

Hidden Permutation Symmetry of Squared Amplitudes in Aharony-Bergman-Jafferis-Maldacena Theory

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We define the “squared amplitudes” in planar Aharony-Bergman-Jafferis-Maldacena theory, analogous to those in $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory. Surprisingly, the n -point L -loop integrands with fixed $N := n + L$ are unified in a single generating function. Similar to the SYM four-point half-Bogomol’nyi-Prasad-Sommerfield correlator integrand, the generating function enjoys a hidden S_N permutation symmetry in the dual space, allowing us to write it as a linear combination of weight-3 planar f graphs. Remarkably, through Gram identities it can also be represented as a linear combination of “bipartite” f graphs that manifest the important property that no odd-multiplicity amplitude exists in the theory. The generating function and these properties are explicitly checked against squared amplitudes for all n with $N = 4, 6, 8$. By drawing analogies with SYM, we conjecture some “graphical rules” the generating function satisfies, and exploit them to bootstrap a unique $N = 10$ result, which provides new results for $n = 10$ squared tree amplitudes, as well as integrands for $(n, L) = (4, 6), (6, 4)$. Our results strongly suggest the existence of a “bipartite correlator” in Aharony-Bergman-Jafferis-Maldacena theory that unifies all squared amplitudes and satisfies physical constraints underlying these graphical rules.

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Introduction—Planar $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory with gauge group $SU(N_c)$ is a rare example of a four-dimensional integrable theory, enabling the nonperturbative study of conformal data and many off-shell observables [1–5]. On the other hand, as the supersymmetric cousin of the phenomenologically interesting QCD, scattering amplitudes [6] in SYM represent another class of important observables. For amplitudes, integrability is reflected in the dual superconformal and Yangian symmetry [7–11]. Although infrared divergences [12–15] obscure dual conformal invariance (DCI), the loop “integrand” (well-defined in the planar limit) manifests the Yangian symmetry [16]. Physically, dual superconformal symmetry of amplitudes is understood as ordinary superconformal symmetry of null polygonal Wilson loops through the amplitude-Wilson loop-correlator triality [17].

In particular, it is possible to introduce the “squared amplitude,” a bosonic object simpler than the (super) amplitude itself, which is believed to encode all information of the amplitude [18,19]. Moreover, the squared amplitude and the analogous squared form factor [20] are key ingredients in perturbative calculations such as those for energy correlators [21]. At the integrand level, the triality relates the squared amplitude to the adjoint Wilson loop and the lightlike limit of the four-point half-Bogomol’nyi-Prasad-Sommerfield correlator. The method of Lagrangian insertion [22] reveals a hidden permutation symmetry [23] of the correlator integrand, which enables the efficient bootstrap [24–27] of the correlator integrand (and hence the squared amplitude) using the so-called f graphs.

The Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [28] is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons (CS) matter theory, originally introduced to study the dynamics of M2-branes. It was soon realized that the theory is very similar to SYM in the sense that it is also integrable in the planar limit [29]. In particular, the (integrands of) scattering amplitudes also enjoy dual superconformal and Yangian symmetry [30–36]. However, the physical origin of DCI is not as clear as in SYM. Although there are some hints of duality to the Wilson loop at four points [37–39], attempting to identify a dual “super Wilson loop” [necessary at higher points due to a lack of bosonic maximally helicity violating (MHV) sectors in the theory]

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is met with difficulties. The most natural candidate super Wilson loop at weak coupling fails to be dual to the superamplitude [40,41], and the nonexistence of self-T-duality at strong coupling deprives us of string-informed guesses [42,43]. Also, explicit results of ABJM amplitudes are not as satisfying as in SYM. Using generalized unitarity [44] and geometry [45,46], it is possible to obtain the two-loop eight-point integrand [47–51] and the five-loop four-point integrand [52].

In this Letter, we take a major step toward completing the parallel story between ABJM and SYM by providing evidence of a duality between the squared amplitude and some “correlator,” sidestepping the super Wilson loop corner in the triality. Specifically, by carefully defining the (integrand of) squared amplitude, we observe that it can be written as the lightlike limit of a “generating function” that also enjoys hidden permutation symmetry. Moreover, the f -graph representation is manifestly “bipartite, echoing previous studies of the (logarithm of) amplitude [51,52]. This strongly suggests the interpretation of the generating function as some “bipartite correlator with Lagrangian insertions.” Based on the physical picture and drawing analogies from SYM, we conjecture some graphical rules the generating function should satisfy. Together with the bipartite property, this enables us to bootstrap the generating function at ten points, obtaining new results for tree-level ten-point, two-loop eight-point, and four-loop six-point squared amplitudes, as well as the six-loop four-point (unsquared) amplitude.

Squared amplitudes in SYM—Let us briefly review the SYM story. In chiral on-shell superspace [16,53], Poincaré supersymmetry implies that the superamplitude $\mathcal{A}_n(\lambda, \tilde{\lambda}, \tilde{\eta}) = \delta^4(P)\delta^8(Q)A_n(\lambda, \tilde{\lambda}, \tilde{\eta})$ satisfies $\tilde{Q}A_n = 0$, where $(\partial_{iA} := \partial_{\tilde{\eta}_i^A})$

$$P^{\alpha\dot{\alpha}} = \sum_{i=1}^n \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}, \quad Q^{\alpha A} = \sum_{i=1}^n \lambda_i^\alpha \tilde{\eta}_i^A, \quad \tilde{Q}^{\dot{\alpha}}_A = \sum_{i=1}^n \tilde{\lambda}_i^{\dot{\alpha}} \partial_{iA}. \quad (1)$$

Parity maps the superamplitude $\mathcal{A}_n(\lambda, \tilde{\lambda}, \tilde{\eta})$ to a differential operator $\tilde{\mathcal{A}}_n(\tilde{\lambda}, \lambda, \partial_{\tilde{\eta}}) = \delta^4(P)\delta^8(Q)\tilde{A}_n(\tilde{\lambda}, \lambda, \partial_{\tilde{\eta}})$ on chiral superspace. The squared amplitude is then defined on the support of $\delta(P)$ as

$$M_n = \frac{1}{2} \tilde{A}_n(\tilde{\lambda}, \lambda, \partial_{\tilde{\eta}}) A_n(\lambda, \tilde{\lambda}, \tilde{\eta}) \Big|_{\tilde{\eta}, \partial_{\tilde{\eta}}=0}. \quad (2)$$

This can be computed by writing $A_n = \sum f(\lambda, \tilde{\lambda}) \prod \tilde{\eta}$ as a polynomial in $\tilde{\eta}$. In $\tilde{A}_n A_n$, all cross terms vanish upon setting $\tilde{\eta}, \partial_{\tilde{\eta}} = 0$, and $M_n = \frac{1}{2} \sum f(\tilde{\lambda}, \lambda) f(\lambda, \tilde{\lambda})$ is simply the sum of the squared coefficients, which is parity-even and hence a function of Mandelstam variables only.

Note that it is necessary to strip off $\delta(P)\delta(Q)$ in (2) because $\tilde{\mathcal{A}}_n A_n \propto \tilde{Q}A_n = 0$ trivially. Moreover, (2) is well-defined in the sense that M_n is invariant under shifting $A_n \mapsto A'_n = A_n + QX$ keeping $\tilde{Q}A'_n = 0$, because the would-be cross terms vanish due to $\tilde{Q}A'_n = 0$.

In the planar limit $N_c \rightarrow \infty$ with fixed 't Hooft coupling $a = g_{\text{YM}}^2 N_c$, it is possible to define a loop integrand $A_n^{(L)}(\lambda, \tilde{\lambda}, \tilde{\eta}; y_1, \dots, y_L)$ where the loop momenta are specified by the dual variables $y_\ell \in \mathbb{R}^4$,

$$A_n = \sum_{L=0}^{\infty} a^L \int d^{4L} y A_n^{(L)}, \quad (3)$$

where $A_n^{(L)}$ is defined to be symmetric under permuting $y_1, \dots, y_L$ because the integration region is symmetric. The integrand $M_n^{(L)}$ of squared amplitudes is given by

$$M_n^{(L)} = \frac{1}{2} \sum_{\ell=0}^L \frac{1}{L!} \sum_{\sigma \in S_L} A_n^{(\ell)}(\tilde{\lambda}, \lambda, \partial_{\tilde{\eta}}; y_{\sigma(1)}, \dots, y_{\sigma(\ell)}) \times A_n^{(L-\ell)}(\lambda, \tilde{\lambda}, \tilde{\eta}; y_{\sigma(\ell+1)}, \dots, y_{(L)}) \Big|_{\tilde{\eta}, \partial_{\tilde{\eta}}=0}. \quad (4)$$

Introducing dual points $x_i \in \mathbb{R}^4$ such that $x_i - x_{i+1} = \lambda_i \tilde{\lambda}_i$, we can rewrite $M_n^{(L)}(x_i, y_\ell)$ as a rational function of the dual points. Remarkably, there exists a single object $F_N(x_1, \dots, x_N)$ with $N := n + L$ that packages $M_n^{(L)}$ with different n, L through n -gon lightlike limits [25],

$$\lim_{x_{12}^2, \dots, x_{n1}^2 \rightarrow 0} \sigma_n F_N = M_n^{(L)}(x_1, \dots, x_N), \quad (5)$$

where $\sigma_n := x_{12}^2 x_{23}^2 \cdots x_{n1}^2$. Moreover, F_N is invariant under permutation of $x_1, \dots, x_N$, unifying “loops and legs” and allowing the introduction of weight-4 f graphs [23].

We remark that in the literature, the SYM squared amplitude is usually defined [17,25] in terms of the stripped N^k MHV amplitudes $R_{n,k}^{(L)} := \langle 12 \rangle \cdots \langle n1 \rangle A_n^{(L)}|_{\tilde{\eta}^{4k}}$, which agrees with (2) up to a Parke-Taylor factor,

$$\tilde{M}_n^{(L)} = \frac{1}{2} \sum_{k=0}^{n-4} \sum_{\ell=0}^L \frac{R_{n,k}^{(\ell)} R_{n,n-4-k}^{(L-\ell)}}{R_{n,n-4}^{(0)}} = M_n^{(L)} \prod_{i=1}^n s_{i,i+1}. \quad (6)$$

Squared amplitudes in ABJM—In ABJM, the (integrand of) squared amplitudes can be defined in a similar fashion. Under the gauge group $U(N_c)_{k\text{CS}} \times U(N_c)_{-k\text{CS}}$, the two on-shell superfields [53] transform as $\Phi \in (\mathbf{N}_c, \bar{\mathbf{N}}_c)$ and $\Psi \in (\bar{\mathbf{N}}_c, \mathbf{N}_c)$. In the planar limit $N_c \rightarrow \infty$ with fixed 't Hooft coupling $a = N_c/k_{\text{CS}}$, the color-ordered superamplitude is defined as

$$\mathcal{A}_n(\lambda, \eta) = \mathcal{A}_n(\Psi_1 \Phi_2 \cdots \Psi_{n-1} \Phi_n), \quad (7)$$

where only an even number $n = 2k$ of particles have nonvanishing amplitudes. Poincaré supersymmetry implies that $\mathcal{A}_n(\lambda, \eta) = \delta^3(P)\delta^6(Q)A_n(\lambda, \eta)$ satisfies $\tilde{Q}A_n = 0$, where $A_n \sim \eta^{3(k-2)}$ and $(\partial_{iI} := \partial_{\eta_i^I})$,

$$P^{\alpha\beta} = \sum_{i=1}^n \lambda_i^\alpha \lambda_i^\beta, \quad Q^{\alpha I} = \sum_{i=1}^n \lambda_i^\alpha \eta_i^I, \quad \tilde{Q}_I^\alpha = \sum_{i=1}^n \lambda_i^\alpha \partial_{iI}. \quad (8)$$

Here, $I = 1, 2, 3$ manifests the $SU(3)$ subgroup of the R-symmetry $SU(4) \cong SO(6)$. Parity is trivial in three

dimensions and there is only one type of spinor λ . In order to define the squared amplitude, the correct prescription is charge conjugation [54], which maps $\mathcal{A}_n(\lambda, \eta)$ to a differential operator $\bar{\mathcal{A}}_n(\lambda, \partial_\eta) = \delta^3(P) \delta^6(\tilde{Q}) \bar{\mathcal{A}}_n(\lambda, \partial_\eta)$. As charge conjugation changes $k_{CS} \leftrightarrow -k_{CS}$, the integrand $\bar{A}_n^{(L)}$ is related to $A_n^{(L)}$ through an additional $(-)^L$ factor,

$$\bar{A}_n^{(L)}(\lambda, \partial_\eta) = (-)^L A_n^{(L)}(\lambda, \eta) \Big|_{\eta \rightarrow \partial_\eta}. \quad (9)$$

The squared amplitude is defined as in (2), which at the integrand level reads

$$M_n^{(L)} = \frac{1}{2} \sum_{\ell=0}^L \frac{1}{L!} \sum_{\sigma \in S_L} (-)^\ell A_n^{(\ell)}(\lambda, \partial_\eta; y_{\sigma(1)}, \dots, y_{\sigma(\ell)}) \times A_n^{(L-\ell)}(\lambda, \eta; y_{\sigma(\ell+1)}, \dots, y_{\sigma(L)}) \Big|_{\eta, \partial_\eta=0}. \quad (10)$$

Because of the $(-)^L$ factor, $M_n^{(L)}$ is only nonzero if L is even. We will see that an S_N permutation invariant generating function $F_N(x_1, \dots, x_N)$ with $N := n + L$ unifies $M_n^{(L)}$ in the same way (5) as in SYM. As the simplest case, the four-point tree amplitude $A_4^{(0)} = (1/\langle 12 \rangle \langle 23 \rangle)$ leads to

$$M_4^{(0)} = \frac{1}{2x_{13}^2 x_{24}^2} = \lim_{x_{12}^2, \dots, x_{41}^2 \rightarrow 0} \sigma_4 \underbrace{\left(\frac{1}{2x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2} \right)}_{F_4}. \quad (11)$$

$N = 6$: A first indication—Let us proceed to the $N = 6$ case. For $(n, L) = (4, 2)$, the amplitude A_4 is η -independent. Hence, (10) reads

$$M_4^{(2)} = A_4^{(0)} \left(\frac{A_4^{(2)}(56) + A_4^{(2)}(65)}{2} - \frac{1}{2} A_4^{(1)}(5) A_4^{(1)}(6) \right), \quad (12)$$

where $A_4^{(L)}(5, \dots, 4+L)$ denotes the integrand with dual loop momenta $x_5, \dots, x_{4+L}$, whose explicit expressions can be found in [33]. Explicit computation gives, up to an overall normalization,

$$M_4^{(2)} = \left[\frac{1}{x_{15}^2 x_{25}^2 x_{56}^2 x_{36}^2 x_{46}^2} + \text{cyc}(1234) \right] - \left[\frac{x_{13}^2}{x_{15}^2 x_{16}^2 x_{25}^2 x_{35}^2 x_{36}^2 x_{46}^2} + (13 \leftrightarrow 24) + (5 \leftrightarrow 6) \right] - \left[\frac{x_{13}^2/x_{24}^2}{x_{15}^2 x_{16}^2 x_{35}^2 x_{36}^2 x_{56}^2} + (13 \leftrightarrow 24) \right] + \frac{1}{2} \left[\frac{x_{13}^2 x_{26}^2 x_{45}^2/x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{56}^2 x_{36}^2 x_{46}^2 x_{16}^2} + \text{cyc}(1234) \right] + \frac{1}{2} \frac{x_{13}^2 x_{24}^2 x_{56}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2}. \quad (13)$$

Throughout this Letter, we ignore the overall normalization of $M_n^{(L)}$, since conventions differ in the literature and cannot

be straightforwardly aligned. However, we believe there exists a good convention such that (5) holds exactly in ABJM. By dividing σ_4 , one can easily obtain F_6 as a sum of weight-3 f graphs, whose leading term in the four-gon lightlike limit $x_{12}^2, \dots, x_{41}^2 \rightarrow 0$ coincides with $M_4^{(2)}$,

$$F_6 = \text{graph 1} - \text{graph 2} + \frac{1}{2} \text{graph 3}. \quad (14)$$

Here, each f graph denotes a sum of permutation inequivalent monomials where each solid line denotes a denominator $1/x_{ij}^2$ and each dashed line denotes a numerator x_{ij}^2 . The conformal weight of each vertex (the number of solid edges minus the number of dashed lines) is always 3, as expected from DCI in three dimensions. Each f graph is S_N -permutation invariant. For example,

$$\text{graph 1} = \frac{x_{56}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2 x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{12}^2 x_{34}^2} + \text{perm.} \quad (15)$$

45 = 6!/16 terms in total,
where 16 is the symmetry factor of the graph

Monomials surviving the four-gon lightlike limit correspond to solid line four-cycles. Note that all f graphs in (14) are planar in the sense that the solid line subgraphs are planar. This is not surprising, as the integrands in [33] are planar Feynman diagrams.

For $(n, L) = (6, 0)$, the amplitude $A_6 \sim \eta^3$ [30,32],

$$A_6^{(0)} = f_+(\lambda) \delta^3 \left(\sum_{i=1}^6 \alpha_+^i(\lambda) \eta_i \right) + f_-(\lambda) \delta^3 \left(\sum_{i=1}^6 \alpha_-^i(\lambda) \eta_i \right). \quad (16)$$

According to the definition,

$$M_6^{(0)} = \frac{1}{2} \sum_{i_1 i_2 i_3=1}^6 (f_+ \alpha_+^{i_1} \alpha_+^{i_2} \alpha_+^{i_3} + f_- \alpha_-^{i_1} \alpha_-^{i_2} \alpha_-^{i_3})^2 = \frac{2}{x_{14}^2 x_{25}^2 x_{36}^2} = \lim_{x_{12}^2, \dots, x_{61}^2 \rightarrow 0} \sigma_6 2 \text{graph 4}. \quad (17)$$

This f graph is not planar; however, it is bipartite (also referring to the solid line subgraph). It looks different from (14), but in three dimensions, due to the vanishing of the conformal Gram determinant $\det[x_{ij}^2]_{i,j=1}^6 = 0$ [55],

$$2 \text{graph 4} = \text{graph 1} - \text{graph 2} + \frac{1}{2} \text{graph 3}. \quad (18)$$

This demonstrates that both $M_4^{(2)}$ and $M_6^{(0)}$ are unified in the same generating function F_6 , which can be represented as a linear combination of either planar f graphs or bipartite f graphs, related through three-dimensional Gram identities. Incidentally, directly using the integrands for $\log(A_4/A_4^{(0)})$ [52], we would obtain [56] $M_4^{(2)} = \frac{2}{x_{15}^2 x_{35}^2 x_{56}^2 x_{26}^2 x_{46}^2} + (5 \leftrightarrow 6) = \lim \sigma_4 2 \triangleleft$, which manifests the bipartite pole structure. Note that it is natural to consider bipartite f graphs instead of just planar f graphs in ABJM, since these contain no odd-length cycles and manifest the vanishing of odd- n (squared) amplitudes through $\lim \sigma_n F_N = 0$ [57].

$N = 8$: *Bipartite vs planar*—The lesson learned at $N = 6$ applies to $N = 8$. For example, the six-point L -loop amplitudes can be written as [49]

$$A_6^{(L)} = I_6^{(L)} A_6^{(0)} + J_6^{(L)} A_{6,\text{shift}}^{(0)}. \quad (19)$$

Here, $I_6^{(L)}$ and $J_6^{(L)}$ are purely bosonic functions, and all the η dependence is captured by the tree amplitude $A_6^{(0)}$ and its cyclic-by-one image,

$$A_{6,\text{shift}}^{(0)}(\Psi_1 \Phi_2 \cdots \Psi_5 \Phi_6) := A_6^{(0)}(\Psi_2 \Phi_3 \cdots \Psi_6 \Phi_1). \quad (20)$$

Computation shows that

$$\bar{A}_6^{(0)}(\partial_\eta) A_6^{(0)}(\eta) = \bar{A}_{6,\text{shift}}^{(0)}(\partial_\eta) A_{6,\text{shift}}^{(0)}(\eta) = 2M_6^{(0)}, \quad (21)$$

$$\bar{A}_6^{(0)}(\partial_\eta) A_{6,\text{shift}}^{(0)}(\eta) = \bar{A}_{6,\text{shift}}^{(0)}(\partial_\eta) A_6^{(0)}(\eta) = 0. \quad (22)$$

Therefore, cross terms $I \times J$ do not contribute to $M_6^{(2)}$,

$$\frac{M_6^{(2)}}{M_6^{(0)}} = I_6^{(2)}(78) + I_6^{(2)}(87) - I_6^{(1)}(7)I_6^{(1)}(8) - J_6^{(1)}(7)J_6^{(1)}(8). \quad (23)$$

Similarly, we compute $M_4^{(4)}$ [52] and $M_8^{(0)}$ [48] and check numerically that $M_4^{(4)}, M_6^{(2)}, M_8^{(0)}$ are all unified in a generating function F_8 through (5), where

$$F_8 = 8 \triangleleft - 4 \triangleleft + 2 \triangleleft. \quad (24)$$

Once again, we see that F_8 can be represented graphically as a linear combination of bipartite f graphs.

Let us now discuss the relation between planar f graphs and bipartite f graphs in more detail. It is easy to generate all planar or bipartite f graphs using software such as PLANTRI [58] and NAUTY [59]. At $N = 8$, there are 61 planar f graphs and four bipartite f graphs. Apart from $f_8^{1,2,3}$ shown in (24), the fourth bipartite f graph is

$$f_8^4 = \triangleleft. \quad (25)$$

Among the bipartite f graphs $f_8^2 = \triangleleft$, is special in that it is both planar and bipartite. More generally, through Gram identities, certain linear combinations of planar f graphs coincide with linear combinations of bipartite f graphs, as we have seen for F_6 . In other words, given the vector spaces $\mathcal{B}, \mathcal{P}$ generated by bipartite and planar f graphs, we have $F_6 \in \mathcal{B} \cap \mathcal{P}$. We check numerically that $F_8 \in \mathcal{B} \cap \mathcal{P}$ as well.

With the above observation, we turn the logic around to bootstrap F_N , assuming $F_N \in \mathcal{B} \cap \mathcal{P}$. We can numerically identify all linear relations relating planar f graphs to bipartite f graphs, which generate the subspace $\mathcal{B} \cap \mathcal{P}$. This fixes $F_6 \propto \triangleleft$ and constrains F_8 to be

$$F_8 = c \left(4 \triangleleft + \triangleleft \right) + c' \triangleleft + 0. \quad (26)$$

In particular, the coefficient of (25) in F_8 is fixed to be 0. Note that planarity only implies $F_8 \in \mathcal{P}$, which leads to a 61-term ansatz, while the bipartite property drastically reduces the ansatz size down to 2! This demonstrates the impressive constraining power of the $\mathcal{B} \cap \mathcal{P}$ bootstrap.

Apart from the interclass Gram identities generating $\mathcal{B} \cap \mathcal{P}$, there could be intraclass Gram identities within $\mathcal{B}$ or $\mathcal{P}$. For example, at $N = 8$, even though there are 61 planar f graphs, actually $\dim \mathcal{P} = 60$; the unique linear relation (involving 43 planar f graphs) is recorded in an ancillary file. As a result, although F_8 has a unique expansion onto bipartite f graphs (24), the expansion onto planar f graphs is not unique.

$N = 10$: *Putative graphical rules*—Seeing that $F_8 \in \mathcal{B} \cap \mathcal{P}$ already fixes F_8 down to two coefficients (only one if we ignore the overall normalization), it is natural to ask whether we can fix F_8 completely.

Let us recall how F_N is bootstrapped in SYM. There, by studying the asymptotic behavior of the correlator under the cusp [26,60] (operator product expansion [55]) limit, one can extract the double-triangle (triangle) rule that relates F_N and F_{N-1} [61]. Notably, it is possible to impose these constraints BEFORE identifying linear combinations of f graphs related by Gram identities, which makes them graphical rules.

Drawing analogies, it is natural to look for graphical rules relating F_N and F_{N-2} in ABJM since N is even. Since the number of bipartite f graphs is much smaller than that of planar f -graphs, we look for graphical rules in the former representation. Only even-length solid line cycles are

allowed in bipartite f graphs, so instead of triangles, we look for quadrangles, which motivates the following putative graphical rules. Given $F_N = \sum_i c_i f_N^i$ and $F_{N-2} = \sum_j b_j f_{N-2}^j$, where f_N^i, f_{N-2}^j are the bipartite f graphs, the “quadrangle rule” reads [Fig. 1(a)]

$$\alpha \sum_i c_i \sum_{\square \in f_N^i} \frac{\mathcal{Q}(f_N^i, \square)}{|f_N^i|} = 2 \sum_j b_j \sum_{\square \in f_{N-2}^j} \frac{\mathcal{E}(f_{N-2}^j, \square)}{|f_{N-2}^j|}. \quad (27)$$

Here, $f_N^i, \square$ means f_N^i with a (solid line) quadrangle highlighted (i.e., the corresponding $x_{ij}^2 = 0$), and similarly for the edge-highlighted f_{N-2}^j ; $|f_N^i|$ and $|f_{N-2}^j|$ denote the symmetry factors of the highlighted f graphs. Since the graphical rule is putative, we allow for an arbitrary overall factor α to be determined by consistency. $\mathcal{Q}$ shrinks the quadrangle and $\mathcal{E}$ shrinks the edge down to a point,

$$\mathcal{Q} : \text{quadrangle} \mapsto \text{point}, \quad \mathcal{E} : \text{edge} \mapsto \text{point}. \quad (28)$$

Comparing Fig. 1(a) with the correct results (24) and (17), we see that choosing $\alpha = 1$ is sufficient to fix the relative coefficient $c' = -2c$ in (26). The double-quadrangle rule reads [Fig. 1(b)]

$$\beta \sum_i c_i \sum_{\diamond \in f_N^i} \frac{\mathcal{D}(f_N^i, \diamond)}{|f_N^i|} = \sum_j b_j \sum_{\diamond \in f_{N-2}^j} \frac{f_{N-2}^j}{|f_{N-2}^j|}, \quad (29)$$

where β is another overall factor and $\mathcal{D}$ pinches the highlighted double quadrangle to a highlighted cusp,

$$\mathcal{D} : \text{double quadrangle} \mapsto \text{cusp}. \quad (30)$$

Comparing Fig. 1(b) with (24) and (17), we see that it is consistent to choose $\beta = \alpha/2 = 1/2$. In fact, a weaker version of the double-quadrangle rule exactly corresponds to the double-soft limit [62] of the squared amplitude [63].

With these putative graphical rules, we bootstrap $F_{10} \in \mathcal{B} \cap \mathcal{P}$. There are 19 741 planar and 120 bipartite f graphs in total, and we numerically find $3 = \dim \mathcal{B} \cap \mathcal{P}$ linear relations relating them. Finally, the quadrangle and double-quadrangle rules uniquely fix the answer in the bipartite representation modulo Gram identities, which we present in Supplemental Material [61].

It is extremely nontrivial that the putative graphical rules have a unique consistent solution F_{10} . We have numerically checked that $M_{10}^{(0)} = \lim_{\sigma_{10}} F_{10}$ matches the correct result obtained from the Grassmannian formula [34] (see Supplemental Material for details [61]). We have also extracted $A_4^{(6)}$ by subtracting off $A_4^{(\ell)} A_4^{(6-\ell)}$ ($1 \leq \ell \leq 5$) [52] from $M_4^{(6)}$ and verified that $A_4^{(6)}$ has correct unitarity cuts [64]. We leave the check against $M_8^{(2)}$ [50] as well as further studies of these new data at $L = 4, 6$ to future works.

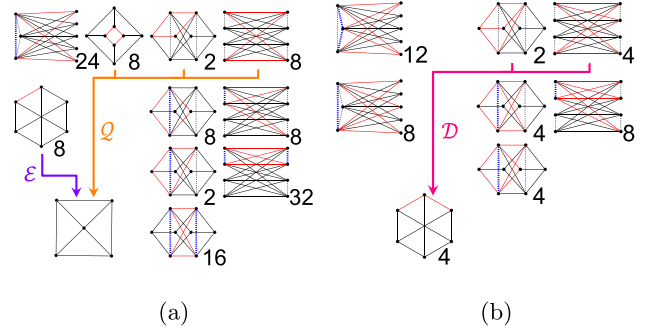


FIG. 1. The double-quadrangle and quadrangle rules relating F_8 and F_6 , with the symmetry factors shown beside the highlighted f graphs. The highlighted f graphs without a connecting arrow are mapped to 0 by $\mathcal{Q}$ or $\mathcal{D}$ due to the numerators colored in blue; hence they do not contribute to the constraining equations. Note that there is no way of highlighting a double quadrangle in $f_8^2 = \text{quadrangle}$. (a) Quadrangle rule: $\alpha(\frac{c_2}{8} + \frac{c_3}{8} + \frac{c_4}{8}) = 2(\frac{b_1}{8})$ and (b) Double-quadrangle rule: $\beta(\frac{c_3}{8} + \frac{c_4}{8}) = (\frac{b_1}{4})$.

Conclusion—In this Letter, we observed that the (planar integrands of) ABJM squared amplitudes, $M_n^{(L)}$, can be unified in an S_N -permutation-invariant generating function F_N ($N := n + L$), which satisfies planarity as well as a beautiful bipartite property. These two basic properties combined already strongly constrain F_N ; paralleling the amplitude-correlator duality in SYM, we proposed putative graphical rules that we used to bootstrap it up to F_{10} providing valuable perturbative data for squared amplitudes. Our results strongly suggests the existence of a bipartite correlator dual to amplitudes in ABJM, which should exhibit universal behaviors under operator product expansion and cusp limits similar to the half-Bogomol’nyi-Prasad-Sommerfield correlators in SYM.

Conceptually, it is extremely important to search for the exact physical observable behind the generating function F_N . The permutation symmetry unifying loops and legs suggests it should be related to “Lagrangian insertion” in ABJM. However, since k_{CS} is quantized, there are no exactly marginal scalar operators respecting $\mathcal{N} = 6$ superconformal symmetry [65], and the SYM mechanism $\int dy \langle \cdots \mathcal{L}(y) \rangle = a \partial_a \langle \cdots \rangle$ likely has to be modified. On the other hand, the manifest $SU(3)$ R symmetry in $\mathcal{A}_n$ suggests that F_N might be related to correlators in mass-deformed ABJM [66] which breaks superconformal symmetry to $\mathcal{N} = 2$ but exhibits a global $SU(3)$ symmetry. Identifying the correlator behind F_N is necessary if we wish to justify the graphical rules and push the bootstrap to higher N like in SYM. It will also shed light on properties of ABJM and its relation with SYM.

One immediate question is how to efficiently extract the (unsquared) amplitudes from $M_n^{(L)}$ for $n \geq 6$ along the lines of [19]. Geometrically, it is interesting to look for the three-dimensional “squared amplituhedron” and “correlahedron” [67–70] in ABJM. From our results, one can also extract

the collinear splitting functions in ABJM (analogous to [71] for SYM), which can be exploited for other important quantities such as energy correlators.

Last but not least, the weight- $3f$ graphs are also mathematically interesting since they generate a vast collection of interesting three-dimensional DCI integrals, e.g., some of these are known to evaluate to elliptic functions (see Ref. [72]). As graphical functions [73], it is also worth studying their periods

$$\mathcal{P}_{f_N^i} \propto \int \frac{d^3x_1 \cdots d^3x_N}{\text{SO}(5)} f_N^i, \quad (31)$$

which would provide valuable data in number theory [74–80] and regarding the corresponding “integrated correlators” (see Refs. [81–83] for SYM case) in ABJM theory.

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