

Emergent Spacetime Supersymmetry at 2D Fractionalized Quantum Criticality

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While experimental evidence for spacetime supersymmetry (SUSY) in particle physics remains elusive, condensed matter systems offer a promising arena for its emergence at quantum critical points (QCPs). Although there have been a variety of proposals for emergent SUSY at symmetry-breaking QCPs, the emergence of SUSY at fractionalized QCPs remains largely unexplored. Here, we demonstrate emergent spacetime SUSY at a fractionalized QCP in the Kitaev honeycomb model with Su-Schrieffer-Heeger (SSH) spin-phonon coupling. Specifically, through numerical computations and analytical analysis, we show that the anisotropic SSH-Kitaev model hosts a fractionalized QCP between a Dirac spin liquid and an incommensurate or commensurate valence-bond-solid phase coexisting with $\mathbb{Z}_2$ topological order. A low-energy field theory incorporating phonon quantum fluctuations reveals that this fractionalized QCP features an emergent $\mathcal{N} = 2$ spacetime SUSY. We further discuss their universal experimental signatures in thermal transport and viscosity, highlighting the concrete lattice realization of emergent SUSY at a fractionalized QCP in 2D.

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Introduction—Supersymmetry (SUSY), a fundamental spacetime symmetry relating bosons and fermions [1–3], has been extensively explored in high-energy physics, but has not been observed in nature at accessible energy scales. Condensed matter systems, however, provide a promising alternative platform for exploring SUSY [4–18]. Crucially, spacetime SUSY can emerge dynamically at quantum critical points between symmetric and symmetry-broken phases [19–29], providing a natural arena for studying its consequences by tuning microscopic parameters. Despite this progress, SUSY in fascinating fractionalized settings, which typically involve topological order with deconfined gauge fields and fractionalized particles [30–33], remains largely unexplored.

More specifically, achieving emergent SUSY at fractionalized quantum critical points (QCPs) [34–39] has been an open problem and poses significant theoretical challenges, owing to the intrinsic difficulties of reliably treating deconfined gauge fields and frustrated spin interactions

at phase transitions between quantum spin liquids (QSLs) and symmetry-breaking phases. However, the Kitaev honeycomb model [40] (and other related models [41–55]) offers a promising platform to overcome these challenges. As a paradigmatic solvable model hosting $\mathbb{Z}_2$ QSLs with potential material realizations [56–60], the Kitaev model has conserved deconfined $\mathbb{Z}_2$ gauge fields. While additional interactions (e.g., Heisenberg couplings) typically endow these gauge fields with dynamics, we find that certain spin-phonon couplings can preserve the $\mathbb{Z}_2$ gauge structure while simultaneously driving a fractionalized QCP with emergent SUSY via a spin-Peierls instability [61–76]. Crucially, the QCPs of this instability remain far less understood in high dimensions than the resulting ordered phases. Moreover, spin-phonon coupling in the Kitaev model has attracted significant interest beyond the spin-Peierls instability [77–83], as it is believed to underlie several experimental observations in Kitaev materials, such as the thermal Hall conductivity [84,85].

In this Letter, we propose that the Kitaev honeycomb model with spin-phonon coupling, which modulates the strength of Kitaev exchange interactions in proportion to phonon displacements, can host emergent SUSY at fractionalized quantum criticality. Combining Lieb’s theorems and large-scale numerical computations, we first obtain the phase diagram in the adiabatic limit, revealing transitions from a Dirac QSL to incommensurate or commensurate valence-bond-solid (VBS) order coexisting with $\mathbb{Z}_2$ topological order. Significantly, incorporating phonon quantum

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fluctuations away from the adiabatic limit, we construct a low-energy theory of Dirac fermions coupled to the VBS order parameter and show that these transitions flow to an emergent $\mathcal{N} = 2$ spacetime SUSY* fixed point. This provides a concrete lattice realization of SUSY at a fractionalized QCP, with universal signatures in thermal transport and viscosity. Our proposal of SUSY* fixed points is intrinsically different from previous ones [21,86], since our model neither resides at the boundary of a topological phase nor requires nonlocal interactions. Furthermore, unlike Ref. [21], we do not need to impose particle-hole symmetry of the order-parameter fluctuations to avoid a multicritical point, since it is automatically enforced by an inversion symmetry in our model.

Model—We consider the following Su-Schrieffer-Heeger (SSH)–Kitaev model on the honeycomb lattice:

$$\hat{H} = \sum_{\langle ij \rangle \in \mu} (J_\mu + g\hat{X}_{\langle ij \rangle})\tau_i^\mu \tau_j^\mu + \sum_{\langle ij \rangle} \frac{\hat{P}_{\langle ij \rangle}^2}{2m} + \frac{k}{2}\hat{X}_{\langle ij \rangle}^2, \quad (1)$$

where each bond $\langle ij \rangle$ is labeled with $\mu = x, y, z$ according to its direction, as illustrated in the inset of Fig. 1(a). $\hat{X}_{\langle ij \rangle}$ is the phonon field on the nearest-neighbor bond $\langle ij \rangle$ and $\hat{P}_{\langle ij \rangle}$ is the conjugate phonon momentum. Here, we consider the simplest Einstein phonon with phonon frequency $\omega_{\text{ph}} = \sqrt{(k/m)}$. Using the Majorana fermion representation $\tau_i^\mu = i\hat{c}_i^\mu \hat{c}_i$ [40], where $\hat{c}_i^\mu, \hat{c}_i$ are Majorana fermions, the Hamiltonian can be rewritten as

$$\hat{H}_f = \sum_{\langle ij \rangle \in \mu} \left[\hat{u}_{\langle ij \rangle} (J_\mu + g\hat{X}_{\langle ij \rangle}) \right] (i\hat{c}_i \hat{c}_j) + \sum_{\langle ij \rangle} \frac{\hat{P}_{\langle ij \rangle}^2}{2m} + \frac{k}{2}\hat{X}_{\langle ij \rangle}^2. \quad (2)$$

The $\mathbb{Z}_2$ gauge fields $\hat{u}_{\langle ij \rangle \in \mu} = i\hat{c}_i^\mu \hat{c}_j^\mu$ are conserved, implying that the model's dynamics is governed solely by itinerant Majorana fermions and phonon fields. To realize a SUSY critical point, the model should exhibit a continuous phase transition with an equal number of Dirac cones and complex bosons, which is fulfilled here as we demonstrate below.

To gain intuition, we start from the C_3 symmetric point $J_x = J_y = J_z$. In the adiabatic limit $m \rightarrow \infty$, the phonon field $\hat{X}_{\langle ij \rangle}$ becomes a classical field, and there is a phase transition between the gapless quantum spin liquid ($g = 0$) and the $\mathbb{Z}_2$ topological order ($g \rightarrow \infty$) with the increase of g . The $\mathbb{Z}_2$ topological order phase also exhibits a valence-bond-solid long-range order with a Kekulé pattern, which breaks the C_3 rotation symmetry and gaps out the Dirac cone located at the corner $\mathbf{K}$ of the Brillouin zone. This phase transition can be deduced to be continuous in the adiabatic limit from a Landau-Ginzburg free energy analysis, where the cubic term of the order parameter $X_{\mathbf{K}}^3 + \text{H.c.}$

(the order parameter $X_{\mathbf{K}}$ is the Fourier component of $X_{\langle ij \rangle}$ at momentum $\mathbf{K}$) is overcome by the nonanalytic term $|X_{\mathbf{K}}|^3$. The nonanalytic term $|X_{\mathbf{K}}|^3$ arises from integrating out gapless fermions $\hat{c}_i$, analogous to the mechanism of the previously investigated fermion-induced QCP [94–96]. However, with the inclusion of quantum fluctuations, or equivalently a large but finite m , the cubic term becomes relevant at a presumably continuous phase transition [94], which means the phase transition becomes first order with an infinitesimal deviation from the $m \rightarrow \infty$ limit.

Building on insights from the isotropic limit, we focus on the anisotropic regime to eliminate the cubic term $X_{\mathbf{K}}^3 + \text{H.c.}$ and thus access a continuous phase transition beyond the adiabatic limit. We choose the coupling constants as: $J_x = J_y = J > 0$ and $J_z = aJ$ with $0 < a < 1$, which produces a gapless spin liquid when $g = 0$. We first analyze its phase diagram in the adiabatic limit, which paves the way for further controlled analysis of quantum fluctuations. After a rescaling of the phonon field: $X_{\langle ij \rangle} \rightarrow X_{\langle ij \rangle} J/g$, $\hat{H}_f$ (with $J = 1$) can be simplified to a form with only one tuning parameter $\lambda = (g^2/k)$:

$$\hat{H}_f = \sum_{\langle ij \rangle \in \mu} \hat{u}_{\langle ij \rangle} (a_\mu + X_{\langle ij \rangle}) (i\hat{c}_i \hat{c}_j) + \sum_{\langle ij \rangle} \frac{X_{\langle ij \rangle}^2}{2\lambda}, \quad (3)$$

where $a_x = a_y = 1$ and $a_z = a$ represent the anisotropy of the Kitaev couplings.

The quantum phase diagram—We begin by analyzing the phase diagram of the model (3) in the adiabatic limit. Now determining the ground state reduces to minimizing the energy over both phonon and flux configurations. This problem is greatly simplified by invoking two Lieb's theorems which are based on reflection positivity [87–89]. The relevant symmetry required by Lieb's theorems is the reflection symmetry with mirror planes $\mathcal{M}$ bisecting the z type of bonds (namely, $y \rightarrow -y$ under the reflection). Owing to these theorems, the ground state lies in the zero flux sector and the most general configuration of the phonon fields $X_{\langle ij \rangle}$ in the ground state is symmetric under $\mathcal{M}$. Importantly, Lieb's theorems guarantee zero flux of the total hopping amplitude $t_{\langle ij \rangle} = \hat{u}_{\langle ij \rangle} (a_\mu + \hat{X}_{\langle ij \rangle})$, not necessarily zero flux of the gauge field $\hat{u}_{\langle ij \rangle}$ itself. Consequently, two scenarios arise: (1) if all the phonon fields $\hat{X}_{\langle ij \rangle}$ are small enough such that all hoppings $a_\mu + \hat{X}_{\langle ij \rangle}$ are positive, then the $\hat{u}_{\langle ij \rangle}$ flux is also zero; (2) if some of the $\hat{X}_{\langle ij \rangle}$ are negative enough to induce π flux in the hoppings $a_\mu + \hat{X}_{\langle ij \rangle}$ on certain plaquettes, then the corresponding $\hat{u}_{\langle ij \rangle}$ flux should also be π to preserve zero net flux of $t_{\langle ij \rangle}$.

Here, we go beyond Lieb's theorems and prove that the flux of $u_{\langle ij \rangle}$ is actually zero in the ground state sector. This result is established through a proof by contradiction. If the flux of $u_{\langle ij \rangle}$ is π around a plaquette p in the ground state

sector: $\Pi_{\langle ij \rangle \in p} u_{\langle ij \rangle} = -1$, then some of the phonon fields $X_{\langle ij \rangle}$ and $a_\mu + X_{\langle ij \rangle}$ ($\langle ij \rangle \in p$) must be negative, since the total flux of $\Pi_{\langle ij \rangle \in p} [u_{\langle ij \rangle} (a_\mu + X_{\langle ij \rangle})]$ is required to be zero by Lieb's theorems. Then we can find new phonon and gauge field configurations on these bonds with strictly lower energy: $\tilde{X}_{\langle ij \rangle} = -2a_\mu - X_{\langle ij \rangle}$, $\tilde{u}_{\langle ij \rangle} = -u_{\langle ij \rangle}$. This configuration preserves the hopping amplitudes: $\tilde{t}_{\langle ij \rangle} = (a_\mu + \tilde{X}_{\langle ij \rangle})\tilde{u}_{\langle ij \rangle} = (a_\mu + X_{\langle ij \rangle})u_{\langle ij \rangle} = t_{\langle ij \rangle}$, and thus the fermion energy is unchanged. However, the new phonon configuration yields a strictly lower phonon potential energy, since $|\tilde{X}_{\langle ij \rangle}|^2 - |X_{\langle ij \rangle}|^2 = 4a_\mu(a_\mu + X_{\langle ij \rangle}) < 0$. This process can be iterated until all plaquettes satisfy $\Pi_{\langle ij \rangle \in p} \tilde{u}_{\langle ij \rangle} = 1$, which has strictly lower energy than any state with π flux of $u_{\langle ij \rangle}$, which contradicts our initial assumption. This completes the proof. Hence, we set all the $\mathbb{Z}_2$ gauge fields $\hat{u}_{\langle ij \rangle} = 1$ in the following discussions. A direct implication of our proof is the effective $\tilde{J}_\mu = a_\mu + X_\mu \geq 0$ in the ground state sector.

Furthermore, Lieb's theorems do not address whether the ground state exhibits spontaneous symmetry breaking. Consequently, there are two possible scenarios in principle: (1) For sufficiently small λ , it is expected that the phonon fields $X_{\langle ij \rangle}$ preserve translation symmetries; that is, $X_{\langle ij \rangle \in \mu}$ satisfies $X_{\langle ij \rangle \in \mu} = X_\mu$, resulting in an itinerant fermion spectrum that resembles the pure Kitaev honeycomb model with anisotropic couplings $\tilde{J}_\mu = a_\mu + X_\mu$. However, in this scenario, the ground state always remains a gapless spin liquid with a single Dirac cone and can never enter a gapped phase. This is because the hopping amplitudes satisfy $\tilde{J}_x = \tilde{J}_y$ in the ground state sector, as follows directly from Lieb's theorems [87–89] together with the zero flux condition for $\hat{u}_{\langle ij \rangle}$ we proved above, and are bounded within $0 \leq \tilde{J}_z/\tilde{J}_x < 1$ for any λ , preventing the system from reaching the anisotropic limit $|\tilde{J}_z| > \tilde{J}_x + \tilde{J}_y$ where the Kitaev model becomes gapped. $\tilde{J}_z$ can never be larger than $\tilde{J}_x$, since the Dirac cone can only move along the k_x axis with $k_x \in [\pi, \frac{4\pi}{3}]$, where $(\tilde{J}_z/\tilde{J}_x) \rightarrow 1$ as k_x approaches $\frac{4\pi}{3}$. However, the end point $k_x = \frac{4\pi}{3}$, which would imply isotropic $\tilde{J}_x = \tilde{J}_y = \tilde{J}_z$, is not attainable, as the energy minimum results in a C_3 symmetric $X_x = X_y = X_z$ phonon configuration, which contradicts with the isotropic $\tilde{J}_\mu$ coupling. (2) For a sufficiently large λ beyond a critical value λ_c , the phonon fields $X_{\langle ij \rangle}$ break the translation symmetry. Driven by the Peierls instability, the momentum of this $X_{\langle ij \rangle}$ configuration is expected to match the intravalley scattering momentum of the Dirac cone, when the coupling λ just exceeds the critical λ_c . So, the ground state typically has an incommensurate VBS order coexisting with $\mathbb{Z}_2$ topological order (in the sense the collective Goldstone mode is neglected) in this phase.

As a result, the phase diagram is governed by spontaneous breaking of the translation symmetry as the

spin-phonon coupling λ increases. We obtain the global quantum phase diagram through large-scale numerical simulations, as is illustrated in Fig. 1(a), which agrees well with the previous theoretical analyses. In particular, all the phases depicted in Fig. 1 are quantum spin liquids. This is due to the persistence of exact anomalous 1-form symmetry and deconfined fermions in all phases [97,98]. Although the strong-coupling columnar VBS topological order phase in Fig. 1(a) is compelling, our primary focus is on the QCP between the gapless spin liquid and the incommensurate VBS spin liquid phases. Interestingly, a vortex in the incommensurate VBS spin liquid can trap a Majorana zero mode [99], which in this context behaves as an Ising anyon. As shown in Fig. 1(b), this phase transition is continuous in the adiabatic limit. Through field theory analysis, we demonstrate that there is an emergent $\mathcal{N} = 2$ spacetime SUSY at these QCPs, when the quantum fluctuations of the phonon fields are further included. Additionally, we establish that the transitions into commensurate VBS topological order phases belong to the same supersymmetric universality class, provided that their commensurability exceed four.

Field theory in QCPs—We now develop the field theory for the quantum critical points between the gapless spin liquid and the topological ordered VBS. Given that the gauge fields $\hat{u}_{\langle ij \rangle}$ are always conserved, only the itinerant fermions $\hat{c}_i$ and the phonon fields are dynamic degrees of freedom, so we fix the gauge $\hat{u}_{\langle ij \rangle} = 1$ hereafter [100]. Here, we incorporate quantum fluctuations of the phonon field by taking a finite yet sufficiently large phonon mass m , rendering the phonon dynamics a perturbative factor that does not qualitatively alter the phase diagram and the ground state still lies in the zero $\mathbb{Z}_2$ flux sector. This assumption is supported by our perturbative calculations at finite phonon frequency [87], and the finite ω_{ph} phase diagram is consistent with Fig. 1(c).

The critical field theory contains three parts: $S = S_f + S_b + S_{\text{int}}$, where we should retain all the relevant terms allowed for symmetry. $S_f = \int d\tau d^2x \psi^\dagger (\partial_\tau - i v_x \partial_x \sigma_y - i v_y \partial_y \sigma_x) \psi$ is the action of a Dirac cone obtained by expanding the Majorana fermions $\hat{c}_i$ at the band touching point $\tilde{\mathbf{K}}$ [101]: $[\hat{c}_A(\mathbf{r}_i), \hat{c}_B(\mathbf{r}_i)]^T \approx \hat{\psi}(x) e^{i\tilde{\mathbf{K}} \cdot \mathbf{r}_i} + \text{H.c.}$ Here, we take the two nearest-neighbor sites in a z bond as a unit cell labeled by $\mathbf{r}_i$.

S_b describes the quantum fluctuation of the order parameter $\phi(\vec{x}, \tau): S_b = \int d\tau d^2x |\partial_\tau \phi|^2 + \sum_{i=x,y} v_{b,i}^2 |\partial_i \phi|^2 + V(\phi, \phi^*)$, where $\phi(\vec{x}, \tau)$ is a complex bosonic field with momentum $-2\tilde{\mathbf{K}}$ under the lattice translation: $\hat{T}_{\mathbf{a}_i} \phi(\vec{x}, \tau) \hat{T}_{\mathbf{a}_i}^{-1} = e^{-2i\tilde{\mathbf{K}} \cdot \mathbf{a}_i} \phi(\vec{x}, \tau)$, where $\mathbf{a}_{i=1,2}$ are the unit vectors of the honeycomb lattice. If the ordered VBS phase is incommensurate, then the translation symmetry becomes an emergent $U(1)$ symmetry and the symmetry allowed potential $V(\phi, \phi^*)$ can only depend on the module of $\phi(\vec{x}, \tau): V(\phi, \phi^*) = u \int d\tau d^2x |\phi(\vec{x}, \tau)|^4$, where we tune

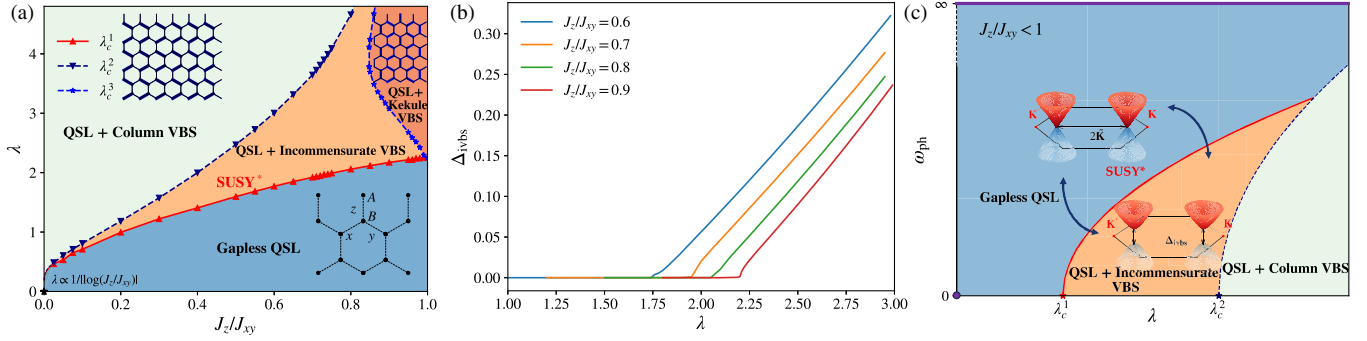


FIG. 1. (a) The quantum phase diagram in the adiabatic limit. The horizontal and vertical axes represent the anisotropy parameter J_z/J_{xy} and the dimensionless spin-phonon coupling λ , respectively. For weak coupling $\lambda < \lambda_c^1$ (red line), the ground state is a gapless spin liquid with a single Dirac cone with both positive- and negative-energy branches. As λ increases, the system first enters an incommensurate VBS phase with topological order, and eventually transitions into a columnar VBS phase with period two when $\lambda > \lambda_c^2$ (dark blue dashed line). In the incommensurate regime, there inevitably exist regions of commensurate VBS phases whose phase boundaries are difficult to resolve numerically when the periodicity is large. Here, we explicitly show the case of period three, corresponding to a Kekulé VBS order, with λ_c^3 marking its phase boundary. The numerical calculations are performed on a finite lattice with $2 \times 240 \times 120$ sites, and phase boundaries $\lambda_c^{2,3}$ are identified by comparing the energies of competing phases. λ_c^1 are identified by the linear extrapolation of the order parameter Δ_{ivbs} [87]. (b) The incommensurate VBS order parameter $\Delta_{ivbs} \equiv |(1/N) \sum_{i \in A} e^{-2i\tilde{\mathbf{K}} \cdot \mathbf{r}_i} X_{(ij) \in z}|$ near the critical point λ_c^1 , which clearly exhibits a continuous phase transition. Here, $\tilde{\mathbf{K}}$ denotes the momentum of the Dirac cone, which is illustrated as two red positive-energy branches in the full Brillouin zone in (c). The linear onset indicates the presence of $|\Delta_{ivbs}|^3$ terms in the free energy, originating from the Dirac cone. (c) Schematic phase diagram at a finite phonon frequency ω_{ph} which is supported by perturbative calculations [87]. The fractionalized quantum critical line between the gapless QSL and the incommensurate VBS QSL belongs to the SUSY* universality class. In the limit $\omega_{ph} \rightarrow \infty$, the spin degrees of freedom are decoupled from phonons.

the mass term of $\phi(\vec{x}, \tau)$ to be zero, since we are considering a critical point. On the other hand, if the VBS order is commensurate with the lattice, which means that $2n\tilde{\mathbf{K}} \equiv 0 \pmod{\text{reciprocal momentum}}$, then the lattice symmetry also allows for an additional term $r_n \int d\tau d^2x [\phi^n + (\phi^*)^n]$ in the potential $V(\phi, \phi^*)$. In addition, although $i\phi^* \partial_\tau \phi$ is typically allowed in the complex boson kinetic part, it is prohibited by an inversion symmetry $\mathcal{I}$ here: $\psi^T(\vec{x}, \tau) \rightarrow \tilde{\psi}(-\vec{x}, \tau)\sigma_y$, $\phi(\vec{x}, \tau) \rightarrow \phi^*(-\vec{x}, \tau)$. Here, ϕ becomes its complex conjugate under inversion since it carries finite momentum. Finally, the symmetry-allowed interaction S_{int} is a Yukawa-type coupling: $S_{\text{int}} = g \int d\tau d^2x (\phi \psi^T \sigma_y \psi + \text{H.c.})$.

Microscopically, we can derive the Yukawa coupling from the Su-Schrieffer-Heeger (SSH) coupling by relating the bosonic field $\phi(\vec{x}, \tau)$ to the phonon field $\hat{X}_{(ij) \in \mu}$ on the lattice. In each unit cell, there are three phonon fields labeled by μ (corresponding to the μ -type bonds in the Kitaev interaction). Crucially, we only retain those phonon modes with momenta near $\pm 2\tilde{\mathbf{K}}$ at the critical point, since only these modes couple with the low-energy fermions. For each bond type μ , the field $\hat{X}_{(ij) \in \mu}$ can therefore be approximated as $\hat{X}_{(ij) \in \mu} \approx e^{-2i\tilde{\mathbf{K}} \cdot \mathbf{r}_i} \hat{\phi}^\mu(\vec{x}) + \text{H.c.}$, where $\hat{\phi}^\mu(\vec{x})$ is a slowly varying field compared to the lattice constant (where $\mathbf{r}_i$ denotes the position of site i in the A sublattice) and $\hat{\phi}(\vec{x})$ is the linear combination of $\hat{\phi}^\mu$: $\sum_\mu e^{i\tilde{\mathbf{K}} \cdot \mathbf{e}_\mu} \hat{\phi}^\mu(\vec{x}) = \hat{\phi}(\vec{x})$, where $\mathbf{e}_x = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$,

$\mathbf{e}_y = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, and $\mathbf{e}_z = (0, 0)$ since it is the intra-unit-cell vector. Taking this continuum limit, the SSH spin-phonon coupling reduces to the Yukawa coupling described by the action S_{int} . Furthermore, the inversion symmetry $\mathcal{I}$ of the continuum theory directly inherits from the lattice inversion symmetry of the original SSH-Kitaev model.

Emergent SUSY at fractionalized QCPs—In this section, we investigate the emergent supersymmetry at the fractionalized QCP described by the above critical field theory S . We begin with the phase transition between the gapless spin liquid and the incommensurate VBS topological order, where the potential $V(\phi, \phi^*)$ depends solely on the modulus of $\phi(\vec{x}, \tau)$. Renormalization group calculations [20,21,23,87] indicate that the critical field theory S flows toward a supersymmetric fixed point, specifically the $\mathcal{N} = 2$ supersymmetric Wess-Zumino model. This emergent SUSY has also been corroborated through sign-problem-free quantum Monte Carlo simulations, achieved by discretizing the single Dirac cone on a lattice using “SLAC” fermions with long-range hoppings [86].

Then we move to the fractionalized QCP between the gapless spin liquid and the commensurate VBS topological order. The additional term, $r_n \int d\tau d^2x [\phi^n + (\phi^*)^n]$, can be treated as a perturbation to the supersymmetric Wess-Zumino fixed point. Owing to the supersymmetry, the scaling dimension of the field ϕ^n is exactly known as $(2n/3)$ [102]. As a result, if the period n of the VBS order

($2n\tilde{\mathbf{K}} \equiv 0 \bmod$ reciprocal momentum) satisfies $n \geq 5$, then ϕ^n is irrelevant in the renormalization group sense and the fixed point remains the supersymmetric Wess-Zumino model. A schematic phase diagram with a finite phonon frequency is illustrated in Fig. 1(c).

More precisely, our fractionalized QCP should belong to the SUSY* universality class since the constituent fermions come from the fractionalization of physical spin operators. This SUSY* not only shares the general properties of fractionalized QCPs, such as the appearance of multiple copies of the standard SUSY spectrum when placed on a torus [36,39], but also exhibits intrinsically new physical consequences arising from SUSY which are absent in conventional unfractionalized SUSY QCPs. In particular, while SUSY usually maps local bosonic to local fermionic operators, here it relates the local bosonic field $\phi(\vec{x}, \tau)$ to a nonlocal fermionic operator dressed with a $\mathbb{Z}_2$ gauge string connected to infinity. This arises because physical operators and states must be $\mathbb{Z}_2$ gauge-invariant, whereas the itinerate Majorana fermions $\hat{c}_i$ alone are not. This “local-nonlocal” SUSY correspondence is also manifest in correlation functions. Emergent SUSY enforces identical anomalous dimensions $\eta_f = \eta_b = \frac{1}{3}$. Although η_b appears in a local phonon correlation, η_f is only revealed in a gauge-invariant nonlocal string operator $\langle \Pi_{\langle lm \rangle \in L}(\tau_l^\mu \tau_m^\mu) \rangle \propto (1/|r_i - r_{i'}|^{2+\eta_f})$, where L is a path connecting sites i and i' and μ is the label of the bond $\langle lm \rangle$ in the Kitaev interaction. This string correlator is precisely the gauge-invariant version of the dynamical Majorana two-point function dressed by a $\mathbb{Z}_2$ gauge string: $\langle \hat{c}_i(\hat{u}_{\langle ij \rangle} \hat{u}_{\langle jk \rangle} \dots \hat{u}_{\langle h'i' \rangle}) \hat{c}_{i'} \rangle$. This is an intrinsically fractionalized feature with no analog in unfractionalized QCPs, where η_f is directly visible in a local fermion correlator.

Finally, the appearance of η_f in a nonlocal string operator is not a fine-tuned property of our Kitaev-SSH model, but rather a general property of the SUSY* fractionalized QCP between two $\mathbb{Z}_2$ spin liquid phases with deconfined $\mathbb{Z}_2$ gauge fields, although the microscopic form of the string operator may become more extended or “thicker” in a perturbed model (e.g., with the inclusion of a small Heisenberg term).

Experimental signatures of SUSY fixed points—Possible experimental signatures of the supersymmetric QCP between the gapless spin liquid and topological order phases are provided by universal scaling exponents of physical quantities. Since the system is an electric insulator, a typical physical quantity of transport is the longitudinal thermal conductivity, which can be obtained from the Kubo formula through analytical continuation: $\kappa^{ii}(\omega) = \kappa_{\text{Kubo}}^{ii}(\omega_n)|_{\omega_n \rightarrow -i\omega + \delta}$, where $\kappa_{\text{Kubo}}^{ii}(i\omega_n) = (1/\omega_n) \langle J_Q^i(\omega_n) J_Q^i(-\omega_n) \rangle$ with $\omega_n = 2\pi nT$ being the Matsubara frequency and $J_Q^i(\omega_n)$ the heat current operator in the spatial direction i . In the high-frequency regime $\hbar\omega \gg k_B T$ and neglecting the contributions of gapped flux excitations, we can show that $\kappa^{ii}(\omega)$ scales as: $\kappa^{ii}(\omega) \propto (i\omega)^{2-\Delta} T^\Delta$ [87] using the

operator product expansion (OPE) method in [90], where $\Delta = 3 - (1/\nu) \approx 1.9098$ is the scaling dimension of the bosonic field $|\phi|^2(\vec{x}, \tau)$ at the critical point [91,92]. In addition, the zero-temperature dynamical shear viscosity $\eta(\omega, T=0)$ also takes a universal form $\eta(\omega, T=0) = \eta_\infty \omega^2 \hbar$ at the SUSY critical point [92], where $\eta_\infty \approx 5.68 \times 10^{-3}$.

Discussions and concluding remarks—In conclusion, we have shown that coupling a Kitaev quantum spin liquid to phonons via an SSH-type interaction yields a rich sequence of fractionalized phases and continuous transitions from a Dirac QSL to incommensurate or commensurate VBS coexisting with $\mathbb{Z}_2$ topological order. Using Lieb’s theorems, large-scale numerical computations, and low-energy field theory, we have demonstrated that quantum phonon fluctuations drive these fractionalized quantum critical points to an emergent $\mathcal{N} = 2$ spacetime SUSY* fixed point. This provides a rare and concrete lattice realization of SUSY in a strongly correlated setting with deconfined fractionalized particles. We further identify universal signatures in thermal transport and shear viscosity that can serve as experimental probes in Kitaev-like materials [56–60] with strong spin-lattice coupling, highlighting spin-phonon interactions as a promising route for engineering emergent supersymmetry in two dimensions.

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Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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