

Complete Function Space for Planar Two-Loop Six-Particle Scattering Amplitudes

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We derive the full system of canonical differential equations for all planar two-loop massless six-particle master integrals and determine analytically the boundary conditions. This fully specifies the solutions, which may be written as Chen iterated integrals. We argue that this is sufficient information for evaluating any scattering amplitude in four dimensions up to the finite part. We support this claim by reducing, for the most complicated integral topologies, integrals with typical Yang-Mills numerators. We use the analytic solutions to the differential equations, together with dihedral symmetry, to provide the full solution space relevant for two-loop six-particle computations. This includes the relevant function alphabet, as well as the independent set of iterated integrals up to weight four. We also provide the answer for all master integrals in terms of iterated integrals that can be readily evaluated numerically. As a proof of concept, we provide a numerical implementation that evaluates the integrals in part of the Euclidean region and validate this against numerical evaluation of the Feynman integrals. Our result removes the bottleneck of the Feynman integral evaluation, paving the way for future analytic evaluations of six-particle scattering amplitudes.

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Introduction—About ten years ago, obtaining phenomenological predictions for two-to-three collider processes beyond the next-to-leading order was widely considered to be impossible. However, this changed with breakthroughs in our ability to evaluate the complicated two-loop Feynman integrals and to think about the results in terms of Chen iterated integrals [1]. This involved many steps: deriving suitable differential equations for the master integrals, solving them in the relevant physical regions, assembling full scattering amplitudes leveraging finite field ideas, etc. This is nicely reviewed in a recent thesis [2]. Indeed, the last years have seen tremendous progress: analytical calculations for two-loop five-point massless integrals have been achieved [3–7], enabling important phenomenological applications, as demonstrated in Refs. [8–11]. Furthermore, recent efforts have extended these findings to processes involving one off-shell leg, with notable studies such as [12–15] for the master integrals and [16] for scattering amplitudes. In this Letter, we take an

important step into the next frontier: we provide the full information on the master integrals and their analytic solutions relevant for planar, massless two-loop six-particle scattering processes. We also provide a proof-of-principle numerical evaluation.

Beyond its phenomenological implications, this analytic information is also of great theoretical interest. There has been a resurgence of interest and new mathematical ideas in the Landau method to predict the singular locus of Feynman integrals [17–19], which is a way of constraining the function space. Moreover, starting from results in maximally supersymmetric Yang-Mills theory (sYM), several studies have looked into possible cluster algebra interpretations of function spaces [20–22]. Our six-particle results for the function alphabet provide a highly nontrivial data point and benchmark for these ideas. Moreover, work in sYM has also revived interest in the analytic structure of amplitudes, such as the Steinmann relations [23], adjacency properties, crossing relations [24], questions of factorization in certain channels [25], and intriguing antipodal relations [26]. By providing the complete function space, our Letter opens the possibility of investigating similar questions for generic Yang-Mills theories.

We use the fact that all integrals relevant to Yang-Mills amplitudes can be reduced to scalar integrals. These scalar integrals are further reduced, using integration-by-parts (IBP) identities, to a set of basis integrals, which we call

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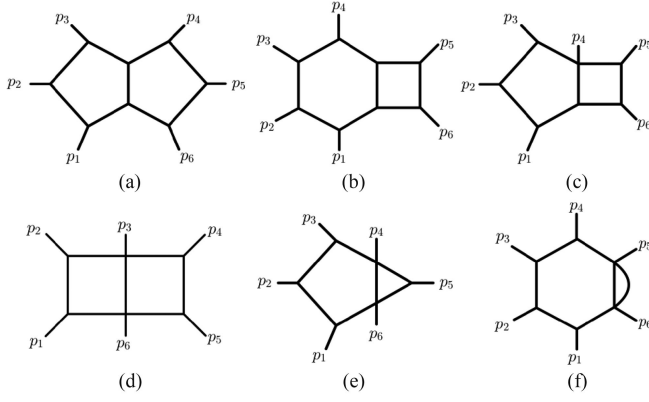


FIG. 1. Two-loop planar Feynman diagrams with generic six-particle kinematic dependence.

master integrals [27,28]. The computation of these integrals is performed using differential equations [29–31]. The latter take a greatly simplified, canonical form [32], if one chooses a basis of uniform transcendent (UT) weight integrals [33].

We use this approach for most of the relevant integrals. However, for the most complicated integrals, of the type shown in Fig. 1(a), we proceed differently. Specifically, we show that the relevant integrals for computing two-loop amplitudes up to and including the finite part can be related to integrals known in the literature.

Notation and kinematics—We consider six-particle scattering in $D = 4 - 2\epsilon$ dimensional Minkowski space $\mathbb{M}^D$. The six external momenta $p_i \in \mathbb{M}^D$, $i = 1, \dots, 6$, are on shell, i.e., $p_i^2 = 0$ and satisfy momentum conservation, $\sum_{i=1}^6 p_i = 0$. For planar integrals, it is useful to introduce dual coordinates. The momenta form a closed polygon, whose vertices are dual coordinates z_i . In other words,

$$z_{i+1} - z_i = p_i, \quad (1)$$

with $z_7 \equiv z_1$. Moreover, we introduce dual coordinates y_1, y_2 corresponding to the loop momenta. Any scalar planar two-loop hexagon integral can then be written as

$$I_{\vec{a}} = e^{2\epsilon\gamma_E} \int \frac{d^D y_1}{i\pi^{D/2}} \frac{d^D y_2}{i\pi^{D/2}} \frac{1}{[-(y_1 - y_2)^2]^{a_{13}}} \times \frac{1}{\prod_{j=1}^6 [-(z_j - y_1)^2]^{a_j} [-(z_j - y_2)^2]^{a_{j+6}}}. \quad (2)$$

Here γ_E is the Euler-Mascheroni constant, and $\vec{a}$ is a vector of 13 elements that correspond to the powers of the propagator factors. The latter can take positive or negative values. To translate back to usual momentum space, one may fix, e.g., $z_1 = 0$, and use $z_1 - y_1 = l_1$, $z_1 - y_2 = l_2$ for the loop momenta.

For kinematics in D dimensions, the integrals $I_{\vec{a}}$ depend on nine independent scalar Poincaré invariants, which we chose as

$$\{z_{13}^2, z_{24}^2, z_{35}^2, z_{46}^2, z_{15}^2, z_{26}^2, z_{14}^2, z_{25}^2, z_{36}^2\}, \quad (3)$$

where we abbreviated $z_{ij} \equiv z_i - z_j$. For example, $z_{13} = (p_1 + p_2)^2$ and $z_{14}^2 = (p_1 + p_2 + p_3)^2$.

In the following, we will take the external kinematics to lie in a four-dimensional subspace of D . In this case, the five momenta $p_1, \dots, p_5$ become linearly dependent, which translates into a constraint on the invariants [Eq. (3)]. It is advantageous to use momentum twistor variables [34], which are unconstrained four-dimensional variables (i.e., they also solve the on-shell and momentum conservation conditions). We choose the particular momentum twistor parametrization of section 2 of [35], where the eight independent variables are denoted by $\vec{x}$.

There is also one pseudoscalar Poincaré invariant, which we introduce in order to fully specify the scattering kinematics. We choose it as

$$\epsilon_{1234} \equiv 4\sqrt{-1}\epsilon_{\mu_1\mu_2\mu_3\mu_4}p_1^{\mu_1}p_2^{\mu_2}p_3^{\mu_3}p_4^{\mu_4}, \quad (4)$$

where ϵ is a totally antisymmetric tensor. Moreover, we define the Gram determinant

$$G\left(\begin{matrix} u_1 \dots u_n \\ v_1 \dots v_n \end{matrix}\right) = \det(2u_i \cdot v_j), \quad (5)$$

where the right side is the determinant of the $n \times n$ matrix with entries $2(u_i \cdot v_j)$, $1 \leq i, j \leq n$. We also introduce the abbreviation

$$G(i_1, \dots, i_k) \equiv G\left(\begin{matrix} p_{i_1} \dots p_{i_k} \\ p_{i_1} \dots p_{i_k} \end{matrix}\right). \quad (6)$$

Integral family and the canonical differential equation—The full set of planar two-loop integral families with genuine six-particle kinematics is displayed in Fig. 1. All planar two-loop six-point integrals belong to the double pentagon (DP) in Fig. 1(a) or the hexagon box (HB) in Fig. 1(b), or to permutations thereof.

The state-of-the-art method for evaluating such integrals involves choosing a basis of master integrals $\vec{I}$ such that they satisfy canonical differential equations [32], i.e.,

$$d\vec{I}^{\text{fam}}(\epsilon, \vec{x}) = \epsilon d\tilde{A}^{\text{fam}}(\vec{x}) \cdot \vec{I}^{\text{fam}}(\epsilon, \vec{x}), \quad (7)$$

where fam denotes the integral family and $\tilde{A}^{\text{fam}}$ is a matrix whose dimension depends on the number of master integrals. It is a 267×267 matrix for fam = DP and a 202×202 matrix for fam = HB. Moreover, it has a block structure, as shown in Fig. 2.

In our previous work [35], the integral families shown in Figs. 1(d)–1(f), corresponding to the orange block with diagonal lines in Fig. 2, were evaluated. Additionally, in Ref. [36], the canonical basis for integral families shown in

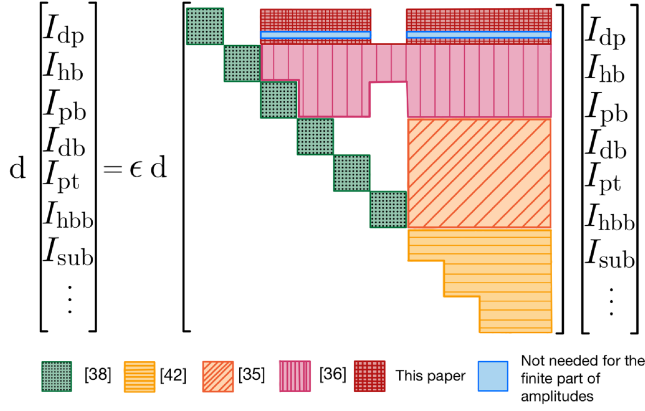


FIG. 2. Schematic representation of the block structure of the differential equation. Blank spaces indicate zero entries. The colored blocks of the matrix were considered previously in the respective references.

Figs. 1(b) and 1(c) were determined by verifying the ϵ -dependence of Eq. (7), but the functional dependence of the DE matrix was not yet reconstructed. This is represented with the pink blocks with vertical lines in Fig. 2. In the present Letter, we determine the UT weight integral basis for the full DP sector and present the analytic solution up to weight four for the DP and HB sectors. We derive the differential equations using momentum twistor variables in the form of [37] and verify the ϵ -factorized form of Eq. (7).

In order to achieve the canonical form [Eq. (7)], we chose the top sector for the DP family as in Ref. [38]. For the HB and pentagon-box (PB) families, we used the Baikov analysis in order to choose a simpler basis compared to the one used in [38] (see Ref. [36]). To achieve this, we employed a private code, included as an ancillary file, based on the ideas discussed in Ref. [39].

Let us assume that the connection matrix of the canonical differential equations [Eq. (7)] is expanded in terms of logarithms, i.e.,

$$\tilde{A}(\vec{x}) = \sum_j c_j \log[\alpha_j(\vec{x})], \quad (8)$$

where $c_j \in \mathbb{Q}$ and the so-called alphabet letters $\alpha_j(\vec{x})$ are algebraic functions of the kinematic variables.

In principle, one could perform a full analytic IBP-reduction to determine matrix $\tilde{A}$ in Eq. (7). However, if one knows the form of the entries of this matrix in advance, one may use finite field methods [40,41] to reconstruct them, as in our previous work [35]. This significantly lowers the computational cost and speeds up the computation. Our ansatz for the alphabet involves known letters from the literature for five-point one-mass integrals [42], supplemented with additional algebraic letters that we constructed using the method outlined in [2,36,43], using the computer algebra implementation [44]. Under these assumptions, we can profit from finite field methods to obtain the

coefficients c_j . The idea is to perform the IBP reductions for several numerical values of the external invariants, i.e., as many as the number of unknown coefficients, modulo several choices of prime numbers and fit the coefficients c_j over the corresponding prime fields. We then use Wang's rational reconstruction algorithm and the Chinese remainder theorem to obtain the full set of coefficients c_j over the rational field.

In this way, we were able to construct the full matrix $\tilde{A}^{\text{fam}}$ for the DP and HB families, except for entries associated to one six-dimensional double-pentagon integral (blue line in Fig. 2). This validates the assumed form of Eq. (8) for those cases. The UT integral basis and the canonical differential equation matrix are provided as auxiliary files. The missing matrix elements, which may or may not be of the form of Eq. (8), could in principle be reconstructed using a full functional reconstruction. However, as we discuss presently, the information that we have already computed is all that is needed for two-loop amplitude computations up to and including the finite part.

Relevant functions for 4D Yang-Mills scattering amplitudes—Let us explain why no new information is needed for the most complicated double pentagon sector shown in Fig. 1(a). There are five master integrals [38], i.e., in principle up to five new functions to be computed. However, our goal is to compute two-loop amplitudes up to the finite part only. This allows us to truncate the expansion in ϵ of the basis integrals. The argument uses the following ingredients: (1) two-loop amplitudes in four dimensions are widely believed to involve terms up to transcendental weight four only [45] and (2) the canonical form of the differential equations makes the transcendental weight of the basis integrals manifest.

We argue that it is sufficient to expand the basis integrals up to weight four. Note that, for a random basis, this is in general not enough as the IBP-reduction coefficients may in general involve poles in ϵ , which in turn would require us to expand the basis integrals to higher orders (cf., e.g., the discussion in Ref. [46]). However, we argue that this does not occur for uniform weight integrals (that have the same weight as the amplitude): indeed, if a UT integral with a nonzero ϵ^k order meets a pole $\mathcal{O}(1/\epsilon^k)$, this would yield a contribution of weight higher than four to the finite part of the amplitude, in contradiction with point (1) above. We support this statement by computing IBP-reductions to master integrals for a complete set of integrals up to rank six that are expected to appear in Yang-Mills theories and reconstructing their dependence on the dimensional regulator ϵ on the maximal cut. We have also extended this analysis without cut for a selection of typical integrals contributing to Yang-Mills amplitudes. We found that all coefficients of these IBP-reductions are finite in the limit $\epsilon \rightarrow 0$, except for coefficients multiplying the six-dimensional double pentagon master integral defined in Ref. [38], whose coefficients are generally of $\mathcal{O}(1/\epsilon)$.

These can however be shown to be harmless since this integral is $\mathcal{O}(\epsilon^2)$; hence its contributions to the finite part of an amplitude vanish, even when multiplied by a pole in the dimensional regulator.

So let us turn to the basis integrals employed in Ref. [38] and take their four-dimensional limit. This is straightforward due to the good choice of local integrands [33] made in Ref. [38]. In fact, the UT integral I_4^{DP} , a finite six-dimensional double pentagon integral, starts contributing at weight six only, i.e., at order ϵ^2 , so we can discard it [47]. Similarly, I_1^{DP} , a parity-odd combination of dual-conformal-invariant integrals, is known to be evanescent [48]. The integral I_3^{DP} is evanescent due to a Gram determinant that vanishes in four dimensions and due to its good infrared behavior. So these integrals can also be dropped.

It turns out that the remaining integrals are just the parity-even and parity-odd parts of well-known integrals! Indeed, we have

$$I_2^{\text{DP}} = -2\tilde{\Omega}_{\text{odd}} + \mathcal{O}(\epsilon), \quad (9)$$

$$I_5^{\text{DP}} = 2\Omega_{\text{even}} + \mathcal{O}(\epsilon), \quad (10)$$

where Ω and $\tilde{\Omega}$ are the finite, dual conformal double pentagon integrals defined in [33] and computed in [48]. So the double pentagon top sector UT integrals are either evanescent or known from the studies of $\mathcal{N} = 4$ sYM.

Furthermore, we find that $\tilde{\Omega}_{\text{odd}}$ and Ω_{even} can also be represented as subsector integrals (up to higher weight terms). We discovered these relations from the symbol results (see discussion below) and then verified them numerically. For $\tilde{\Omega}_{\text{odd}}$, there is a simple relation

$$\begin{aligned} \tilde{\Omega}_{\text{odd}} = & \frac{\Delta_6 G \left(\begin{smallmatrix} l_2 - p_5 - p_6 & p_5 & p_6 & p_2 & p_3 \\ l_1 + p_5 + p_6 & p_5 & p_6 & p_2 & p_3 \end{smallmatrix} \right)}{4\epsilon G(p_5, p_6, p_2, p_3)} \quad \begin{array}{c} p_3 \\ p_4 \\ p_5 \\ p_2 \\ p_1 \end{array} \\ & - \frac{\Delta_6 G \left(\begin{smallmatrix} l_2 - p_6 & p_5 & p_4 & p_2 & p_1 \\ l_1 + p_6 & p_5 & p_4 & p_2 & p_1 \end{smallmatrix} \right)}{4\epsilon G(p_5, p_4, p_2, p_1)} \quad \begin{array}{c} p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_1 \end{array} + \mathcal{O}(\epsilon), \end{aligned} \quad (11)$$

where the dot indicates a power of $a_{13} = 2$. The weight-four reduction relation for Ω_{even} is provided in an auxiliary file.

Analytic results for the master integrals—The canonical differential Eqs. (7) and (8) determine the complete analytic structure of the integrals. Let us now discuss their structure, limiting ourselves to the part up to weight four.

We showed that the top sector of the DP family is not needed for results up to and including weight four in scattering amplitudes. So let us focus on the differential equations for HB and for all subsectors of DP, as well as any integrals related to them by dihedral symmetry [49].

We find that a total of 245 distinct linearly independent letters appear in them.

However, as we show below, when solving these differential equations, only a subset of 167 letters appears up to a transcendental weight four. The majority of these letters come from five-point subsector integrals, known from Ref. [42], while only 13 of them are genuine six-particle letters. Starting at weight four, one new letter (α_{138}) appears in the sector shown in Fig. 1(f), while 12 new letters ($\alpha_{100}, \dots, \alpha_{105}, \alpha_{242}, \dots, \alpha_{247}$) appear in the sector shown in Fig. 1(e).

We use the symbol [1] to describe the full six-particle function space up to transcendental weight four. The symbol map can be defined through its action on the Chen iterated integrals where it maps a k -fold iterated integral to a k -fold tensor product [1]. On the one hand, symbols are a useful tool for organizing the function space, cf. Refs. [4,50], and on the other hand, they are a crucial ingredient for bootstrap approaches. [51] It is straightforward to extract the symbol from the differential equation [Eq. (7)] at each transcendental weight. Taking into account dihedral symmetry and modding out linear dependencies, we find the full set of symbols relevant up to weight four. We also include symbols that appear from products of one-loop integrals [52], as such terms are relevant for studies of infrared properties and of the exponentiation of amplitudes. The results are summarized in Table I and are given in ancillary files. The first row in this table corresponds to the total number of independent symbols at each transcendental weight, including products of symbols appearing in one-loop integrals (the number of which is given in the second row). The next to the last row corresponds to symbols from previously known two-loop five-point one-mass symbols, while the last row corresponds to genuine two-loop six-point symbols [53].

Having identified the complete alphabet relevant for four-dimensional Yang-Mills scattering amplitudes, it is interesting to compare it with information that can be obtained from different approaches. References [18,19] use methods from computational algebraic geometry to analyze the locus of singularities of Feynman integrals. In comparison with our results, we find that this method fails to identify 18 distinct components of the singular locus. Specifically, 15 of these correspond to integrals of the type shown in Fig. 1(e) and to its subsectors, and the remaining three are leading singularities of the top sector

TABLE I. Counting of independent symbols.

Transcendental weight	1	2	3	4
Symbols	9	62	319	945
One-loop squared symbols	9	59	221	428
Two-loop five-point symbols	9	59	263	594
Genuine two-loop six-point symbols	0	0	3	45

integrals of HB, PB, and DB. Reference [54] is based on the Baikov representation of Feynman integrals. We find that it produces all letters except the letter α_{100} that appears in the family in Fig. 1(e).

Let us now give more details on the analytic solutions of the differential equations (7). We write them as a series expansion around $\epsilon = 0$:

$$\tilde{I}(\epsilon, \vec{x}) = \sum_{k=0}^{\infty} \epsilon^k \tilde{I}^{(k)}(\vec{x}), \quad (12)$$

where $\tilde{I}^{(k)}(\vec{x})$ denotes functions of transcendental weight k given as k -fold iterated integrals

$$\tilde{I}^{(k)}(\vec{x}) = \tilde{I}^{(k)}(\vec{x}_0) + \int_{\gamma} dA(\vec{x}') \tilde{I}^{(k-1)}(\vec{x}'), \quad (13)$$

where $\tilde{I}^{(k)}(\vec{x}_0)$ is the boundary value at reference point $\vec{x}_0$. The boundary values are analytically determined at the reference point $\vec{x}_0 = \{-1, 1, 0, 0, 1, 1, 0, 0\}$ in momentum twistor space that corresponds to all Mandelstam variables of Eq. (3) being equal to -1 . We do so by imposing the regularity of our basis integrals throughout the Euclidean region [55–57]. At transcendental weights one and two, the solutions are expressed in terms of a basis of classical polylogarithms. We provide them in the ancillary files `Weight2Solutions_<fam>.m`. At transcendental weights three and four, we express the solutions in terms of one-fold integrals over the weight-two functions [4,58]:

$$\begin{aligned} \tilde{I}^{(3)}(\vec{x}) &= \tilde{I}^{(3)}(\vec{x}_0) + \int_0^1 dt \frac{dA}{dt} \tilde{f}^{(2)}(t), \\ \tilde{I}^{(4)}(\vec{x}) &= \tilde{I}^{(4)}(\vec{x}_0) + \int_0^1 dt \left(\frac{dA}{dt} \tilde{f}^{(3)}(\vec{x}_0) \right. \\ &\quad \left. + [A(1) - A(t)] \frac{dA}{dt} \tilde{f}^{(2)}(t) \right), \end{aligned} \quad (14)$$

where t parametrizes a straight line between boundary point $\vec{x}_0$ and point $\vec{x}$ [59]. This representation is used for numerical verification of the master integrals at four different points in the momentum twistor parameterization, within the bulk of the Euclidean region:

$$\begin{aligned} \vec{x}_1 &= \left\{ -\frac{299}{300}, \frac{221}{200}, \frac{1}{300}, \frac{1}{200}, \frac{53}{50}, \frac{21}{20}, \frac{1}{100}, \frac{1}{700} \right\}, \\ \vec{x}_2 &= \left\{ -\frac{31}{30}, \frac{161}{150}, \frac{43}{875}, -\frac{17}{630}, \frac{10153}{9450}, \frac{243}{250}, -\frac{22}{75}, \frac{1}{30} \right\}, \\ \vec{x}_3 &= \left\{ -\frac{17}{10}, \frac{51}{50}, \frac{109}{3670}, -\frac{197}{5505}, \frac{3799}{3670}, \frac{774}{815}, -\frac{3}{20}, \frac{1}{25} \right\}, \\ \vec{x}_4 &= \left\{ -1, \frac{9021}{8950}, \frac{7}{1250}, -\frac{17}{250}, \frac{2003}{1790}, \frac{466}{475}, -\frac{637}{3580}, \frac{5}{179} \right\}. \end{aligned} \quad (15)$$

The 40-digit results that we computed from one-fold integration agree with those obtained using AMFlow [60,61].

For the DP family, all subsector integrals overlap with other families, while the two top sector integrals contributing at weight four are known from the studies of $\mathcal{N} = 4$ sYM; see Eqs. (9) and (10). Therefore, we use integral representations of Ω_{even} and $\tilde{\Omega}_{\text{odd}}$ from [48] for numerical validation of the two integrals at the point $\vec{x}_1$. We find perfect agreement with numerical results obtained using AMFlow [62].

Conclusion—In this Letter, we analytically calculated all two-loop six-point planar massless integrals and determined the corresponding function space, up to weight four. This is a breakthrough in the field of analytic Feynman integral computation that provides the relevant information for the calculation of planar amplitudes of $2 \rightarrow 4$ massless QCD processes. These results allow the community to initiate research on next-to-next-to-leading order precision physics with four massless final states.

Building upon recent previous work [35,36,38], we provided the complete UT basis for all planar two-loop six-point integral families. We determined the analytic boundary values and expressed the UT integrals as Chen iterated integrals. Furthermore, we showed that the double pentagon top sector's UT integrals, up to weight four, can be directly obtained from known functions [48] in the maximally supersymmetric theory. The complete alphabet and the symbol space for two-loop six-point massless integrals, which can be used for future bootstrap applications, are provided. The fully analytic information that we provide can also be used to study interesting physical limits of six-particle scattering processes, such as multi-Regge [63] or collinear limits (see, e.g., [64]), or the double parton scattering limit of Ref. [65], for example.

We provided a proof-of-concept evaluation of the integral in the Euclidean region. A natural follow-up to this, relevant for phenomenological applications, is to develop a dedicated code for numerical evaluation in physical scattering regions, e.g., following Refs. [13,15].

Another possibility opened up by our Letter, together with recent progress on three-loop five-point Feynman integrals [66], is to envisage next-to-next-to-next-to-leading order (N3LO) cross section computations for three final states. Our integrals contribute to the double-virtual single-real contribution to observables with five massless external particles at N3LO. Based on the argument presented in [67], one may expect that the weight four results presented here are sufficient for this purpose.

Finally, we focused on reconstructing the part of the differential equations matrix relevant for computing all functions up to and including the finite part, cf. Fig. 2. It would be interesting to reconstruct the remaining matrix elements, e.g., to see whether they correspond to novel singularities. Determining those terms would allow us to expand the solutions to the differential equations to any order in the dimensional regulator. This could be of interest

for studying the infrared structure of scattering amplitudes at higher loop orders.

Note added—while this manuscript was in the final stage of preparation, the preprint [68] appeared, which overlaps with some of the results presented here. As far as we can tell, the results are in agreement.

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Views and opinions expressed are, however, those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

Data availability—The data that support the findings of this Letter are openly available [72], embargo periods may apply.

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