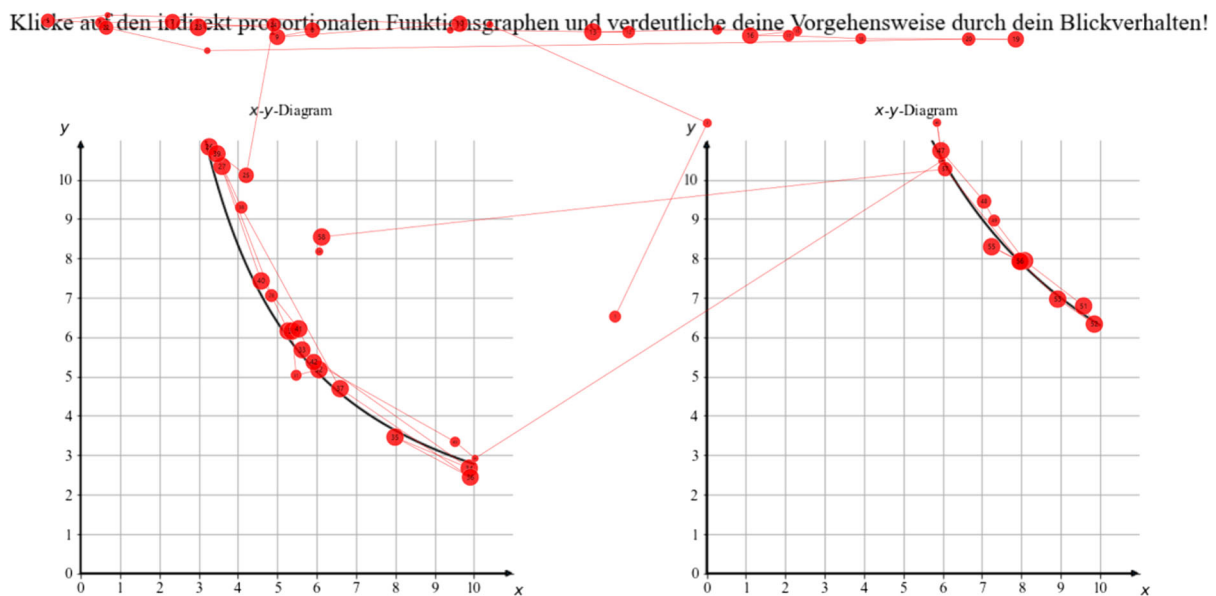


FIG. 8. Exemplary gaze path for the strategy focused on the graph of function, as illustrated in example 2 of an intuitive strategy (participant: JA, JU, LA, 05). Task: click on the graph of the indirectly proportional function and demonstrate your process through your eye movements!

For the changing strategies, participants shifted between focusing on the graph and focusing on the value pairs to solve the three tasks in the mathematical and thermodynamic contexts, as shown by the visual patterns.

The interview data revealed that the strategies used were largely consistent with those employed in Fisher's study [21]. However, the proportional approach also played a role. Specifically, when focusing on the graph, attention was also given to a rectangular hyperbola.

Additionally, participants were asked about the reasons for their change in strategy. It became apparent that uncertainties regarding the current strategy or the different appearance of the graphs were the main reasons for the shift.

The following quotes illustrate how uncertainties and visual similarities between the graphs influence strategies.

I was a bit unsure, and because it's so far away from it, I took a closer look at the axes. (Participant: IL, OK, MA, 04)...

I found it obvious which one was indirectly proportional and which wasn't. But here, it wasn't obvious to me because the two look quite similar. So, through the values, specifically the x - and y -values... (Participant: AL, AP, QU, 07)

Uncertainties lead participants to focus more on details like the axes, while visual similarities between the answer options cause them to rely on numerical values and not only consider the graph when making decisions.

B. Response frequency

Table IV presents the arithmetic mean test scores for three strategies across the two contexts. In the thermodynamic context, the mean score for focusing on the value pairs is 2.38, while in the mathematical context, a higher mean score of 2.60 is observed. The focus on graphs shows a mean score of 1.81 in the thermodynamic context, compared to 1.64 in the mathematical context. Regarding the changing strategy, the thermodynamic context has a mean score of 1.47, whereas a higher score of 1.70 is found in the mathematical context.

No significant differences were found for the same strategies across the contexts. The highest response frequency was observed for the strategy "focus on the value pairs," particularly in the mathematical context, indicating that this approach was the most commonly used.

In contrast, the lower response frequencies for the strategies "focus on the graph of function" and "changing strategies" suggest that these were less frequently applied and were linked to a higher number of incorrect answers.

C. Analyzing strategy differences within and across contexts

In this section, we analyze the differences between various strategies within each context and compare how the same strategies differ between the two contexts. For this purpose, we utilize saccade directions and transitions.

1. Saccade directions

Figure 9 shows the kernel density estimations (KDE) of saccade directions in polar diagrams for different strategies in thermodynamic and mathematical contexts for all items.

The KDE of saccade directions highlights preferred gaze directions and their frequency during the tasks. Saccade directions are plotted as angles, with KDE values plotted along the radius.

The upper row of the figure displays the KDEs of saccade directions for the strategies focus on the value pairs, focus on the graph of function and "changing processing strategies" within the thermodynamic context. In the first polar diagram, for the focus on the value pairs strategy, the KDE shows predominantly horizontal and vertical saccades.

The second polar diagram, illustrating the focus on the graph of function strategy, highlights primarily diagonal saccades at approximately $120^\circ (\pm 40^\circ)$ and $300^\circ (\pm 40^\circ)$. The third diagram, representing changing processing strategies, also shows mostly diagonal saccades, with some horizontal saccades as well.

The lower row, which shows the saccade directions in the mathematical context, follows similar patterns. The focus on value pairs strategy predominantly exhibits horizontal and vertical saccades. In the focus on the graph of function strategy, diagonal saccades around $120^\circ (\pm 50^\circ)$ and $300^\circ (\pm 50^\circ)$ dominate. Finally, the changing processing strategies approach again shows mostly diagonal saccades, though horizontal saccades are also present.

The directional tendencies of the saccades visually align in both contexts.

In summary, the results show that the strategy of focusing on the value pairs is associated with horizontal and vertical saccades, the strategy of focusing on the graph with diagonal saccades, and the strategy of changing strategies with

TABLE IV. Arithmetic mean test scores for the strategies in thermodynamic and mathematical contexts. Values in parentheses are adjusted for guessing, with any negative values set to zero to prevent negative scoring. The values refer to the 80 participants.

Context	Focus on the value pairs	Focus on the graph of function	Changing strategies
Thermodynamic context	2.38 (2.17)	1.81 (1.33)	1.47 (1.00)
Mathematical context	2.60 (2.40)	1.64 (1.14)	1.70 (1.38)

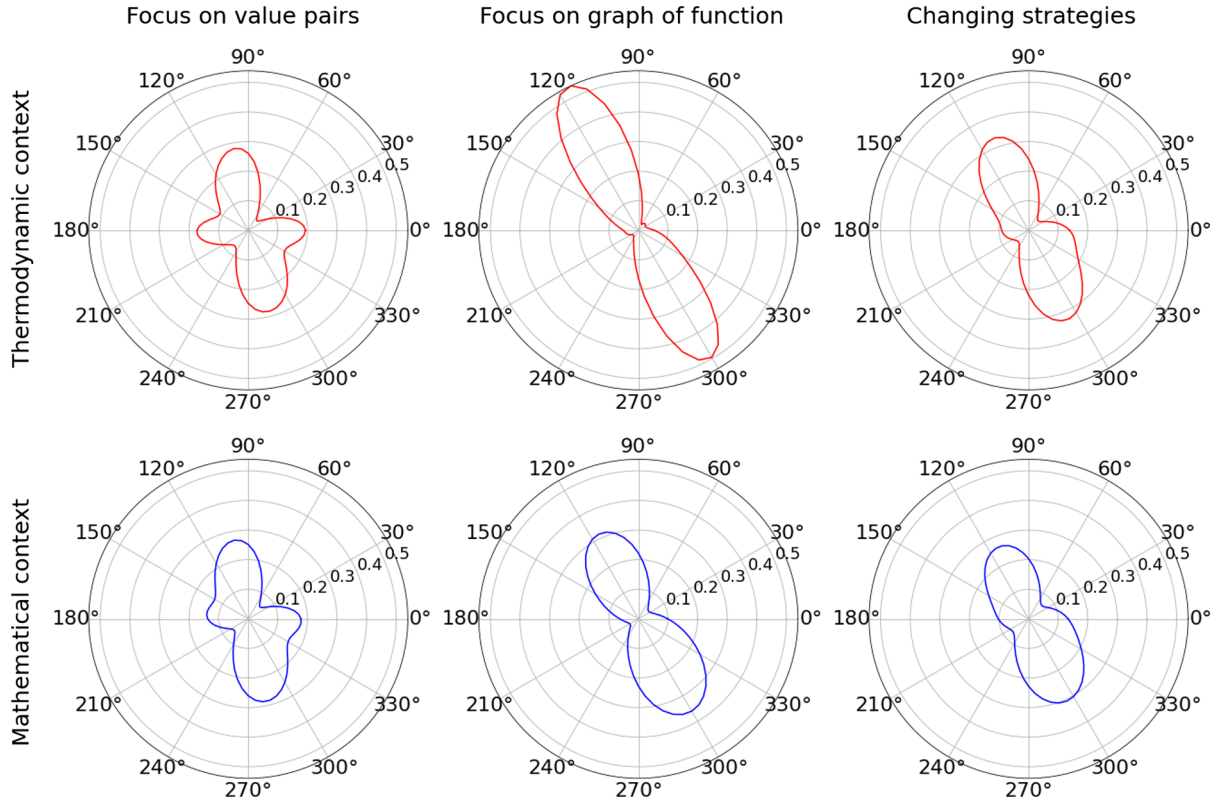


FIG. 9. Polar diagrams of KDEs of saccade directions for different strategies (focus on the value pairs, focus on the graphs, and changing strategies) in thermodynamic (red) and mathematical (blue) contexts. The values refer to the 80 participants.

diagonal saccades that are slightly shifted toward vertical saccades, along with weaker horizontal saccades.

2. Transitions

Figure 10 shows the relative frequency of these transitions across different strategies. The relative frequencies of the transitions within the AOIs were represented in matrices, where the vertical axis shows the start AOIs and the horizontal axis shows the landing AOIs.

In the top row of Fig. 10, the transitions in the thermodynamic context are displayed, while the bottom row shows the transitions in the mathematical context.

In thermodynamics, it becomes clear that when focusing on the value pairs, transitions primarily occur within the areas of the graph. However, there are also notable transitions between the graph areas and the axes, as well as between the axes and their corresponding values. This is illustrated in Fig. 6, which exemplifies this strategy and helps explain the horizontal and vertical saccade directions discussed earlier (see Fig. 9). For the strategy focus on the graph of function, transitions predominantly occur within the graph areas, as shown in Fig. 8.

This explains the diagonal saccade directions associated with this strategy, as depicted in Fig. 10.

In changing strategies, there is a broader distribution of transitions, particularly between the graph, the abscissa, and the points on the graph.

In thermodynamics, the analysis of the most common transitions between the three strategies shows that several transitions exhibit significant differences between the groups.

The results of the permutation test show that transitions with significant differences include “graph of function to x value” ($T_{\text{var}} = 0.000034$, $p_{\text{adj.}} = 0.008991$; see Table A in Supplemental Material Sec. G [31]), which frequently occurs in strategies focused on the value pairs (see Fig. 10). Another exemplary transition is “ x value to abscissa” ($T_{\text{var}} = 0.000157$, $p_{\text{adj.}} = 0.003814$; see Table A in Supplemental Material Sec. G [31]), also relevant in this context. Additionally, “ x value to graph of function” ($T_{\text{var}} = 0.000133$, $p_{\text{adj.}} = 0.003497$; see Table A in Supplemental Material Sec. G [31]) frequently associates with the emphasis on value pairs.

These transitions serve as illustrative examples of those that play a significant role in understanding value pairs.

The transition “point on graph of function to graph of function” ($T_{\text{var}} = 0.002048$, $p_{\text{adj.}} = 0.005994$; see Table A in Supplemental Material Sec. G [31]) and graph of function to graph of function ($T_{\text{var}} = 0.13331$, $p_{\text{adj.}} = 0.138$) shows significant differences and primarily occurs in the context of focus on the graph of the function (see Fig. 10).

In contrast, some transitions, such as “abscissa to abscissa” ($T_{\text{var}} = 0.000228$, $p_{\text{adj.}} = 0.283217$; see Table A in Supplemental Material Sec. F [31]) and “function

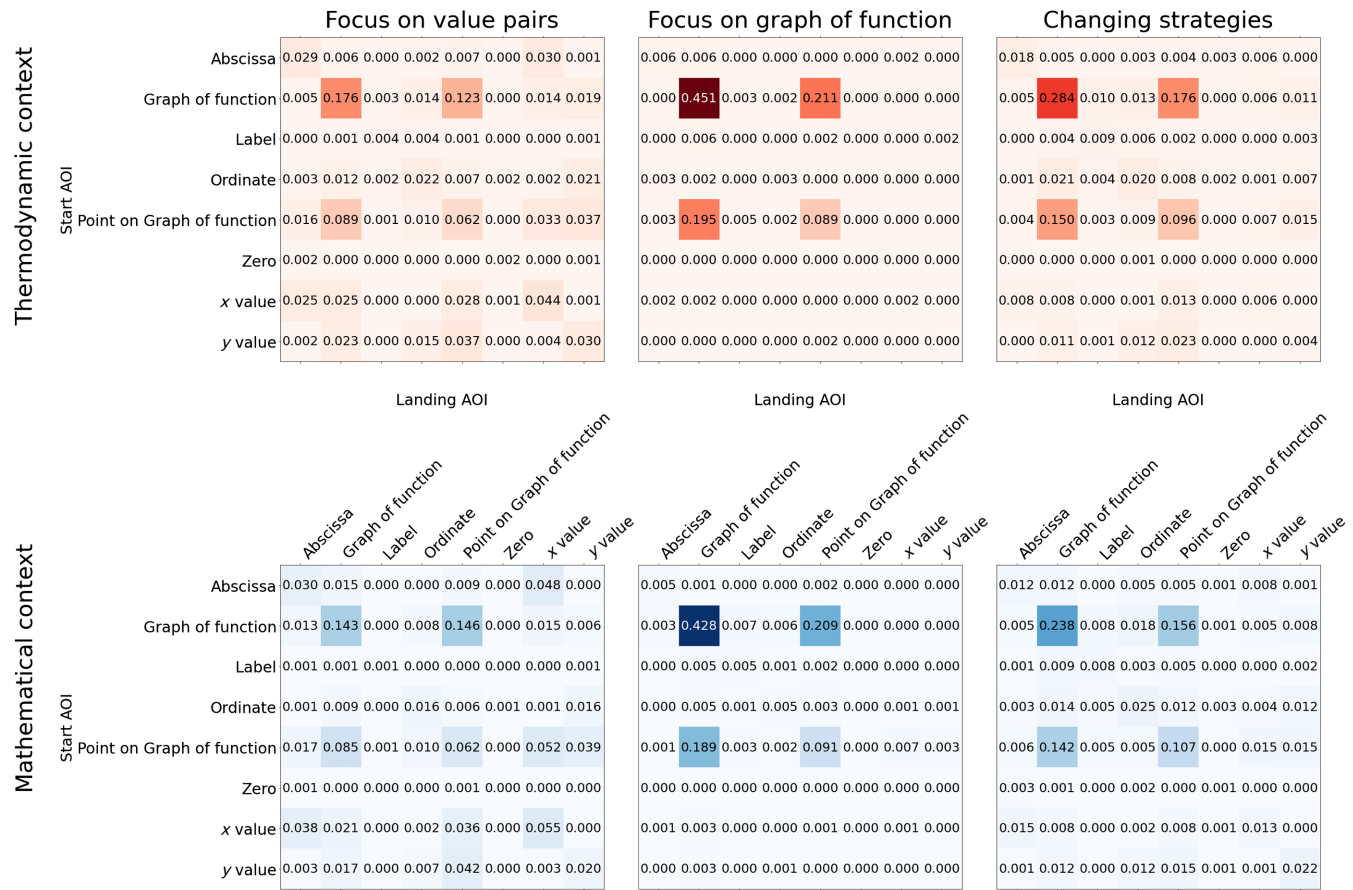


FIG. 10. Transition matrices showing the relative frequency of transitions for different strategies, with darker colors indicating higher frequencies. The values refer to the 80 participants.

value to ordinate” ($T_{\text{var}} = 0.000148$, $p_{\text{adj.}} = 0.297702$; see Table in Supplemental Material Sec. F [31]), do not exhibit significant differences. This suggests that the strategies in these specific areas are relatively homogeneous. The complete table of permutation statistics and corrected p values provided in Supplemental Material [31] offers a comprehensive overview of all analyzed transitions and significantly contributes to understanding the differences in the groups’ approaches.

In the mathematical context, the same transitions are most frequently observed across the strategies (see Fig. 9). Additionally, the permutation test reveals nearly identical significant differences in transitions among the strategies.

Examples of significant transitions include graph of function to x value ($T_{\text{var}} = 0.000092$, $p_{\text{adj.}} = 0.0084$; see Table A in Supplemental Material Sec. G [31]), x value to abscissa ($T_{\text{var}} = 0.000361$, $p_{\text{adj.}} = 0.0070$; see Table in Supplemental Material Sec. G [31]), and x value to graph of function ($T_{\text{var}} = 0.000081$, $p_{\text{adj.}} = 0.0060$; see Table A in Supplemental Material Sec. G [31]).

These transitions are examined significantly longer in the strategy focused on the graph of function (see Fig. 10). The complete results for all transitions can be found in Supplemental Material Sec. F [31].

The results of the correlations between transitions and solution frequencies in thermodynamics show that certain transitions in strategies are closely linked to success. The transition “ x value to x value” has the strongest correlation ($\rho = 0.4308$, $p_{\text{adj.}} = 0.0014$). The transitions “ x value to point on graph of function” and “abscissae to x value” also show significant positive correlations ($\rho = 0.3776$; $\rho = 0.3637$), suggesting that these transitions also play an important role in success. Moderate correlations are observed with graph of function to x value ($\rho = 0.3020$) and x value to graph of function ($\rho = 0.2879$), which are also significant but show a weaker link to solution success.

In the mathematical context, no significant results were found. A detailed analysis of all results is provided in Supplemental Material Sec. H [31].

The same strategies in both contexts show no significant differences in the transitions. The results of the permutation test are presented in Tables B and C in Supplemental Material Sec. G [31].

VI. DISCUSSION

The present study demonstrates that learners use different strategies to identify indirectly proportional graphs in both mathematical and thermodynamic contexts.

Cluster analysis of the gaze data revealed two main strategies, which were further refined through the integration of interview data. This supports the assumption underlying RQ 1 that learners do not rely on a uniform approach but rather apply distinct strategies depending on how they process the graph.

One of these strategies focuses on the analysis of numerical value pairs. Learners using this approach identify specific points on the graph and examine the corresponding values on the coordinate axes. Interview data suggest that this strategy is based on recognizing a structural relationship between the function value and its argument. When the argument of one point and the function value of another point have the same magnitude, learners frequently classify the function as indirectly proportional.

According to the theoretical considerations outlined in Eq. (3) and based on the proportional formula provided by Fisher [21], this strategy does not rely on verifying the equality of products but rather on a structural matching of values. This strategy can thus be seen as a preliminary form of mathematical reasoning, in which structural similarities are recognized without explicitly examining the underlying multiplicative relationship—that is, the constancy of the product. It is therefore categorized under the “algebra” strategy type, as described by Fisher [21].

Despite the differences, this strategy is considered valid, as it reflects a meaningful understanding of the functional relationship. This is supported by the high number of correct responses among participants using this strategy.

In both contexts, focusing on the value pairs led to the highest success rates across all three tasks. In the thermodynamic context, this strategy was further supported by transitions from the graph to the values on the axes, which correlated positively with task success.

The second strategy places a strong emphasis on the graph itself. Interview data indicate that this strategy often relies on visual impressions, such as shape, curvature, and position of the graph, while neglecting mathematical properties—particularly those of the rectangular hyperbola and its functional relationships. This predominantly visual interpretation may reflect a more intuitive and less formalized understanding.

Its widespread use could be attributed to the perceived immediacy and simplicity of comparing overall graph shapes, which may seem sufficient in the absence of explicit prompts for numerical analysis. However, this strategy carries a higher risk of misinterpretations, as it overlooks critical information such as axis scaling, coordinate values, and the underlying function type. According to Fisher [21], this approach is classified as an “intuitive” strategy within the broader category of “false strategies,” as these strategies are far removed from mathematical reasoning. Thus, this strategy is associated with lower success rates across both contexts, which may be explained by the lack of understanding.

Some learners do not use the same strategy for the three tasks in each context but instead switch between focusing on the graph and focusing on the value pairs. This shift in strategy suggests that learners interpret the requirements of the tasks differently and adjust their approach. At the same time, this could also indicate uncertainties. Additionally, the appearance of the graph can influence the change in strategy. For instance, when learners quickly recognize a specific graph shape, such as a rectangular hyperbola indicating indirect proportionality, they may choose to forgo a detailed analysis of the value pairs. These assumptions are also supported by selected quotes.

The mid-range response frequency, particularly among groups that frequently changing strategies or focused on the function graph in the mathematical context, may indicate a lack of systematic reasoning and uncertainty in interpreting functional relationships. These learners may rely more on visual impressions or intuitive judgments rather than on the consistent application of formal mathematical strategies.

Furthermore, it cannot be ruled out that as-yet unexamined characteristics of the tasks themselves may contribute to such shifts in strategy. For instance, a greater distance between the graph and the coordinate axes may hinder the identification of coordinate pairs, thereby encouraging a shift from a value-pair-focused strategy to one centered on visual features of the graph.

The strategies and their respective details from the interview data align well with the descriptions provided by Fisher [21] and are evident in both contexts, albeit with varying response frequencies.

The analysis of the strategies in the two contexts (RQ 2.1) reveals differences in saccade directions and transitions between strategies. These differences align with the hypothesis that each strategy has characteristic gaze patterns, as they are associated with specific cognitive processes [26,27].

In the focus on value pairs strategy, gaze movements are predominantly horizontal and vertical. This corresponds to the previously described features of this strategy, which involve transitions between the graph and the axes. These observations support the transitions that occur significantly more frequently within this strategy.

Just and Carpenter [25] showed that eye fixations are closely linked to the cognitive processing of the fixated point. Carpenter and Shah [39] showed that fixations in diagrams typically shift between patterns, axis labels, and numerical values, indicating a continuous integration of different information areas and forming part of a successful inspection process. This observation was also confirmed in this study through the analysis of transitions.

The transitions between axes and graph suggest that continuously updating and linking information is essential for precise data processing and a comprehensive understanding of the presented content.

In the focus on the graph of function strategy, gaze movements tend to be diagonal, reflecting the characteristics

of this strategy, which involve following the course of the graph. This observation correlates with the transitions that occur more frequently within this strategy.

Carpenter and Shah [39] describe an inspection process that requires precise consideration of x - and y -values for a full understanding of graphs. In contrast, a strategy in which only the shape of the graph is considered leads to poorer test results, as was shown in this study.

Focusing on the graph alone is therefore insufficient in most cases and can quickly lead to errors. Misunderstandings often arise because this approach overlooks critical details such as scale, axis intervals, and specific values, which are essential for accurate interpretation. Without these elements, viewers may misjudge the relationships or proportions represented in the graph, leading to faulty conclusions.

The change between strategies, focusing on both value pairs and the graph during task solving, is characterized by predominantly diagonal gaze movements, but also includes slight horizontal movements. This is due to the simultaneous use of both strategies, bringing the characteristics of both approaches to bear. This is also evident in the transitions.

In both contexts, the same strategies were observed, albeit with differing frequencies. The saccade directions within each strategy also showed similarities across the two contexts. This discussion relates to research question 2.2. No significant differences were found in the transitions of the same strategies between contexts. Nearly identical transitions were observed within each strategy across both contexts, as well as similar frequency trends. While the lack of significant differences could be influenced by the surface similarity of the tasks, it also suggests that learners apply similar strategies regardless of the context. This indicates that the cognitive processing of graph-based tasks may be less context-sensitive than previously assumed, at least when the visual and structural elements of the tasks are closely aligned. The absence of significant differences in strategy transitions can be explained by several factors. First, the learning environments in both contexts are highly similar, providing learners with comparable conditions that promote consistent strategies. Greeno *et al.* [40] argue that such similar affordances can lead to identical or similar action schemas, provided these affordances are recognized. As a result, learners' cognitive processes may operate in parallel across both contexts, leading to the same transitions in identifying indirect proportionality. Salomon and Perkins [41] describe this phenomenon as "low transfer," which occurs when the conditions in the transfer context closely resemble those of the original learning context, thereby triggering well-automated or semiautomated response patterns.

VII. LIMITATIONS

Several limitations should be considered when interpreting the results. First, the sample size and demographic

composition may limit generalizability. Although 80 students participated in the study, the sample was drawn from a specific age group and region and may not be representative of broader or more learner populations. Future studies with larger and more heterogeneous samples are needed to assess the robustness and applicability of the results across educational contexts.

Second, the tasks focused exclusively on mathematical and thermodynamic representations of indirectly proportional functions. It remains unclear to what extent the identified strategies can be transferred to other scientific or technical disciplines, as disciplinary conventions and contextual features may lead to alternative approaches not captured in this study.

Nonetheless, the methodological approach remains applicable regardless of both the disciplinary context and the type of function represented. Whether dealing with linear, quadratic, or other functional relationships, it is possible to systematically identify whether learners shift between graph and axis values, focus primarily on the graph, or change strategies over time. The observed transitions and eye movement patterns would simply align with the specific features of the graph and value structures in question.

The controlled testing environment may have influenced the participants' strategic behavior. Participants might have adopted strategies specific to the test situation rather than reflecting their reasoning in authentic classroom environments. Future studies could address this limitation by incorporating more ecologically valid settings that mirror real instructional contexts.

Despite these limitations, the study provides important insights into learners' strategic processing of graphical representations and offers a foundation for further research on domain-specific and cross-contextual reasoning.

VIII. CONCLUSIONS

The present study provides novel insights into the gaze-based characteristics of strategies used to identify indirectly proportional graphs across two distinct contexts. Although differences in saccade directions and transitions between the strategies were observed, they were not context dependent, as initially assumed.

A key contribution of this study lies in the visual characterization of specific strategies through eye-tracking data. These gaze patterns offer a deeper understanding of learners' cognitive processes and provide a foundation for developing adaptive support systems that respond to individual gaze behavior.

In particular, the strategy of focusing on the graph of the function was often associated with attention to irrelevant visual features rather than meaningful mathematical relationships. This finding, supported by interview data, highlights the potential of visual analysis to uncover learning obstacles that may not be verbally expressed.

The results suggest that learners' gaze patterns and strategies are valuable indicators of cognitive processing and conceptual understanding.

Future research should build on these findings to develop adaptive learning systems that respond in real time to individual gaze patterns. Such systems could enable the early identification of misconceptions and provide tailored feedback. Ultimately, this could significantly enhance the understanding and handling of indirectly proportional graphs in both mathematical and science.

IX. IMPLICATIONS

Teachers can help students in developing effective graph-reading strategies by explicitly explaining the relationship between the variables on the axes and the graph itself. Structured instruction should help learners move beyond surface features and develop an understanding of how changes in one variable affect the other.

A challenge is that students tend to focus on irrelevant areas of a graph, either due to intuitive heuristics or a lack of strategic awareness. Therefore, it is important to address these aspects deliberately. By learning how to handle irrelevant details, students sharpen their analytical skills and are able to direct their attention to the key features. This can be achieved through visual aids, well-structured tasks, and comparative exercises that guide focus toward the relevant elements. Such approaches not only enhance students' analytical abilities but also deepen their understanding of effective strategies.

The findings of the study would not have been possible without the recording of gaze data, as many students often focus on areas of a graph without being aware of it. The gaze data revealed hidden patterns that would have otherwise gone unnoticed. Often, students struggle to articulate their strategies, which is further exacerbated by the lack of

technical language. This leads to misunderstandings and a distorted assessment of their understanding.

The combination of gaze data and verbal interview data provides a powerful tool for diagnosing misconceptions and learning processes.

X. OUTLOOKS

The analysis of these strategies and their corresponding eye movements highlights the potential for targeted adaptive support. When different visual patterns are identifiable across various tasks, this can indicate uncertainties or difficulties. Additionally, visual patterns that focus primarily on the graph are often indicative of intuitive strategies, as suggested by the interview data, which can point to existing challenges. When such patterns are identified, an adaptive learning system can respond individually to the specific difficulties faced by learners. This allows for the targeted identification of inadequate and intuitive strategies, as well as the provision of tailored support that deepens understanding and enhances the processing of visual information.

B. W.: conceptualization, supervision, writing—review and editing, resources, software; P. F.: Data curation, formal analysis, investigation, methodology, project administration, software, visualization, writing—original draft, writing—review and editing, conceptualization; S. B.-G.: resources, writing—review and editing.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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