Editors' Suggestion

# Extending the action, process, object, schema theory to physics education: A genetic decomposition of the two-dimensional heat equation

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In this paper, we describe our extension of the action, process, object, schema (APOS) theory to capture the complex interplay of mathematics and physics. We do this in the context of the 2D heat equation. We describe a hypothetical learning trajectory of the 2D heat equation, a preliminary genetic decomposition that stresses the conceptual understanding of its mathematical formulation. In our extension of APOS to physics education, we highlight how our approach precisely captures the interplay between mathematics and physics in different mental constructions of the framework. In addition to this core focus, we consider other layers when designing the preliminary genetic decomposition: the key concepts and the necessary depth of students' prerequisite knowledge of the chosen context, the challenges students commonly face with these concepts, and the use of multiple external representations. We conclude by giving an overview of future research to be done after the design of the preliminary genetic decomposition.

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## I. INTRODUCTION

### A. Mathematics-physics interplay

The intertwined evolution of mathematics and physics reveals a history of mutual inspiration: from Kepler's orbits of planets to Newton's gravity and calculus, from Fourier's modelling of heat transfer to Fourier analysis and its later development in wave theory, and from Maxwell's equations describing electric and magnetic fields to the development of vector calculus and later, a systematic mathematical framework for electromagnetism. This intertwined reasoning should not dismiss the unique viewpoints of the two disciplines [1], but rather extend beyond the idea that mathematics is merely a tool for physics and physics merely a context for mathematics, reaching a point where reasoning becomes inseparable. While mathematics can provide structural insight into physical meaning, its abstract nature contrasts with the concreteness of physics.

In practice, the perception of the interplay between mathematics and physics in an educational context is quite different. The teaching and learning of mathematics and physics are typically separated, with mathematics courses usually preceding the physics courses in which the mathematical content is applied. However, studies have shown that combining mathematics and physics remains a challenge for learners at all levels of education, even for those with good mathematical understanding [2–4]. The interplay of mathematics and physics has been viewed through several lenses in education research, for example: mathematical modeling [5,6], cognitive semantics and language [7–10], and symbolic representations [11]. However, theoretical frameworks commonly used in mathematics education research are rarely used in physics education research, and vice versa. An example is the research of Doughty *et al.* [12] who have applied the concept image framework from mathematics education research [13] to the concept of integration in physics. Karam *et al.* [14] have argued that many difficulties arise from students not recognizing the strong connections between mathematics and physics, and Pospiech *et al.* [1] have emphasized the importance of an integrated approach from early stages in education. We argue that at more advanced levels, an integrated approach is necessary.

In this paper, we use an existing mathematics education research framework, APOS theory [15], in physics education research. We have chosen APOS because it allows us to look at the teaching-learning process at any level of granularity we desire. That said, we have found it necessary to extend APOS theory to allow us to investigate the interplay of mathematics and physics. APOS describes how a concept is constructed in the mind of an individual from a cognitive perspective in terms of actions, processes, objects, and schemas. It is built on Piaget's notion of

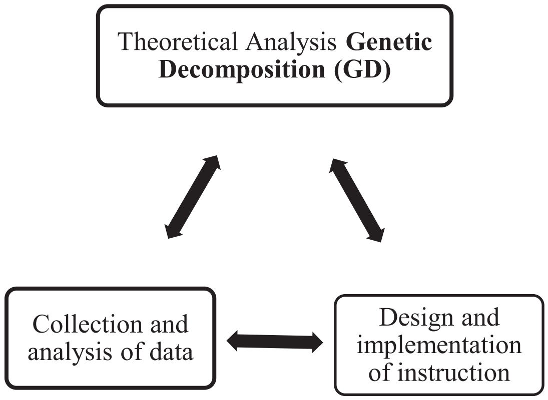


FIG. 1. Research cycle in APOS [17].

reflective abstraction [16]. APOS is a theoretical framework that combines three aspects (see Fig. 1): (i) theoretical analysis; (ii) design of instruction; and (iii) data gathering. These three aspects give APOS the capacity to combine the investigation of students' reasoning and the development of topic-based research-validated instructional material. In this paper, we show how we extended APOS theory to allow it to capture the complex interplay of mathematics and physics. We focus solely on the theoretical analysis aspect of the framework. Using the context of the two-dimensional heat equation, we illustrate our approach but believe that it applies beyond this context. We will discuss the data collection and design of instruction in a separate paper.

In Sec. I B, we dive into the complexity of the interplay of mathematics and physics in the context of the two-dimensional (2D) heat equation. It includes a discussion of the use of multiple external representations to depict specific concepts. In Sec. II, we elaborate on the APOS theoretical framework as it is used in mathematics education research and illustrate its different mental structures and mechanisms with examples. Furthermore, we explain the main tool in APOS, the genetic decomposition, a learning trajectory consisting of specific mental constructions a learner could carry out to understand a concept. In Sec. III, we explain how we extend APOS to describe the interplay of mathematics and physics with specific examples in the context of the 2D heat equation. We describe a hypothetical learning trajectory of the 2D heat equation (in APOS-terminology, a preliminary genetic decomposition) that stresses the conceptual understanding of its mathematical formulation while highlighting the interplay of mathematics and physics. We conclude with an overview of future research to be conducted following the design of the preliminary genetic decomposition of the 2D heat equation.

### B. Reasoning with the 2D heat equation: The complexity of the mathematics-physics interplay

The broad aim of this project is to help students understand the interplay between mathematics and physics in the context of the 2D heat equation. We investigate how students assign physical meaning to the equation's mathematical structure and how they use mathematical structures to interpret physical principles. Since our extension of the APOS framework is best explained in context, we describe in detail how we would like our students to view the 2D heat equation. Earlier work investigated this interplay in the context of the 1D heat equation [2,17–19] and in electromagnetism using other frameworks [20].

We are not concerned with the solution procedure of the 2D heat equation but rather with the conceptual understanding of its mathematical formulation and how this formulation captures the spatial and temporal evolution of both the temperature and the heat flow in a two-dimensional region of space. We represent the 2D heat equation as

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha \nabla^2 T(x,y,t). \quad (1)$$

This equation describes how the temperature distribution on a 2D plate evolves over time due to heat conduction. Equation (1) can be used at any point on a 2D plate with a given temperature distribution, not just to calculate but also to understand how the time rate of variation of temperature at this point, $\frac{\partial T}{\partial t}$, depends on the Laplacian of the temperature, $\nabla^2 T$ (alternatively denoted by $\vec{\nabla} \cdot \vec{\nabla} T$ and $\Delta T$). The Laplacian represents and connects several mathematics and physics concepts. It may be calculated as the sum of the spatial second partial derivatives (second PDs) of temperature and as the divergence of the temperature gradient at a point. The Laplacian of the temperature is related to the net heat flow at a point and to the average temperature difference between a point and its surroundings.

Reasoning with the Laplacian of the temperature in these different ways involves different external representations: Cartesian graphs, gradient vector fields, linguistic descriptions, and symbolic representations (Figs. 2–5). Conceptual understanding can be enhanced through the use of multiple external representations (MERs). In her DeFT framework, Ainsworth argues that MERs serve many pedagogical functions, including deepening students' understanding [21]. A well-designed purpose for the use of MERs, which takes into consideration design parameters such as the number of these representations, the information they

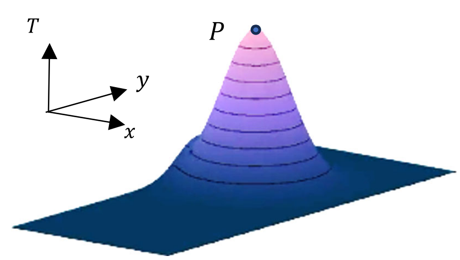


FIG. 2. Temperature distribution on a 2D plate at time $t = t_1$.

display, and their interconnections, while managing the cognitive load of the student, may improve learning outcomes. In the following paragraphs, we will elaborate on different aspects of the Laplacian of the temperature and how they are depicted by multiple external representations.

On a Cartesian graph of the temperature distribution (Fig. 2), the temperature at a point can be compared visually to the average temperature of its surroundings. For example, heat will flow out of point $P$ to regions of lower temperature and toward $P$ from regions of higher temperature in accordance with Fourier's law of heat conduction. Since point $P$ in Fig. 2 has a higher temperature than the average temperature of the points surrounding it, more heat will flow out than in, and the *net* heat flow into this point is negative. This causes a decrease in temperature at $P$ as time passes. This is reflected in the heat equation, where the Laplacian at $P$ is negative. While the sign of the Laplacian is not always obvious from a Cartesian graph, it can easily be obtained for point $P$, as we will show now.

Mathematically, the Laplacian of the temperature at a point on a 2D plate can be algebraically expressed as

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}. \tag{2}$$

To visualize this, consider the second PD with respect to $x$ and $y$. These second PDs describe how the *spatial* rate of change of temperature varies with respect to each spatial variable (since they represent the rate of change of the rate of change). Specifically, they describe the concavity along $x$ and $y$. Moving in the direction of increasing $x$ or $y$ around $P$ in Fig. 2 and comparing the slopes of successive tangents, the spatial rate of change of temperature is decreasing at $P$ with respect to both $x$ and $y$. The Laplacian, being the sum of the two second PDs, is therefore negative, and, as Eq. (1) states, the temperature at point $P$ will decrease with respect to time.

Generally, at any point, the concavity with respect to $x$ and $y$ and the "average bending"[1] with respect to $x$ and $y$ indicate the direction and magnitude of heat flow. The greater concavity at $P$ in Fig. 3(a) reflects a larger average temperature difference between $P$ and its surroundings, and therefore a greater net heat flow at $P$. In contrast, the smaller concavity at $P$, as shown in Fig. 3(b), indicates a lower average temperature difference between $P$ and its surroundings, and therefore a smaller net heat flow at $P$.

Moreover, on a gradient vector field plot of the temperature distribution, the direction and magnitude of the heat flow can be visualized. According to Fourier's law of heat conduction, the heat flux density $\vec{q}$ is proportional to the negative of the temperature gradient $\vec{\nabla}T$; the constant of proportionality is the thermal conductivity of the material, $k$:

$$\vec{q} = -k\vec{\nabla}T. \tag{3}$$

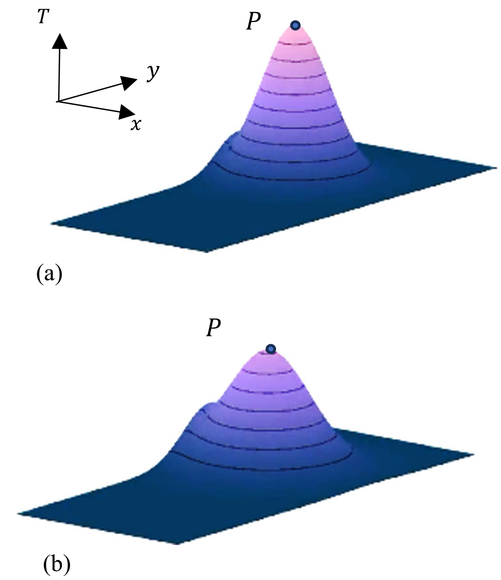


FIG. 3. Concavity at point $P$ (a) at instant $t_1$ and (b) at a later instant $t_2$.

The temporal rate of change of temperature at each point is proportional to the net heat flow into the point, and therefore it is also proportional to the divergence of the heat flux density vector. Mathematically, this relates to the Laplacian of the temperature, which can be obtained using the divergence of the gradient of the temperature:

$$\nabla^2 T = \vec{\nabla} \cdot (\vec{\nabla}T). \tag{4}$$

The gradient vector field plot may be used in two ways. Following a differential approach (Fig. 4), the divergence of the gradient may be deduced by identifying how $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$, the first spatial partial derivatives of the temperature, change with respect to each spatial direction. This observation links Eq. (4) with Eq. (2).

Following an integral approach (Fig. 5), the Laplacian of the temperature at a point can be obtained by analyzing the flux of the temperature gradient out of a small closed surface surrounding this point. A negative flux of the temperature gradient at a point will result in $\vec{\nabla} \cdot (\vec{\nabla}T) < 0$ and therefore a negative Laplacian.

Thus, to develop a conceptual understanding of the mathematical formulation of the heat equation, students need a physical understanding of temperature distribution, heat transfer, and Fourier's law. They also need a mathematical understanding of concepts such as the first and second partial derivatives, gradient, divergence, and

[1]We avoid the term "curvature," since we are not interested in numerical values, merely in associating greater bending with greater heat flow.

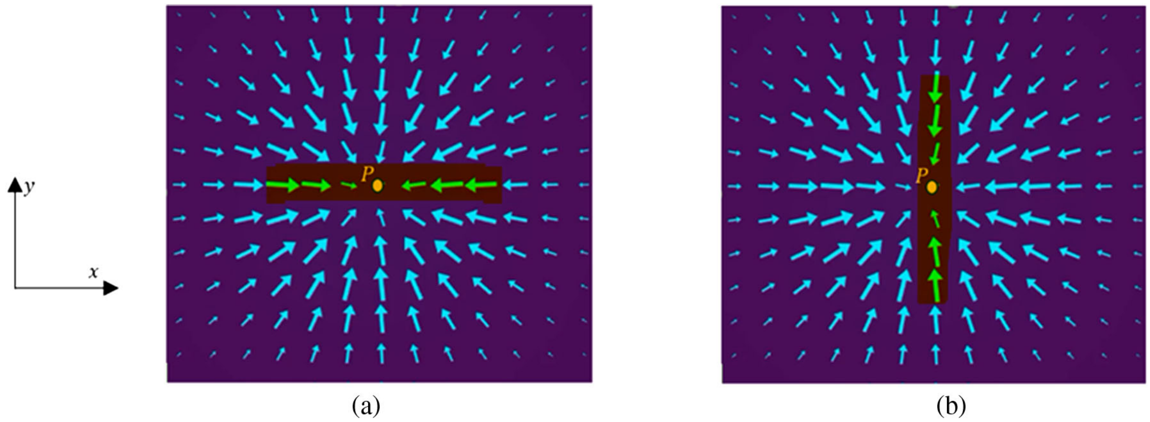


FIG. 4. Temperature gradient vector field. Using a differential approach, it can be seen that in the neighborhood of point $P$, (a) $\frac{\partial T}{\partial x}$ is decreasing along the $+x$ direction and (b) $\frac{\partial T}{\partial y}$ is decreasing in the $+y$ direction.

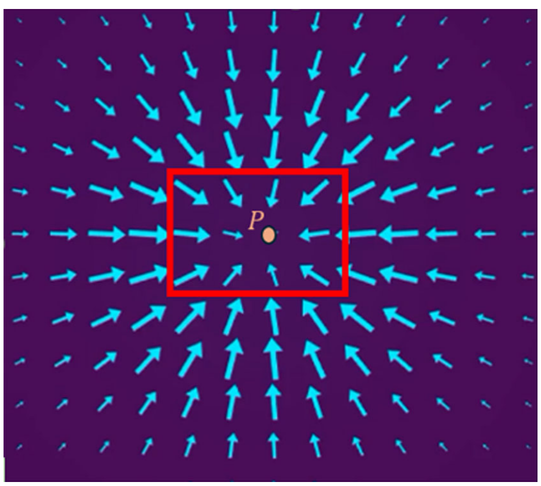


FIG. 5. Gradient vector field of the temperature distribution on a 2D plate. Using an integral approach, the net flux of the temperature gradient through a small surface surrounding a point is seen to be negative, resulting in $\overrightarrow{\nabla} \cdot (\overrightarrow{\nabla} T) < 0$.

Laplacian, including their linguistic descriptions and graphical interpretations (Table I).

## II. APOS FRAMEWORK

### A. Overview of APOS in mathematics education research

Action, process, object, schema (APOS) theory is a theoretical framework from mathematics education research that explores the cognitive development of concepts based on Piaget's concept of reflective abstraction.

We illustrate APOS as it is used in mathematics education with our own description of how the fraction concept could be learnt. For example, in the context of cutting pizza into equal parts and understanding that one, two, three … parts form a fraction of a whole, a student may contemplate this example to understand that different fractions can represent equivalent quantities of the pizza. For instance, two parts out of four are equivalent to one part out of two, as both represent half of the pizza. This abstraction should be distinguished from empirical abstraction, where the subject extracts features from an object or

TABLE I. Concepts related to the understanding of the 2D heat equation.

| Concept | Importance of the 2D heat equation |
|---|---|
| First partial derivative with respect to position or time | Spatial or temporal rate of change of temperature |
| Laplacian of temperature | How heat diffuses in 2D space. Net heat flow |
| Temperature gradient | Steepest spatial rate of change of temperature at a point. Perpendicular to isotherms. Has a direction opposing heat flux density |
| Divergence of T gradient | Divergence of heat flux. Derivation of the heat equation: Divergence theorem relates behavior of heat flux inside a volume to net heat flow at boundaries |
| Fourier's law of heat conduction | Links temperature gradient and heat flux density |
| Multivariable functions of temperature and 3D space | Representation for $T$ distribution |

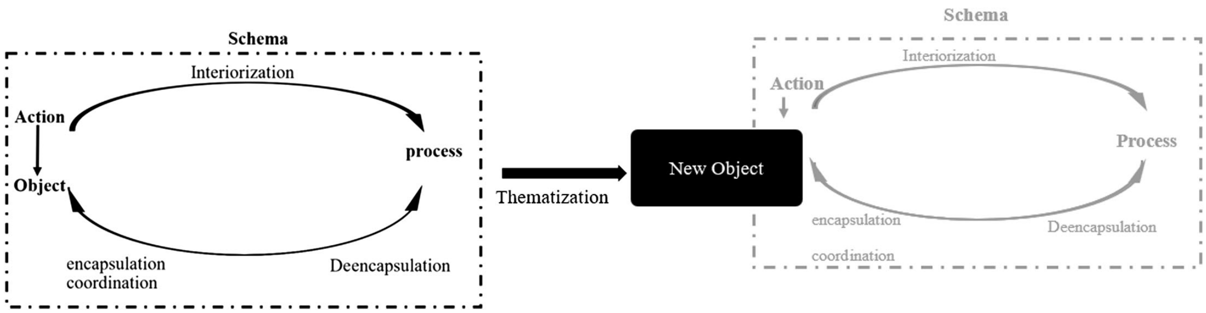


FIG. 6. Part of mental structures and mechanisms and how they are related [22].

experience, such as size or color. APOS theory describes mathematical development in terms of mental structures and mental mechanisms. Mental structures—classified as actions, processes, objects, or schemas—are stable and are used to make sense of a situation. Mental mechanisms—including interiorization, encapsulation, deencapsulation, coordination, and thematization—are the means through which a structure changes in the student's mind [22]. Mechanisms and structures are related, as shown in Fig. 6.

In an action conception, the subject follows external instructions. Using the pizza and fractions example, this means the subject is cutting the pizza into equal parts while observing the slices.

In a process conception, the student reflects on their understanding, recognizes a pattern, and interiorizes the actions. The student can now interiorize that cutting the pizza into 2, 3, or 4 equal parts results in each slice forming one-half, one-third, or one-quarter of a whole, respectively.

In an object conception, the student has encapsulated the process into a mental object. This means that the student can recognize $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ as mathematical objects. The student has encapsulated the dynamic process of representing each pizza slice as part of a whole into a static structure, the fraction itself. The student can apply actions on the object, e.g., add fractions with the same denominator. To add fractions with different denominators, a student needs to de-encapsulate the mathematical object, the fraction in this case, into the process that it arises from. For example, to add $\frac{1}{3}$ and $\frac{1}{4}$, a student needs to coordinate the process understanding of fractions as "part of a whole" (specifically, that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$ … and $\frac{1}{4}$ is equivalent to $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$ …) with the process understanding of finding a common denominator to obtain the least common multiple. The coordinated processes then get encapsulated into a new object conception of a fraction: a static structure arising from the combination of two or more fractions, in this example, $\frac{7}{12}$ representing the equivalence of $\frac{1}{3}+\frac{1}{4}$.

A schema conception is developed when all that preceded is coherently connected in one structure, a coherent mental framework of fractions. This framework involves reasoning with fractions as part of a whole, equivalent fractions, fraction addition, and fractions as a static structure. The coherence of the mental framework gives a student the flexibility to extend their understanding of fractions beyond a specific example, like pizza cutting, to other contexts and situations. The subject can then possibly decide whether a problem falls within this schema. Finally, the thematization mechanism takes place when a student can perform actions on the schema so that it is now transformed into an object. For example, the learner can apply their knowledge about fractions to percentages. Thematization also involves treating the schema as a dynamic structure where related objects and schemas get assimilated into it—for example, improper fractions, mixed numbers, and other fraction operations, such as multiplication and subtraction, into the fraction schema.

Readers may be familiar with Zandieh's framework [23], which analyzes mathematical understanding through a binary distinction between process and object. In contrast, APOS theory provides a more elaborate development of a collection of different mental structures—action → process → object → schema—through mental mechanisms (encapsulation, thematization …). In Zandieh's framework, there is less emphasis on how an object is used dynamically in future construction, while in APOS, objects have a dynamic nature; they can be acted upon through specific mechanisms. They can be deencapsulated into processes and integrated into broader schemas, which can then be thematized to produce new objects. Overall, Zandieh's framework emphasizes classification of student conceptions, while APOS provides a more recursive and mechanism-rich account of mathematical learning, capable of describing how concepts are transformed, built upon, and structurally interrelated.

A researcher can hypothesize a model of mental constructions (structures and mechanisms) that should be developed to learn the given concept. This hypothetical model is called the preliminary genetic decomposition (PGD), which is a main tool in the APOS framework.

It starts with a preliminary theoretical analysis (preliminary genetic decomposition, PGD) of the concept. This corresponds to the initial stage of genetic decomposition (GD1 in Fig. 7). This PGD is tested experimentally by using learning materials specifically designed to probe predicted mental constructions. Data are gathered using

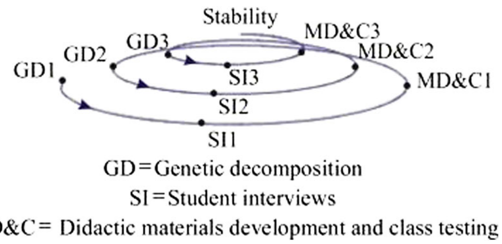


FIG. 7. Designing a stable genetic decomposition in APOS [24].

interviews, tests … Students could show evidence of carrying out the predicted mental constructions. They could show difficulty in carrying out certain mental constructions as well, or doing other, unpredicted mental constructions. This then leads to an alteration in the preliminary theoretical analysis to better reflect what students do that leads them to a better learning of the concept. It also leads to the design of learning material to support students in carrying out difficult mental constructions. Next, the learning material is tested in class. Further cycles of genetic decomposition validation are to take place until a stable genetic decomposition is reached [21,22].

As an example, the middle column of Table III can be considered a simple preliminary genetic decomposition of fraction multiplication. It describes a roadmap of mental constructions to develop an understanding of multiplying fractions. In more advanced topics such as the 2D heat equation, a genetic composition includes multiple schemas, objects, and processes (Supplemental Material [25]).

### B. How do we operationalize APOS to describe students' understanding of mathematics and physics in the context of the heat equation?

In our project, we aim to investigate students' conceptual understanding of the 2D heat equation, which is a partial differential equation representing heat conduction on a 2D plate.

Literature shows that even with mathematical understanding, physical understanding is not guaranteed [2–4], and physical understanding does not mean students can skillfully represent physical concepts mathematically. Thus, students' integration of mathematics and physics is crucial [26]. To describe this integration, we extend APOS to also include physics concepts. We illustrate this extension using the 2D heat equation. To reach a conceptual understanding of the 2D heat equation (Fig. 8), we predict, using the preliminary genetic decomposition (Sec. III), that integration of the two disciplines occurs not merely at the schema level, but already within object and process structures, which we will argue in the following paragraph.

In APOS terminology (Table II), a coherent collection of different actions, processes, and objects, and the mechanisms that relate them, constitutes a schema conception. Developing a schema of a concept can allow the individual to connect it to other concepts, bridging mathematics and physics. Integration of mathematics and physics at the schema conception is typically left to the students. They follow a mathematics course, and they are expected to make connections to other contexts, such as physics, independently, a path that is proven to be ineffective, which reinforces the idea that physics is a context for mathematics and mathematics is a tool for physics.

However, integration of mathematics and physics at the process and object conceptions is hypothesized to be more fruitful as it intertwines mathematics and physics earlier on in the learning experience. This results in a teaching and learning sequence, where instructors provide tasks that support students in the interplay of the two disciplines. For example, providing a task in which students are asked to consider the consequences of the concavity of the temperature distribution at a point for the subsequent evolution of the temperature could enrich students' learning experience in both disciplines. Typically, in prior mathematics courses, students would merely have focused on obtaining analytical

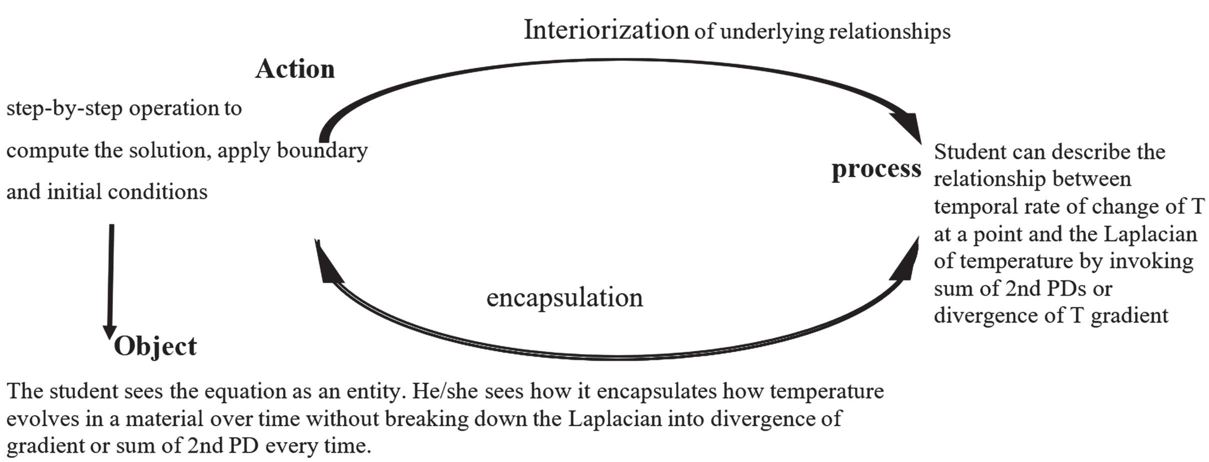


FIG. 8. APOS mental constructions of the statement of the 2D heat equation.

TABLE II. Summary of mental structures in APOS.

| Mental structures | APOS definitions and keywords |
|---|---|
| Action conception | Procedural understanding, step by step, follow instructions. |
| Process conception | Reflect on action, interiorize an action, anticipate a result without performing calculations or following step by step explicitly, a deencapsulated object. |
| Object conception | Apply transformation on a process, encapsulate, thematize a schema, coordinate 2 processes. |
| Schema conception | Coherent collection of processes, objects, actions, and relative mechanisms. Connect 2 concepts. Coherence could possibly lead to a decision whether a problem falls within the schema. |

solutions for $t \to \infty$ through separation of variables and uniqueness of solutions.

Following APOS terminology, an object conception is either an encapsulation of a process or processes or a thematized schema (Table II). In our extension of APOS, we predict the mathematical and the physical processes that, when coordinated, will lead to the development of an interdisciplinary object conception in which mathematics and physics are integrated, a *physicomathematical object*. A thematized schema would correspond to understanding how to apply a concept learned in a mathematics course in physics, and in some instances, this is an appropriate approach. In other cases, the encapsulation and coordination of processes from both disciplines are critical for students to develop an integrated awareness of specific mathematical and physical reasoning within the context of the 2D heat equation.

Furthermore, according to APOS theory, a process conception is an interiorized action or a deencapsulated object. We predict the mathematical, physical, and physicomathematical objects that, when deencapsulated, will lead to process conceptions. By achieving this deencapsulation, students can engage with the underlying principles of both disciplines, recognizing their intertwined roles when reasoning in the context of the 2D heat equation.

By the extension of APOS to include physics, we stress the integration of mathematics and physics in different structures (process, object, and schema conceptions), and, hence, not only in the schema conception. Since APOS is a teaching and learning framework, this integration can feed into the design of instructional material (Fig. 7) and, when used in a particular context such as the 2D heat equation, can aid students in executing an interplay of mathematics and physics specific to the 2D heat equation. This reinforces our previously mentioned point regarding the role of instructors in forging this interplay through targeted teaching and learning activities.

In Fig. 8, we describe the mental constructions needed to understand the statement of the 2D heat equation for each conception. Table III elaborates this in more detail. As previously stated, an advanced topic such as the 2D heat equation needs a nuanced genetic decomposition. It should consider the conceptual depth and interconnectedness of concepts. It should account for the complexity of hierarchical learning that considers specific prerequisites. It should align with our main goal, which is describing the interplay of mathematics and physics within the context of this topic. In the next section, we explain our preliminary genetic decomposition of the 2D heat equation.

## III. PRELIMINARY GENETIC DECOMPOSITION OF THE 2D HEAT EQUATION

### A. Considering what students need to know and do

To develop a conceptual understanding of the heat equation, a student needs to understand (i) the causal relationship between the Laplacian of the temperature and the time rate of temperature variation at a point, and (ii) that the magnitude of the temporal rate of change of temperature depends on a material property called the thermal diffusivity.

The student needs to integrate the physical characterization of the situation with mathematical concepts: a two-dimensional function to describe the temperature distribution, the first partial derivative with respect to time to describe the temporal rate of change at a point, the Laplacian of the function to quantify the difference of the temperature at a point with the average temperature of the surroundings, and a proportionality constant to account for the material property.

Cartesian graphs that show $T$ as a function of $x$ and $y$ are increasingly used as a visual aid to track the temporal evolution of a temperature distribution [19]. We hypothesize that this graphical representation can greatly aid students' understanding of how the temperature on a plate evolves. Local maxima and minima in the temperature are easily observed in Cartesian graphs; they can be linked, with relative ease, to the mathematical concepts of concavity, first and second spatial derivatives, and the Laplacian [see Eq. (2)]. At points where concavity shows that the second partial derivatives have opposite signs, it is not straightforward to visually determine the sign of the Laplacian, but a good sense of the direction and relative magnitude of the gradient can often easily be obtained by inspecting the graph. Based on earlier research on students' understanding of divergence in Maxwell's equations [20], we surmise that a vector field plot of the gradient of the temperature could aid students' understanding of the

TABLE III. Mental construction of fraction multiplication and 2D heat equation.

| | Fraction multiplication | 2D heat equation |
|---|---|---|
| Action conception | An action conception of fraction multiplication as the calculation of the area of rectangle having two sides as fraction for example $\frac{2}{5}$ and $\frac{3}{4}$. Students follow a step-by-step procedure. They may compute $\frac{2\times 3}{5\times 4}$ or draw a rectangle, divide its length into 5 equal parts of which they take 2, divide its width into 4 equal parts of which they take 3, which results in 6 parts out of 20: $\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$. | Students have a procedural understanding of the 2D heat equation statement. They follow a step-by-step operation to compute the solution using separation of variables without considering or understanding heat flow, how temperature evolves, etc. |
| Process conception and interiorization | A process conception of fraction multiplication where students have interiorized this action. They do not need to perform step-by-step operations every time they perform a multiplication. They can imagine the drawing and the taking parts pattern while performing the calculation. | Students interiorize step-by-step procedures and underlying relationships into an anticipation or a reflection so that they can describe the relationship between the temporal rate of change of $T$ at a point to the Laplacian of the temperature by invoking the sum of second partial derivatives or the divergence of the temperature gradient. |
| Object conception and encapsulation | An object conception of fraction multiplication is obtained by encapsulating the above process into an entity that students can reflect on. For example, they can compare different multiplications of two fractions. They can, e.g., reflect on why multiplying two fractions results in a smaller number if the fractions are less than 1, and a bigger number if the fractions are bigger than 1. | Students recognize the equation as an entity. Students can reason with this entity in heat diffusion situations. They see how it encapsulates how temperature evolves in a material without breaking down the Laplacian into divergence of gradient or sum of second partial derivatives every time. They are aware of the previously defined process of the Laplacian, so they recognize that the Laplacian of the temperature is an indication of net heat flow at a point, and of the temperature difference between that point and its surroundings. They can act on the equation as an entity when the thermal diffusivity constant changes and see how it affects the rate of evolution of temperature. |
| Schema conception | A schema conception of fraction multiplication is achieved when actions, processes, and objects are coherently connected. Students can then determine whether a problem requires fraction multiplication. | Students coherently connect actions, processes, and objects. They can determine whether a problem falls within the 2D heat equation schema. They may be able to identify and tackle similar situations, such as the wave equation. |

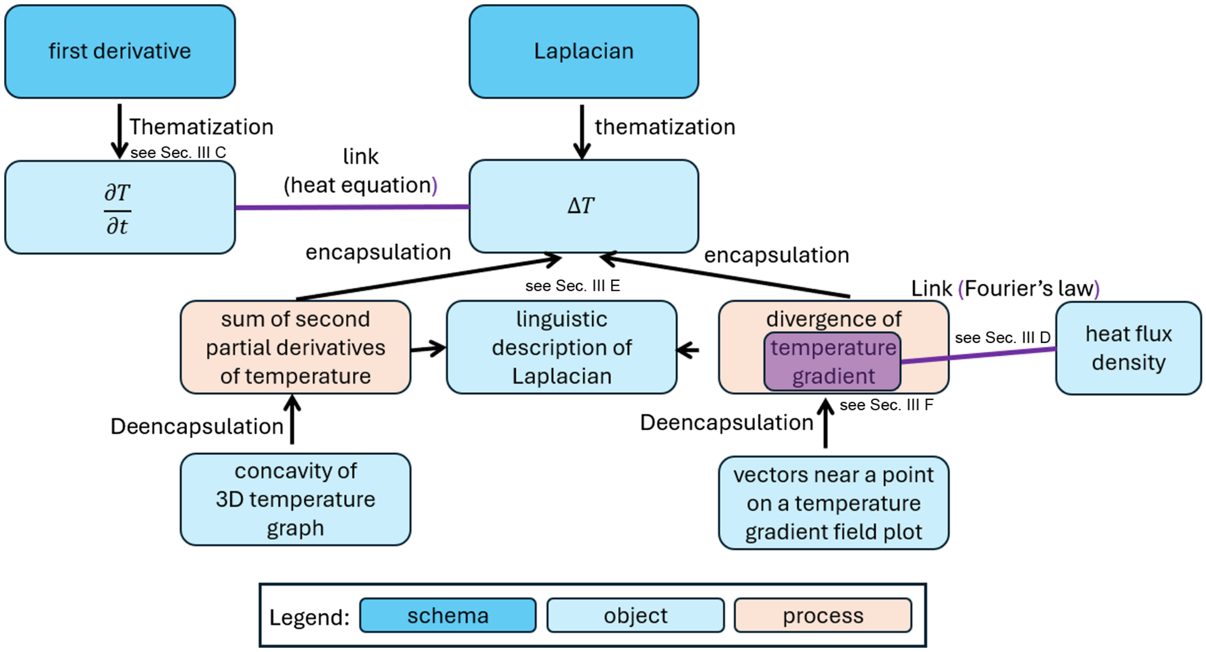


FIG. 9. Preliminary GD of the heat equation.

average temperature difference with the surroundings as the divergence of the temperature gradient [see Eq. (4)]. We predict that understanding how Cartesian graphs and gradient vector plots reflect the Laplacian of the temperature will aid students' general understanding of this concept to the extent that they would later be able to articulate a linguistic description of the concept (see Sec. III E).

### B. Hierarchical structure

We now turn to the hierarchical structure of concepts. Effective learning of the 2D heat equation depends on the successful integration of prerequisite concepts. The 2D heat equation links the Laplacian of a function representing the temperature in a region of space to the first partial derivative of that same function with respect to time. In Fig. 9, we show that even at this level, students need to coordinate schemas, objects, and processes. However, we do not expect students to enter classes with a fully formed schema understanding of, e.g., the Laplacian. Furthermore, concepts such as the first partial derivative, the Laplacian, and the heat flux density are taught in different courses, while the topic of the 2D heat equation is learned in one course. For this reason, the PGD of the 2D heat equation (Fig. 9) also includes PGDs of other concepts that would normally just be referred to as prerequisites. As will become apparent, these prerequisite concepts are crucial for the integration of mathematics and physics. When designing the PGD for the heat equation, we realized that there are many concepts to be taken into consideration. Although they are crucial, we had to draw a line on how far in the prerequisites we want to go. For example, including different coordinate systems in learning the Laplacian could lead to a different learning trajectory.

The complete PGD of the 2D heat equation, including a complete list of concepts and their representations, is given in Supplemental Material [25]. In the preliminary genetic decomposition, we hypothesize specific mental mechanisms, such as encapsulation and coordination, that a learner needs to carry out to develop a specific mental structure (process, object, or schema) of the concepts mentioned. Supplemental Material [25] lists similar preliminary genetic decompositions of prerequisite concepts. These include an understanding of first and second partial derivatives with their different representations, including slope of tangent, rate of change, concavity at a point, rate of change of rate of change, multivariable functions, and 3D space; procedural understanding students can reflect upon for divergence, Laplacian, and gradient. Concepts related to heat conduction, such as heat, temperature, thermal equilibrium, and Fourier's law [see Eq. (3)], are also considered crucial prerequisites.

Each preliminary genetic decomposition in Supplemental Material [25] highlights the integration of mathematics and physics as discussed previously in Sec. II B. Our decision about which aspects of the PGD to describe in detail in this article is guided by two objectives: (i) extending APOS theory into physics education and showing the mathematics-physics interplay, and (ii) identifying key concepts for understanding the heat equation. In the following subsections, we illustrate how we developed the preliminary genetic decompositions by showing several examples. In Secs. III C–III E, we explain how we extended APOS to integrate physics concepts in the different structures and mechanisms, where we discuss one example about the thematization of a schema, two examples about processes, and two examples about *physicomathematical* objects. In Sec. III F, we explain how we integrated multiple representations in the (extended) APOS framework and discuss two examples. This is a generic approach to describe APOS using the 2D heat equation as an example. This paper mainly focuses on the theoretical analysis; probing specific mental structures and mechanisms will be addressed in a separate publication.

### C. Thematization of a schema

In this subsection, we give one example from our PGD (Supplemental Material [25]) of how we extended APOS to include physics in the thematization of a schema. As previously mentioned, the thematization mechanism takes place when the learner performs actions on a schema so that it is transformed into an object. In the case of the 2D heat equation, a student has to be able to thematize the first partial derivative schema, i.e., they flexibly apply it to the variables temperature and time (Fig. 10). In our PGD, we hypothesize that most students will be able to do this effectively. The student can then reason with the temporal rate of change of temperature throughout a 2D region of space as a first partial derivative with temperature as the dependent variable and time as the independent variable. Similarly, a student must be able to thematize the Laplacian of a generic function to apply it to a 2D temperature distribution. However, such thematization alone does not guarantee that they can interpret the Laplacian, as our data illustrate in Ref. [27]. For this reason, we do not solely propose thematization; we also propose the construction of the corresponding *physicomathematical* object, as described below in Sec. III E.

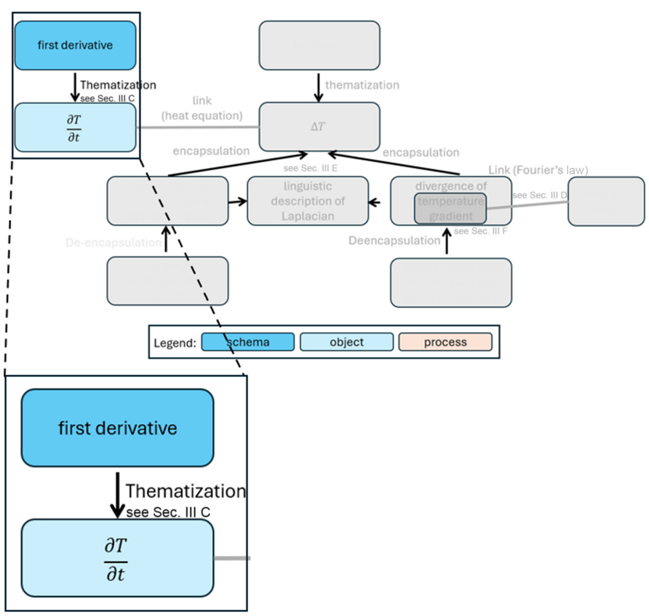


FIG. 10. Thematization example from preliminary genetic decomposition of 2D heat equation.

### D. Process

In this subsection, we give two examples from our preliminary genetic decomposition (Supplemental Material [25]), where we infused mathematical and physical understanding to reach a process conception. In both examples, we stress the relational aspect between quantities as a core idea.

The first example describes a process conception of the Fourier's law [see Eq. (3)] part of the heat conduction schema (see Fig. 11). It is important that students understand that heat conduction is represented by the proportional relationship between the heat flux density and the gradient of the temperature $\vec{\nabla} T$ and recognize the interdependence of the two quantities to develop a conceptual understanding of the 2D heat equation. Therefore, we predict that students need the following process conception of Fourier's law:

> *A process conception of Fourier's law so that a student can predict the heat flow (flux) based on the temperature difference between two points (temperature gradient) without the need for calculation.*

This prediction aligns with the APOS description of process given in Table II, where a student can recognize a pattern without performing a calculation. It emphasizes the connections between quantities, which is achieved by conceptualizing how the equation quantifies these relationships, often through mathematical operations like multiplication, division … Recognizing the pattern is, thus, the recognition of these relationships.

The second example is developing a process conception of the physical concepts of heat and temperature (Fig. 12), also part of the prerequisite heat conduction schema. It is well known in literature that many students confuse the two concepts [28–32]. In our preliminary genetic decomposition (PGD), we predict that for a student to differentiate between heat and temperature, they need to know how both concepts are related to thermal energy. We hypothesize this happens when students have a process conception of heat and temperature.

Brahmia [33] defines quantification as "the process of conceptualizing the attribute of an object such that it has a unit of measure that has a proportional relationship with its magnitude." She discusses how understanding physical quantities requires recognizing relationships through understanding of ratios, differences, summation, rates, etc.

We applied Brahmia's definition to our PGD by treating the physics concepts of temperature and heat as quantifications *related* to thermal energy, which allowed us to incorporate their quantification into a process conception. So, to differentiate heat and temperature, our detailed prediction is as follows:

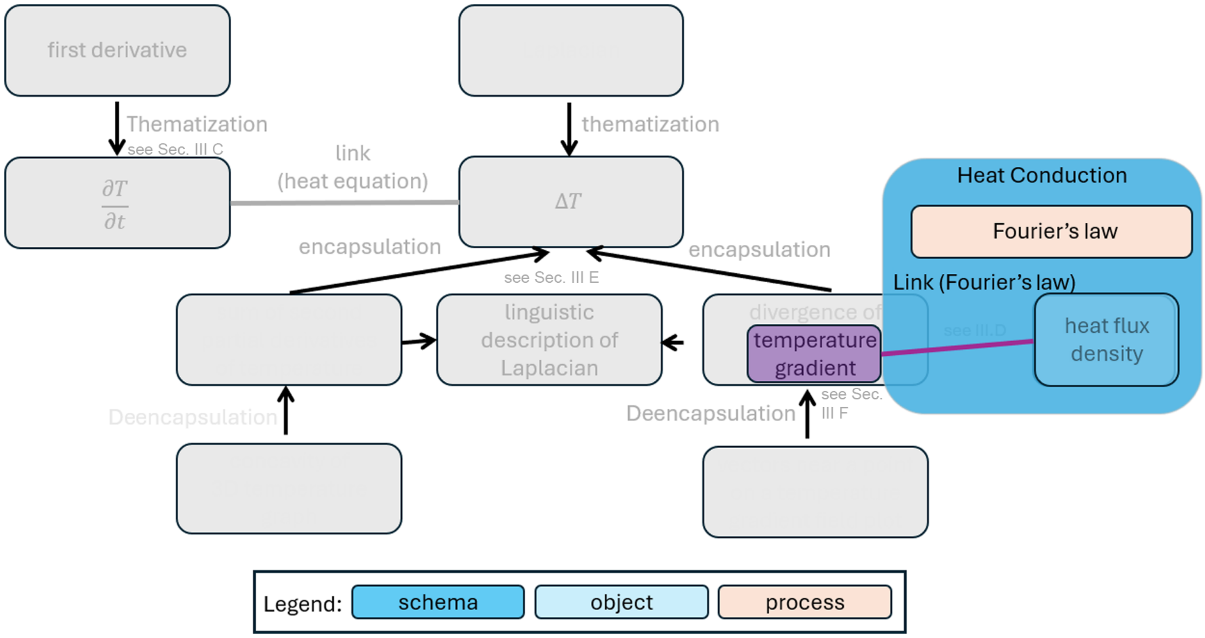


FIG. 11. Process conception of Fourier's law.

*A process conception of temperature as a measure of thermal energy and a process conception of heat as a transfer of thermal energy allow students to differentiate the 2 concepts.*

By employing quantification of physical concepts and stressing relational aspects of quantities as seen in the previous two examples, we aim to meet our previously mentioned goal of integrating mathematics and physics in the process conception.

### E. A physicomathematical object

In this subsection, we give two examples of how we integrate mathematical and physical understanding to come to a *physicomathematical object* conception. The first

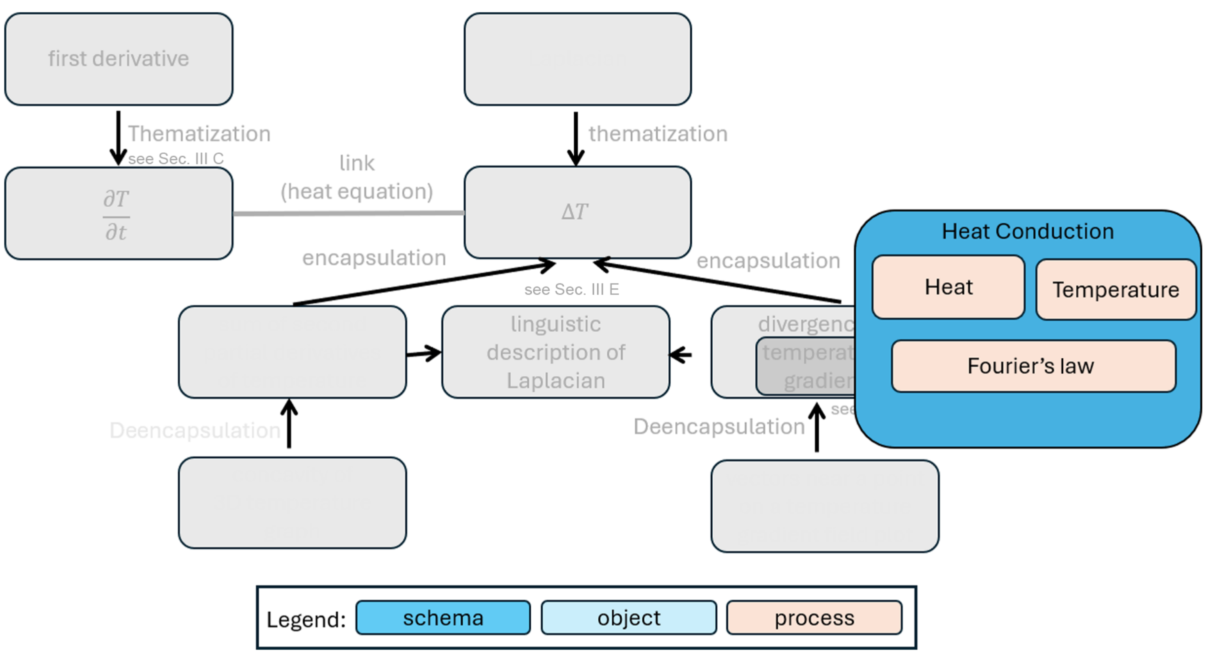


FIG. 12. Process conception of heat and temperature.

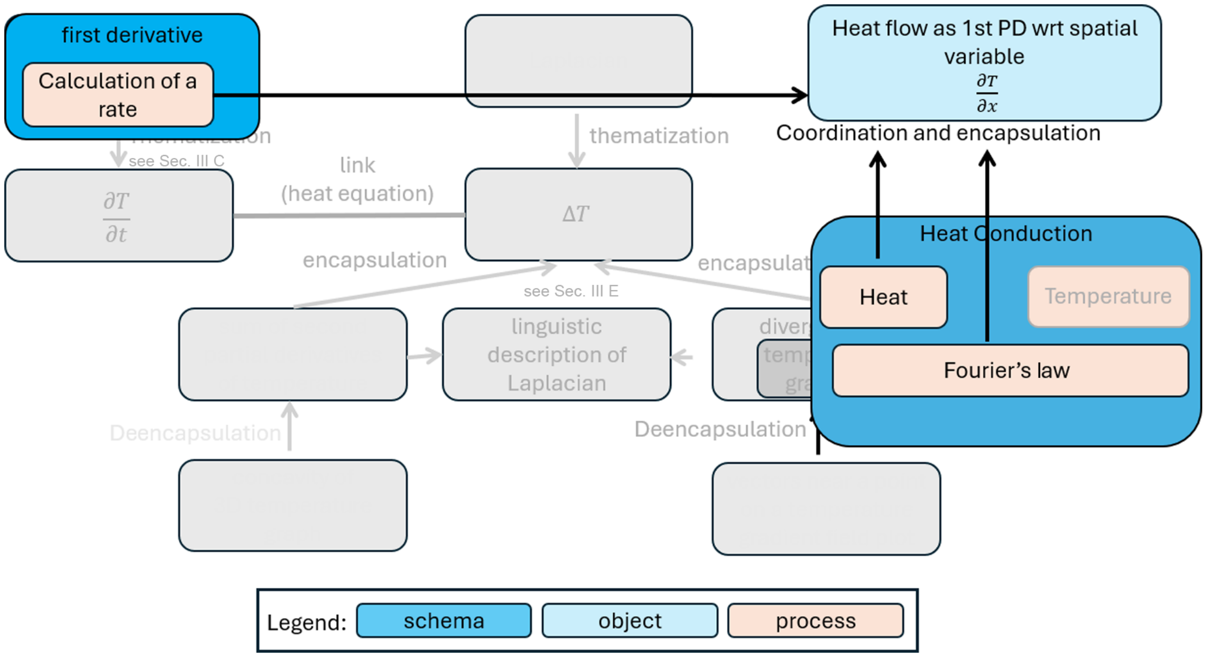


FIG. 13. Predicted mental constructions to form a *physicomathematical object* understanding of heat flow as spatial rate of change of temperature arising from mathematical and physical processes.

example is related to students' understanding of heat flow as a spatial rate of change of temperature (Fig. 13), and the second example is related to students' understanding of the Laplacian of the temperature (Fig. 14).

Starting with the first example, students tend to give an incorrect mathematical formulation of boundary conditions in the context of the heat equation [2,28]. Specifically, some students confuse the meanings of partial derivatives, spatial rate of change of temperature, $\frac{\partial T}{\partial x}$, and the time rate of change of temperature, $\frac{\partial T}{\partial t}$.

They do not connect heat flow to a spatial partial derivative of temperature. We took this finding into consideration when developing our preliminary genetic decomposition. We hypothesize that students need an object conception of heat flow that involves recognizing

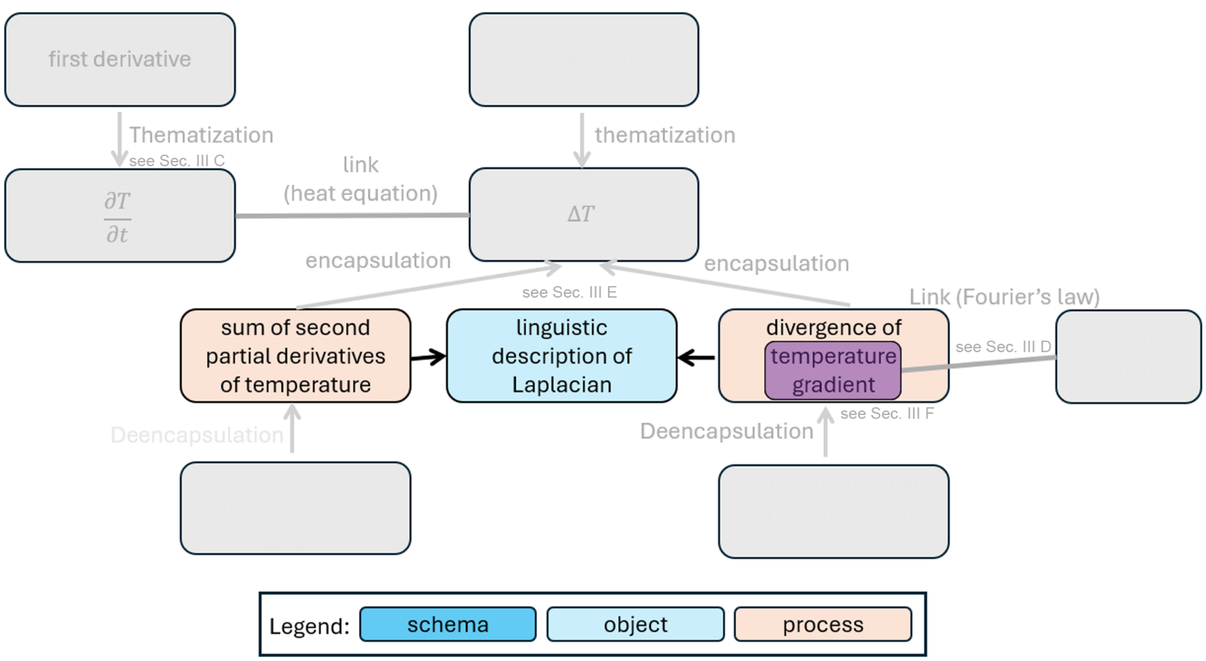


FIG. 14. Predicted mental constructions to form an object understanding of the Laplacian of a temperature distribution.

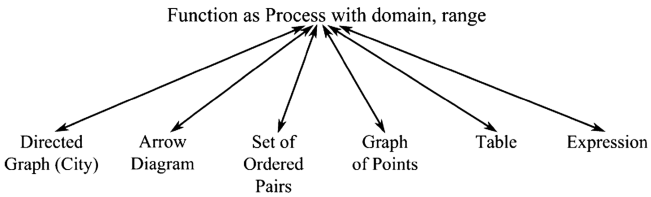


FIG. 15. Integration of multiple representations within APOS for the function concept [22].

it as a quantity involving a first partial derivative with respect to a spatial variable arising from mathematical and physical processes. It is crucial to acknowledge that heat flux density is the precise term. Heat flux density describes the rate of heat transfer per unit area and is mathematically proportional to the gradient of temperature, a concept that aligns with the formalism of partial derivatives. However, we hypothesize that guiding students from their intuitive notions where they are generally more familiar with the idea of heat "flowing" from a hot object to a cooler one, an accessible and familiar concept, would be helpful to overcome this difficulty and aid them in gaining the specific *physicomathematical* understanding. To obtain this object conception, we suggest that students need to coordinate three process conceptions: heat as the transfer of thermal energy, Fourier's law as relating heat flow and the temperature gradient, and first partial derivatives as the process of calculating a rate of change of a dependent variable with respect to an independent variable. Coordinating these three processes means that students reflect on them simultaneously. Recognizing the proportional relationship between temperature gradient and heat flux density happens simultaneously with reflecting on the process of calculating partial derivatives, and on heat as the transfer of thermal energy between two positions. They can now encapsulate the coordinated processes in a new *physicomathematical object* conception of heat flow as the *spatial* rate of change of temperature $\frac{\partial T}{\partial x}$, distinguishing it from a temporal rate of change of temperature in the context of boundary conditions of the heat equation. We hypothesize that this *physicomathematical object* facilitates a correct interpretation of partial derivatives in the context of the heat equation.

The second example relates to the understanding of the Laplacian of the temperature. In a previous study, we investigated students' conceptual understanding of the Laplacian both in mathematics and physics contexts [34]. We found that students' linguistic and graphical understanding of the concept is rather limited in both contexts. They used similar terminology to describe the Laplacian, the derivative, and the gradient, and tended not to mention the concavity of the graph surrounding a point. We took these findings into consideration when developing the preliminary genetic decomposition.

Starting from the prerequisite that heat flows from higher temperature to lower temperature, we hypothesize that students can build a solid understanding of the Laplacian of the temperature distribution by coordinating two key processes (Fig. 14). First, reasoning with the Laplacian of the temperature distribution as the sum of second partial derivatives with respect to spatial variables helps them see it as a measure of concavity or average bending. Second, interpreting the Laplacian of the temperature distribution as the divergence of the temperature gradient connects it to heat flux density. By reflecting on these two processes simultaneously, students can reason with the differential approach to the divergence of the temperature gradient's $x$ component or $y$ component (Fig. 4) as a spatial second PD with respect to $x$ or $y$. As a result, students can see how heat flux density and concavity or average bending are related. They can now encapsulate this into a *physicomathematical object* conception of the Laplacian of the temperature. They can linguistically describe it as an entity characterizing net heat flow, as well as the temperature difference between a point and the average of its surroundings. This allows them to reason with it as a mathematical operator for analyzing the spatial temperature difference between a point and its surroundings and as a physical entity related to the net heat flow at a point.

One could argue that the thematization of the Laplacian schema to form an object understanding of the Laplacian of temperature serves the same purpose. However, we did not adopt this approach since we hypothesize that the coordination and encapsulation mechanisms would help students to connect a physical meaning to a mathematical structure more than thematization. Thematization, as described in Sec. III C, means the student can perform actions on the schema. In the case of the Laplacian, students might be able to manipulate the mathematical structure of the Laplacian in a new context; however, we doubt they will be able to give an interpretation for this concept as discussed in Ref. [35].

### F. Multiple representations of a concept

In Ref. [22], Arnon *et al.* discuss how multiple representations are included in the APOS framework (Fig. 15). They highlight the importance of the genetic decomposition in supporting students in transforming from one representation of a concept to another. The students need to cognitively grasp the underlying process before they are able to switch between representations.

From the literature, we know that multiple representations affect students' performance and understanding of concepts [36–39]. Therefore, the approach we followed in our PGD aims to support students by including specific representations of concepts as objects encapsulating specific processes.

For example, in the second partial derivative schema (Fig. 16), obtaining the sign of the second partial derivative by comparing slopes of tangents at neighboring points is considered a process. This process can be encapsulated in three objects: (i) graphical representation in the form of

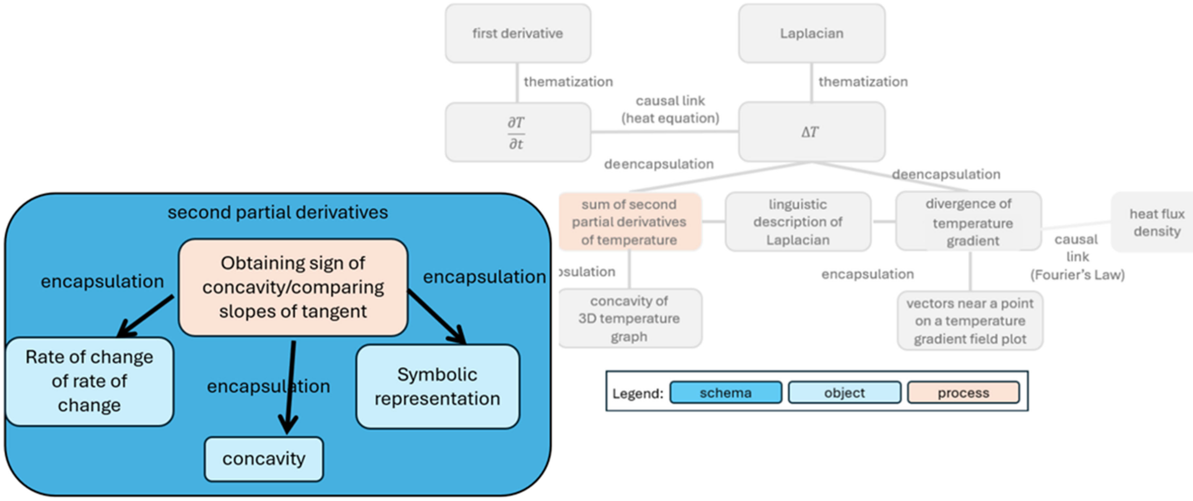


FIG. 16. Schema of second PD in prerequisites.

concavity at a point, (ii) symbolic representation, and (iii) linguistic description as the rate of change of the rate of change.

Similarly, in the schema of the temperature gradient, we included three representations of this concept as objects (Fig. 17). In a previous study [27], we found that students have difficulty interpreting the gradient as a vector with varying magnitude depending on the steepest ascent at a given point. In addition, they find it challenging to interpret the gradient's magnitude as the highest rate of change at a point. Since understanding the gradient is crucial to understanding the 2D heat equation—where the Laplacian represents the divergence of the gradient—we decided to include this in the PGD. The representation of the gradient as a vector field is considered an object conception. This object is to be deencapsulated into a process of obtaining the magnitude and direction of steepest ascent at a point.

The latter process is to be coordinated with the process of assigning a symbolic variable to a spatial variable to produce a mathematical symbolic expression of the gradient (symbolic representation) as first partial derivatives with spatial independent variables.

The gradient of the temperature can be thought of as a process of obtaining the spatial rate of the temperature with respect to $x$ and $y$. Coordinating this process with the process of obtaining the magnitude and direction of the steepest ascent will result in an object conception of the temperature gradient as the *highest* spatial rate of change of the temperature distribution at a point (linguistic description).

## IV. CLOSING REMARKS AND FURTHER RESEARCH

Our goal is to investigate the interplay of mathematics and physics in the context of the 2D heat equation. For this purpose, we used the APOS theoretical framework from

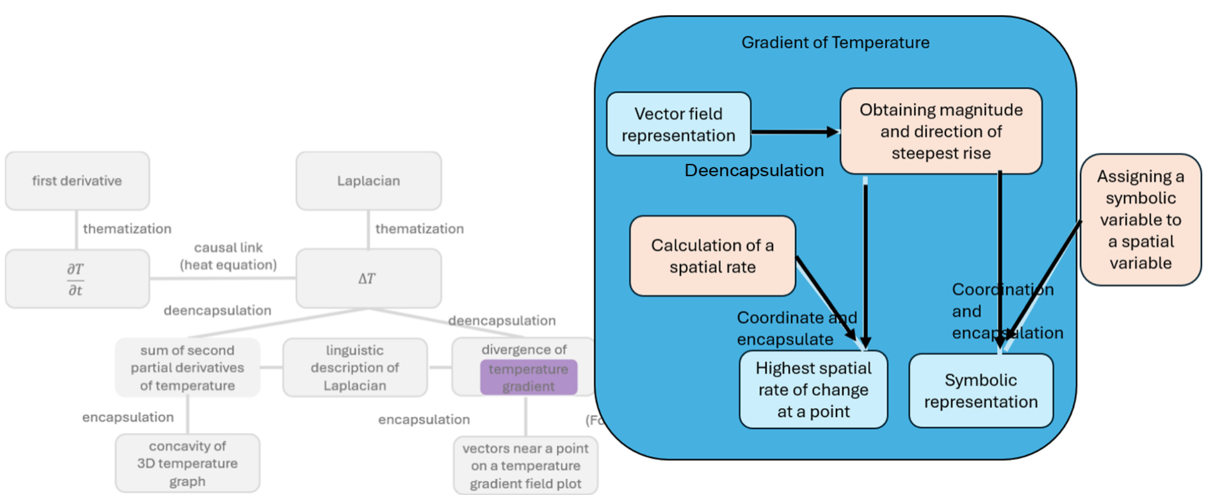


FIG. 17. Schema of temperature gradient.

mathematics education research. We employed its main tool, the genetic decomposition. We designed a preliminary genetic decomposition (PGD), a hypothetical trajectory, aimed at a conceptual understanding of the 2D heat equation. This model takes into consideration the concepts of heat diffusion and conduction, in addition to the mathematical formulation of the partial differential equation. In this paper, we introduced APOS terminology and extended it to a topic where the interplay between mathematics and physics is highly important.

To design this PGD, we took multiple layers into consideration: (i) concepts and how far to go into prerequisites; (ii) previously known challenges to students with these concepts; (iii) multiple external representations of these concepts; and (iv) APOS in a new context. In our proposal, we hypothesize a fine-grained trajectory of cognitive constructions that precisely captures the interplay of mathematics and physics. We hypothesize that this approach enhances clarity and guidance by ensuring that different understandings from both disciplines are structured and concrete.

An adaptation of APOS to physics education has been proposed before by Rebello and Rebello in the context of equations [40]. At the process level, e.g., they state, "A student sees the equation as a *relationship* that allows them to predict, without the need for explicit calculation, the behavior of a physical quantity given the behavior of another physical quantity." The authors also connected physics principles to the object conception. While we extend physics to different mental constructions, our extension goes beyond the mere use of mathematics and emphasizes the structural integration of mathematics and physics. In the process conception, we use the idea of relations and apply it to physical concepts such as heat and temperature, where their quantifications of thermal energy are highlighted. In the object conception, we propose a *physicomathematical object* that precisely captures the coordination and encapsulation of physical and mathematical processes. Multiple external representations of a concept (graphical, linguistic description …) appear as objects in our preliminary genetic decomposition. It is important to highlight that awareness and conversion of the processes play a key role in the construction of objects [41,42], an approach we adopt in our extension. Dubinsky [41] defines encapsulation as "conversion of a (dynamic) process into a (static) object." Oktaç [42] highlights that evidence of object construction is based on both awareness of processes and the ability to act on processes and transform them.

In the next step of this project, we will follow the cycle of testing and designing a stable genetic decomposition in APOS, as shown in Fig. 7. We will test this PGD to validate and/or review its mental constructions. We designed questions to probe these different predicted mental constructions and used these questions in interviews. Findings and results from these interviews will be reported in a separate paper.

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

The data that support the findings of this article are not publicly available because of legal restrictions preventing unrestricted public distribution. The data are available from the authors upon reasonable request.


[1] G. Pospiech, M. Michelini, and B.-S. Eylon, *Mathematics in Physics Education*, 1st ed. (Springer International Publishing AG, Cham, 2019), 10.1007/978-3-030-04627-9.

[2] S. Van Den Eynde, B. P. Schermerhorn, J. Deprez, M. Goedhart, J. R. Thompson, and M. De Cock, Dynamic conceptual blending analysis to model student reasoning processes while integrating mathematics and physics: A case study in the context of the heat equation, Phys. Rev. Phys. Educ. Res. **16**, 010114 (2020).

[3] J. R. Thompson, B. R. Bucy, and D. B. Mountcastle, Assessing student understanding of partial derivatives in thermodynamics, AIP Conf. Proc. **818**, 77 (2006).

[4] J. Tuminaro, *A Cognitive Framework for Analyzing and Describing Introductory Students' Use and Understanding of Mathematics in Physics* (University of Maryland, College Park, MD, 2004).

[5] E. F. Redish, Problem solving and the use of math in physics courses, arXiv:physics/0608268.

[6] O. Uhden, R. Karam, M. Pietrocola, and G. Pospiech, Modelling mathematical reasoning in physics education, Sci. Educ. **21**, 485 (2012).

[7] G. Fauconnier and M. Turner, Conceptual integration networks, Cognit. Sci. **22**, 133 (1998).

[8] E. F. Redish, A theoretical framework for physics education research: modeling student thinking, arXiv:physics/0411149.

[9] E. F. Redish and E. Kuo, Language of physics, language of math: Disciplinary culture and dynamic epistemology, Sci. Educ. **24**, 561 (2015).

[10] T. J. Bing and E. F. Redish, Analyzing problem solving using math in physics: Epistemological framing via warrants, Phys. Rev. ST Phys. Educ. Res. **5**, 020108 (2009).

[11] B. L. Sherin, How students understand physics equations, Cognit. Instr. **19**, 479 (2001).
[12] L. Doughty, E. McLoughlin, and P. van Kampen, What integration cues, and what cues integration in intermediate electromagnetism, Am. J. Phys. **82**, 1093 (2014).
[13] D. Tall and S. Vinner, Concept image and concept definition in mathematics with particular reference to limits and continuity, Educ. Stud. Math. **12**, 151 (1981).
[14] R. Karam, O. Uhden, and D. Höttecke, The "math as prerequisite" illusion: Historical considerations, and implications for physics teaching, in *Mathematics in Physics Education*, edited by G. Pospiech, M. Michelini, and B.-S. Eylon (Springer International Publishing, Cham, 2019), pp. 37–52, 10.1007/978-3-030-04627-9_2.
[15] E. Dubinsky and P. Lewin, Reflective abstraction and mathematics education: The genetic decomposition of induction and compactness, J. Math. Behav. **5**, 55 (1986), https://psycnet.apa.org/record/1987-29647-001.
[16] D. Tall, Advanced mathematical thinking, in *Mathematics Education Library* (Kluwer, Dordrecht, 1991).
[17] M. Asiala, A. Brown, D. Devries, D. Mathews, and K. Thomas, A framework for research and curriculum development in undergraduate mathematics education, in *Research in Collegiate Mathematics Education. II* (American Mathematical Society, Washington, DC, 1996), pp. 1–32.
[18] Van Den Eynde Sofie, *Blending of mathematics and physics: Undergraduate students' reasoning in the context of the heat equation* (KU Leuven, 2021) [Online]. Available: https://kuleuven.limo.libis.be/permalink/32KUL_KUL/cro8e8/alma9993054108401488%0A.
[19] S. Van den Eynde, M. Goedhart, J. Deprez, and M. De Cock, Role of graphs in blending physical and mathematical meaning of partial derivatives in the context of the heat equation, Int. J. Sci. Math. Educ. **21**, 25 (2023).
[20] L. Bollen, P. van Kampen, C. Baily, and M. De Cock, Qualitative investigation into students' use of divergence and curl in electromagnetism, Phys. Rev. Phys. Educ. Res. **12**, 020134 (2016).
[21] S. Ainsworth, DeFT: A conceptual framework for considering learning with multiple representations, Learn. Instr. **16**, 183 (2006).
[22] I. Arnon *et al.*, *APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education* (Springer Science + Business Media, New York, 2014).
[23] M. Zandieh, A theoretical framework for analyzing student understanding of the concept of derivative, in *CBMS Issues in Mathematics Education*, edited by E. Dubinsky, A. H. Schoenfeld, and J. KAput (American Mathematical Society, 2000), Vol. 8, pp. 103–127, 10.1090/cbmath/008.
[24] R. Martínez-Planell and M. Trigueros, Using cycles of research in APOS: The case of functions of two variables, J. Math. Behav. **55**, 100687 (2019).
[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/1wc9-xgdj for the complete PGD of the 2D heat equation and its prerequisite concepts.
[26] F. B. Kneubil and M. R. Robilotta, Physics teaching: Mathematics as an epistemological tool, Sci. Educ. **24**, 645 (2015).
[27] M. Al Dehaybes, J. Deprez, P. van Kampen, and M. De Cock, Students' understanding of two-variable calculus concepts in mathematics, and physics contexts. II. The gradient and the Laplacian, Phys. Rev. Phys. Educ. Res. **21**, 010132 (2025).
[28] S. Van den Eynde, J. Deprez, M. Goedhart, and M. De Cock, Undergraduate students' difficulties with boundary conditions for the diffusion equation, Int. J. Math. Educ. Sci. Technol. **53**, 2176 (2022).
[29] K. Carlton, Teaching about heat and temperature, Phys. Educ. **35**, 101 (2000).
[30] F. N. Nabilah, J. Maknun, M. Muslim, A. Samsudin, L. Hasanah, and A. Suhandi, Eleventh-grade student's conceptions about temperature and heat, J. Phys. Conf. Ser. **1280**, 052067 (2019).
[31] J. Yeo, E. Lim, K. C. D. Tan, and Y. S. Ong, The efficacy of an image-to-writing approach to learning abstract scientific concepts: Temperature and heat, Int. J. Sci. Math. Educ. **19**, 21 (2021).
[32] D. T. Brookes and E. Etkina, The importance of language in students' reasoning about heat in thermodynamic processes, Int. J. Sci. Educ. **37**, 759 (2015).
[33] S. M. Brahmia, Mathematization in introductory physics, Ph.D. thesis, The State University of New Jersey, 2014.
[34] M. Al Dehaybes, J. Deprez, P. van Kampen, and M. De Cock, Students' understanding of Laplacian, and gradient in mathematics, and physics context, in *The Learning, and Teaching of Calculus Across Disciplines*, edited by T. Dreyfus, A. S. Gonzalez-Marin, J. Monaghan, E. Nardi, and P. W. Thompson (The Centre for Research, Innovation and Coordination of Mathematics Teaching at the University of Agder, Bergen, 2023), pp. 85–88 [Online]. Available: https://matriccalcconf2.sciencesconf.org/.
[35] M. Al Dehaybes, J. Deprez, P. Van Kampen, and M. De Cock, Students' understanding of two-variable calculus concepts in mathematics and physics contexts. II. The gradient and the Laplacian, Phys. Rev. Phys. Educ. Res. **21**, 010132 (2025).
[36] M. Trigueros and R. Martínez-Planell, Geometrical representations in the learning of two-variable functions, Educ. Stud. Math. **73**, 3 (2010).
[37] R. Martínez-Planell and M. Trigueros Gaisman, Students' understanding of the general notion of a function of two variables, Educ. Stud. Math. **81**, 365 (2012).
[38] R. Martínez-Planell and M. T. Gaisman, Graphs of functions of two variables: Results from the design of instruction, Int. J. Math. Educ. Sci. Technol. **44**, 663 (2013).
[39] M. De Cock, Representation use and strategy choice in physics problem solving, Phys. Rev. ST Phys. Educ. Res. **8**, 020117 (2012).
[40] C. M. Rebello and N. S. Rebello, Adapting a theoretical framework for characterizing students' use of equations in physics problem solving, AIP Conf. Proc. **1413**, 311 (2012).
[41] E. D. Dubinsky, Reflective abstraction in advanced mathematical thinking, in *Advanced Mathematical Thinking* (Springer, Dordrecht, 1991), pp. 95–126.
[42] A. Oktaç, What's new with APOS theory? A look into levels and totality, Av. Invest. Educ. Mat. **21**, 9 (2022).