

Metarepresentational competence in quantum mechanics change of basis problems

Idris Malik^{*} and Warren Christensen[†]

Department of Physics, North Dakota State University, Fargo, North Dakota 58108, USA



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Metarepresentational competence (MRC) is a theoretical framework that is used to analyze how people create and interact with external representations. Within quantum mechanics, problems may be approached or perceived differently based on the notation used (such as Dirac, Matrix, or Spinor notation) in the problem statement or by the person working on the problem. Semistructured interviews were conducted to present physics students with change of basis content problems in the context of quantum mechanics. These content questions were paired with MRC-focused follow-up questions for the explicit purpose of asking students about MRC concepts directly. Student statements were coded to confirm previously identified MRC concepts and to identify MRC concepts that are novel in this data. Our analysis of three student interviews demonstrates an array of MRC concepts, the usefulness of asking direct questions about MRC concepts as part of an interview, and solidifies MRC as a useful lens for investigating student thinking.

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I. INTRODUCTION

The learning of physics has been described as developing a toolbox of conceptual and mathematical skills [1]. In this model, expertise is identified as building up a wide array of skills and having sufficient skill at efficiently selecting the appropriate tools given a novel problem or situation. The fluid use of multiple representations is an essential skill within this toolbox [2]. A tool from a physics toolbox might be suitable for a number of different problems, although it might not be the most efficient or effective tool depending on the parameters of the problem. Consider a physical analogy: A flathead screwdriver is an excellent tool for installing a standard screw. Although it might also be possible to use a butter knife for the same task (perhaps a proper screwdriver is not freely available for a quick repair), it might not be as effective, or it might make the task a bit more difficult, but it could work. Now consider that you would like to spread some butter on a piece of bread. Although a butter knife would be ideal, one could, in principle, use a flathead screwdriver for this task instead. It would be inefficient, messy, and perhaps unsanitary, but likely possible. In physics, there are many such tools, and specifically representations, that have varying affordances and constraints when it comes to

solving particular problems. This work aims to cast light on analyzing students' understanding of the affordances and constraints of different representations by direct investigation, in the context of change of basis problems in quantum mechanics. We seek to answer the following two research questions:

1. Can an interview specifically designed to investigate students' MRC concepts be fruitful? And by doing so—
2. To what extent can we identify novel MRC concept codes from our data and confirm MRC concept codes from a previous study [3]?

II. LITERATURE REVIEW

Notions involving representation and notation in quantum mechanics have often been investigated in physics education research (PER). Fredlund *et al.* [4] noted that “the affordances of different representations determine the role they can play in communication, and thus in the sharing of knowledge.” These affordances, or the “inherent potential of that representation to provide access to disciplinary knowledge,” are said to “enable certain representations to become legitimate within a discipline such as physics. Physics learning, then, involves coming to appreciate the disciplinary affordances of representations” [4]. While PER literature has examined representations in quantum mechanics in the past decade, it often did so without leveraging diSessa's metarepresentational competence (MRC) theoretical framework [5,6].

It is important to make a distinction between the metarepresentational competence theoretical framework, and the concepts or statements that could be analyzed by the MRC theoretical framework. In this paper, we use

^{*}Contact author: idris.malik@ndsu.edu

[†]Contact author: warren.christensen@ndsu.edu

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the abbreviation “MRC framework” to reference the theoretical framework that guided the development of our interview protocol and our analysis of the student interview data. The terms “MRC concept” or “MRC statement” denote a concept or statement that could be analyzed or interpreted by the metarepresentational competence theoretical framework.

We provide a summary of three works that form the basis of our current research interest and study design focus involving MRC: Gire and Price [7], Wawro *et al.* [3], and Corsiglia *et al.* [8]. We intend to build upon the theoretical basis and content areas covered in these three works with the aim of extending the MRC framework and its specific relevance to Physics Education Research. Gire and Price [7] sought to characterize the representations used in quantum mechanics, which they name “Dirac notation, algebraic wave-function notation, and matrix notation.” Gire and Price characterize four structural features of quantum notations, which they note as: individuation, degree of externalization, compactness, and symbolic support for computational rules. The authors conducted eight student interviews to characterize each representation and examine how the features of each representation support or hinder students’ computation. The findings include the characterization of the four structural features applied to the three different representations, and how student statements support the characteristics of the representations and how their computational experiences interact with the theorized affordances and constraints of each representation. The Gire and Price structural features framework [7] was adapted by Schermerhorn *et al.* [9] into a “Computational Features Framework,” in order to “answer questions regarding how and why students use different methods.”

Investigations using the MRC framework have been conducted in the context of quantum mechanics, specifically in units involving “spin,” which tend to use eigen-theory concepts. In a 2020 study, Wawro *et al.* enumerated the different MRC concept-related codes (seen in Fig. 1) (also Fig. 1 in [3]) that arose in their interviews, which were structured around eigentheory concepts [3]. These codes were in part informed by the “Computational Features Framework” developed by Schermerhorn *et al.* [9], along with additional codes that were organized and explained through the MRC framework. The authors used the MRC framework to analyze student statements that compared

matrix and Dirac representations in the context of expectation value calculations. Wawro *et al.* conducted end-of-course interviews with eight students to learn how students reasoned about linear algebra concepts (particularly normalization, basis, and eigentheory) and how students reasoned with these concepts as they discussed quantum mechanics concepts [3]. The MRC framework was applied after the interviews were conducted, and was not used when designing the study. The authors noted how “computational competence” was related to student preferences for each notation in context. The authors also were interested in exploring how computational competence was related to the amount of MRC statements students used. Figure 1 is a table of MRC statement codes from Wawro *et al.* [3], which captures student thinking and perspectives on different representations in 15 codes organized into three different categories: Value-based Preferences, Problem-based Preferences, and Purpose and Utility Awareness statements. This MRC statement code bank formed our initial MRC statement code bank and inspired the design of our present interview study.

Corsiglia *et al.* investigated student thinking about change of basis problems in quantum mechanics, including how changing basis affects a quantum state, how notation could relate to basis, how basis expansion is interpreted, and how students attempt to carry out a change of basis [8]. The study used a phenomenographical approach to explore student thinking on different representations of state vectors in different bases (e.g., the z -basis, the x -basis, an n -basis, etc.). The questions mainly used Dirac notation and did not focus on how students viewed Dirac notation, matrix notation, or other notations as being different or having varied utility. We felt this same content area—change of basis problems in quantum mechanics—would be an excellent space for using an MRC framework since there are various notations with each notation having context dependent affordances and constraints. We, therefore, adapted some of the survey questions from this study and sought to identify and refine codes from Wawro *et al.* [3] [Fig. 1] (also Fig. 1 in [3]) as well as identify new codes that were emergent in our data.

III. THEORETICAL FRAMEWORK

We employed Andrea diSessa’s metarepresentational competence at the forefront of our experimental design and to interpret student reasoning regarding representation [5]. Metarepresentational competence (MRC) “... [describes] the full range of capabilities that students (and others) have concerning the constructions and use of external representations” [10]. The MRC framework has significant use in the research in undergraduate math education (RUME) community [11–15], but has seen little use in physics education research. Representation and notation in quantum mechanics have been investigated in physics education research (PER), [16–19], but only one paper incorporated metarepresentational competence as a framework [3]. In Wawro *et al.*, MRC statement codes were

(a) Value-based Preference	(b) Problem-based Preference
1. Clarity	1. Useful in calculations
2. Speed	2. Makes more direct use of given relationships
3. Familiarity	3. Needs less information
4. “Likeability”	4. Compactness
5. Ease of writing	5. Individuation
	6. Externalization
(c) Purpose and Utility Awareness	
1. Has freedom to choose symbols	
2. Has an ease with notation, writes symbols to mean what is personally desired	
3. Aware of one’s own progress in notation use	
4. Able to “step back” and weigh options to decide which notation system is best	

FIG. 1. Figure 1 from Wawro *et al.* [3], which presents their MRC codes and categories.

created based on students' statements on representation when performing expectation value calculations (see Fig. 1) and proved to be very fruitful in parsing their data [3]. However, the theoretical framework was applied *post hoc*. This work uses an interview protocol designed to specifically parse out student statements regarding MRC concepts in the context of change of basis of quantum spin state vectors, a closely related yet distinct content area.

IV. BACKGROUND

In quantum mechanics, spin is a property of some particles, often described as analogous to orbital angular momentum in classical mechanics [20]. This intrinsic angular momentum can be observed when particles interact with a magnetic field (such as a Stern-Gerlach device) [21]. This often confusing concept is better understood mathematically through the use of eigentheory. Eigentheory involves sets of vectors ("eigenvectors") that are unchanged by the application of a mathematical operator beyond being multiplied by a certain constant (an "eigenvalue"). Spin states of a particle or system are represented by Kets ($|\psi\rangle$), which function similarly to vectors. For spin- $\frac{1}{2}$ particles, the most common type of particles investigated in an introductory quantum mechanics course, applying a directional spin operator can only result in one of the two possible eigenvalues and its corresponding eigenvector. The result is either "spin-up" (a "+" eigenvector with a corresponding eigenvalue of +2) or "spin-down" (with a "-" eigenvector with eigenvalue of -2). In a certain coordinate direction (namely z , x , or y), a spin- $\frac{1}{2}$ particle's spin state can be described as a linear combination of these $|+\rangle$ "spin-up" and "spin-down" $|-\rangle$ kets.

These state vector kets, defined as linear combinations of eigenvector kets, can also be written in terms of 2×1 matrices. Here, the first entry represents the $|+\rangle$ ket (usually in the z direction), and the second entry represents the $|-\rangle$ ket. Introductory units of instruction may present the same information in both Dirac and matrix form so that students can see how one notation is translated to the other. Additionally, "Spinor" notation is sometimes used to write statements without explicitly using a Dirac ket or a column vector for the eigenvectors.

In any notation, the relations among the eigenvector kets, the spin operators, and the eigenvalue kets can be written in eigen-equation form: $\hat{S}_z |+\rangle = 2 |+\rangle$. Figure 2 includes an example of the three notations used in the context of a state vector equation:

When applying an operator to a corresponding eigenvector, the result will involve the eigenvalue for that

$$\text{Spinor: } \chi = \frac{3}{5} \cdot \chi_+ - \frac{4}{5} \cdot \chi_- \quad \text{Dirac: } |\psi\rangle = \frac{3}{5} \cdot |+\rangle - \frac{4}{5} \cdot |-\rangle \quad \text{Matrix: } \chi = \frac{3}{5} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{4}{5} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

FIG. 2. Equivalent state vectors, or wave-function kets, written in different notations.

eigenvector. This is most easily calculated when the operator and the state vector are in the same basis. Changing basis can aid in "diagonalizing" the matrix for a given operator or to make it easier to see what the physical result of an observation of a physical system, such as the measurement of a beam of particles using a Stern-Gerlach device oriented in a certain direction. Change of basis problems of this kind are often included in first-semester quantum mechanics courses and involve state vectors and operators such as spin- $\frac{1}{2}$ systems.

The authors identify Dirac notation and matrix notation as the main representations taught in undergraduate quantum courses. Dirac notation is prominently featured in quantum courses, and quantum mechanics is likely the first course in which most students are introduced to Dirac notation. While matrices are often used in mathematical contexts such as linear algebra and multidirectional or vector calculus, physics students in a quantum course may have only encountered matrices when solving systems of equations or in rotation units such as during cross product (torque) computations. The combination of vertical bars and angle brackets in Dirac notation is largely unique to quantum mechanics, especially when combined with the greek letter psi (ψ) often reserved for quantum state vectors or wave functions. Since spinor notation uses greek letters in manners similar to Latin alphabet variables commonly used in math contexts without additional novel and distinct markings, students may not identify spinor notation as distinct and prominent to quantum mechanics as matrix and Dirac notation.

V. METHODS

We designed a one-on-one, semistructured interview protocol [22] that provided participants content questions and opportunities to ask follow-up questions about MRC concepts directly. Content questions were adapted from questions from the Corsiglia *et al.* study, which focused on change of basis and included concept-focused and computational-focused questions [8]. The questions were adapted for our interview to provide a context for discussing MRC concepts, rather than pure tests of conceptual understanding. We chose to focus on change of basis problems because it is a rich and authentic topic that commonly uses three different notations. When answering change of basis problems, students can work with all three notations (and we can observe their thinking and MRC statements) as they approach the problems conceptually or while carrying out computational steps. For some of these content questions, multiple versions were created using different representations to provide students with an opportunity to comment on the differences between the representations (see Fig. 2). After each set of content questions, one or more verbal follow-up questions were asked. To avoid dead ends due to content issues, hints were either structured into the content questions or made available by the interviewer.

1. Consider a spin- $\frac{1}{2}$ state vector written in terms of the z -basis:

$$\chi = b_1 \cdot \chi_+ + b_2 \cdot \chi_-$$

- Suppose we want to write this state in terms of the x -basis vectors:

Given the coefficients in all options are correct, which option is closest to how you would choose to label the state vector?

- $\chi = -\frac{1}{5\sqrt{2}} \cdot \chi_+^{(x)} + \frac{7}{5\sqrt{2}} \cdot \chi_-^{(x)}$
- $\chi_x = -\frac{1}{5\sqrt{2}} \cdot \chi_+^{(x)} + \frac{7}{5\sqrt{2}} \cdot \chi_-^{(x)}$
- $\phi = -\frac{1}{5\sqrt{2}} \cdot \chi_+^{(x)} + \frac{7}{5\sqrt{2}} \cdot \chi_-^{(x)}$
- $\phi_x = -\frac{1}{5\sqrt{2}} \cdot \chi_+^{(x)} + \frac{7}{5\sqrt{2}} \cdot \chi_-^{(x)}$

2. Consider a spin- $\frac{1}{2}$ state vector written in terms of the z -basis:

$$|\psi\rangle = b_1 \cdot |+\rangle + b_2 \cdot |-\rangle$$

- Suppose we want to write this state in terms of the x -basis vectors:

Given the coefficients in all options are correct, which option is closest to how you would choose to label the state vector?

- $|\psi\rangle = -\frac{1}{5\sqrt{2}} \cdot |+\rangle_x + \frac{7}{5\sqrt{2}} \cdot |-\rangle_x$
- $|\psi\rangle_x = -\frac{1}{5\sqrt{2}} \cdot |+\rangle_x + \frac{7}{5\sqrt{2}} \cdot |-\rangle_x$
- $|\alpha\rangle = -\frac{1}{5\sqrt{2}} \cdot |+\rangle_x + \frac{7}{5\sqrt{2}} \cdot |-\rangle_x$
- $|\alpha\rangle_x = -\frac{1}{5\sqrt{2}} \cdot |+\rangle_x + \frac{7}{5\sqrt{2}} \cdot |-\rangle_x$

3. Consider a spin- $\frac{1}{2}$ state vector written in terms of the z -basis:

$$\chi = b_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Suppose we want to write this state in terms of the x -basis vectors:

Given the coefficients in all options are correct, which option is closest to how you would choose to label the state vector?

- $\chi = -\frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\chi_x = -\frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\phi = -\frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\phi_x = -\frac{1}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{7}{5\sqrt{2}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. Consider a spin- $\frac{1}{2}$ electron prepared in a quantum state represented by the following state vector:

$$|\phi\rangle = \frac{1}{2} \cdot |+\rangle_z - \frac{\sqrt{3}}{2} \cdot |-\rangle_z$$

- What do the coefficients $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ say about the state?

- What would be the units of ${}_x\langle +|\phi\rangle$?

- What does the expression in part (b.) mean?

- Could you write ${}_x\langle +|\phi\rangle$ in a different way?

- If you wanted to compute ${}_x\langle +|\phi\rangle$, which way would you choose to compute it?

Why?

Could you try to compute it?

5. For the following problem, use whichever notation you prefer:

Consider a spin- $\frac{1}{2}$ particle prepared in the quantum state represented by χ , written in the z -basis as:

$$\chi = \frac{1}{\sqrt{2}} \cdot \chi_+ + i \frac{1}{\sqrt{2}} \cdot \chi_-$$

- Write the state vector χ in the n -basis, given:

$$\chi_+^{(n)} = \frac{1}{\sqrt{3}} \cdot \chi_+ + i \frac{\sqrt{2}}{\sqrt{3}} \cdot \chi_- \quad \text{and} \quad \chi_-^{(n)} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \chi_+ + i \frac{1}{\sqrt{3}} \cdot \chi_-$$

- Hint: Solve for the values of a and b such that $\chi = a \cdot \chi_+^{(n)} + b \cdot \chi_-^{(n)}$

- By writing the state vector χ in the n -basis, have the probabilities for measuring along the z -direction changed?

- By representing the state vector χ in the n -basis, does χ in the n -basis represent the same physical quantum state?

- Did rewriting the state in the n basis create a new quantum state?

7. Consider a spin- $\frac{1}{2}$ particle prepared in the quantum state represented by $|\psi\rangle$, written in the z -basis as:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \cdot |+\rangle + i \frac{1}{\sqrt{2}} \cdot |-\rangle$$

$$\text{Where } |+\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |-\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Write the state vector $|\psi\rangle$ in the n -basis, given:

$$|+\rangle_n \doteq \frac{1}{2} \cdot \begin{bmatrix} 1 \\ i\sqrt{3} \end{bmatrix} \quad \text{and} \quad |-\rangle_n \doteq \frac{1}{2} \cdot \begin{bmatrix} \sqrt{3} \\ i \end{bmatrix}$$

- Hint: Solve for the values of a and b such that $|\psi\rangle = a \cdot |+\rangle_n + b \cdot |-\rangle_n$

- By writing the state vector $|\psi\rangle$ in the n -basis, have the probabilities for measuring along the z -direction changed?

- By representing the state vector $|\psi\rangle$ in the n -basis, does $|\psi\rangle$ in the n -basis represent the same physical quantum state?

- Did rewriting the state in the n -basis create a new quantum state?

8. • Can you think of a problem where you would choose to use Matrix notation?

- What specific features of the problem would influence your decision?

- Can you think of a problem where you would choose to use Dirac notation?

- What specific features of the problem would influence your decision?

FIG. 3. Interview questions 1, 2, 3, 4, 5, 7, and 8, from our semistructured interview protocol. The questions include all the text presented to students without the spaces provided for written work. These questions were based on questions from the Corsiglia *et al.* study [8]. Question 6 is not included, as we did not elect to give it to students in any of our interviews.

The focus of our interview was to investigate the landscape of student thinking and utterances related to MRC concepts in the context of quantum mechanics. This interview structure was selected to organically ease students into talking about MRC concepts without directly

explaining the MRC theoretical framework. The follow-up questions were intended to directly ask students about how they viewed and compared different representations. At first, the prompts were content-focused and considered MRC concepts on a surface level. As the interview

progressed, the follow-ups asked students deeper questions surrounding MRC concepts.

Three student interviews were conducted at a midwestern, land-grant research university, each lasting about one hour. Our interview sample consisted of three individuals: One undergraduate student (pseudonym UG), one student in an accelerated master's program (pseudonym AM), and one Ph.D. student (pseudonym PD). All three had taken at least two semesters of quantum mechanics. All three participants identify as white men. All three were recruited via an email that was sent to students who had taken at least two semesters of quantum mechanics and were present on campus during the summer when the interviews were conducted. We present analysis across student interviewees after the MRC concept code-focused Results section.

Author I. M. designed the interview protocol, served as the lead co-interviewer for all three interviews, transcribed each interview, and was the main coder. Author W. C. oversaw the study design and recruitment of participants, edited the interview protocol, served as the supporting co-interviewer, and was a secondary coder. Transcripts were coded for MRC statements. The initial codebook was guided by the codes delineated in Ref. [3]. Both authors independently coded each transcript, then met to discuss each code instance and deliberate the chosen code and its definition, with reference to transcripts and videos, until consensus was reached [23]. We created new codes to categorize one or more MRC statements connected to a strong idea that were determined to not fit under any of the preexisting codes from Wawro *et al.* [3]. In most cases, a code needed to appear more than once to warrant identifying it as a new MRC code. When defining a new code, we picked a name and definition that was reflective of our data, prioritizing the words chosen by student interviewees.

A. Interview questions

Here we include our interview questions as presented to our interview participants (see Fig. 3). Questions 1, 2, and 3 were adapted from Corsiglia *et al.* [8]. Questions 1, 2, and 3 (Q1, Q2, and Q3) were given to provide students with a chance to see spinor, Dirac, and matrix notation and respond to them. Question 4 (Q4) was adapted from Corsiglia *et al.* questions 8 and 9 and was meant to gauge a students' conceptual content knowledge and how the notation may have interacted with their explanations [8]. Questions 5, 6, and 7 (Q5, Q6, and Q7) were adapted from Corsiglia *et al.* question 7 so that students could work through a complete change of basis problem, with the problem statement presented in spinor, Dirac, and matrix notation, respectively [8] (see Fig. 3).

Q1, Q2 and Q3 were given in order and covered how to label a state vector after its basis was changed. After these three questions, we asked verbal follow-up questions for students to reflect on how the questions were similar and different (see Fig. 4). We then asked students Q4, which

asks about basis vector coefficients and inner products of basis vectors. Next, we asked students to work on Q5 (spinor notation). Depending on how students approached and responded to Q5, we would give students either Q6 (Dirac notation) or Q7 (matrix notation). For all three of our student interviews, we chose to ask Q7. For student UG, who struggled with the computation in Q5, we hoped the matrix notation in Q7 would make things easier for them, which it did. For student PD, we wanted to see if they would rewrite the matrix notation in Q7 as Dirac notation given their expressed preference for Dirac notation, and they did. Student AM made many comments on Dirac notation while answering Q5 and did not remember exactly how the inner product identities worked, so we gave them Q7 to see what comments they made about matrix notation and to give them a better opportunity at calculating an answer. Q8 provided us with an opportunity to talk about their notation use and conceptions in a more general sense and give students an opportunity to share their thinking by extending to different contexts (see Fig. 4.).

1. Follow-up questions

Our follow-up questions were designed to specifically probe into potential MRC concepts that students may have used or thought about but did not articulate while initially working through the problem. Direct follow-up questions on MRC concepts were not part of the Wawro *et al.* study [3]; therefore, they are unique to this study.

- Reflect on this most recent question (*perhaps Q1, 2, 3*) and the one before it:
 - How similar are these questions?
 - Does one representation make more sense to you?
 - Is one representation "**better**" than the other one?
 - Would you say one is more comfortable than the other? Why?
 - Have you thought about these comparisons before?
- If you were given a problem that might be doable using either Matrix representation or Dirac representation, what would you consider when choosing a representation to use?
- Do you feel differently about the two representations? How would you describe your relationships with both of them?
- Does the comparison and contrast of Matrix and Dirac representation remind you of anything else?

Prompt

 - * Maybe a different situation?
 - * Maybe a different type of question?
- Are there other times were you make similar decisions in physics or math, or even other contexts?

FIG. 4. Verbal follow-up questions from our semistructured interview protocol. These questions asked students to reflect on the problems they worked on, and share their thinking on MRC concepts.

While we wanted to ask students questions regarding MRC concepts, we did not want to lead students to merely confirm our expertlike expectations regarding the different representations. We sought authentic student thoughts and beliefs about what each notation meant to them and how they felt about using it depending on the question being asked. Wawro *et al.* did not have an opportunity to ask these follow-up questions because their goals were to investigate student conceptual and mathematical reasoning and they did not approach their data collection with MRC in mind. We were able to observe MRC concepts in students' initial responses, in circumstances similar to those in which Wawro *et al.* identified MRC concepts [3]. Our follow-up questions afforded us additional insight into their use of these different representations. We include some of our prepared follow-up question prompts in Fig. 4.

VI. RESULTS

We present evidence for the existing MRC concept codes that were most prominent in our interviews, as well as evidence for six novel codes that were not identified in the Wawro *et al.* study [3]. We share our definitions for each code and comment on the multidirectionality of codes. Some of the MRC concept codes were connected to statements that were spoken organically while the interviewees were working through a particular problem (Fig. 3), while others were captured during a directed MRC-focused follow-up question (Fig. 4) from the interview protocol that was designed to explicitly elicit statements that could be identified as MRC concept codes. Each participant articulated different preferences in notation throughout the interview. We will conclude with features within interviews and comments on conceptual understanding as they pertain to MRC concept fluency.

A. Metarepresentational competence codes

A result of our research is an updated bank of MRC codes, which we have defined based on previous work by Gire and Price [7], Wawro *et al.* [3], and on data and analysis that we present in this paper (see Table I). Wawro *et al.* note that categories A and B fall under “critique and compare the adequacy of representations and judge their suitability for various tasks” [6] (p. 293). Codes in category A are related to “related to preferences students expressed related to some overarching value” [3]. Codes in category B include “preferential statements students made that seemed more intrinsically tied to particular problems or contexts” [3] and include the structural features from Gire and Price [7]. Category C codes relate to the general purposes of representations and demonstrate understanding of why and how representations are useful to those who use them [3]. Note that MRC codes are not mutually exclusive. Student statements can involve multiple MRC codes.

We attempted to be specific about what utterances corresponded to each code.

B. Code multidirectionality

We view the MRC codes as dimensions or ways of thinking and commenting on ideas surrounding representation. We do not intend for our identified MRC codes to be a list of ideas or beliefs students must hold in order to “hold expertlike views.” We want our MRC code bank to cover a breadth of statements that individuals make when thinking about their use of notations from mathematics and physics and when comparing different forms of representation. In support of this, we define our codes as explicitly being not unidirectional. By this, we mean that a statement that takes a certain judgmental opinion (positive, negative, ambiguous, etc.) is still coded as belonging to the same category as statements that are in conflict with it. For example, if a student commented that they “don’t like matrix representation,” this would still be coded as *Likeability*, even though the student “doesn’t like” the representation.

Here are some examples of cases in our data where we have applied the multidirectionality of codes. When student PD was talking through their thinking on Q1, they went through each multiple-choice response and commented on their suitability. We coded the statement, “Like you go from labeling the state. With chi or whatever that variable is called. Well you start playing with fire—it’s just confusing,” as “Has freedom to choose symbols,” because it expresses the idea that there should not be the freedom to relabel a state vector with a new variable because it would be conceptually confusing. In a follow-up to the first three questions, student PD said, “Working with matrices is also just a pain if you’re trying to—to write like a really long problem.” We coded this as *Useful in calculations*, although the statement comments that matrix notation is not useful in calculations. Student AM made a similar statement in their interview, claiming “It’s harder to have to go through and multiply the matrices every time, even if it’s a very simple matrix operation,” which we also classified as *Useful in calculations*. Student PD claimed that “You can’t imagine like Cartesian graph, like R 3 or something,” when using Dirac notation, so this was coded as a statement on *Visualizability*. On Q8 of the interview, student AM made a statement about the compactness of matrix notation:

AM: So when you’re multiplying matrices, dealing with like higher level linear algebra, you want to a lot of the times represent larger matrices as sums so you don’t have to write out every single one and you can have infinitely large matrices: which are annoying.

This statement indicates that matrix notation is not compact, as summation notation is needed to be more compact.

TABLE I. Our list of MRC codes used in our data analysis. Our definitions are informed by Gire and Price [7], Wawro *et al.* [3], and analysis of our interview data. Novel codes have been marked with an asterisk (*).

Code category	Code	Our definitions (informed by Gire and Price [7], Wawro <i>et al.</i> [3], and our interview data)
A. Value-based Preferences	Clarity	Statements involving a notation being clearer to read and interpret
	Speed	Statements involving a notation being preferred because it is faster to write than another
	Familiarity	Statements involving a notation being more familiar to work with compared to another
	“Likeability”	Statements made simply because an individual “just likes it”
	Ease of writing	Statements involving a notation being easier to write than another
B. Problem-based Preferences	Useful in calculations	Statements involving that a notation is useful or not useful in a calculation
	Makes more direct use of given relationships	Statements involving a notation being preferred because it makes more direct use of relationships given in the problem statements
	“Overkill” / Needs less information	Statements involving the degree to which a representation contains information that is excess or superfluous for a certain purpose. For example, if a representation uses 2 markings to indicate the same thing, then 1 of those markings is “overkill.”
	Compactness	Statements about a notation having a smaller overall size because there are fewer things to write down
	Individuation	Statements about the degree to which important features are represented as separate and elemental
	Externalization	Statements about the degree to which elements and features are externalized with markings included in the representation
	Hierarchy*	Statements that indicate or reference the idea that one or more representations has a directed relationship to one or multiple other representations. A directed relationship could mean that one representation is “a shorthand for” / “Is better than” / “Is more generic than” / etc. another representation.
	Abstractability*	Statements that involve the idea that the notation can represent ideas with a high level of abstraction, and that symbols could be changed without a loss of conceptual meaning
	Visualizability*	Statements that comment on how a notation may extend itself to a visual medium such as a graph, illustration, 3-D model, or other figure
C. Purpose and utility awareness	Freedom to choose symbols	Statements that explicitly comment on the freedom to choose any symbol they want to use in a particular context. This includes statements that claim that there is not freedom to choose any symbol.
	Has an ease with notation, writes symbols to mean what is personally desired	Statements that describe an ease with making a particular notation mean what an individual wants it to mean. This includes when individuals demonstrate agency to choose and edit symbols with what they want to communicate.
	Aware of one’s own progress in notation use	Statements of explicit student reflection on their personal progress (or lack thereof) in using a particular notation
	Able to “step back” and weigh options to decide which notation system is best	Statements made when individuals “step back” from the work they are doing to consider which notation might be best for a given problem. This includes an individual rewriting a problem in a different notation than what is given in a problem statement.
	Notation system as part of conceptual reasoning*	Statements that indicate that an individual is using a representation to help them understand or explain their understanding of a conceptual idea

(Table continued)

TABLE I. (Continued)

Code category	Code	Our definitions (informed by Gire and Price [7], Wawro <i>et al.</i> [3], and our interview data)
	Notation as a discipline*	Statements that claim that notation is a discipline that is distinct from other practices or features of a given discipline
	Reflects on learning*	Statements that acknowledge the learning process of those new to a notation. This could include statements involving learning a notation as being accessible or difficult, or being useful or not very useful.

C. Occurrence of MRC concept codes

We identified 111 MRC statements across the three student interviews. Of these, 19 statements were made by students while working on content questions (Q1, Q2, Q3, Q4, Q5, Q7 in Fig. 3) without any direct prompting from the interviewers. Ten MRC statements were made while working on content questions after a semistructured interview prompt. This includes interviewers asking a student to explain a choice of words or further clarification on their work. The remaining 82 statements were made as a response to MRC-focused follow-up questions (including Q8 in Fig. 3), including prepared reflective prompts (Fig. 4), and further prompts as a feature of semistructured interviews on these reflective questions.

D. Preexisting codes

Before introducing our rationale and results for the creation of new MRC codes, we examine two codes from Wawro *et al.* that featured prominently in our dataset [3]. The analysis of the MRC codes, *Able to “step back” and weigh options* and *Useful in Calculations*, strengthens the work of Wawro *et al.* and demonstrates the viability of this framework for analyzing student work [3].

1. *Able to “step back” and weigh options (to decide which notation system is best)*

The most frequent code in our interview data was *Able to “step back” and weigh options (to decide which notation system is best)*. It should be noted that our interview design (see Fig. 3) and follow-up questions (see Fig. 4) were both developed to directly prompt students to do this. We assert that the prevalence and depth to which all three interviewed students interfaced with this MRC code is evidence that students can actively discuss MRC concepts in the context of quantum mechanics (a goal of RQ1).

Many instances of the MRC code *Able to “step back” and weigh options* were made with a positive connotation. Students express different levels of complexity with this code. In a follow-up to Q8, student UG states: “So I guess, any time you’re not changing basis I would probably prefer Dirac notation.” About 15 min into their interview, in a

follow-up to Q3, student PD talks explicitly on a case where they might use matrix or Dirac notation:

PD: If you got a ton of matrices with a bunch of numbers and like, and then with the spin, it’s like, **if you’re not working on a two-dimensional problem, like are you going to do that?**

PD makes a similar statement in response to Q8.

PD: Definitely like the symmetry of the problems, like, and by ‘symmetry’ I mean like, you know, **if everything’s orthogonal, orthonormal, then just use Dirac notation like for sure**, there’s no reason to not, in my opinion. But if it’s like messy with matrix notation, I guess.

In another instance, when asked about what they would do for a problem that might be doable in either the matrix or the Dirac notation (Q4) PD noted:

PD: “Definitely the bra-ket, Dirac. Like if you can, just because uh ... you’re not doing the matrix multiplication and I’m like **if things are orthogonal**, like if you know for a fact about your system, the basis is like the orthogonal, the state vectors, or not the state vectors, what do you call them? But. The basis vectors are orthogonal, **then you don’t do matrix multiplication.**”

PD elaborated on the MRC concept that we coded as *Able to “step back” and weigh options*, and when we asked them if it reminded them of other contexts or analogies, they provided a math context and an illustrative example about housework:

PD: So when you were choosing the matrix representations like basis and stuff, in my opinion, **that’s super, super similar to picking a coordinate system**. Like if you’re going to pick a coordinate system like we’re going to—**like spherical coordinates or cylindrical coordinates**—because sometimes, for example, **if you picked a good**

basis to work in, you know, all your stuff will fly out really easy ... But if you pick something really horrible basis, it's going to be nasty to work in. And the same thing if you're like working on a problem that's like really spherically symmetric and you're working in Cartesian coordinates, [...]

PD: [...]

trying to do some like work on the house or something and you try to use the wrong. Tool. You know [...] If you're trying to like. **Open like a screwdriver, like a Phillips screw, with like a butter knife or something like that. It's way harder, you know. But then you try to butter your toast with a Phillips screwdriver. (group laugh) That might not work.**

Student AM made a statement when reflecting on Q4 which included a sentiment about which representation would be better for people who did not know any quantum mechanics but did have knowledge of matrices: “So, what I would say is if I were taught this way to just do the operation and then taught this—Sorry if you're pulling people off the street that have a knowledge of matrices. Right, this is the way that you want to do it.” This reflection on what representation would be better for themselves vs others that have a different content background reflects an understanding of the learning process needed for individuals to be comfortable with using different representations. This instance reflects the MRC code of *Able to “step back” and weigh options*, and this, along with AM's directly following statement motivates the creation of a *Reflects on learning* code: “If you're trying to teach it in a class, it's easier to learn the identities [For ket inner products] and go through those because it's a lot more clear of a notation. [Dirac notation] gives you a lot more information.”

2. Useful in calculations

Another prominent, previously identified code was *Useful in calculations*, which was evident in statements that comment on whether a notation is useful or not useful in a calculation. This code captures statements that are made about a given specific case or a more general category of cases. When reflecting on Q8, student UG stated, “If they're asking for like values, I guess I would probably prefer a matrix notation.” Student PD talked about the usefulness of Dirac notation in a follow-up question to Q7:

PD: But there's tons of tricks about the [gestures Dirac notation angle brackets] that's why such a good notation you can like, [pushes angle brackets together like a half clap] Put them together, you know.

When working on Q5 and Q7, student AM faced difficulty in calculating and answering because they were unsure of the inner product identities used in Dirac

notation. Reflecting on this, student AM said about Dirac notation, “If I knew the bra-ket relation it would be very convenient to just go through and say, OK, I know that this bra and this ket go together and that has an identity, and that makes it one.” We also coded this statement as *Makes more direct use of given relationships* because it comments on how Dirac notation leverages the inner product identities for basis kets. Earlier, AM restated how Dirac notation is useful after working on Q4:

AM: “Here's what I'm gonna say here: The one on the left [(Dirac notation)] if I—if I come in here, this is [the] is the obvious orientation to me. [Holding up page and pointing to Dirac notation] So this is how I will be able to solve [Q4] 10 out of 10 times. If I come in here knowing that identity. This is easier 10 out of 10 times.”

E. Modification of existing codes

Instances of existing codes within our dataset led us to modify the definitions of some codes in Wawro *et al.* [3]. When modifying definitions, we gave preference to the words chosen by our interviewees.

We chose to add the quoted term “*Overkill*” to the *Needs Less Information* code. The definition we use for this code is: “Statements involving the degree to which a representation contains information that is excess or superfluous for a certain purpose.” For example, if a representation uses 2 markings to indicate the same thing, then 1 of those markings is “overkill.” When interviewee UG was working on Q2, they noted regarding the x subscripts on the left-hand side of question choices “b” and “d,” “but I don't think you need that. At least for like, maybe like, it's kind of like overkill notation because you have the x basis.” This code instance motivated us to include the word “overkill” because it captured a very similar idea well and uses an interviewee's natural language. Another instance of this code, when UG was answering Q8, is:

UG: Yeah, because I'm thinking like as long as you're not changing the basis, **there's not too much [you] need to know.** Like you don't really need to write once this whole [rigamarole] One i square root of three. Like if you're not changing basis it's, it makes sense to just use the Dirac spin up in the N basis.

Here, UG notes that Dirac notation does not include information that is unnecessary to consider in a problem that does not involve a change of basis. Hence, this statement is an instance of a representation being not *Overkill* compared to other possible representation choices. The multidirectionality of code instances further motivated

our addition of the word *Overkill* to the *Overkill / Needs Less Information* code.

F. Creation of new codes

Generally, we created a new code if there was more than one instance of a statement that we felt were closely aligned with one another and were not well identified among the existing codes presented by Wawro *et al.* [3]. We also identified a new code if a single instance occurred where the interviewee presented a cogent and clear explanation for use of a particular notation or representation. When identifying a new code, we provided a definition reflective of the usage in our data set and picked a name that was reflective of our data and the words chosen by our interviewees. We also categorized them into one of the three code categories from Wawro *et al.* [3]: “A. Value-based Preferences,” “B. Problem-based Preferences,” or “C. Purpose and Utility Awareness.” We have identified three new B. Problem-based Preferences codes and three new C. Purpose and Utility Awareness codes.

1. Hierarchy

Within the first 10 min of the interview, student UG made a statement involving their belief in a “Hierarchy” of notations. In a follow-up to Q3, UG stated that Dirac (and potentially spinor as well) notation is just a “shorthand” for the matrix notation. Much later in the interview, UG made a similar remark regarding the spinor notation in Q5, after being given Q7 (emphasis added in bold):

- UG: There’s a reason that I already like this notation way better, I got to be honest.
- INT: Why do you like it better?
- UG: But again, you know, **it’s like, hey, spin up–1 0, spin down–0 1**, and like, like I said before, I just **I like the given—what exactly given values for just like shorthand** [points at Q5] like Psi, Psi plus that’s the spin-up, and you know, I kinda get, like all screwed up in my mind, like—how do I do this again?
- UG: **Where I can actually, like ... work out the numbers?** Like, [points at Q5 page] can one over square root of three be multiplied times Chi spin up? **Whereas here [Q7] it’s like, I know I can, you know?**

It is possible that UG views this hierarchy as matrix notation being the “real” notation, with Dirac and spinor notation as “shorthands” because they prefer doing change of basis computations in matrix notation. UG’s consistent statements about the hierarchical relationships of the notations motivates *Hierarchy* as an MRC statement code, as this idea was not well captured by any existing code. We have chosen to categorize this code under B. Problem-based preferences, because it considers the relationship

between notations in the context of how they are used in solving problems.

2. Abstractability

Student PD made many statements involving concepts of *Abstractability* and *Visualizability*, which we identify as new MRC codes. PD viewed Dirac notation as being more “abstract” in nature relative to matrix notation. PD stated that the Dirac kets could represent anything or be written as “smiley faces” or “frowny faces” without any loss of meaning (we also coded this statement as *Has freedom to choose symbols*)

- PD: So I think when you’re doing bra-ket notation, it’s a little more abstract because, you might uh ... **You need to think of it as a more abstract vector space**, because you just have two vectors that are labeled, you can label [the basis vectors] with whatever you want, you know. You can call them smiley faces and frowny faces, and it will still be okay.

In a statement that also touches on *Reflects on Learning* (another new code included in Table I), PD says, “... but like, it’s better to like learn how to do things without more abstractly, you know, that was like definitely a big jump. Like, it was hard to learn that.” Additionally, PD notes that spinor notation was more abstract than Dirac notation, “I think [Spinor notation] is even more abstract, maybe because, like, it’s not even clear, like taking inner products and stuff like that.” This statement involves the code *Clarity*, and PD continues to use the word “abstract” and compares notations in this sense. Because of PD’s focus on these statements, and due to their emphasis on this wording, we have defined *Abstractability* as a new code in our MRC code bank (see Table I). We have chosen to categorize this code under B. Problem-based Preferences, because it considers how notations are connected to conceptualizations useful for thinking through problems and operations used in problems. Additionally, we identify *Abstractability* more similar in scope to other B. Problem-based Preferences codes rather than C. Purpose and Utility Awareness codes.

3. Visualizability

Student PD commented frequently on the idea of “*Visualizability*,” in a manner distinct to existing MRC codes. PD used hand gestures to set up a three-dimensional coordinate system when talking about state vectors. PD then mentioned understanding matrix notation as similar to applying a linear transform to a vector, to change the direction of the state vector. In a follow-up to Q3, PD said:

- PD: But then the way that they labeled [in Q3] the basis vectors of the same vector spaces in the same

problem with matrices. That gives you more of a **sort of geometric understanding of what's going on. You can visualize**, I mean, it's nice because it's problems in two dimensions. So you can like literally visualize like [pointing in perpendicular directions with fingers] this sort of thing going on now.

Student PD soon followed with this explanation:

PD: So you can visualize. The two vectors, you know, like they've got these two matrices here, which are basis vectors and so **you can visualize like an i vector pointing in the x direction**. And a j vector which is orthogonal to the [air quotes] y direction, if you want to call it that. But that is also normal which you can see from the matrices. So like, it's more natural, to like work with that because you can do each. Even though it's like the exact same thing.

While working on Q8 near the end of the interview, PD noted that this conceptualization might differ for others, "Maybe for some other people, that are like less visual or more visual or something— I mean, that's how I think about it." Because of PD's focus on these statements, and due to their emphasis on this wording, we define *Visualizability* as a new code in our MRC code bank (See Table I). This code is similar in scope to *Abstractability*, so we also categorize *Visualizability* as B. Problem-based Preferences.

4. Notation system as part of conceptual reasoning

There are points in the interview where students use terms and symbols that are specific to one notation system as part of their reasoning or elaboration on conceptual understanding. While this is present throughout our data to different degrees, the best example of this is when student AM notes in a follow-up to Q1, "So what you can do, then is take the complex conjugate and make it into a ket. Well, you're not making it a ket, you're thinking about the ket." Student AM specifically notes that while they are not exactly making the complex conjugate of a state vector into a ket, the idea is connected to their understanding of kets. Here, Dirac notation is allowing student AM to process their thinking in a nuanced way. This instance is our motivation for *Notation system as part of conceptual reasoning* as its own code. We categorize this code as C. Purpose and Utility Awareness because it comments on the purpose of representations beyond working on specific problems.

5. Notation as discipline

Student AM stated early in their interview that notation is a discipline that is distinct from physics as a discipline. In reference to Q1, Q2, and Q3, student AM commented that:

AM: They're identical questions, but some of them are more general than others. In my head, **these aren't physics questions. They're notation questions**. Which are heavily related, but entirely separate, separate disciplines, like when I was writing the paper that I published [earlier], we had to take an entire like week to go over notation. And that was separate from what the physics was. We had to look through and see what other people had notated. Try and keep it consistent, so that's part of why I'm going with these answers. Here is part of. It is because this makes the most sense to me. The other part is that it's consistent.

This idea is one that we think is important to note despite only a singular statement to warrant its identification, because it was unique and distinct from other viewpoints on notation we observed. This code acknowledges that notation choices have historical backgrounds and current implications for how scientists communicate as valid and interesting topics for students and others to consider. Viewing *Notation as a discipline* (or not) may be an insightful activity for a quantum classroom, where students can discuss notational choices that may be confusing without exploring their context. We categorize this code as C. Purpose and Utility Awareness, because it comments on the purpose of representations in a general sense.

6. Reflects on learning

While there are many instances in our interviews where students reflected on their *Familiarity* with a notation, or was *Aware of one's own progress in notation use*, there were also places where students commented on the process or value of learning about a notation. During Q4, Student AM noted that:

AM: If you're trying to teach it in a class, it's easier to learn the [Dirac inner product] identities and go through those because it's a lot more clear of a notation. It gives you a lot more information.

When talking about Dirac notation in a follow-up to Q7, Student PD talked about the learning process (and *Abstractability*), "but like it's better to like learn how to do things without more abstractly, you know, that was definitely a big jump, like it was hard to learn that." While PD calls back to their own learning process, they make a statement of the value of learning to use Dirac notation. We code this as distinct, for example, from PD's statement a minute before the previous quote coded as *Aware of one's own progress in notation use*:

PD: I feel like. Like learning this stuff, when starting off, I had way less of an intuition on the Dirac, like bra-ket, because you like take Linear Algebra and

stuff and learn about like ... moving or like linear transformations and matrices and stuff

This clear distinction from *Aware of one's own progress in notation use* motivates our use of *Reflects on learning* as a novel code. We categorize this code as C. Purpose and Utility Awareness, the same category as *Aware of one's own progress in notation use*, as it concerns the value of representations in the context of learning content.

G. Overarching results and student interview comparisons

There are some characterizations and trends from our three student interviews that suggest a relationship between content knowledge and facility with MRC concepts. We present the data here and further explore this potential relationship in the discussion section.

Undergraduate student UG made fewer MRC comments in total, and it took them longer to “warm up” to speaking freely about MRC in depth. UG was effective in elaborating on their thought process while working on content questions when prompted. Additionally, most of the MRC statements UG made were “Value-based Preferences” (Fig. 1), which are potentially less nuanced than “Problem-based Preferences” and “Purpose and Utility Awareness” statements.

Student UG spent significant time on Q5 but was not able to complete it. UG expressed that they did not have enough equations to solve for the constants they wanted. When given Q7, which was identical to Q5 but used matrix notation, UG responded with the statements from the Hierarchy section above. Even though Q7 was similar to Q5, UG was able to complete Q7. Afterward, UG made conflicting statements regarding their success in Q7 compared to Q5. Initially, they made statements indicating that they understood the question better after they saw it presented in matrix notation. However, a few minutes later, UG said that they could not be sure that the different notation made it easier. In UG's opinion, their success may have been due to just seeing the same question a second time.

Ph.D. student PD, who performed extremely well on content questions, spoke fluently about how they represented their work verbally, on the page through equations, and by using hand gestures. Relative to UG (and to a lesser extent AM), PD made MRC statements earlier and more often over the course of their interview. Many of these statements involved a preference for Dirac notation, and of the 18 statements regarding Dirac notation, 4 were made while working on content questions without any form of prompting, while 14 were made on reflective follow-up questions. While UG made few statements regarding Purpose and Utility Awareness (see Fig. 1), PD made many distinct statements in this category; the first of which came during an early follow-up question. Most frequent

were statements that would fall under *Aware of one's own progress in notation use*, which encompassed Dirac, matrix, and spinor notation. PD also made many statements consistent with the *Able to “step back” and weigh options* code after one of the verbal follow-up questions and while considering how to begin working on Q5.

Accelerated masters student AM's interview experience was somewhere in between that of students UG and PD. AM began making many MRC statements fairly early on in the interview. While working on Q1, AM noted that the interview questions focused on notation more so than on the content covered in the questions. AM then made statements we coded as *Notation as discipline* due to their belief that this was distinct from the broad umbrella of “Physics” in this context and in their prior experience. AM made many statements in all three categories of Value-based Preferences, Problem-based preferences, and Purpose and Utility Awareness, even though they had less computational success and slightly less complex elaborations on the different representations compared to PD.

Taken as a whole, we observed a direct relationship between education, experience, success and MRC codes. While we are not able to apply causation to any of these relationships, we believe that it is plausible that there could be a connection to developing MRC and an associated improvement in performance.

VII. DISCUSSION

Throughout the interviews, interviewees made a number of unprompted comments that were coded as MRC statements ($n = 19$). Interviewees made additional comments ($n = 10$) while working on content questions when asked to further elaborate on an idea or thought process, in a manner characteristic of semistructured interviews [22]. Additional MRC statements were made when interviewees were directly prompted to comment on the representation or notation within a problem via follow-up questions ($n = 82$). Some of these prompted comments were similar to what the interviewee had previously said, but there were a number of instances involving additional MRC statements that the student had not previously articulated or that had not previously been coded. This confirms that analysis using the MRC framework to analyze data *post hoc* (as in Wawro *et al.* [3]) is possible and valuable, while also demonstrating that a protocol specifically designed to encourage students to talk about MRC can provide both overlapping and additional insight into student thinking about representation.

A. Findings that relate to Corsiglia *et al.* [8]

Given the different goals of the Corsiglia *et al.* [8] and our present study, there is not a great deal of overlap in the findings of our study. However, we did see evidence in our interview sample, such as students understanding that

changing the basis does not change the state of the system. There was mixed success among our interview sample on questions relating to basis expansion coefficients (Q5 and Q7, in Fig. 3.). While PD was able to navigate these problems successfully, both AM and UG struggled with substituting in the change of basis and completing the task.

B. Potential relationship between metarepresentational competence and computational skill

MRC concepts, which involve the capability to use external representations [10], are likely related to skills developed by individuals by the time they are considered an “expert” in a particular type of problem. Our findings on this idea are limited due to our sample size of three interviews, and given that our interview protocol did not have a focus on capturing the breadth of a student’s conceptual knowledge or so called “level of metarepresentational competence.” However, there were discernible differences between students’ success on the interview problems and the extent to which they made statements about representation that we coded as MRC statements.

For example, interviewee PD performed extremely well on content questions, and successfully represented their work verbally, on the page through equations, and with hand gestures. Interviewee UG struggled with several content questions and did not articulate as many MRC statements as PD, outside of the Value-based Preference category. Interviewee AM, who articulated MRC statements less frequently than PD but more frequently than UG, was successful on more questions than UG but struggled at times, especially with Q7. This does suggest a direct relationship between student understanding and MRC proficiency; however, it is unclear to what extent a causal relationship is at play.

How and when MRC concepts are developed and the role they play alongside content understanding in upper-division physics, are still unclear. Future research is necessary to shed additional light on the relationship between students’ use of different representations, how they articulate their understanding of using different representations, and their performance on content questions in upper-division physics.

C. Discussion of MRC framework in PER

At the beginning of this paper, we presented the analogy of a screwdriver and a butter knife. Both tools have similar features and are adept at several tasks, although they are not always acceptable substitutes on tasks for which one is perfectly suited. Similarly, notations used in change of basis problems have unique affordances and constraints for solving different problems. Using an MRC framework and

an interview protocol specifically designed to elicit MRC statements can help us understand how students think about using different notations and why they prefer one notation over another. It is plausible that instruction that includes MRC concepts could be beneficial for improving student understanding, although we do not have sufficient evidence to declare the exact mechanisms that connect the two at this time.

While this study focused on change of basis problems in quantum mechanics, there are a number of concepts that use different representations (e.g., kinematics equations, a vs t , v vs t and x vs t graphs, pictures/sketches) where investigations using MRC could be illustrative of student thinking. We believe that the MRC code bank presented here, while perhaps imperfect or incomplete, is not specific to the content area we studied and could be used across many more physics content areas.

Finally, it is worth noting that students do make statements that are identifiable as MRC statements while talking about problems they are solving, and, at times, make additional statements relevant to representation when directly asked about it. Research methods, such as those articulated above, can be employed to further investigate the nuances of student thinking about content and metarepresentational competence.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [24].

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