

3D Conformal Field Theories with $\mathrm{Sp}(N)$ Global Symmetry on a Fuzzy Sphere

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The quest to discover new 3D CFTs has been intriguing for physicists. For this purpose, fuzzy sphere regularization that studies interacting quantum systems defined on the lowest Landau level on a sphere has emerged as a powerful tool. In this Letter, we discover a series of new CFTs with global symmetry $\mathrm{Sp}(N)$ in the fuzzy sphere models that are closely related to the $\mathrm{SO}(5)$ deconfined phase transition, and are related to a $\mathrm{Sp}(N)/[\mathrm{Sp}(M) \times \mathrm{Sp}(N-M)]$ nonlinear sigma model with a Wess-Zumino-Witten term. We numerically verify the emergent conformal symmetry by observing the integer-spaced conformal multiplets, the quality of conformal generators, and studying the finite-size scaling of the conformality. We discuss possible candidates for these newly discovered CFTs, the most plausible ones being Chern-Simons-matter theories which have N flavors of gapless bosons or fermions coupled to a non-Abelian [*viz.* $\mathrm{Sp}(1)$, $\mathrm{Sp}(2)$, etc.] Chern-Simons gauge field. Our Letter provides new avenues for studying interacting CFTs in 3D, possibly facilitating the nonperturbative study of critical gauge theories and previously undiscovered CFTs.

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Introduction—Conformal field theory (CFT) is one of the central topics of modern physics. It has provided invaluable insights into critical phenomena in condensed matter physics [1,2], string theory and AdS/CFT correspondence in quantum gravity [3], and enhanced our understanding of the renormalization group [4] and other fundamental structures and dynamics of quantum field theory (QFT). In 2d, many CFTs are well understood thanks to their integrability [5,6]. Going to higher dimensions, CFTs are much less well-studied due to a much smaller conformal group. Existing approaches, including Monte Carlo lattice simulation and numerical conformal bootstrap [7,8], despite having achieved many successes, can only handle a limited number of CFTs and obtain a limited number of conformal data.

Therefore, the quest to discover new CFTs in $d \geq 3$ has been particularly intriguing to physicists. A virgin land on this quest is the parity-breaking CFTs. In 3D, the Chern-Simons-matter theory stands out as the most well known and possibly the only known type of parity-breaking CFTs. These theories involve gapless bosonic or fermionic fields coupled to a Chern-Simons gauge field. The Chern-Simons-matter theories play a crucial role in understanding phase transitions of topological orders [9–15]. They are also conjectured to exhibit interesting field theory dualities

[16–20]. Despite their fundamental and practical importance, there has been no nonperturbative study for these parity-breaking theories, primarily due to the intrinsic sign problem they pose for Monte Carlo simulations.

Recently, fuzzy sphere regularization has emerged as a new powerful method to study 3D CFTs [21]. By studying interacting quantum systems on the fuzzy (noncommutative) sphere [22], the method realizes $(2+1)$ D quantum phase transitions on the geometry $S^2 \times \mathbb{R}$. Compared with conventional methods that involve simulating lattice models, this approach offers distinct advantages including exact preservation of rotation symmetry, direct observation of emergent conformal symmetry, and the efficient extraction of conformal data. In the fuzzy sphere method, the state-operator correspondence [23,24] plays an essential role. Specifically, there is a one-to-one correspondence between the eigenstates of the critical Hamiltonian on the sphere and the CFT operators, where the energy gaps are proportional to the scaling dimensions. The power of this approach has been demonstrated in the context of the 3D Ising transition [21,25–28], where the presence of emergent conformal symmetry has been convincingly established and a wealth of conformal data has been accurately computed. The study has been extended to the magnetic line defect [29–31], various conformal boundaries in 3D Ising CFT [32,33], conformal generators [34,35], Wilson-Fisher theory [36], and $\mathrm{SO}(5)$ deconfined criticality [37].

Among these researches, the $\mathrm{SO}(5)$ deconfined quantum critical point (DQCP) [38,39] is in particular noticeable. This pioneering example of phase transitions beyond the Landau paradigm can be described by the nonlinear sigma

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model (NL σ M) on the Grassmannian $\text{Sp}(2)/[\text{Sp}(1) \times \text{Sp}(1)] \equiv S^4$ with a level-1 Wess-Zumino-Witten (WZW) term [40]. This theory has several dual descriptions, *viz.* $\text{SU}(2)$ QCD with two flavors of fermions, and $\mathbb{CP}^1$ theory, *i.e.*, scalar QED with two flavors of complex boson [38,41–43]. It can be realized on the fuzzy sphere by half filling four flavors of fermions and restricting the symmetry to $\text{SO}(5) \equiv \text{Sp}(2)/\mathbb{Z}_2$ [37,44–47]. As the parity symmetry is realized on the fuzzy sphere as the particle-hole symmetry, a generalization of this model that breaks the particle-hole symmetry is a promising candidate for realizing parity-breaking CFTs. Specifically, we consider $2N$ flavors of fermions, fill $2M$ of these flavors, and break the symmetry from $\text{SU}(2N)$ to $\text{Sp}(N)$ [48]. This model should be described by a NL σ M on Grassmannian $\text{Sp}(N)/[\text{Sp}(M) \times \text{Sp}(N-M)]$ with a level-1 WZW term. This WZW-NL σ M is known to be closely related to Chern-Simons-matter theories, particularly when $N \neq 2M$ [49]. Although the NL σ M itself is nonrenormalizable, it can be used as a tool to help determine the phase diagram. By matching the global symmetry and the anomaly, one can figure out the natural candidate IR fates, including CFTs and TQFTs, that the fuzzy sphere model could flow to. Different values of N and M correspond to different unexplored CFTs.

In this Letter, we study these partially filled $\text{Sp}(N)$ models with $(N, M) = (3, 1), (4, 1)$ and half-filled theories $(N, M) = (2, 1), (4, 2)$ numerically. By tuning the interaction strength, we find a parameter region for each (N, M) conformal symmetry, with evidence such as conserved currents, integer-spaced conformal multiplets, and conformal generators with good quality. By defining a cost function, we further find hint that the spectrum scales toward conformality with currently available system sizes for the $\text{Sp}(3)$ and $\text{Sp}(4)$ models, which is distinct from the case of $\text{Sp}(2)$ —the likely pseudocritical $\text{SO}(5)$ DQCP [43,50,51]. These evidences show that each of the (N, M) fuzzy sphere models at the particular parameters is described by a CFT. We extract and analyze one of the most important properties of the CFT—the scaling dimensions of primary operators—through state-operator correspondence. Different values of (N, M) flows to different CFTs, and these CFTs are likely to be unexplored previously. By matching the anomaly through the corresponding NL σ M, we show that the most plausible candidates are the Chern-Simons-matter theories which have N flavors of gapless bosons or fermions coupled to a non-Abelian [*viz.* $\text{Sp}(1), \text{Sp}(2)$, etc.] Chern-Simons gauge field [49]. Our Letter provides new avenues for studying interacting CFTs in 3D, possibly facilitating the nonperturbative study of critical gauge theories and previously undiscovered CFTs.

Model—We consider $2N$ flavors of fermions $\psi_a(\mathbf{r})$ ($a = 1, \dots, 2N$) moving on a sphere in the presence of a $4\pi s$ monopole at its center. Due to the presence of the monopole,

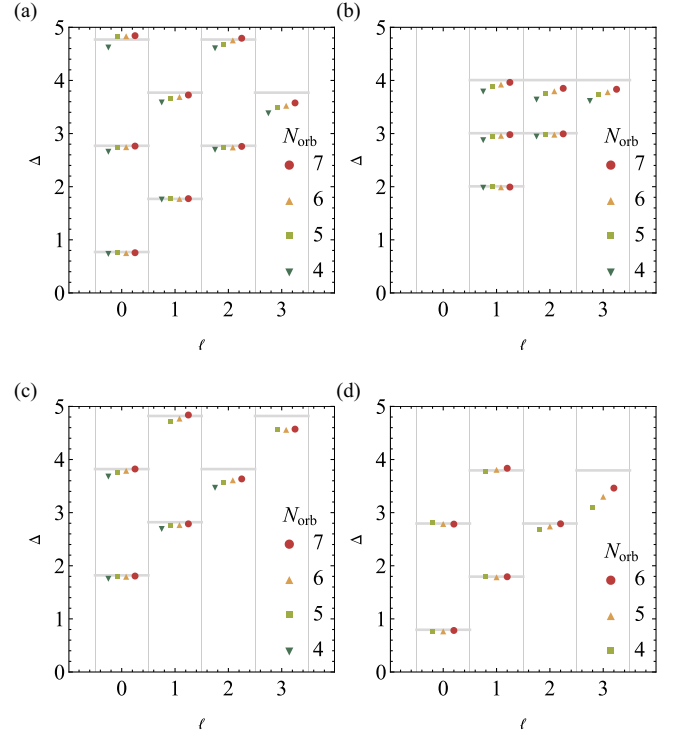


FIG. 1. The scaling dimensions of the conformal multiplet of (a) $\phi_{(1,1)}$, (b) $J_{(2)}^\mu$, (c) $t_{(2,2)}$ in the $(N, M) = (3, 1)$ theory and (d) $\phi_{(1,1)}$ in the $(N, M) = (4, 1)$ theory at different spin ℓ and system size N_{orb} . The horizontal gray bars denote the anticipated values based on the integer-spaced level from the conformal symmetry. The optimal points at the maximal sizes are taken for the calculation.

the fermion single-particle eigenstates form highly degenerate quantized Landau levels. The ground state, *i.e.*, lowest Landau level (LLL), has a degeneracy $N_{\text{orb}} = 2s + 1$ for each flavor. We partially fill the LLL and set the gap between the LLL and higher Landau levels to be much larger than other energy scales in the system. In this case, we can effectively project the system into the LLL. After the projection, the fermion operator $\psi_a(\mathbf{r})$ ($a = 1, \dots, 2N$) can be expressed in terms of the creation and annihilation operators on the LLL $\psi_a(\mathbf{r}) = \sum_{m=-s}^s Y_{sm}^{(s)} c_{ma}$. Furthermore, we fill $2M$ out of the $2N$ flavors to obtain the desired NL σ M.

To construct an interaction Hamiltonian that breaks the maximal flavor symmetry of $\text{SU}(2N)$ down to $\text{Sp}(N)$, we consider the $\text{Sp}(N)$ -invariant fermion bilinears: $\psi_a^\dagger \psi_a$ is $\text{SU}(2N)$ invariant; $\Omega_{aa'} \psi_a \psi_{a'}$ where $\Omega = \begin{pmatrix} 0 & \mathbb{I}_N \\ -\mathbb{I}_N & 0 \end{pmatrix}$ is $\text{Sp}(N)$ invariant but not $\text{SU}(2N)$ invariant. Consequently, we construct

$$H = \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \left(U(\mathbf{r}_{12}) \psi_a^\dagger(\mathbf{r}_1) \psi_b^\dagger(\mathbf{r}_2) \psi_b(\mathbf{r}_2) \psi_a(\mathbf{r}_1) - \frac{1}{2} V(\mathbf{r}_{12}) \Omega_{aa'} \Omega_{bb'} \psi_a^\dagger(\mathbf{r}_1) \psi_{a'}^\dagger(\mathbf{r}_2) \psi_{b'}(\mathbf{r}_2) \psi_b(\mathbf{r}_1) \right). \quad (1)$$

The potential functions $U(\mathbf{r}_{12})$ and $V(\mathbf{r}_{12})$ can be parametrized by the Haldane pseudopotentials [52] U_l and V_l ($l = 0, \dots, 2s$). Only the V_l with even l keeps the $\text{Sp}(N)$ symmetry. This model is a direct generalization of the $\text{SO}(5)$ DQCP on the fuzzy sphere or Landau levels [37,44–47], and for detailed construction of the invariant fermion bilinears, the pseudopotentials and the projection onto LLL, see Supplemental Material (SM) [53]. In the calculations below, we set $U_0 = 1$ as the energy unit and vary U_1, U_2, V_0, V_2 as the tuning parameters.

Results—We perform an exact diagonalization (ED) calculation for the lowest 800 eigenstates with a maximal size $N_{\text{orb}} = 7, 6, 9, 5$ for $(N, M) = (3, 1), (4, 1), (2, 1), (4, 2)$, respectively, using our open source Julia package FuzzifiED [54]. As stated by the state-operator correspondence [23,24], for a CFT, each eigenstate corresponds to a scaling operator. Its scaling dimension is proportional to the excited energy $\Delta_i = \alpha(E_i - E_0)$, where the undetermined coefficient α depends on the microscopic model. Its representation under $\text{SO}(3)$ rotation symmetry can be measured by the expectation value of the total angular momentum $\langle L^2 \rangle = \ell(\ell + 1)$, and its representation under $\text{Sp}(N)$ global symmetry can be measured by the expectation value of the quadratic Casimir C_2 [53,55].

The spectrum of a CFT is characterized by (1) conserved symmetry currents J^μ and stress tensor $T^{\mu\nu}$ with scaling dimensions $\Delta_{J^\mu} = 2$ and $\Delta_{T^{\mu\nu}} = 3$; (2) conformal descendants with integer spacing from the primary operator. To analyze an energy spectrum described by a CFT, we follow the recipe below [21]: (1) we select the lowest state in each representation and identify it as a primary; (2) we identify its descendants by matching the quantum numbers ℓ, C_2 and Δ ; and (3) we then remove the identified conformal multiplet from the spectrum and repeat the process. To search for a parameter point with conformal symmetry, we define a cost function Q as the root mean square of the deviations of the two conserved currents and six conformal descendants with $\Delta < 3$ from the prediction of conformal symmetry. We minimize the cost function with respect to the parameters U_1, U_2, V_0, V_2 in the Hamiltonian and the coefficient α to find the ideal point for the calculation. We take the optimal point at the maximal size for future calculation [53].

The good conformal symmetry is evidenced by the existence of integer-spaced conformal multiplets. For a scalar primary Φ , its descendants are of the form $\partial^\ell \square^n \Phi$ with spin ℓ and $\Delta = \Delta_\Phi + n + 2l$; for the conserved current J^μ , it has two forms of descendants $\square^n \partial^{\mu_1} \dots \partial^{\mu_{\ell-1}} J^{\mu_\ell}$ and $\square^n \partial^{\mu_1} \dots \partial^{\mu_{\ell-1}} \partial^\nu \epsilon_{\mu_\ell \nu \rho} J^\rho$ [21]. All the operators in the spectrum can be organized into multiplets. We take the lowest three primaries as an example, viz. the lowest operator $\phi_{(1,1)}$ in the $\square$ representation of $\text{Sp}(N)$, the lowest operator $t_{(2,2)}$ in the $\boxplus$ representation of $\text{Sp}(N)$, and the conserved symmetry current $J_{(2)}^\mu$, where the subscripts describe the Young diagrams for the representations, and

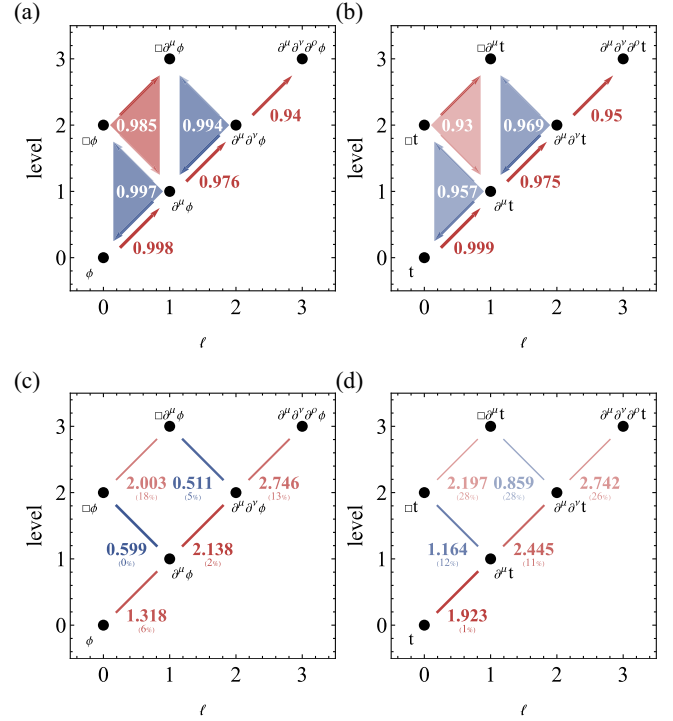


FIG. 2. (a),(b) The total squared overlap of the normalized state $(\Lambda^\mu |\partial^n \Phi, \ell\rangle)_{\ell'}$ with $|\partial^{n+1} \Phi, \ell'\rangle$ and $|\partial^{n-1} \Phi, \ell'\rangle$ for $\ell' = \ell \pm 1$ for the multiplets of (a) $\Phi = \phi_{(1,1)}$ and (b) $\Phi = t_{(2,2)}$. Numbers closer to unity represent better conformal symmetry. The tail of the arrow represents the state that Λ^μ acts upon and the head represents the state with which the overlap is taken. (c),(d) The matrix elements $\langle \partial^{n'} \Phi, \ell' | \Lambda^\mu | \partial^n \Phi, \ell \rangle$ and their deviation from conformality. The opacity and thickness represent the closeness to conformality. The colors are for better readability. The calculations are done for $(N, M) = (3, 1)$ theory and $N_{\text{orb}} = 6$.

these primaries will be explained in detail later. We identify all of their lowest-lying descendants with $\Delta < 5$ and $\ell \leq 3$ in the $(N, M) = (3, 1)$ and $(4, 1)$ theories, and the scaling dimensions are very close to the integer spacing with the primary operator (Fig. 1).

Further evidence for conformal symmetry comes from the conformal generators on the fuzzy sphere [34,35]. The conformal generator $\Lambda^\mu = P^\mu + K^\mu$ on the states is the Hamiltonian density $\mathcal{H}(\mathbf{r})$ integrated against $l = 1$ spherical harmonics Y_{1m} [53]. For a state $|\partial^n \Phi, \ell\rangle$ in the multiplet of primary Φ , the action of Λ^z brings it to the superposition of $|\partial^{n\pm 1} \Phi, \ell' = \ell \pm 1\rangle$, we decompose $\Lambda^z |\partial^n \Phi, \ell\rangle$ into angular momentum sectors and check the overlap

$$\frac{|\langle \partial^{n+1} \Phi, \ell' | (\Lambda_z | \partial^n \Phi, \ell \rangle)_{\ell'}|^2 + |\langle \partial^{n-1} \Phi, \ell' | (\Lambda_z | \partial^n \Phi, \ell \rangle)_{\ell'}|^2}{\|(\Lambda_z | \partial^n \Phi, \ell \rangle)_{\ell'}\|^2} \quad (2)$$

for $\Phi = \phi_{1,1}$ and $t_{2,2}$ [Figs. 2(a) and 2(b)]. We find that most numbers are close to unity as predicted by conformality,

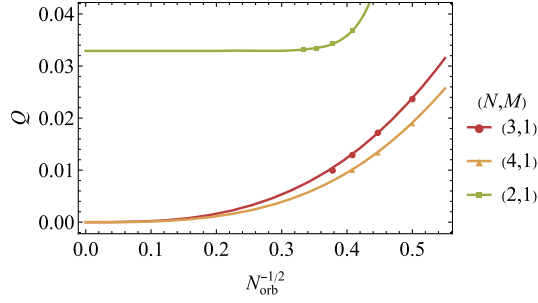


FIG. 3. The minimal cost function $Q_{\min}$ as a function of system size N_{orb} in $(N, M) = (3, 1), (4, 1), (2, 1)$ models. The solid lines, fitted with the ansatz $Q(N_{\text{orb}}) = Q_0 + Q_1 N_{\text{orb}}^{-\alpha}$ ($Q_0 > 0$), are just for an eye guidance.

with a maximal deviation of 7%. We also compute the matrix element $\langle \partial^{\mu'} \Phi, \ell' | \Lambda^z | \partial^{\mu} \Phi, \ell \rangle$ and compare it with the value in CFT [Figs. 2(c) and 2(d)]. The agreement is good up to the second-order descendants. Higher descendants are subjected to larger deviations because the states receive corrections due to the irrelevant perturbations and may be mixed with states in the same symmetry sectors with similar energy.

To confirm whether or not this point describes a genuine CFT, we study how the quality of conformal symmetry scales with system size. In the $(N, M) = (3, 1)$ and $(4, 1)$ models, the minimal cost function $Q_{\min}$ decreases rapidly as the system size increases (Fig. 3) and shows a trend of scaling to 0 in the thermodynamic limit, indicating perfect conformal symmetry. In contrast, $Q_{\min}$ has little size dependence in the $(N, M) = (2, 1)$ model. This may suggest that the nature of the conformal region of $(3, 1)$ and $(4, 1)$ is qualitatively different from $(2, 1)$: the $(3, 1)$ and $(4, 1)$ models are likely to describe genuine CFTs, while the $(2, 1)$ model that describes the $\text{SO}(5)$ DQCP is likely to be pseudocritical, although due to the size limit of our calculation, it is difficult to rule out the possibilities that $(3, 1)$ and $(4, 1)$ models could be pseudocritical or the $(2, 1)$ model could be genuinely critical at the current stage.

We add that for the case of $N = 2M$, the model has an extra particle-hole symmetry in the UV, and it becomes the parity symmetry of the CFT in the IR [37], so one would naturally expect the parity symmetry to be broken in the CFT if $N \neq 2M$. Nevertheless, to conclude if this is true, one needs to examine the existence of parity-odd structures in the CFT correlators of spinning operators, e.g., three-point correlators of conserved current $\langle J^{\mu} J^{\nu} J^{\rho} \rangle$. We leave this for future exploration.

We then take a closer look at the operator spectra in the $\text{Sp}(N)$ CFTs. The low-lying primary operators are reported in Table I. The reported numbers are raw data at the optimal point and the maximal size. Since they may receive correction from irrelevant perturbations, the value in the CFT may differ from our raw data. The parameter dependence of these scaling dimensions and the full

spectrum can be found in SM [53]. In the following, we explain some notable primaries [56]. (1) $\phi_{(1,1)}$ is a traceless antisymmetric rank-2 $\text{Sp}(N)$ tensor $\boxminus$. In the proximate ordered phase with symmetry-breaking into target space $\text{Sp}(N)/(\text{Sp}(N-M) \times \text{Sp}(M))$, this operator serves as an order parameter. (2) $t_{(2,2)}$ is a rank-4 tensor. This operator can break the global symmetry to $\text{SU}(N) \times \text{U}(1)$. (3) The global symmetry current $J_{(2)}^{\mu}$ is a spin-1 symmetric rank-2 tensor $\boxplus$ with exactly $\Delta = 2$. (4) S is the lowest symmetry singlet. We find it slightly relevant in both $(N, M) = (3, 1)$ and $(4, 1)$ theories, which may suggest that this CFT describes a phase transition, and slightly irrelevant in the $(4, 2)$ theory. We also need to note that the scaling dimension is subject to correction from irrelevant operators, given Δ_S is very close to 3, its relevance remains to be settled. In the half-filled models, it is parity-even. (5) $J_{(2,1,1)}^{\pm, \mu}$ are spin-1 rank-4 tensors $\boxplus$. This operator is new in $(N, M) = (3, 1)$ and $(4, 1)$ theories and does not have correspondence in the $(N, M) = (2, 1)$ theory. For the $(N, M) = (4, 2)$ theory, two operators of such kind with close scaling dimensions exist, respectively parity-odd and -even. (6) The stress tensor $T^{\mu\nu}$ is a spin-2 singlet with exactly $\Delta = 3$. (7) S' is the second lowest symmetry singlet. In the $N = 2M$ theories, it is parity-odd and corresponds to mass $\bar{\psi}\psi$ in the QCD description. (8) $t_{(1,1,1,1)}$ and $t_{(2,2,1,1)}$ are higher-rank tensors that exist only at $N = 4$. Their scaling dimensions are high in $(4, 1)$ theory but fall to very low in $(4, 2)$ theory. We also note that the scaling dimensions for the $(N, M) = (2, 1)$ theory are different from our former work [37], it is either due to the dimension drift from the pseudocriticality, or we have found a distinct fixed point in the phase diagram. It would be interesting to study how this result fits into the widely studied $\text{SO}(5)$ DQCP [57–65].

Candidate theories—Having presented the numerical observations of the new series of CFTs with $\text{Sp}(N)$ symmetry, we now discuss the possible candidates for the CFTs that could theoretically exist in the phase diagram of our model. One way to answer this question is to consider the possible UV completion of the possible symmetry-broken phases. The simplest symmetry-broken phase would be filling the first $2M$ flavors out of the $2N$ flavors. It has a residual symmetry of $\text{Sp}(M) \times \text{Sp}(N-M)$ describing the rotation within the filled flavors and the empty flavors. Hence, it is described by the $\text{NL}\sigma\text{M}$ living on a Grassmannian

$$\mathcal{G} = \frac{\text{Sp}(N)}{\text{Sp}(M) \times \text{Sp}(N-M)}.$$

Furthermore, there is a Wess-Zumino-Witten (WZW) term with level $k = 1$, as a natural generalization of the scenario of $(N, M) = (2, 1)$ [37]. This symmetry-broken phase can be considered as a multiflavor generalization of quantum

TABLE I. The scaling dimension and quantum numbers for the low lying primary operators obtained from state-operator correspondence at different (N, M) . For the case of $M = N/2$, the theory has an extra parity symmetry $\mathcal{P}$. The reported numbers are raw data at the optimal point and the maximal size. Since they may receive correction from irrelevant perturbations, the value in the CFT may differ from the raw data. The scaling dimensions are calibrated to minimize the cost function, which is different from our previous paper [37]. We believe the $(2,1)$ theory is not truly conformal. The full spectrum can be found in SM [53]. Operators that do not exist are denoted by “...”

Operator	Spin	Representation	Δ				$\mathcal{P}$
			(3,1)	(4,1)	(2,1)	(4,2)	
$\phi_{(1,1)}$	0	$\square$	0.770	0.795	0.702	0.782	–
$t_{(2,2)}$	0	$\square\square$	1.818	1.798	1.787	1.767	+
$J_{(2)}^\mu$	1	$\square$	2.005	2.008	2.029	2.002	+
$J_{(2,1,1)}^{+\mu}$	1	$\square\square$	2.521	2.586	...	2.554	+
$J_{(2,1,1)}^{-\mu}$	1	$\square\square$	...	...	...	2.563	–
S	0	$\bullet$	2.979	2.987	2.797	3.028	+
$T^{\mu\nu}$	2	$\bullet$	2.998	3.000	2.956	2.992	+
$m_{(3,3)}$	0	$\square\square\square$	3.079	2.959	3.170	2.916	+
$J_{(3,1)}^\mu$	1	$\square\square$	3.269	3.204	3.345	3.165	–
$t_{(1,1,1,1)}$	0	$\square\square\square$	...	3.408	...	1.172	+
$t_{(2,2,1,1)}$	0	$\square\square\square$	...	4.144	...	2.160	–
S'	0	$\bullet$	4.584	4.428	4.370	4.378	–

Hall ferromagnet, and the existence of WZW term directly follows a well-established result of quantum Hall ferromagnet; namely, its Skyrmion carries original fermion charge [66–68].

As discussed in Ref. [49], one can write down several UV Lagrangians that share the same global symmetry and anomaly as this WZW-NL σ M, hence serve as the natural candidates for the $\text{Sp}(N)$ CFTs we have discovered [53]. (1) $\text{Sp}(M)_1$ with $N\Phi$: N flavours of critical scalar fields (in the fundamental representation) coupled to an $\text{Sp}(M)_1$ Chern-Simons field. (2) $\text{Sp}(N-M)_{-1}$ with $N\Phi$: N flavours of critical scalar fields (in the fundamental representation) coupled to an $\text{Sp}(N-M)_{-1}$ Chern-Simons field. (3) $\text{SU}(2)_{N/2-M}$ with $N\Psi$: N flavours of critical fermions (in the fundamental representation) coupled to an $\text{Sp}(1)_{N/2-M} \cong \text{SU}(2)_{N/2-M}$ Chern-Simons field.

Semiclassically, all three theories describe phase transitions of topological orders, where the $\text{Sp}(N)$ singlet mass of bosons or fermions is the tuning operator for the transition. Specifically, $\text{Sp}(M)_1$ with $N\Phi$ describes the transition between the $\text{Sp}(M)_1$ Chern-Simons theory and the symmetry breaking phase; $\text{Sp}(N-M)_{-1}$ with $N\Phi$ describes the transition between the $\text{Sp}(N-M)_{-1}$ Chern-Simons theory and the symmetry breaking phase; $\text{SU}(2)_{N/2-M}$ with $N\Psi$ describes the transition between the $\text{Sp}(M)_1$ and $\text{Sp}(N-M)_{-1}$ Chern-Simons theories (Fig. 4). For the case

of $N = 2M$, $\text{SU}(2)_{N/2-M}$ with $N\Psi$ describes a stable critical phase rather than a phase transition because it has an extra parity symmetry that forbids the fermion mass term. We also note that $\text{Sp}(1)_1 \cong \text{SU}(2)_1$ is the familiar $\nu = 1/2$ bosonic Laughlin state (semion topological order), while $\text{Sp}(M)_1$ for $M > 1$ corresponds to non-Abelian topological orders. In particular, $\text{Sp}(2)_1$ and $\text{Sp}(3)_1$ contain Ising anyon and Fibonacci anyon, respectively.

Interestingly, our numerical data do not align well with the semiclassical expectation of any of the three candidate theories. In particular, the lowest singlet in Table I is very close to 3, which is in strong contrast to the semiclassical expectation that all three theories will have a relevant singlet (i.e., mass term) with scaling dimension $\Delta \approx 2$, serving as a tuning operator for the phase transition. Indeed, because of the almost marginal lowest singlet, it is difficult to determine whether our discovered CFTs describe a phase transition or a stable conformal phase. Some finite-size scaling results away from the conformal fixed point is given in SM [53]. Clearer evidence requires a more careful analysis by conformal perturbation theory [69]. Moreover, all three candidate theories are proximate to topological orders, but we have not found any clear signature of topological order in our phase diagram at accessible system sizes, including chiral edge modes and topological degeneracy on the torus. However, we remark that these tensions do not necessarily mean that our discovered CFTs are not one of the three Chern-Simons-matter theories, since the semiclassical expectations may not be valid at small N and M we are studying here.

Discussion—We have presented clear evidence for the emergence of a series of CFTs in the $\text{Sp}(N)$ Grassmannian WZW-NL σ M on the fuzzy sphere. The phase diagram of our $\text{Sp}(N)$ model can contain a wealth of CFT candidates, including three different critical non-Abelian Chern-Simons-matter theories. It will be interesting to determine if our discovered CFTs are one of the three Chern-Simons-matter theories, and to explore a larger phase diagram to find all the three Chern-Simons-matter theories in the model. It is also intriguing to explore the topological orders in the model, particularly since most of the candidate topological orders will be non-Abelian. In addition to the bulk properties, it is also appealing to study the conformal defects and boundaries of these CFTs and explore the consequences of the anomalies. At last, extending our Letter to other NL σ M [49,70–73] may lead to many fruitful results with important applications to quantum criticality and critical spin liquid.

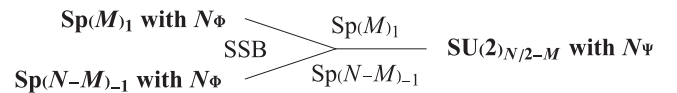


FIG. 4. A phase diagram of the three candidate critical Chern-Simons-matter theories and the gapped and symmetry-broken phases in their vicinity.

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