

Anomalous Diffusion and Run-and-Tumble Motion of a Chemotactic Particle in Low Dimensions

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We study the stochastic dynamics of a symmetric self-chemotactic particle and determine the long-time behavior of its mean squared displacement (MSD). The attractive or repulsive interaction of the particle with the chemical field that it generates induces a nonlinear, non-Markovian effective dynamics, which results in anomalous diffusion for spatial dimensions $d \leq 2$. In one spatial dimension, we map the case of repulsive chemotaxis onto a run-and-tumble-like dynamics, leading to an MSD which, as a function of the elapsed time t , grows superdiffusively with exponent $4/3$. In the presence of attractive chemotaxis, instead, the particle exhibits a slowdown, with the MSD growing logarithmically with time. In $d = 2$, we find logarithmic aging of the diffusion coefficient, while in $d = 3$ the motion reverts to standard diffusive behavior with a renormalized diffusion coefficient.

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Introduction—Chemotaxis—the ability to move in response to chemical gradients—is a fundamental mechanism of motion in biological and synthetic systems, governing processes such as bacterial and cellular migration [1,2], the motion of catalytic colloids and swimming droplets [3–10], and even the activity of individual enzymes [11–13]. In these systems, the environment is continuously reshaped by the particles themselves, as they emit the chemical species that drive their own motion and create an evolving chemical landscape. This self-chemotactic behavior represents a paradigmatic example of active matter [14], and it has been extensively studied for its ability to induce self-propulsion when the particle maintains a chemical gradient along its surface [3,15]. Janus colloids, i.e., particles engineered with asymmetric surface activity, represent a standard route to generate such gradients [15–18], but this built-in asymmetry is not essential. In fact, spontaneous symmetry breaking can occur when the interaction between the colloid and the released chemical is repulsive: in this case, the chemical accumulation around the particle destabilizes its stationary configuration, resulting in autonomous propulsion. In the steady state and in the absence of thermal fluctuations, the particle moves with a constant speed that can be predicted analytically [19–22]. However, because of their microscopic size, chemotactic particles are affected significantly by

TABLE I. Long-time behavior of the MSD $\langle r^2(t) \rangle$ depending on the dimensionality d and on the sign of the interaction α with the field. $D(\cdots)$ is an unknown function.

Chemotaxis	MSD		
	$d = 1$	$d = 2$	$d = 3$
Repulsive ($\alpha > 0$)	$t^{4/3}$	$D(\ln t)t$	t
Attractive ($\alpha < 0$)	$\ln t$	$t/(\ln t)^2$	

stochastic forces. Accordingly, in order to understand their transport properties, it is essential to characterize their mean squared displacement (MSD), defined as the mean $\langle r^2(t) \rangle$ of the particle position $\mathbf{r}(t)$, as a function of time t . If the chemical produced by the colloid decays in time, the temporal correlations of the particle's velocity vanish rapidly and the particle behaves diffusively at long times with modified diffusion coefficient [23,24]. However, most chemotactic particles emit chemical species with negligible or no decay on the timescale of their dynamics. In this case, modifications of the chemical landscape by the particle motion induce persistent, nonlinear memory effects on its dynamics. Persistent memory typically leads to anomalous diffusion, as demonstrated in various low-dimensional systems, including trail-interacting particles [25], self-interacting random walks [26–30], and tracers in both passive [31–33] and active baths [34]. Moreover, particles in (possibly nonreciprocal) interaction with thermally fluctuating fields are likewise expected to exhibit anomalous diffusion [35,36]. Existing studies of low-dimensional chemotactic dynamics mostly focus on determining the effective diffusion coefficients [35–39], leaving the precise functional form of the MSD at long times largely

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unexplored. In this Letter, we aim at filling this gap by analyzing a paradigmatic model of a self-chemotactic particle and systematically studying the influence of the spatial dimensionality d and the sign of the (attractive or repulsive) particle-field interaction α on their long-time dynamics. Using a scaling analysis, we identify a critical spatial dimension $d_c = 2$ separating distinct dynamical regimes. For $d > d_c$, the particle exhibits normal diffusion at long times with renormalized diffusion coefficient. For $d \leq d_c$, instead, we find anomalous diffusion: superdiffusion for repulsive interactions and subdiffusion for attractive interactions. To further characterize these behaviors, we map the dynamics onto simpler effective models—a generalized Langevin equation with memory for the attractive case and a biased run-and-tumble model for the repulsive one. These mappings allow us to determine analytically the asymptotic behavior of the MSD, yielding logarithmic growth for the attractive case and superdiffusive growth scaling as $t^{4/3}$ in $d = 1$. These predictions are summarized in Table I.

The model—The dynamics of the system is determined by the evolution of the position $\mathbf{r}$ of the particle and of the concentration field $\phi(\mathbf{x}, t)$ of the produced chemical. The particle, with overdamped dynamics, is subject to a phoretic force proportional to the gradient of the chemical field at its current position, to a viscous friction with coefficient γ , as well as to a stochastic force due to its interaction with a bath in equilibrium at temperature T . The field, instead, diffuses in space with a point source located at the colloid position, due to the chemical emission from the particle. The resulting equations are

$$\gamma \dot{\mathbf{r}}(t) = -\alpha \nabla \phi_{t_R}(\mathbf{r}(t), t) + \boldsymbol{\zeta}(t), \quad (1a)$$

$$\partial_t \phi(\mathbf{x}, t) = \mathcal{D} \nabla^2 \phi(\mathbf{x}, t) + \delta^d(\mathbf{x} - \mathbf{r}(t)), \quad (1b)$$

where $\mathcal{D}$ is the diffusivity of the chemical field, α the phoretic constant, and the noise $\boldsymbol{\zeta}(t)$ has zero mean and variance $\langle \zeta_i(t) \zeta_j(t') \rangle = 2T \delta(t - t') \delta_{ij}$. A similar set of equations appears in diffusion through random media, studied for both passive [40,41] and active [35,36] particles. Here, however, chemical production and the absence of decay leads to anomalous diffusion at all parameter values. Note that, in the presence of a point source, $\nabla \phi$ actually diverges at the particle position for $d > 1$. Accordingly, one needs to regularize the divergence: this introduces a timescale t_R , indicated as a subscript of ϕ in Eq. (1a), which is assumed to be smaller than any other scale of the system. Common regularization schemes are smoothing the δ source in Eq. (1b) with a Gaussian [39,40] or assuming that the chemotactic force at time t is actually the one generated by the chemicals produced up to an earlier time $t' < t$ [37]. Changing scheme does not alter the dynamics (up to numerical factors), provided that the scheme itself does not break the rotational symmetry of the problem. We regularize Eqs. (1) as follows: being $G_d(\mathbf{x}, t) = (4\pi \mathcal{D} t)^{-d/2} e^{-x^2/(4\mathcal{D}t)}$

the solution of the diffusion equation, the field generated at position $\mathbf{x}$ by the colloid with trajectory $\{\mathbf{r}(t'), 0 \leq t' \leq t\}$ is $\phi(\mathbf{x}, t) = \int_0^t dt' G_d(\mathbf{x} - \mathbf{r}(t'), t - t')$. Then we define

$$\phi_{t_R}(\mathbf{x}, t) = \int_0^t dt' G_d(\mathbf{x} - \mathbf{r}(t'), t + t_R - t'), \quad (2)$$

i.e., the field ϕ at position $\mathbf{x}$ but at a later time $t + t_R$, assuming that no chemical is released from t to $t + t_R$. Considering $\phi_{t_R}(\mathbf{x}, t)$ for $t' \simeq t$, it can be shown that $\nabla \phi_{t_R}(\mathbf{x}, t)$ diverges at $\mathbf{x} = \mathbf{r}(t)$ as $t_R \rightarrow 0$ for $d \geq 2$. In particular, this divergence is proportional to the particle velocity: $\nabla \phi_{t_R}(\mathbf{r}, t) \rightarrow g(t_R) \dot{\mathbf{r}}(t) + \text{finite}$, where the finite part is independent of t_R , while g is given by

$$g(t_R) = \begin{cases} (\ln t_R)/(8\pi \mathcal{D}^2) & \text{for } d = 2, \\ -1/(12\pi^{3/2} \mathcal{D}^2 \sqrt{\mathcal{D} t_R}) & \text{for } d = 3. \end{cases} \quad (3)$$

Scaling analysis—To study the long-time behavior of the system we perform a scaling analysis of Eqs. (1) by applying the transformations $\mathbf{r} \rightarrow b\mathbf{r}$ and $\mathbf{x} \rightarrow b\mathbf{x}$ to the space variables with a scale parameter $b \simeq 1$. Time, field, and the phoretic constant are correspondingly scaled as $t \rightarrow b^z t$, $\phi \rightarrow b^{d_\phi} \phi$, $\alpha \rightarrow b^{d_\alpha} \alpha$, where z , d_ϕ , and d_α are determined such that Eqs. (1) are invariant under the scale transformation. Note that the scale transformation reduces the cutoff t_R to t_R/b^z . We use Eq. (3) to restore the original cutoff, by noticing that for $b \simeq 1$ one has $\nabla \phi_{t_R/b^z} = \nabla \phi_{t_R} - z t_R [\partial g / \partial t_R] \dot{\mathbf{r}}(t) \ln b$. The rescaled equations of motion are

$$\gamma b^{1-z} \dot{\mathbf{r}} = -b^{d_\alpha + d_\phi - 1} [\alpha \nabla \phi_{t_R} - \beta_\gamma \dot{\mathbf{r}} \ln b] + b^{-z/2} \boldsymbol{\zeta}(t), \quad (4a)$$

$$b^{d_\phi - z} \partial_t \phi(\mathbf{x}, t) = b^{d_\phi - 2} \mathcal{D} \nabla^2 \phi(\mathbf{x}, t) + b^{-d} \delta^d(\mathbf{x} - \mathbf{r}(t)), \quad (4b)$$

with $\beta_\gamma(t_R) = \alpha z (\partial g / \partial \ln t_R)$ for $d = 3$ or 2 and $\beta_\gamma = 0$ for $d = 1$. Imposing that Eqs. (4) are the same as Eqs. (1) requires $z = 2$, $d_\phi = 2 - d$, $d_\alpha = -d_\phi$, but this is not sufficient. We also transform the viscosity as $\gamma \rightarrow \gamma + \beta_\gamma(t_R) \ln b$ to account for the short-scale correction to the gradient due to the change in cutoff. Since Eqs. (1) are unchanged under the aforementioned transformations, also $\langle \mathbf{r}^2 \rangle$ has to be unchanged. Imposing its invariance for $b \simeq 1$ gives the Callan-Symanzik equation

$$[-2 + \alpha d_\alpha \partial_\alpha + z t \partial_t + \beta_\gamma(t_R) \partial_\gamma] \langle \mathbf{r}^2 \rangle = 0. \quad (5)$$

Using the method of characteristics, we solve Eq. (5) to express the MSD as $\langle \mathbf{r}^2(t) \rangle = D_E(\alpha_E(t), \gamma_E(t), t_R) t$, where the effective diffusivity D_E depends on the effective coupling $\alpha_E(t) = \alpha(t/t_R)^{-d_\alpha/z}$ and viscosity [42]:

$$\gamma_E(t) = \begin{cases} \gamma - \frac{\alpha}{8\pi \mathcal{D}^2} \ln(t/t_R) & \text{for } d = 2, \\ \gamma - \frac{\alpha}{12\pi^{3/2} \mathcal{D}^2 \sqrt{\mathcal{D} t_R}} & \text{for } d = 3. \end{cases} \quad (6)$$

For $d < 2$, $|\alpha_E|$ grows with time and thus the MSD depends on the asymptotic form of D_E at large α , which is unknown from scaling alone. This case will be discussed separately below. For $d \geq 2$ and $\alpha < 0$, Eq. (6) shows how the chemical cloud surrounding the particle generally slows down its dynamics by increasing the effective viscosity $\gamma_E(t)$ of the medium compared to its bare value γ . Moreover, for $d > 2$, α_E vanishes at long times: then, assuming that D_E is a continuous function of α for $\alpha \rightarrow 0$ from the left (as heuristically expected in the attractive case), D_E can be evaluated from the Stokes-Einstein relation $D_E(0, \gamma_E, t_R) = 2dT/\gamma_E^2$ (for $d = 2$ this is an approximation which additionally requires small α). In particular, for $d = 3$, only the chemicals produced at times close to t affect the dynamics of the colloid and thus the effective viscosity approaches a constant value which is function of the bare coupling α . This is shown in Fig. 1(a). By contrast, in $d = 2$, the dynamics is affected by the chemical production at early times, causing a logarithmic growth of the effective viscosity γ_E and thus aging of the system. The corresponding reduction in the effective diffusivity upon increasing time is shown in Fig. 1(b). Finally, with $\alpha > 0$, γ_E in Eq. (6) vanishes at $\alpha = \alpha_c = 12\pi^{3/2}\mathcal{D}^2\sqrt{\mathcal{D}t_R\gamma}$ for $d = 3$ or generically at sufficiently long times for $d = 2$, causing a divergence of D_E . This change of sign corresponds to the onset of the self-propelling regime of the chemotactic particle, similarly to other active systems [34,45–47]. In this regime, the fact that the free theory (i.e., that with $\alpha = 0$) is ill-defined for $\gamma < 0$ signals a discontinuity of D_E as a function of α , which occurs from the right at $\alpha = 0$ and $d = 2$. Therefore, in this regime, the Stokes-Einstein relation cannot account for the dependence of D_E on $\ln t$. The numerical investigation of this case is also difficult, due to the long timescales required to resolve a logarithmic growth, and it is left for future works. For $d = 3$ and $\alpha > 0$, the scaling analysis predicts diffusive dynamics independently of the functional form of D_E . As in two

dimensions, the Stokes-Einstein relation necessarily breaks down for $\gamma_E < 0$, while the numerical simulations in Fig. 1(a) show that this disagreement actually emerges already in the region $0 < \alpha < \alpha_c$.

Attractive case in one dimension—In $d = 1$ the chemical interaction generates non-Markovian effects which affect more strongly the particle dynamics. Note that, now, we can set $t_R = 0$ without encountering divergences. Without loss of generality, we also set $\mathcal{D} = 1$ and $\gamma = 1$. The dependence on $\mathcal{D}$ and γ can be recovered by applying the transformations: $\alpha \rightarrow \alpha/(\gamma\mathcal{D}^2)$, $T \rightarrow T/(\gamma^2\mathcal{D}^2)$, $t \rightarrow t/\mathcal{D}$, $\phi \rightarrow \phi/\mathcal{D}$. Since for $d < 2$ the scaling analysis is inconclusive, a separate study of the attractive and repulsive interactions is required. Consider, first, the attractive case $\alpha < 0$: as a subdiffusive growth of the MSD is expected, the displacement $[r(t) - r(t')]^2$ grows slower than the elapsed time $t - t'$ if the latter is large and/or T is low. Accordingly, we can approximate $\nabla G_1(x, t) \simeq -(1/4\sqrt{\pi})(x/t^{3/2})$ in the expression of $\nabla\phi_{t_R=0}$ following from Eq. (2), to linearize the memory kernel induced by the self chemotaxis. This leads to the generalized Langevin equation for the colloid:

$$\dot{r}(t) - \frac{\alpha}{2\sqrt{\pi}} I_t[\dot{r}] = \zeta(t), \quad (7)$$

where I_t is an effective retarded friction due to chemotaxis, given by $I_t[f] = \int_0^t dt' [(t-t')^{-1/2} - t^{-1/2}] f(t')$. The solution of Eq. (7) (see Ref. [42] for details) implies, at long times,

$$\langle r^2(t) \rangle \simeq \frac{8\pi T}{\alpha^2 a^2(0)} \ln t + c, \quad (8)$$

where $a(0) = 1.39\dots$ while c is an undetermined constant. Numerical simulations confirm this prediction [42].

Repulsive case in one dimension—For $\alpha > 0$ both nonlinearity and memory play a crucial role, such that, contrary to passive systems, the dynamics does not admit a representation in terms of an effective generalized Langevin equation. In the noiseless limit $T \rightarrow 0$, the particle exhibits self-propulsion: in fact, by inserting the ansatz $r(t) = \pm vt$ into Eqs. (1), we find two steady-state solutions with speed $v = \alpha/2$, introducing a natural time-scale $\tau_\alpha = 4/v^2 = 16/\alpha^2$. At small but finite temperature T , the noise induces tumbling events between these two states, inverting the direction of motion. When a tumble occurs, the particle visits again previously explored regions and interacts with the residual chemical field it left behind. This self-generated chemical memory modifies the dynamics and plays a key role in determining its long-time behavior. To investigate this, one needs to understand both the mechanism of tumbling and how it is influenced by the chemical memory. Note that these tumblings—defined as changes in the sign of the velocity—cannot be straightforwardly identified from $\dot{r}(t)$, which is nondifferentiable due to the white noise. To single out the slow dynamics, we

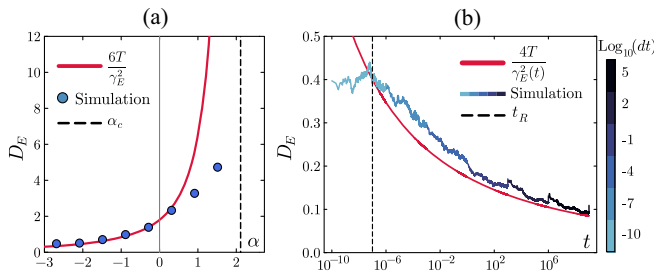


FIG. 1. Effective diffusivity $D_E = \langle r^2(t) \rangle / t$ as a function of (a) α in $d = 3$ at large t for $\mathcal{D} = 1$, $t_R = 10^{-3}$ and (b) t in $d = 2$ with $\alpha = -0.2$, $\mathcal{D} = 0.5$, and $t_R = 10^{-7}$. $\gamma = 1$ in both cases. The red curves follow the Stokes-Einstein relation with effective viscosity γ_E [Eq. (6)], while blue symbols and lines are simulations of Eqs. (1). In panel (b) the time step is increased during the simulation to explore multiple scales [42].

project the trajectory onto low-frequency modes by convolving Eq. (1a) with the exponential kernel $g(t) = \tau^{-1}e^{-t/\tau}$. In particular, we define the coarse-grained position as $r_s(t) = (g * r)(t)$, where $(g * f)(t) = \int_{-\infty}^t dt' g(t-t')f(t')$. We choose the convolution timescale to be $\tau \propto \tau_\alpha$. We assume that fast fluctuations average out and we approximate the convoluted chemotactic force as $g * \nabla\phi(r(t), t) \simeq \nabla\phi(r_s(t), t)$. Under this approximation, $r_s(t)$ evolves according to the same dynamics as $r(t)$ in Eqs. (1), but with the white noise $\zeta(t)$ replaced by the smoothed noise $\zeta_s(t) = g * \zeta(t)$. This filtering allows one to identify the tumblings events from the slow variable $r_s(t)$. To understand how the tumblings occur, we split the chemotactic force $-\alpha\nabla\phi(r_s(t), t)$ into two contributions: one arising from the chemical produced since the last tumble and the other from those released before that time, which we denote by $f_o(r_s(t), t)$. Note that if T is sufficiently low, tumblings are rare and the chemical field has time to relax in the meanwhile. This causes f_o to be a slowly varying function, consistent with the requirement that r_s is also slow. For infinitely slow f_o and ζ_s , the systems admit an adiabatic solution with velocity $\dot{r}_a(t) = \epsilon\alpha/2 + f_o(r_s(t), t) + \zeta_s(t)$, where $\epsilon = \pm 1$ indicates the direction of the self-propelling motion. Expanding $r_s(t) = r_a(t) + \delta r(t)$ and Laplace-transforming the equation for δr we arrive to [42]

$$\mathcal{L}[\delta r](s) = \frac{\alpha\mathcal{L}[\dot{r}_a/v_a^3](s)}{\left[s + \frac{\alpha}{2}v_a - \frac{\alpha}{4}\left(\frac{2v_a^2 + 4s}{\sqrt{v_a^2 + 4s}}\right)\right]}, \quad (9)$$

where $\mathcal{L}$ denotes the Laplace transform and $v_a = |\dot{r}_a(t)|$. The tumbling instability happens when at least one pole of $\mathcal{L}[\delta r](s)$ has a positive real part, leading to exponential growth of $\delta r(t)$. According to Eq. (9) this requires that the particle is slowed down to $v_a < (\sqrt{5} - 2)\alpha$ and therefore that $-\epsilon[f_o(r_s, t) + \zeta_s(t)]$ exceeds the critical force $f_c = (5/2 - \sqrt{5})\alpha$. We numerically verified this tumbling condition in the deterministic case [42]. The rate $\lambda^\epsilon(f_o)$ with which fluctuations in ζ_s cause $f_o + \zeta_s$ to satisfy the tumbling condition are given by [42]

$$\lambda^\epsilon(f_o) \simeq \frac{\alpha(f_c + \epsilon f_o)B}{4\sqrt{2\pi TA}} \exp\left[-8A \frac{(f_c + \epsilon f_o)^2}{\alpha^2 T}\right]. \quad (10)$$

Here, A and B are two fitting parameters: A optimizes the scale separation between fast and slow dynamics (thus fixing the coarse-grained timescale $\tau = A\tau_\alpha$), while B accounts for subleading corrections of order $O(T)$ in f_c , v_a , and τ_α . Their values are determined by measuring the rate $\lambda_0 = \lambda^\pm(0)$ of the first tumbling from simulations of Eqs. (1) at various α and T and fit Eq. (10). To conclude the derivation of the effective run-and-tumble dynamics we express $f_o(x, t)$ in terms of the positions at which previous tumblings occurred. As at low T the separation between

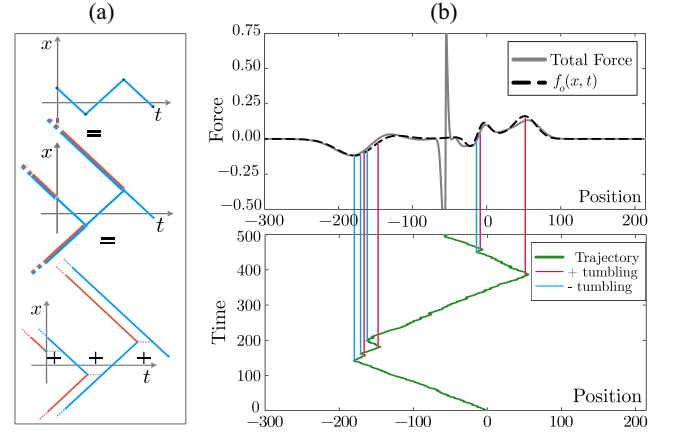


FIG. 2. (a) The field generated by a run-and-tumble trajectory in $d = 1$ (blue line, upper plot) can be decomposed as the sum of those generated by the segments of trajectories coming from $\pm\infty$ (central plot). Red trajectories generate a field with opposite sign than the blue and they can be paired as shown in the lower plot. (b) Top panel: force field $\alpha\nabla\phi(x, t)$ (solid line) from numerical integration of Eq. (2) along a trajectory generated by simulating Eqs. (1), compared with f_o (dashed line) from Eq. (11), as functions of x at $t = 500$. Bottom panel: corresponding particle trajectory (green): it clearly emerges that tumblings are point sources for f_o .

two consecutive tumblings is large compared to the typical length scale $l = \mathcal{D}/v$ of the dynamics, we can study f_o in the limit $l \rightarrow 0$. We start by considering the gradient $\partial_x \phi_\pm$ of the chemical field $\phi_\pm$ produced by a colloid moving with constant velocity $\pm v$ up to time t . As $\phi_\pm$ changes significantly only close to the particle position $r(t)$, at large scales one has $\partial_x \phi_\pm \simeq \mp l^{-1} \delta(x - r(t))$, see Ref. [42]. The chemical field generated by a run-and-tumble trajectory, with tumblings occurring at the space-time points (x_i, t_i) , can then be decomposed as illustrated in Fig. 2(a). For each tumbling, one adds (blue) and subtracts (red) the field ϕ_- or ϕ_+ due to a fictitious particle reaching x_i from $|x| = \infty$ by moving with a velocity opposite to that of the trajectory right before the tumbling. By pairing these contributions, one finds that the i th tumbling generates a field $\epsilon_i(\phi_+ - \phi_-)$, where ϵ_i indicates the sign of the velocity along the trajectory immediately before the tumbling. This field acts as a point source of chemical gradient, causing a chemotactic force field $f_i(x, t) = 4\epsilon_i G_1(x - x_i, t - t_i)$. f_o is then given by the sum of the contributions due to the various tumblings, plus the one of the unpaired trajectory from the initial condition, i.e.,

$$f_o(x, t) = 2\epsilon_1 G_1(x, t) + 4 \sum_{i=1} \epsilon_i G_1(x - x_i, t - t_i). \quad (11)$$

Equations (10) and (11) define an effective run-and-tumble dynamics influenced by the chemotactic memory, which we simulate and compare with the original dynamics, as shown in Fig. 3. In particular, by fitting the measured λ_0

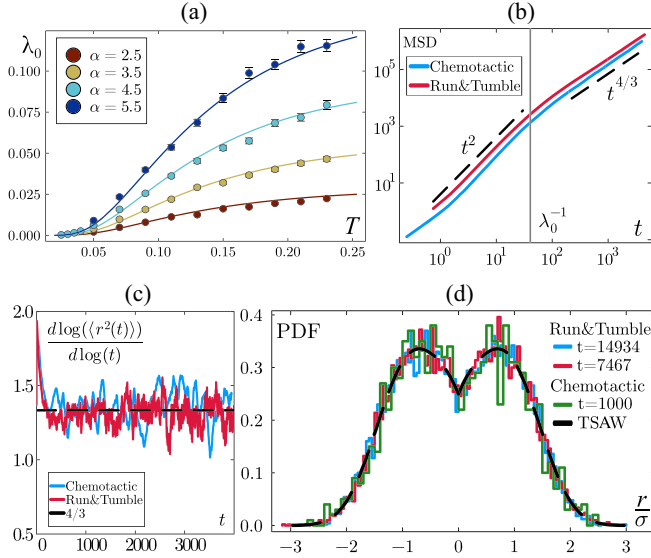


FIG. 3. (a) λ_0 as a function of T at varying α . Symbols: simulations of Eqs. (1); solid lines: fit to Eq. (10). (b) MSD $\langle r^2(t) \rangle$ as a function of t for the chemotactic (blue) and run-and-tumble dynamics (red). Both models display superdiffusive behavior $\propto t^{4/3}$ at long times and ballistic behavior $\propto t^2$ at $t < \lambda_0^{-1}$ (vertical line). (c) Effective exponent $z = [d \log(\langle r^2(t) \rangle) / d \log(t)]$ as a function of t , showing the crossover from ballistic behavior to superdiffusion. (d) Empirical PDF of the rescaled position $r/\sigma(t)$, for the chemotactic (green) and run-and-tumble (red and blue) dynamics, with $\alpha = 3, T = 0.15$, compared with the analytical PDF of the TSAW (dashed line) [50]. See Ref. [42] for details.

with Eq. (10), we obtain $A = 0.42$ and $B = 8.05$. The quality of the resulting fit is shown in Fig. 3(a) at varying T and α . The MSD as a function of t and the probability distribution function (PDF) of $r/\sigma(t)$ [$\sigma^2(t) = \langle r^2(t) \rangle$] at two (large) t , resulting from the effective run-and-tumble dynamics and from the chemotactic Eqs. (1) are compared in Figs. 3(b) and 3(c), respectively, showing good agreement. In particular, the MSD for both models show the same scaling exponent and PDF at long times, which we now derive for the latter. Let $P(r, t)$ denote the probability of finding the chemotactic particle at position r and time t and $\bar{f}(r, t) = \langle f_o(r, t) \rangle$ the force acting on the particle, averaged over all possible past trajectories. Within the mean-field approximation and for large t they satisfy [42]:

$$\partial_t P(r, t) = \frac{\alpha^2}{8\lambda_0} \partial_r^2 P(r, t) + \frac{\alpha\lambda_1}{4\lambda_0} \partial_r [\bar{f}(r, t) P(r, t)], \quad (12a)$$

$$\bar{f}(r, t) = -\alpha \int_0^t dt' \int dr' P(r', t') \partial_r G_1(r - r', t - t'), \quad (12b)$$

with $\lambda_1 = [\partial \lambda^+(f) / \partial f]_{f=0}$. In the first equation, the diffusion term stems from the run-and-tumble motion and competes with the advective term due to f_o . Using the ansatz $P(r, t) = t^{-z} U(r/t^z)$ in Eqs. (12) with $z > 1/2$, one finds $z = 2/3$, such that $\langle r^2(t) \rangle \propto t^{4/3}$ [42]. This prediction is confirmed by the numerical simulations of

both the chemotactic model in Eqs. (1) and the effective run-and-tumble dynamics given by Eqs. (10) and (11), see Fig. 3(b). Remarkably, this exponent coincides with that one of the “true” self-avoiding random walk (TSAW) [26,27,48,49], a random walk in which the probability of visiting a site decreases with the number of previous visits to it. In fact, since the motion of the particle is superdiffusive, at long times the spreading of the chemical by diffusion is negligible compared to the typical particle displacement: the chemical is effectively frozen at its emission points and this fact turns the run-and-tumble into a TSAW. This is confirmed by the agreement between the PDF obtained from both models and the analytical solution of the TSAW [50,51], see Fig. 3(c).

Conclusions—We have shown that non-Markovian, chemically mediated interactions cause anomalous diffusion of self-chemotactic particles for $d \leq 2$. This observation has profound implications for the collective behavior of chemically active matter [52,53]. While most theoretical descriptions of chemotactic particles rely on the assumption of fast chemical diffusion $\mathcal{D} \rightarrow \infty$ [54,55], this assumption breaks down for $\alpha > 0$ and $d \leq 2$, as particles become superdiffusive. In this case, the relevant scaling limit is $\mathcal{D} \rightarrow 0$, suggesting that collective chemotaxis in low dimensions may belong to a fundamentally different universality class than its higher-dimensional counterpart. Understanding these emergent behaviors remains an open and compelling challenge. Another important question concerns the impact of adding nonlinearities in the dynamics of Eqs. (1). Note that in $d = 1$ ($d = 2$), ϕ has a positive (vanishing) scaling dimension: based on power counting, this allows the introduction of an infinite number of relevant (marginal) interaction terms, even in the absence of chemical decay. While these terms do not appear in the renormalization-group flow in Eq. (5), they grow (or remain finite) under scaling if present at the microscopic level and thus potentially affect the dynamics at long times. Beyond these aspects, an interesting generalization of our work involves replacing chemical production in Eq. (1b) with an interaction which conserves its overall mass, encompassing the case of a probe nonreciprocally coupled to a bath [56]. Finally, it would be valuable to consider statistical properties beyond transport, for example examining how memory influences first-passage properties, which are central to search and sensing strategies in active systems.

Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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