

Supporting Information: Thin-film drainage becomes singular at saddles

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EXPERIMENTAL DETAILS

Oil properties. Silicone oils, purchased from Carl Roth (CAS number: 63148-62-9), of kinematic viscosities $\nu = 350 \text{ mm}^2 \text{ s}^{-1}$ and $\nu = 1000 \text{ mm}^2 \text{ s}^{-1}$ (manufacturer specifications: 333–368 and 950–1050 $\text{mm}^2 \text{ s}^{-1}$, respectively) were employed in the experiments. In both cases, the density at 25°C is $\rho = 970 \pm 5 \text{ kg m}^{-3}$ and the surface tension is $\gamma = 21 \pm 0.2 \text{ mN m}^{-1}$ [1].

3D printing.

Three geometrically similar substrates were fabricated in-house with characteristic length scales $L = 15 \pm 0.025$, 30 ± 0.1 , and $40 \pm 0.1 \text{ mm}$, yielding Bond numbers $Bo = \rho g L^2 / \gamma = 102 \pm 1$, $Bo = 408 \pm 5$, and $Bo = 725 \pm 9$, respectively (see Figure S1). The smallest substrate was produced by stereolithography (SLA), which provides a characteristic resolution of approximately $25 \text{ }\mu\text{m}$. Due to the limited build volume of the available SLA system, the two larger substrates were fabricated by fused filament fabrication (FFF), with a characteristic resolution of approximately $100 \text{ }\mu\text{m}$. In all cases, the upper surface of the substrate is described by $\eta(x, y) = L - L \cos(\frac{y}{L}) - \frac{x^2}{2L}$.



FIG. S1: Image of the three substrates used in the experiments. The grey substrate corresponds to $Bo = 102 \pm 1$, while the two black substrates correspond to $Bo = 408 \pm 5$ and $Bo = 725 \pm 9$.

Silicone oil deposition. A large quantity of silicone oil was poured onto the substrate, resulting in an initial condition corresponding to a relatively thick liquid layer, with a thickness of a few mm at the saddle region, from which drainage subsequently develops (see Figure S2).

Imaging method. The draining film was imaged using a Nikon D800 camera equipped with an AF-S Micro Nikkor 105 mm f/2.8G IF-ED macro-lens (spatial resolution $\sim 5 \text{ }\mu\text{m}/\text{pixel}$). A uniform LED panel provided back-illumination (see Figure S3). The camera was focused on the symmetry plane $x = 0$ near the saddle, and the optical axis of the camera was oriented horizontally and intersected the location of the minimum of the valley. A reference image of the dry substrate was first acquired to determine the position and geometry of the substrate in the imaging plane. Time-lapse imaging was initiated at the moment the silicone oil was poured onto the substrate, with the first frame defining the initial time $t = 0$.

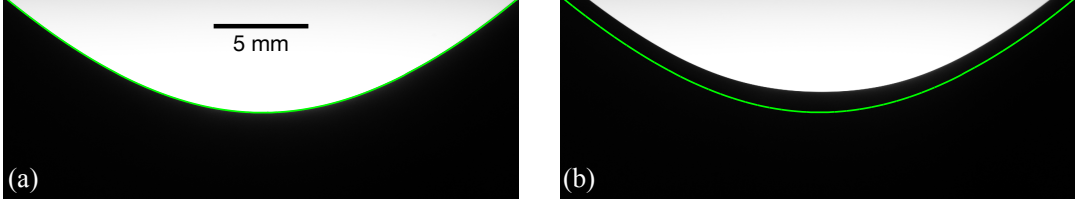


FIG. S2: (a) Image of the dry substrate ($Bo = 102$). (b) Image of the same substrate covered by a silicone oil film immediately after pouring. In both images, the field of view corresponds to the plane $x = 0$, in the vicinity of the saddle point (small y). The green curve indicates the position of the solid-air interface of the dry substrate.

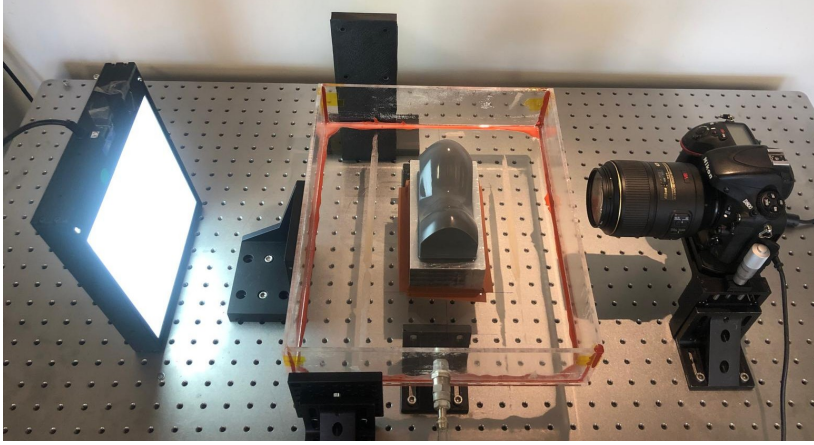


FIG. S3: Image of the experimental set-up.

Movies were recorded by acquiring one image every 5 to 10 s, depending on the oil viscosity and the substrate size.

Thickness extraction. The image of the dry substrate is first converted to grayscale. The solid-air interface is then detected by binarization using a fixed intensity threshold. Specifically, at each horizontal position, the interface is taken as the uppermost detected dark pixel, yielding a discrete curve $y_{\text{sub}}(x)$ which is subsequently interpolated and smoothed. To account for possible misalignment in the measurement plane between the substrate and the imaging axes, the extracted interface is fitted in a rotated coordinate system. Specifically, a rotation angle θ is optimized by minimizing the mean squared deviation between the data and a parabolic fit of the form $y' = ax'^2 + bx' + c$. The optimal parabola is obtained via a least-squares fit in the rotated frame, and the resulting curve is mapped back to the original image coordinates (see Fig. S4). The quadratic coefficient provides an independent estimate of the geometric length scale through $a = 1/(2L)$. Uncertainties in L are evaluated by varying the intensity threshold used for interface detection, leading to variations of up to 5% in the value of L inferred from the procedure. Local normal vectors to the substrate are then computed from the tangent of the fitted parabola in the rotated frame and rotated back to the image frame. The remaining out-of-plane perspective distortions, associated with rotations about the other two axes, were neglected. This is justified by the close proximity of the camera to the substrate and by the small field of view.

In the following, (x_S, y_S) denote coordinates in the image plane. The image horizontal axis corresponds to the physical y -direction, while the vertical axis corresponds to the physical z -direction, within the $x = 0$ plane. Using the local normal vectors, the time-lapse movies are resliced along directions normal to the substrate. At each position (x_S, y_S) sampled uniformly in the vicinity of the parabola minimum, a line is defined along the corresponding unit normal vector $\mathbf{n}(x_S)$. Image intensities are then obtained by bilinear interpolation along trajectories of the form $\mathbf{r}(s) = (x_S, y_S) + s\mathbf{n}(x_S)$, for each frame of the time-lapse sequence. The reslicing procedure transforms the time-lapse movies into spatio-temporal intensity maps, where the horizontal axis corresponds to time and the vertical axis represents image intensity sampled along the local normal direction to the substrate. For each time and position along the substrate, the liquid-air interface is detected as the location of the maximum gradient of intensity along the vertical direction of the resliced image. The solid-liquid interface corresponds to $s = 0$, such that the local film thickness is directly obtained as the distance between the detected interface and the substrate along the normal

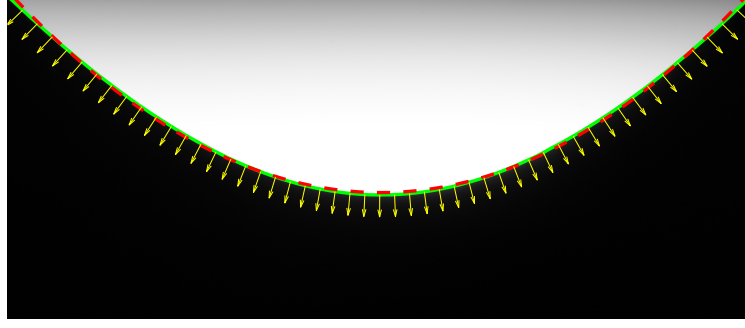


FIG. S4: Image of the dry substrate ($Bo = 102$). The green curve indicates the location of the solid–air interface, obtained by image thresholding. The red dotted line corresponds to the best parabolic fit. The yellow arrows indicate the local surface normal vectors in the $x = 0$ plane.

direction. With this procedure, we experimentally reconstruct the film thickness $\hat{h}(\hat{x} = 0, \hat{y}, \hat{t})$, defined along the local normal direction to the substrate in the $x = 0$ plane, see Figure S5a. Note that measurements are discarded once the thickness $\hat{h}$ becomes comparable to the substrate resolution.

The uncertainty associated with the measured film thickness $\hat{h}$ arises primarily from the calibration factor used to convert pixel distances into physical lengths, as discussed above. An additional contribution stems from the determination of the liquid–air interface position from the maximum of the intensity gradient in the resliced images. Owing to the use of interpolation around the gradient maximum, this localization uncertainty is subpixel. In practice, this contribution remains small compared with the calibration uncertainty for all retained measurements. Accordingly, the total uncertainty on $\hat{h}$ is taken to be dominated by the scale uncertainty.

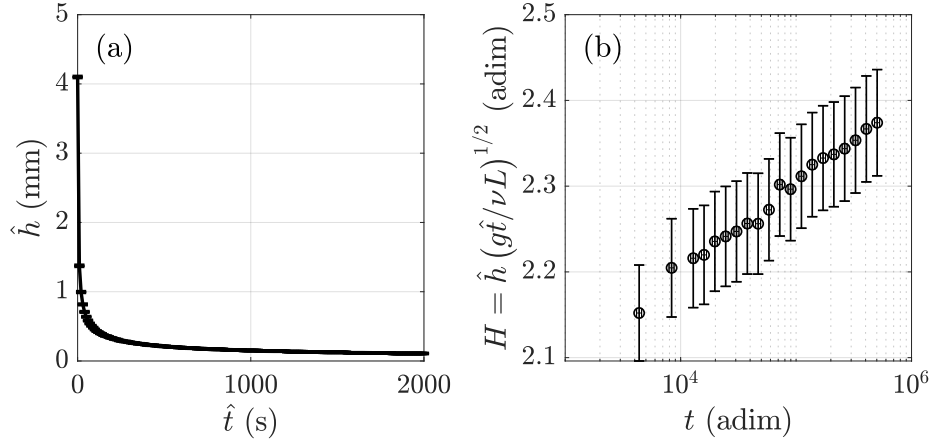


FIG. S5: (a) Liquid film thickness $\hat{h}$ as a function of time $\hat{t}$ measured in the plane $x = 0$ along the local normal to the substrate at the position $y = 0$, for $Bo = 725$ and $\nu = 1000 \text{ mm}^2\text{s}^{-1}$. (b) The corresponding dimensionless quantity $H = \hat{h}(g\hat{t}/\nu L)^{1/2}$ as a function of time.

Comparison with theory. For comparison with the theoretical prediction, we consider the dimensionless quantity

$$H = \hat{h} \left(\frac{g\hat{t}}{\nu L} \right)^{1/2},$$

where $\hat{h}$ is the measured physical thickness and $\hat{t}$ is the physical time.

Since the theoretical prediction concerns the dependence of H on $\log t$, while the time-lapse movies were acquired at constant time intervals, the experimental data are resampled onto a logarithmic time grid. To this end, the variable $\log t$ is divided into N_{bins} equally spaced bins (with $N_{\text{bins}} \sim \sqrt{N}$, where N is the number of data points). Within each bin, the mean value of H is computed, yielding a binned quantity H_{binned} associated with the bin-center time.

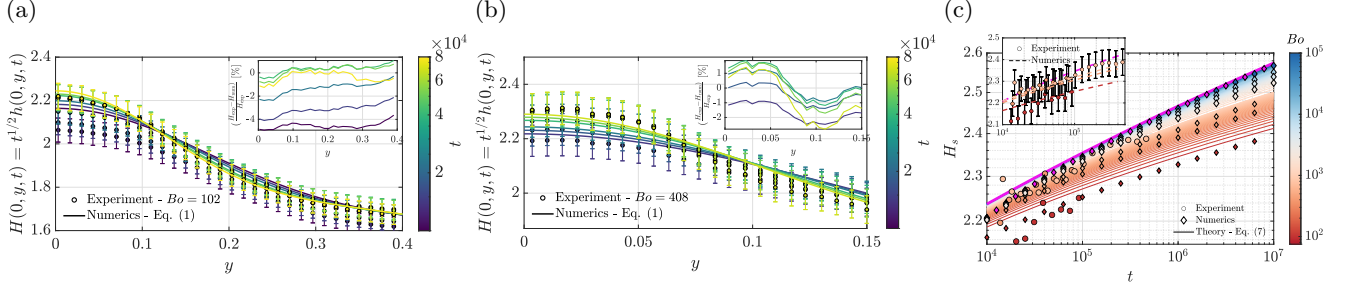


FIG. S6: Additional experiments with the lower-viscosity silicone oil, $\nu = 350 \pm 17 \text{ mm}^2 \text{ s}^{-1}$. (a,b) Rescaled thickness profiles $H = t^{1/2}h$ measured along $x = 0$ for (a) $Bo = 102$ and (b) $Bo = 408$. Symbols denote experimental measurements and solid lines the corresponding numerical solutions; colors indicate increasing nondimensional time. (c) Rescaled saddle thickness $H_s = H(0, 0)$ versus nondimensional time. Diamonds denote numerical solutions with $h_0 = 0.1$ for, from bottom to top, $Bo = 102, 408, 725$, corresponding to the experimental cases, followed by $Bo = 2500, 5000, 10^4$, and 10^5 . Circles denote experimental measurements with $\nu = 0.35 \times 10^{-3} \text{ m}^2 \text{ s}^{-1}$ for $Bo = 102, 408$, and 725 . Colored solid lines are the finite-capillarity saddle-layer predictions, while the magenta solid line is the capillarity-free limit. The inset shows the three experimental datasets with error bars together with the corresponding numerical solutions (dashed lines), while the magenta dashed line denotes the numerical solution without capillarity.

The uncertainty on H arises from three independent contributions. The first is inherited from the uncertainty on the thickness measurement $\hat{h}$, discussed above. Since H depends linearly on $\hat{h}$, this contribution gives a relative uncertainty identical to that of $\hat{h}$, $\frac{\delta H_{\hat{h}}}{H} = \frac{\delta \hat{h}}{\hat{h}}$. The second contribution originates from the uncertainty on the kinematic viscosity ν . Since $H \propto \nu^{-1/2}$, standard error propagation gives $\frac{\delta H_{\nu}}{H} = \frac{1}{2} \frac{\delta \nu}{\nu}$. The third contribution originates from the logarithmic time binning and reflects the statistical dispersion of the data within each bin. It is estimated from the standard error of the mean, $\delta H_{\text{bin}} = \frac{\sigma_H}{\sqrt{N_{\text{bin}}}}$, where σ_H is the standard deviation of the values of H in the bin and N_{bin} is the number of samples in that bin. Assuming these three sources of uncertainty to be independent, the total uncertainty on the binned quantity is obtained from the quadratic sum $\delta H_{\text{tot}} = \sqrt{\delta H_{\hat{h}}^2 + \delta H_{\nu}^2 + \delta H_{\text{bin}}^2}$. The dimensionless time is defined as $t = \frac{gL\hat{t}}{\nu}$, and is therefore also affected by the uncertainty on the viscosity. Since $t \propto \nu^{-1}$, its relative uncertainty is $\frac{\delta t}{t} = \frac{\delta \nu}{\nu}$. The resulting data are represented as H_{binned} versus the logarithmically resampled dimensionless time, with vertical error bars corresponding to δH_{tot} and horizontal error bars corresponding to δt (see Fig. S5b).

ADDITIONAL EXPERIMENTAL DATA

To assess the robustness of the observed drainage mechanism, we report experiments using silicone oil with different kinematic viscosity $\nu = 350 \pm 17 \text{ mm}^2 \text{ s}^{-1}$. Figure S6(a,b) shows the rescaled thickness profiles $H = t^{1/2}h$ measured along $x = 0$, together with the corresponding numerical solutions for $Bo = 102$ and $Bo = 408$. As for the $\nu = 1000 \text{ mm}^2 \text{ s}^{-1}$ experiments, the profiles develop a localized maximum near the saddle while approaching the outer drainage behavior farther away, for large time. Figure S6(c) reports the rescaled saddle thickness $H_s = H(0, 0)$ as in Fig. 3 of the Letter. The same time and Bond-number dependence is observed.

LUBRICATION EQUATION IN GENERALIZED COORDINATES

In this Section, we recall differential geometry concepts [2] to define the lubrication equation on a generic curved substrate, whose derivation is well-established in the literature [3–6].

Differential geometry concepts

We introduce a Cartesian reference frame (x, y, z) . A fluid film of thickness h lies on a substrate of height η . The substrate is identified via a position vector $\mathbf{X}(x^{\{1\}}, x^{\{2\}})$, where $(x^{\{1\}}, x^{\{2\}})$ are the local coordinates used to parameterize the surface. We introduce a local reference frame with two directions lying on the substrate $\mathbf{e}_i = \partial_i \mathbf{X}$ ($i = 1, 2$), not necessarily orthogonal, and the normal-to-the-substrate coordinate vector $\mathbf{e}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}$. We introduce the 2×2 symmetric metric tensor $\mathbb{G}_{ij}$ and the square root of the determinant of the metric tensor w , related to the area element on the surface dA through $dA = w dx^{\{1\}} dx^{\{2\}}$:

$$\mathbb{G}_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \mathbb{G}_{ji}, \quad w = (\det \mathbb{G}_{ij})^{1/2}. \quad (\text{S1})$$

We define the second fundamental form and the curvature tensor, whose components respectively read, following Einstein's notation for the summation:

$$\mathbb{S}_{ij} = \partial_i \mathbf{e}_j \cdot \mathbf{e}_3, \quad \mathbb{K}_i^{\{j\}} = \mathbb{S}_{ik} \mathbb{G}^{\{kj\}}, \quad (\text{S2})$$

where $\mathbb{G}^{\{ij\}}$ is the inverse metric tensor, i.e. $\mathbb{G}^{\{ij\}} = \mathbb{G}_{ij}^{-1}$. The mean $\mathcal{K}$ and the Gaussian $\mathcal{G}$ curvatures read $\mathcal{K} = \text{tr } \mathbb{K}$ and $\mathcal{G} = \det \mathbb{K}$, respectively. A generic vector $\mathbf{f}$ can be written in terms of its covariant and contravariant base, i.e. $\mathbf{f} = f^{\{i\}} \mathbf{e}_i = f_i \mathbf{e}^{\{i\}}$, where $\mathbf{e}^{\{i\}}$ is the covector defined as $\mathbf{e}^{\{i\}} \cdot \mathbf{e}_j = \delta_{ij}$. The two contravariant components, parallel to the substrate, of the gravity vector read $g_t^{\{i\}} = \mathbf{g} \cdot \mathbf{e}^{\{i\}}$, while the normal one reads $g_3 = \mathbf{g} \cdot \mathbf{e}_3$. The gradient of a scalar function f and the divergence of a generic vector $\mathbf{f} = f^{\{i\}} \mathbf{e}_i$ respectively read [2]:

$$\nabla f = \mathbb{G}^{\{ij\}} \partial_j f \mathbf{e}_i = \partial^{\{i\}} f \mathbf{e}_i, \quad \nabla \cdot \mathbf{f} = w^{-1} \partial_i (w f^{\{i\}}). \quad (\text{S3})$$

Complete lubrication model in generalized coordinates

We non-dimensionalize with the characteristic length L of the substrate and with the characteristic time $\tau = \nu/(gL)$. Upon non-dimensionalization, the equation in coordinate-free form reads [3, 4, 7, 8]:

$$\begin{aligned} (1 - \mathcal{K}h + \mathcal{G}h^2) \frac{\partial h}{\partial t} + \frac{1}{3Bo} \nabla \cdot \left[h^3 \left(\nabla \tilde{\kappa} - \frac{1}{2} h (2\mathcal{K}\mathbb{I} - \mathbb{K}) \cdot \nabla \mathcal{K} \right) \right] \\ + \frac{1}{3} \nabla \cdot \left[h^3 \left(\mathbf{g}_t - h \left(\mathcal{K}\mathbb{I} + \frac{1}{2} \mathbb{K} \right) \cdot \mathbf{g}_t + g_3 \nabla h \right) \right] = 0, \end{aligned} \quad (\text{S4})$$

where $\tilde{\kappa} = \mathcal{K} + (\mathcal{K}^2 - 2\mathcal{G})h + \nabla^2 h$ is the free-surface curvature, $Bo = (\rho g L^2)/\gamma$ is the Bond number and $\mathbf{g}_t$ and g_3 are the gravity vector components tangent and normal to the substrate, respectively, normalized by the acceleration of gravity. Equation (S4) is the full model from which Eq. (1) in the main text is obtained by grouping the leading drainage, hydrostatic, and capillary fluxes. The terms in the first and second bracket represent the flux induced by capillary and gravity effects, respectively. At leading order, capillary flow is induced by variations of the mean curvature of the substrate $\mathcal{K}$ and leads to film thinning and thickening in the neighborhood of local maximum and minimum values of the curvature [3]. In addition, we have (i) free-surface curvature variations and (ii) higher order terms of the substrate curvature. Gravity-induced fluxes are instead related to the gravity components tangential to the substrate $\mathbf{g}_t$ and hydrostatic pressure gradients along the thin film. The purely gravity-driven problem is obtained by assuming $Bo \gg 1$ so that terms related to surface tension disappear from the problem.

Monge parameterization of a generic substrate

In the *Monge parameterization*, the substrate position vector is defined as $\mathbf{X} = (x, y, \eta(x, y))$, with $x^{\{1\}} = x, x^{\{2\}} = y$. For ease of notation, we indicate derivatives of substrate-related quantities with subscripts. The tangent and normal vectors to the substrate read:

$$\mathbf{e}_1 = \partial_1 \mathbf{X} = [1, 0, \eta_x], \quad \mathbf{e}_2 = \partial_2 \mathbf{X} = [0, 1, \eta_y], \quad \mathbf{e}_3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|} = \frac{1}{n} [-\eta_x, -\eta_y, 1], \quad (\text{S5})$$

with $n = (1 + \eta_x^2 + \eta_y^2)^{1/2}$. The metric and inverse metric tensors read, respectively:

$$\mathbb{G}_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{pmatrix} 1 + (\eta_x)^2 & \eta_x \eta_y \\ \eta_x \eta_y & 1 + (\eta_y)^2 \end{pmatrix}, \quad \mathbb{G}^{\{ij\}} = \mathbf{e}^{\{i\}} \cdot \mathbf{e}^{\{j\}} = \frac{1}{n^2} \begin{pmatrix} 1 + (\eta_y)^2 & -\eta_x \eta_y \\ -\eta_x \eta_y & 1 + (\eta_x)^2 \end{pmatrix}. \quad (\text{S6})$$

The curvature tensor reads:

$$\mathbb{K}_i^{\{j\}} = \frac{1}{n^3} \begin{pmatrix} \left(1 + (\eta_y)^2\right) \eta_{xx} - \eta_x \eta_y \eta_{xy} & \left(1 + (\eta_x)^2\right) \eta_{xy} - \eta_x \eta_y \eta_{xx} \\ \left(1 + (\eta_y)^2\right) \eta_{xy} - \eta_x \eta_y \eta_{yy} & \left(1 + (\eta_x)^2\right) \eta_{yy} - \eta_x \eta_y \eta_{xy} \end{pmatrix}, \quad (\text{S7})$$

the mean and Gaussian curvature thus read:

$$\mathcal{K} = \frac{1}{n^3} \left(\left(1 + (\eta_y)^2\right) \eta_{xx} - 2\eta_x \eta_y \eta_{xy} + \left(1 + (\eta_x)^2\right) \eta_{yy} \right), \quad \mathcal{G} = \frac{1}{n^4} \left(\eta_{xx} \eta_{yy} - (\eta_{xy})^2 \right). \quad (\text{S8})$$

The gravity vector, which reads $\mathbf{g} = [0, 0, -1]$ in the (x, y, z) coordinate system, is projected on the contravariant base:

$$g_t^{\{1\}} = \mathbf{g} \cdot \mathbf{e}^{\{1\}} = -\eta_x/n^2, \quad g_t^{\{2\}} = \mathbf{g} \cdot \mathbf{e}^{\{2\}} = -\eta_y/n^2, \quad g_3 = \mathbf{g} \cdot \mathbf{e}_3 = -1/n. \quad (\text{S9})$$

These equations close the lubrication model. We considered two different substrates, i.e. (i) the periodic saddle case $\eta = 1 - \cos y - x^2/2$, and (ii) the hyperbolic paraboloid $\eta = y^2/2 - x^2/2$.

NUMERICAL DETAILS

Full model

Numerical simulations of the full model are performed in COMSOL Multiphysics via the *General Form PDE* interface, where the equation is solved in its conservation form. The implementation is based on the definition of the substrate height η in its analytical form, from which all derivatives and differential-geometry quantities are automatically evaluated after discretization. Quadratic Lagrangian elements are exploited for the numerical discretization while the time-marching is performed via the built-in BDF solver, with a relative tolerance of 10^{-5} . Leveraging symmetry, the model is solved for $(x > 0, y > 0)$, imposing zero flux conditions at $x = 0$ and $y = 0, \pi$. The outlet condition is imposed via a sponge that relaxes the thickness to zero. In the periodic saddle case, the sponge extends from $x = 10$ to the outlet boundary at $x = 12$. A convergence study is performed: the model is considered at convergence if further mesh refinement corresponds to a relative variation of the saddle thickness smaller than 0.1%, for fixed and large times. Note that the bottleneck is related to the mesh size at $y = 0$, which requires, for large time, mesh sizes close to the saddle of at most 0.0005, in dimensionless form, to ensure the required convergence.

Drainage problem

Drainage profiles were computed within a Python routine using the method of characteristics on analytically defined surfaces. The equations are solved for $f := H_d^2$. Characteristic curves are initialized from a regular grid of starting points $(\Delta < x_0 < 5, 0 < y_0 < 5)$, with $\Delta = 0.00025$, and integrated forward in the direction of steepest descent using the library *scipy.integrate.solve_ivp* with a Dormand–Prince Runge–Kutta method of order 5(4), with relative and absolute tolerances of 10^{-7} and 10^{-10} , respectively. Integrating backward, each trajectory terminates at the maximum height, here denoted as the pole, where the surface gradient magnitude drops below a fixed threshold (10^{-9}). Along each characteristic, we solve a coupled system of ordinary differential equations for a particular solution $f_p(s)$ and its homogeneous counterpart $f_h(s)$, both initialized with unity. For the saddle $\eta = 1 - \cos y - x^2/2$, the solution on the maxima is known [9] and used to rescale the thickness, which eliminates dependence on arbitrary initial values.

Theoretical saddle solution

The numerical matching procedure of the saddle problem with the drainage solution is performed in MATLAB. We employ the boundary-value problem solver *bvp5c*, with an absolute tolerance of 10^{-8} , with 20000 points, chosen after a convergence analysis. As boundary conditions, we impose zero derivative at $\zeta = 0$. As discussed in the main text, the natural matching with the logarithmic drainage profile for $\zeta \gg 1$ allows for a numerical matching at one point ζ_m . The logarithmic solution becomes singular at $\zeta_b = \exp(C)t^{1/4}$. As discussed in the main text, we impose the second boundary condition at ζ_m , $\mathcal{H}(\zeta_m) = \sqrt{C + \frac{1}{4} \log t - \log(\zeta_m)}$. A variation of the matching point between

$0.5\zeta_b < \zeta_m < 0.9\zeta_b$ leads to a saddle thickness value constant up to four significant digits. For the finite Bond case, the procedure is analogous with, in addition, zero third derivative at $\zeta = 0$ and the matching is performed both for $\mathcal{H}(\zeta_m)$ and $\mathcal{H}'(\zeta_m)$, for the same ζ_m .

DRAINAGE SOLUTION: METHOD OF CHARACTERISTICS

We recast the drainage problem through the Monge parameterization:

$$h_t - \frac{1}{3(1 + \eta_x^2 + \eta_y^2)^{1/2}} \nabla_{xy} \cdot \left[\frac{h^3 \nabla_{xy} \eta}{(1 + \eta_x^2 + \eta_y^2)^{1/2}} \right] = 0, \quad (\text{S10})$$

Upon considering the large-time evolution, at leading order,

$$H_d + \frac{2}{3(1 + \eta_x^2 + \eta_y^2)^{1/2}} \nabla_{xy} \cdot \left[\frac{H_d^3 \nabla_{xy} \eta}{(1 + \eta_x^2 + \eta_y^2)^{1/2}} \right] = 0. \quad (\text{S11})$$

By expanding the divergence, we obtain

$$\nabla_{xy} \eta \cdot \nabla_{xy} H_d^2 + w \left(\frac{2}{3} \mathcal{K} H_d^2 + w \right) = 0, \quad (\text{S12})$$

with the mean curvature $\mathcal{K}$. This equation suggests that the characteristic lines follow the gradient lines of the topography, as the advection of H_d^2 is precisely $\nabla_{xy} \eta$. We solve the problem with the method of characteristics [10]. We introduce the curvilinear coordinate, descending along a gradient line, s such that

$$\begin{aligned} \frac{dx}{ds} &= -\frac{\eta_x}{(\eta_x^2 + \eta_y^2)^{1/2}}, & \frac{dy}{ds} &= -\frac{\eta_y}{(\eta_x^2 + \eta_y^2)^{1/2}}, \\ \frac{dH_d^2}{ds} &= \frac{w}{(\eta_x^2 + \eta_y^2)^{1/2}} \left((2/3) \mathcal{K} H_d^2 + w \right). \end{aligned} \quad (\text{S13})$$

Characteristic (gradient) lines emanate from the stationary points (maxima) of the topography. “Draining” maxima are those of strictly negative mean curvature $\mathcal{K} = \mathcal{K}_p < 0$, where $\mathcal{K}_p$ is the curvature at the maximum of the substrate (in the periodic saddle case, $\mathcal{K}_p = \mathcal{K}(0, y_0 = \pm(2n+1)\pi)$, $n = 0, 1, 2, \dots$), where the thickness has been determined as $H_d^2 = -3/(2\mathcal{K}_p)$ [9].

DRAINAGE SOLUTION APPROACHING THE PRINCIPAL DESCENDING DIRECTION

Approaching $y = 0$, the solution H_d displays a singular behavior of the form

$$H_d(x, y \sim 0) \sim \sqrt{C(x) - B(x) \ln |y|}. \quad (\text{S14})$$

This can be best explained on the symmetry line $x = 0$, where the drainage problem reads

$$\eta_y \frac{dH_d^2}{dy} + w \left(\frac{2}{3} \mathcal{K} H_d^2 + w \right) = 0. \quad (\text{S15})$$

As y goes to 0, approaching the saddle point of zero mean curvature, $\mathcal{K} = 0$, the second term vanishes, while the other two terms with $\eta_y \sim y$ and $w \rightarrow 1$, lead to the singular close-to-the-saddle behavior, as explained in the main text.

Instead, by expanding eq. (S12) for small y ,

$$\eta_x \frac{\partial H_d^2}{\partial x} + \eta_{yy}|_0 y \frac{\partial H_d^2}{\partial y} + \frac{2}{3} \frac{\eta_{xx} + (1 + \eta_x^2) \eta_{yy}|_0}{1 + \eta_x^2} H_d^2 + 1 + \eta_x^2 + \mathcal{O}[y^2] = 0, \quad (\text{S16})$$

and injecting eq. (S14) as an ansatz, two ordinary differential equations for B and C as functions of x may be identified,

$$\eta_x B' + \frac{2}{3} \frac{\eta_{xx} + (1 + \eta_x^2) \eta_{yy}|_0}{1 + \eta_x^2} B = 0, \quad (\text{S17a})$$

$$\eta_x C' - \eta_{yy}|_0 B + \frac{2}{3} \frac{\eta_{xx} + (1 + \eta_x^2) \eta_{yy}|_0}{1 + \eta_x^2} C + 1 + \eta_x^2 = 0, \quad (\text{S17b})$$

with initial conditions for $B(0)$ and $C(0)$. The first equation cancels all terms multiplying $\ln|y|$, while the second one ensures that all other terms, independent of y , vanish. While $B(0)$ is determined directly by the geometry close to the saddle, $C(0)$ is a constant of integration along the inflow characteristic, keeping information of the far field, as explained in the next section. Note that η_x , η_{xx} , and $\eta_{yy}|_0$ are all evaluated at $y = 0$, but remain functions of x .

METHOD FOR DETERMINING THE CONSTANTS IN THE CLOSE-TO-THE-SADDLE BEHAVIOR OF THE DRAINAGE SOLUTION

The purpose of this section is to compute the constant C entering the close-to-the-saddle logarithmic profile of the main text. Let us consider the characteristic line along $0 \leq y \leq y_0 = \pi$, with $x = 0$, that reaches the saddle point at the origin. Along this characteristic, the drainage problem reads

$$\frac{dH_d^2}{dy} = -\frac{w}{\eta_y} \left(\frac{2}{3} \mathcal{K} H_d^2 + w \right), \quad (\text{S18})$$

where all quantities are evaluated along $x = 0$. Introducing $f(y) = H_d^2(0, y)$, the equation can be recast in the linear form $f'(y) + P(y)f(y) + Q(y) = 0$, with $P(y) = \frac{2}{3} \frac{w\mathcal{K}}{\eta_y}$, $Q(y) = \frac{w^2}{\eta_y}$. The general solution reads

$$f(y) = \exp\left(-\int_{y_0}^y P(u) du\right) \left[f_0 - \int_{y_0}^y \exp\left(\int_{y_0}^s P(u) du\right) Q(s) ds \right], \quad (\text{S19})$$

It can be demonstrated that the constant f_0 must vanish, as shown in the next section. We can then rewrite the solution as follows by changing the integral bounds:

$$\begin{aligned} f(y) &= -\exp\left(-\int_{y_0}^y P(u) du\right) \int_{y_0}^y \exp\left(\int_{y_0}^s P(u) du\right) Q(s) ds \\ &= \int_y^{y_0} \exp\left[-\int_{y_0}^y P(u) du + \int_{y_0}^s P(u) du\right] Q(s) ds \\ &= \int_y^{y_0} \exp\left[-\left(\int_0^y P(u) du - \int_0^{y_0} P(u) du\right) + \left(\int_0^s P(u) du - \int_0^{y_0} P(u) du\right)\right] Q(s) ds \\ &= \int_y^{y_0} \exp\left[-\int_0^y P(u) du + \int_0^s P(u) du\right] Q(s) ds \\ &= \exp\left(-\int_0^y P(u) du\right) \int_y^{y_0} \exp\left(\int_0^s P(u) du\right) Q(s) ds. \end{aligned} \quad (\text{S20})$$

Substituting the expressions of f , P , and Q , we obtain

$$H_d^2(0, y) = \exp\left[-\frac{2}{3} \int_0^y \frac{w\mathcal{K}}{\eta_y} du\right] \int_y^{y_0} \exp\left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du\right] \frac{w^2}{\eta_y} ds. \quad (\text{S21})$$

We now isolate the logarithmic singular behavior close to the saddle. Along $x = 0$, for the saddles considered here, $\eta_y \sim s$, $w \rightarrow 1$, $\mathcal{K} \sim 0$, as $s \rightarrow 0$. Moreover, the exponential factor inside the integral tends to one. Therefore,

$$\exp\left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du\right] \frac{w^2}{\eta_y} \sim \frac{1}{s}, \quad s \rightarrow 0. \quad (\text{S22})$$

This $1/s$ term is the only source of the logarithmic divergence. We therefore add and subtract it inside the integral:

$$\begin{aligned} H_d^2(0, y) &= \exp\left[-\frac{2}{3} \int_0^y \frac{w\mathcal{K}}{\eta_y} du\right] \left[\int_y^{y_0} \left\{ \exp\left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du\right] \frac{w^2}{\eta_y} - \frac{1}{s} \right\} ds + \int_y^{y_0} \frac{1}{s} ds \right] \\ &= \exp\left[-\frac{2}{3} \int_0^y \frac{w\mathcal{K}}{\eta_y} du\right] \left[\int_y^{y_0} \left\{ \exp\left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du\right] \frac{w^2}{\eta_y} - \frac{1}{s} \right\} ds + \ln\left|\frac{y_0}{y}\right| \right]. \end{aligned} \quad (\text{S23})$$

The expression inside braces is now regular at $s = 0$, because the singular $1/s$ contribution has been removed. Therefore, in the limit $y \rightarrow 0$, the lower bound of this regularized integral can be replaced by 0. Moreover, the exponential prefactor tends to one. Hence,

$$H_d^2(0, y \sim 0) = \int_0^{y_0} \left\{ \exp \left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du \right] \frac{w^2}{\eta_y} - \frac{1}{s} \right\} ds + \ln |y_0| - \ln |y|. \quad (\text{S24})$$

We identify all finite terms independent of y with the constant C , namely

$$C = \int_0^{y_0} \left\{ \exp \left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du \right] \frac{w^2}{\eta_y} - \frac{1}{s} \right\} ds + \ln |y_0|. \quad (\text{S25})$$

Thus, close to the saddle, $H_d^2(0, y \sim 0) = C - \ln |y|$. The expression above can be directly evaluated if $y = y_0$ corresponds to a pinned contact line. If y_0 is instead a draining maximum (pole), the integral inside the exponential could be singular as $s \rightarrow y_0$ since $\eta_y \sim \eta_{yy}|_{y_0}(s - y_0)$. We now verify that it is bounded instead. We recall the curvature at the draining maximum, $\mathcal{K}_p = \mathcal{K}(0, y_0)$. Near $s = y_0$, one has

$$\frac{w^2}{\eta_y} \sim \frac{1}{\eta_{yy}|_{y_0}(s - y_0)}. \quad (\text{S26})$$

We verify that this singularity is resolved through the exponential term in the nested integral. Inside the u -integral, the singular behavior as $u \rightarrow y_0$ is

$$\frac{w\mathcal{K}}{\eta_y} \sim \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)}. \quad (\text{S27})$$

We therefore add and subtract this singular part inside this integral:

$$\exp \left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du \right] = \exp \left[\frac{2}{3} \int_0^s \left(\frac{w\mathcal{K}}{\eta_y} - \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} + \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} \right) du \right]. \quad (\text{S28})$$

We then separate the two contributions in the exponential:

$$\exp \left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du \right] = \exp \left[\frac{2}{3} \int_0^s \left(\frac{w\mathcal{K}}{\eta_y} - \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} \right) du \right] \exp \left[\frac{2}{3} \int_0^s \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} du \right].$$

The second exponential can be evaluated explicitly:

$$\exp \left[\frac{2}{3} \int_0^s \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} du \right] = \exp \left[\frac{2\mathcal{K}_p}{3\eta_{yy}|_{y_0}} \int_0^s \frac{du}{u - y_0} \right] = \exp \left[\frac{2\mathcal{K}_p}{3\eta_{yy}|_{y_0}} \ln \left(\frac{y_0 - s}{y_0} \right) \right] = \left(\frac{y_0 - s}{y_0} \right)^{2\mathcal{K}_p/(3\eta_{yy}|_{y_0})}.$$

Therefore,

$$\exp \left[\frac{2}{3} \int_0^s \frac{w\mathcal{K}}{\eta_y} du \right] = \exp \left[\frac{2}{3} \int_0^s \left(\frac{w\mathcal{K}}{\eta_y} - \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} \right) du \right] \left(\frac{y_0 - s}{y_0} \right)^{2\mathcal{K}_p/(3\eta_{yy}|_{y_0})}. \quad (\text{S29})$$

Since both $\mathcal{K}_p$ and $\eta_{yy}|_{y_0}$ are negative at a draining maximum, the exponent $\frac{2\mathcal{K}_p}{3\eta_{yy}|_{y_0}}$ is positive. This factor regularizes the behavior of the integrand in the dummy variable s at $s = y_0$ and allows for explicit numerical integration. In the draining-pole case, the constant C is thus evaluated using the regularized expression

$$C = \int_0^{y_0} \left\{ \exp \left[\frac{2}{3} \int_0^s \left\{ \frac{w\mathcal{K}}{\eta_y} - \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(u - y_0)} \right\} du \right] \left(\frac{y_0 - s}{y_0} \right)^{2\mathcal{K}_p/(3\eta_{yy}|_{y_0})} \frac{w^2}{\eta_y} - \frac{1}{s} \right\} ds + \ln |y_0|. \quad (\text{S30})$$

For the periodic saddle topography from the main text, numerical evaluation of this expression gives $C = 1.7949 \dots$.

Constant of integration f_0

We now show that regularity at a draining maximum (pole) imposes the vanishing of the homogeneous integration constant. Let the draining pole be located at $x = 0$, $y = y_0$, and define

$$\beta = \frac{2\mathcal{K}_p}{3\eta_{yy}|_{y_0}}. \quad (\text{S31})$$

Near the draining pole,

$$\eta_y \sim \eta_{yy}|_{y_0}(y - y_0), \quad \frac{w\mathcal{K}}{\eta_y} \sim \frac{\mathcal{K}_p}{\eta_{yy}|_{y_0}(y - y_0)}. \quad (\text{S32})$$

The term proportional to the integration constant is

$$f_h(y) = f_0 \exp\left(-\int_0^y P(u) du\right). \quad (\text{S33})$$

Near the draining maximum,

$$P(y) \sim \frac{\beta}{y - y_0}, \quad \beta = \frac{2\mathcal{K}_p}{3\eta_{yy}|_{y_0}} > 0. \quad (\text{S34})$$

Hence

$$f_h(y) \sim f_0 \left(\frac{y_0 - y}{y_0}\right)^{-\beta}. \quad (\text{S35})$$

This term diverges as $y \rightarrow y_0$ unless $f_0 = 0$. Therefore regularity at the draining maximum imposes $f_0 = 0$.

SYSTEMATIC METRIC EXPANSION CLOSE TO THE SADDLE - PERIODIC SADDLE CASE

We start from the full model with negligible capillary effects:

$$(1 - \mathcal{K}h + \mathcal{G}h^2) \frac{\partial h}{\partial t} + \frac{1}{3} \nabla \cdot \left[h^3 \left(\mathbf{g}_t - h \left(\mathcal{K}\mathbb{I} + \frac{1}{2}\mathbb{K} \right) \cdot \mathbf{g}_t + \mathbf{g}_3 \nabla h \right) \right] = 0 \quad (\text{S36})$$

and introduce a small parameter $\delta = t^{-1/2} \ll 1$ such that $h = t^{-1/2} \mathcal{H} = \delta \mathcal{H}$:

$$-\delta^3(1 - \delta\mathcal{K}\mathcal{H} + \delta^2\mathcal{G}\mathcal{H}^2) \frac{\mathcal{H}}{2} + \frac{\delta^3}{3} \nabla \cdot \left[\mathcal{H}^3 \left(\mathbf{g}_t - \delta\mathcal{H} \left(\mathcal{K}\mathbb{I} + \frac{1}{2}\mathbb{K} \right) \cdot \mathbf{g}_t + \delta\mathbf{g}_3 \nabla \mathcal{H} \right) \right] = 0. \quad (\text{S37})$$

By simplifying the common term δ^3 , at $\mathcal{O}(1)$ one obtains:

$$-\mathcal{H}/2 + \frac{1}{3} \nabla \cdot [\mathcal{H}^3 \mathbf{g}_t] = 0, \quad (\text{S38})$$

i.e., the drainage problem that describes the solution far from the saddle. As shown in the main text, this equation is singular at the saddle, due to the absence of hydrostatic effects.

The problem close to the considered saddle can be obtained as follows. We develop the metric quantities with a Taylor series around $(x = 0, y = 0)$ for $\eta(x, y) = 1 - \cos y = x^2/2$:

$$\begin{aligned} w &= 1 + \frac{x^2 + y^2}{2} + O(r^4), & \mathbf{g}_t &= x\mathbf{e}_x - y\mathbf{e}_y + O(r^3), & g_3 &= -1 + O(r^2), & \mathcal{G} &= -1 + 2x^2 + \frac{5}{2}y^2 + O(r^4) \\ \mathcal{K} &= x^2 - \frac{3}{2}y^2 + O(r^4), & \partial_x \mathcal{K} &= 2x + O(r^3), & \partial_y \mathcal{K} &= -3y + O(r^3), & \partial_x \mathcal{G} &= 4x + O(r^3), & \partial_y \mathcal{G} &= 5y + O(r^3), \\ \mathbb{G}^{\{ij\}} &= \mathbb{I} + \begin{pmatrix} -x^2 & xy \\ xy & -y^2 \end{pmatrix} + O(r^4), & \mathbb{K} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + O(r^2), \end{aligned} \quad (\text{S39})$$

where $r^2 = x^2 + y^2$. We introduce $\nabla = \nabla_{xy} := [\partial_x, \partial_y]$. Upon substitution, one obtains, up to $\mathcal{O}(r)$ in the divergence:

$$-\frac{\mathcal{H}}{2} + \nabla_{xy} \cdot \left[\frac{\mathcal{H}^3}{3} \left(x\mathbf{e}_x - y\mathbf{e}_y + \delta\mathcal{H} \left(\frac{x}{2}\mathbf{e}_x + \frac{y}{2}\mathbf{e}_y \right) - \delta\nabla_{xy}\mathcal{H} \right) \right] = 0. \quad (\text{S40})$$

This is a local Taylor expansion near the saddle. The term $-\delta\nabla_{xy}\mathcal{H}$ is the classical hydrostatic term. The term $\delta\mathcal{H}(x\mathbf{e}_x + y\mathbf{e}_y)/2$ is the leading local curvature-induced hydrostatic correction.

To obtain the equation, we “stretch” along the y direction, consistently with the observed divergence for $y \rightarrow 0$ at $x = 0$, i.e. $y = \delta^\alpha \zeta$. Under the stretching, the transverse drainage term scales as $\mathcal{O}(\delta^\alpha)$, the classical hydrostatic term scales as $\mathcal{O}(\delta^{1-\alpha})$, and the curvature-induced hydrostatic correction scales as $\mathcal{O}(\delta^{1+\alpha})$:

$$\mathcal{H}^3 y \sim \mathcal{O}(\delta^\alpha), \quad \delta\mathcal{H}^3 \partial_y \mathcal{H} \sim \mathcal{O}(\delta^{1-\alpha}), \quad \delta\mathcal{H}^4 y \sim \mathcal{O}(\delta^{1+\alpha}).$$

We obtain

$$-\mathcal{H}/2 + \frac{1}{3} \left[\mathcal{H}^3 \left(x + \delta\mathcal{H} \frac{x}{2} - \delta\partial_x \mathcal{H} \right) \right]_x + \frac{1}{3\delta^\alpha} \left[\mathcal{H}^3 \left(-\delta^\alpha \zeta + \delta^{1+\alpha} \mathcal{H} \frac{\zeta}{2} - \delta^{1-\alpha} \partial_\zeta \mathcal{H} \right) \right]_\zeta = 0. \quad (\text{S41})$$

The asymptotically relevant dominant balance is obtained by balancing the drainage $\delta^\alpha \zeta$ and hydrostatic $\delta^{1-\alpha} \mathcal{H} \zeta$ terms along the y direction, leading to $\delta^\alpha \sim \delta^{1-\alpha} \rightarrow \alpha = 1/2$, that gives $\mathcal{O}(1)$ terms after simplification with term $\delta^{-\alpha}$ outside of the divergence. Eventually, we obtain the following equation, at $\mathcal{O}(1)$:

$$\partial_\zeta \mathcal{H} \mathcal{H}^3 - 3x\mathcal{H}^2 \partial_x \mathcal{H} + 3\mathcal{H}^2 \partial_\zeta \mathcal{H} (\partial_\zeta \mathcal{H} + \zeta) + \frac{3\mathcal{H}}{2} = 0. \quad (\text{S42})$$

Although the width is fixed by the local balance, the saddle problem is not fully self-similar because its ansatz satisfies matching with a solution that retains the explicit dependence on the spatially varying logarithmic profile. We note that the problem after the close-to-the-saddle Taylor expansion and dominant balance coincides with the one obtained by a priori assuming a flat substrate, namely neglecting all metric and curvature effects apart from tangential gravity, as done in the main text. Since the equation is of first order along x , we restrict our PDE to the line $x = 0$, which is also a symmetry plane for $\mathcal{H}$, leading to the final ODE:

$$\partial_\zeta \mathcal{H} \mathcal{H}^3 + 3\mathcal{H}^2 \partial_\zeta \mathcal{H} (\partial_\zeta \mathcal{H} + \zeta) + \frac{3\mathcal{H}}{2} = 0, \quad (\text{S43})$$

subject to the symmetry condition $\mathcal{H}'(0) = 0$ and to the matching with the drainage profile, as detailed in the main.

Capillary effects

For the periodic saddle, i.e., the substrate of elevation $\eta(x, y) = 1 - \cos(y) - x^2/2$, it is observed that the presence of capillary effects (i.e., a finite Bond number $Bo = \rho g L^2 / \gamma$) lets the theoretical profiles deviate from the hydrostatic ridge. To rationalize this, we consider the full model with capillary effects,

$$\begin{aligned} (1 - \mathcal{K}h + \mathcal{G}h^2) \frac{\partial h}{\partial t} + \frac{1}{3Bo} \nabla \cdot \left[h^3 \left(\nabla \tilde{\kappa} - \frac{1}{2} h (2\mathcal{K}\mathbb{I} - \mathbb{K}) \cdot \nabla \mathcal{K} \right) \right] \\ + \frac{1}{3} \nabla \cdot \left[h^3 \left(\mathbf{g}_t - h \left(\mathcal{K}\mathbb{I} + \frac{1}{2} \mathbb{K} \right) \cdot \mathbf{g}_t + \mathbf{g}_3 \nabla h \right) \right] = 0, \end{aligned} \quad (\text{S44})$$

and expand it close to the saddle, where also

$$\tilde{\kappa} = \mathcal{K} + (\mathcal{K}^2 - 2\mathcal{G})h + \nabla^2 h = \left(x^2 - \frac{3}{2}y^2 \right) + (2 - 4x^2 - 5y^2)h + \nabla_{xy}^2 h + \mathcal{O}(r^2),$$

As before, we assume $h = \delta\mathcal{H} = t^{-1/2}\mathcal{H} + o(t^{-1/2})$ and, at $\mathcal{O}(r)$ in the divergence terms, we have (see also next Section):

$$\begin{aligned} -\frac{\mathcal{H}}{2} + \frac{1}{Bo} \nabla_{xy} \cdot \left(\frac{\mathcal{H}^3}{3} \left(2x\mathbf{e}_x - 3y\mathbf{e}_y + \nabla_{xy} \left((2 - 4x^2 - 5y^2)\delta\mathcal{H} + \delta\nabla_{xy}^2 \mathcal{H} \right) + \frac{\delta}{2} \mathcal{H} (-\mathcal{K}_x \mathbf{e}_x + \mathcal{K}_y \mathbf{e}_y) \right) \right) \\ + \nabla_{xy} \cdot \left(\frac{\mathcal{H}^3}{3} \left(x\mathbf{e}_x - y\mathbf{e}_y + \delta\mathcal{H} \left(\frac{x}{2}\mathbf{e}_x + \frac{y}{2}\mathbf{e}_y \right) - \delta\nabla_{xy}\mathcal{H} \right) \right) = 0. \end{aligned} \quad (\text{S45})$$

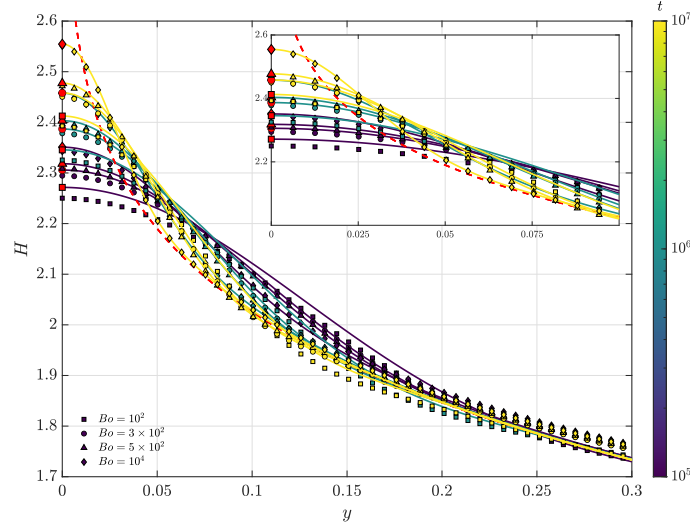


FIG. S7: Numerical profiles of $\mathcal{H} = t^{1/2}h$, with initial condition $h_0 = 0.1$, obtained from the full model close to the saddle for varying Bond number (symbols), increasing time (colormap) and comparison with the theoretical result for varying Bond (lines). The zoom in the inset helps distinguish the curves and the agreement across diverse times and Bond numbers. As the Bond number increases, the ridge maximum height increases.

The balance for the stretched variable $\zeta = y/\delta^\alpha$ now depends on the order of magnitude of the Bond number. If $Bo \gg t^{1/2} = \delta^{-1}$, the term disappears from the previous equation, giving the previously derived pure hydrostatic problem. Capillary effects become more important as time increases since the saddle layer narrows, increasing the order of magnitude of the high derivative terms related to the film curvature. For large-enough time, capillary and hydrostatic term are of the same order of magnitude.

As discussed in the main, $Bo = \mathcal{O}(t^{1/2}) = \mathcal{O}(\delta^{-1}) := Bo^* t^{1/2}$. Therefore, the Bond number now takes into account the hydrostatic thinning with a dynamic length scale $\hat{\ell}_b/L \sim t^{-1/4}$, i.e. $Bo = (\hat{\ell}_b^2/\ell_c^2)t^{1/2}$, and $Bo^* = \hat{\ell}_b^2/\ell_c^2$ is the Bond number referring to this dynamic length scale. Under the combined scaling

$$y = \delta^{1/2}\zeta, \quad Bo^{-1} = \frac{\delta}{Bo^*},$$

the different capillary contributions in Eq. (S45) scale, after the divergence, as

$$\begin{aligned} \frac{1}{Bo} \nabla \cdot (\mathcal{H}^3 \nabla_{xy} \mathcal{K}) &= \mathcal{O}(\delta), & \frac{1}{Bo} \nabla \cdot (\mathcal{H}^3 2\delta \nabla_{xy} \mathcal{H}) &= \mathcal{O}(\delta), \\ \frac{1}{Bo} \nabla \cdot (\mathcal{H}^3 \delta \nabla_{xy} \nabla_{xy}^2 \mathcal{H}) &= \mathcal{O}(1), & \frac{1}{Bo} \nabla \cdot [\delta \mathcal{H}^4 (2\mathcal{K}I - \mathbb{K}) \cdot \nabla_{xy} \mathcal{K}] &= \mathcal{O}(\delta^2). \end{aligned}$$

As a consequence, the scaling is the same as before, i.e. $\zeta = yt^{1/4}$ and, at $\mathcal{O}(1)$, only the highest derivatives of capillarity balance with hydrostatics and drainage:

$$-\frac{\mathcal{H}}{2} + \frac{1}{Bo^*} \partial_\zeta \left(\frac{\mathcal{H}^3}{3} (\partial_\zeta^3 \mathcal{H}) \right) + \partial_x \left(\frac{\mathcal{H}^3}{3} x \right) + \partial_\zeta \left(\frac{\mathcal{H}^3}{3} (-\zeta - \partial_\zeta \mathcal{H}) \right) = 0, \quad (\text{S46})$$

i.e., along $x = 0$,

$$\mathcal{H}'' \mathcal{H}^2 - \frac{1}{Bo^*} (3\mathcal{H} \mathcal{H}' \mathcal{H}'''' + \mathcal{H}^2 \mathcal{H}^{\text{IV}}) + 3\mathcal{H} \mathcal{H}' (\mathcal{H}' + \zeta) + \frac{3}{2} = 0, \quad (\text{S47})$$

subject to the symmetry conditions $\mathcal{H}'(0) = \mathcal{H}'''(0) = 0$ and to the same matching as the pure hydrostatic problem, for the value and first derivative of the logarithmic drainage profile. Consequently, the leading problem is again equivalent to the equation obtained by assuming a flat substrate from the beginning, and hence is generalizable to any substrate presenting a saddle of hyperbolic form, i.e. $\eta \sim y^2 - x^2$. A comparison between the solution of problem Eq. (S47) with numerical simulation profiles is reported in Fig. S7, with a good agreement.

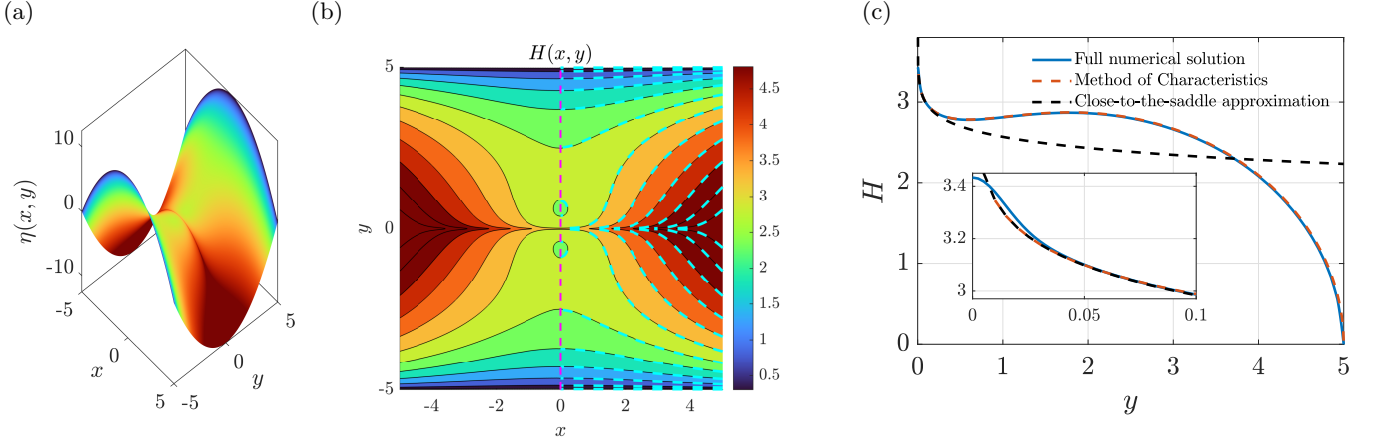


FIG. S8: Hyperbolic paraboloid problem with a pinned contact line at $y = 5$. (a,b) Numerical results ($Bo \rightarrow \infty$) for $h_0 := h(x, y, 0) = 0.1$ and $t = 10^6$: (a) Colormaps of $H(x, y)$ on a surface plot of the substrate height. (b) Colormaps and isocontours (black) of $H(x, y)$. Cyan dashed lines: same isocontours for the drainage solution, reported only for $x > 0$ because of symmetry with respect to $x = 0$ (magenta dashed line). (c) Thickness H as a function of y , for $x = 0$ for $t = 10^8$. The orange dashed line is the $\delta = 0$ solution and the black dash-dotted line its asymptotic behavior close to the saddle. In the inset, a zoom close to the saddle.

Local expansion of the Laplace–Beltrami operator at the saddle

We here clarify how the Laplacian term does not produce any additional term more than the Cartesian form, before the dominant balance. The Laplace–Beltrami operator acting on a scalar field $f(x, y)$ reads

$$\nabla^2 f = \frac{1}{w} \partial_i \left(w G^{\{ij\}} \partial_j f \right),$$

We now substitute the Taylor expansion into the Laplace–Beltrami operator:

$$\nabla^2 f = \frac{1}{w} \left[\partial_x \left(\left(1 - \frac{x^2}{2} + \frac{y^2}{2} \right) \partial_x f + xy \partial_y f \right) + \partial_y \left(xy \partial_x f + \left(1 + \frac{x^2}{2} - \frac{y^2}{2} \right) \partial_y f \right) \right] + O(r^2),$$

Expanding the derivatives yields

$$\nabla^2 f = \frac{1}{w} \left[\left(1 - \frac{x^2}{2} + \frac{y^2}{2} \right) \partial_{xx} f - x \partial_x f + y \partial_y f + 2xy \partial_{xy} f + x \partial_x f + \left(1 + \frac{x^2}{2} - \frac{y^2}{2} \right) \partial_{yy} f - y \partial_y f \right] + O(r^2).$$

Since the linear first-derivative terms cancel exactly, multiplication by $w^{-1} = 1 + O(r^2)$ does not generate linear terms and we can neglect all other quadratic terms. Therefore, the Laplace–Beltrami operator coincides with the Cartesian Laplacian up to linear order near the saddle. In conclusion, no terms of order $O(x)$ or $O(y)$ stem from metric, in the highest-derivative capillary contribution.

THE HYPERBOLIC PARABOLOID PROBLEM

Characteristic lines are such that $xy = \text{const.} := A$. The drainage problem thus becomes

$$\frac{dH_d^2}{dy} = -\frac{2}{3} \frac{A^2 - y^4}{y(A^2 + y^2 + y^4)} H_d^2 - \frac{A^2 + y^2 + y^4}{y^3} \quad (\text{S48})$$

In our case, we impose $H_d(x_0, y_0) = 0$, with $y_0 = 5$ and $x_0 = A/y_0$ with varying A , delimiting the wetted area. This problem is solved in MATLAB via the routine `ode45`, with a relative and absolute tolerance of 10^{-7} and 10^{-10} , respectively. Eq. (S48) can be explicitly integrated for $x = 0$, which implies $A = 0$,

$$H_d^2(x = 0, y) = (1 + y^2)^{1/3} \int_y^{y_0} \frac{(1 + y'^2)^{2/3}}{y'} dy', \quad (\text{S49})$$

which leads to $H_d(x=0, y \rightarrow 0) = \sqrt{C - \ln(y)}$ with

$$C = \ln|y_0| + \int_0^{y_0} \frac{(1+y^2)^{2/3} - 1}{y} dy = 6.5984 \dots \text{ for } y_0 = 5,$$

We compare this drainage solution with the full model with $Bo \rightarrow \infty$, which is obtained by changing the substrate expression in our formulation and by imposing a zero thickness boundary condition (pinned contact line) at $y = 5$, with a sponge method to relax the thickness to zero beyond $x = 6$, while keeping tolerances and mesh the same as the periodic saddle case. As shown in Fig. S8, the same ridge along $y = 0$ as in the periodic saddle case is observed, with the full numerical solution that agrees well with the drainage solution. The same structure is observed for this substrate, confirming that the local saddle topology drives the ridge formation rather than the whole substrate topology.

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