

Generalized Peierls Substitution for Wannier Obstructions: Response to Disorder and Interactions

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We study the interplay between quantum geometry, interactions, and external fields in complex band systems. When Wannier obstructions preclude a description based solely on atomlike orbitals, this complicates the prediction of electromagnetic responses, particularly in the presence of disorder and interactions. In this Letter, we introduce a generalized Peierls substitution framework based on Lagrange multipliers to enforce the constraints of the Wannier obstruction in the band of interest. Thus, we obtain effective descriptions of interactions and disorder in the presence of nontrivial quantum geometry of that band. We apply our approach to examples including the diamagnetic response in flat-band superconductors and delocalization effects in flat-band metals caused by interactions and disorder.

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Introduction—Recent advances in materials science, specially the emergence of moiré materials, challenge our understanding of correlated band systems of nontrivial quantum geometry (QG) [1–11]. These systems have isolated bands that are separated from remote bands by a large band gap Δ_{gap} (see Fig. 1), while the characteristic interaction strengths are much weaker than the band gap. Importantly, QG can manifest through Wannier obstructions, which preclude a description based solely on atomlike orbitals. The geometric contributions can introduce geometric contributions such as Berry curvature governing the topological phase [12–14] and quantum metric encoding the distinguishability of Bloch states [15]. It can reshape the low-energy physics to go beyond knowledge of band dispersion [5,16–31]. This is particularly evident in flat-band superconductors, where the Fermi velocity vanishes, and the quantum metric plays a key role in determining superfluid weight and coherence length [32–56].

One can construct an effective theory through the band projections into the subspace associated with these isolated bands. However, problems can arise when we consider the electromagnetic response. For instance, imposing band

projections directly on both the Hamiltonian and the current operator for a Landau level would yield immobile electrons as the kinetic energy in the Landau level is quenched. However, in actuality, in this problem, it is the projected Coulomb interaction in the lowest Landau level that can control the highly nontrivial charge dynamics [57,58]. In lattice systems, electromagnetic fields are incorporated via the Peierls substitution through attaching phase factors to hopping terms. This method approximates continuum

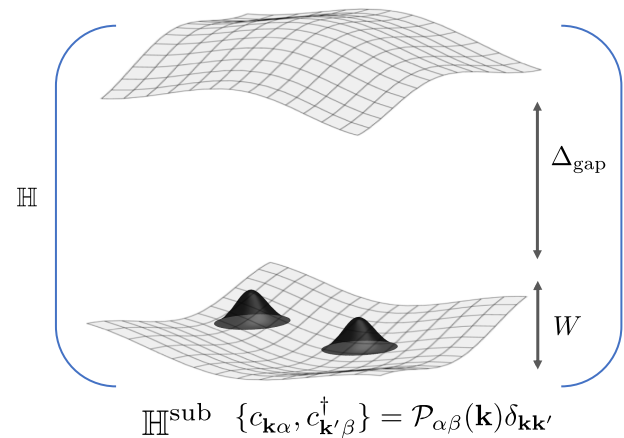


FIG. 1. Schematic illustration of bands of Wannier obstructions. With a large band gap Δ_{gap} , the low-energy physics can be captured in the subspace $\mathbb{H}^{\text{sub}}$. Electrons are not independent across different orbitals $\{c_{\mathbf{k}\alpha}, c_{\mathbf{k}'\beta}^\dagger\} = \mathcal{P}_{\alpha\beta}(\mathbf{k})\delta_{\mathbf{k}\mathbf{k}'}$ in the subspace $\mathbb{H}^{\text{sub}}$. The kernel $\mathcal{P}_{\alpha\beta}(\mathbf{k})$ is the band projector under the orbital basis.

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minimal coupling by assuming exponentially localized Wannier functions, which allows truncating the hopping to nearest neighbors. However, its application is challenged on Wannier-obstructed bands where no complete set of maximally localized Wannier functions exists. The obstructions, exemplified by topological bands, can force tight-binding models to include long-range hopping terms that can produce nonlocal phase factors and thus a nonlocal current operator.

This raises the question: Can we establish a theoretical framework self-consistently within the subspace $\mathbb{H}^{\text{sub}}$? In particular, can we generalize the Peierls substitution to topological bands of Wannier functions to obtain the current operator in the presence of disorder or interactions? Some progress on this has been made in identifying a multiplicative phase factor from the effective projected interaction terms [48,49] in response to an electromagnetic field. However, these approaches lack a general method for deriving the current operator in the subspace $\mathbb{H}^{\text{sub}}$ as they only consider specific cases.

In this Letter, we generalize the Peierls substitution framework to resolve the response to disorder and interactions in bands of Wannier obstructions. We first enforce the band projection by a set of Grassmann-valued Lagrange multipliers through L_λ in Eq. (4) onto the subspace $\mathbb{H}^{\text{sub}}$. The generalization of Peierls substitution is achieved by identifying the gauge coupling through L_λ . We apply our approach to two examples: the diamagnetic response in flat-band superconductors and the delocalization effect in flat-band metals, where the currents from the interaction and disorder terms dominate the low-energy dynamics.

Model setup—We consider a Hamiltonian $H = H_0 + H_1$, where the free part H_0 describes a band structure that possesses isolated bands with a band gap Δ_{gap} as depicted in Fig. 1. The term H_1 includes additional contributions that cannot be represented within the free part H_0 , such as interaction and disorder terms. The energy scales of the additional terms in H_1 are much weaker than the band gap Δ_{gap} . The tight-binding Hamiltonian H_0 contains N_o orbitals,

$$H_0 = \sum_{\alpha, \beta=1, \dots, N_o} \sum_{\mathbf{k}} h_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\beta}, \quad (1)$$

where the operator $c_{\mathbf{k}\alpha} (c_{\mathbf{k}\alpha}^\dagger)$ annihilates (creates) a fermion at an orbital α of the crystal momentum $\mathbf{k}$ and satisfies the anticommutators $\{c_{\mathbf{k}\alpha}, c_{\mathbf{k}'\beta}^\dagger\} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\alpha\beta}$. For simplicity, we assume only one band at the Fermi energy with the wave function $u_{\mathbf{k}\alpha} : \sum_{\beta} h_{\alpha\beta}(\mathbf{k}) u_{\mathbf{k}\beta} = \epsilon_{\mathbf{k}} u_{\mathbf{k}\alpha}$, while these states span the subspace $\mathbb{H}^{\text{sub}}$. We derive a low-energy theory in the subspace $\mathbb{H}^{\text{sub}}$ of the partially filled band for $\Delta_{\text{gap}} \rightarrow \infty$.

Band projection and Lagrange multipliers: As depicted in Fig. 1, we introduce a band projection

$$c_{\mathbf{k}\alpha}^\dagger \rightarrow u_{\mathbf{k}\alpha} f_{\mathbf{k}}^\dagger, \quad (2)$$

with $f_{\mathbf{k}}^\dagger$ satisfying the anticommutation $\{f_{\mathbf{k}}, f_{\mathbf{k}'}^\dagger\} = \delta_{\mathbf{k}\mathbf{k}'}$ and creating a fermion with wave function $u_{\mathbf{k}\alpha}$ in the isolated band. The projection is unique up to an arbitrary phase of wave functions: $u_{\mathbf{k}\alpha} \rightarrow u_{\mathbf{k}\alpha} e^{i\phi(\mathbf{k})}$. Within the sub-Hilbert space $\mathbb{H}^{\text{sub}}$, the projected $c_{\mathbf{k}\alpha}$ fermions obey unconventional anticommutators,

$$\{c_{\mathbf{k}\alpha}, c_{\mathbf{k}'\beta}^\dagger\} = \mathcal{P}_{\alpha\beta}(\mathbf{k}) \delta_{\mathbf{k}\mathbf{k}'}, \{c_{\mathbf{k}\alpha}, c_{\mathbf{k}'\beta}\} = \{c_{\mathbf{k}\alpha}^\dagger, c_{\mathbf{k}'\beta}^\dagger\} = 0. \quad (3)$$

The kernel $\mathcal{P}_{\alpha\beta}(\mathbf{k}) \equiv u_{\mathbf{k}\alpha}^* u_{\mathbf{k}\beta}$ is the band projector under the orbital basis. It is not a constant function of momentum $\mathbf{k}$ when the band possesses the Wannier obstruction [59]. One profound consequence is a generalized Girvin-MacDonald-Platzman algebra [60–62] for the projected density operator $\rho(\mathbf{q}) \equiv \sum_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}+\mathbf{q}\alpha}$.

We highlight two challenges arising from the projection in Eq. (2). First, we cannot independently vary $c_{\mathbf{k}\alpha}$ for different orbitals when, e.g., deriving the dynamics through the variational method. Second, the projection may not commute with the action of an external perturbation. For example, the presence of the electromagnetic field can change the eigenfunction of H_0 , and the projection in Eq. (2) should change accordingly.

To enforce Eq. (3), we introduce the complex Grassmann-valued Lagrange multipliers $\lambda_{\mathbf{k}\alpha}, \lambda_{\mathbf{k}\alpha}^\dagger$ to realize the projection in Eq. (2),

$$L_\lambda = \sum_{\mathbf{k}\alpha} \lambda_{\mathbf{k}\alpha}^\dagger (c_{\mathbf{k}\alpha} - u_{\mathbf{k}\alpha}^* f_{\mathbf{k}}) + \text{H.c.} \quad (4)$$

(For a different scheme, see Ref. [57].) Equation (4) implies a U(1) symmetry in the subspace $\mathbb{H}^{\text{sub}}$. Concretely, with the Wannier function $\varphi_\alpha(\mathbf{r}) = \sum_{\mathbf{k} \in \text{BZ}} e^{i\mathbf{k} \cdot \mathbf{r}} u_{\mathbf{k}\alpha}$ of orbital α centered at the original point, we have in real space

$$L_\lambda = \sum_{\mathbf{r}\alpha} \lambda_{\mathbf{r}\alpha}^\dagger \left[c_{\mathbf{r}\alpha} - \sum_{\mathbf{r}'} \varphi_\alpha^*(\mathbf{r} - \mathbf{r}') f_{\mathbf{r}'} \right] + \text{H.c.}, \quad (5)$$

and then the U(1) symmetry manifests as

$$\begin{aligned} \lambda_{\mathbf{r}\alpha} &\rightarrow \lambda_{\mathbf{r}\alpha} e^{-i\theta}, c_{\mathbf{r}\alpha} \rightarrow c_{\mathbf{r}\alpha} e^{-i\theta}, f_{\mathbf{r}} \rightarrow f_{\mathbf{r}} e^{-i\theta}, \\ \lambda_{\mathbf{r}\alpha}^\dagger &\rightarrow \lambda_{\mathbf{r}\alpha}^\dagger e^{i\theta}, c_{\mathbf{r}\alpha}^\dagger \rightarrow c_{\mathbf{r}\alpha}^\dagger e^{i\theta}, f_{\mathbf{r}}^\dagger \rightarrow f_{\mathbf{r}}^\dagger e^{i\theta}, \end{aligned} \quad (6)$$

where $c_{\mathbf{r}\alpha} = \sum_{\mathbf{k} \in \text{BZ}} c_{\mathbf{k}\alpha} e^{i\mathbf{k} \cdot \mathbf{r}}$ and similar expressions hold for $\lambda_{\mathbf{r}\alpha}$ and $f_{\mathbf{r}}$. The U(1) symmetry in Eq. (6) indicates a Noether current. The formula for multiple bands at the Fermi energy can be expressed similarly [63].

We can summarize the effective theory within the subspace $\mathbb{H}^{\text{sub}}$ with the partition function $Z = \int D\lambda D\lambda^\dagger Dc Dc^\dagger Df Df^\dagger e^{i \int dL}$ with

$$L = \sum_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}^\dagger i \partial_t c_{\mathbf{k}\alpha} - H_0 - H_1 - L_\lambda. \quad (7)$$

After integrating out the Lagrange multipliers, we recover the constraint $c_{\mathbf{k}\alpha} = u_{\mathbf{k}\alpha}^* f_{\mathbf{k}}$. The constraint needs to be varied when we apply external perturbations. In the following, we derive the current response from both Euler-Lagrange equations and the Peierls substitution based on the effective Lagrangian in Eq. (7).

Euler-Lagrange equation and Noether current: The U(1) transformation in Eq. (6) leads to a conserved Noether current. We derive the Euler-Lagrange (EL) equations first and then proceed to derive the continuity equations. For simplicity, we assume that the terms in H_1 , such as the Hubbard interaction, commute with the local density operators in the full Hilbert space $\mathbb{H}$ and therefore do not contribute to the current operator in $\mathbb{H}$. We will identify two contributions to the Noether current, namely, from band dispersion and interactions in the subspace $\mathbb{H}^{\text{sub}}$. The EL equation can be obtained from the functional derivative $(\delta S / \delta \phi) = 0$, $\phi \in \{c, c^\dagger, \lambda, \lambda^\dagger, f, f^\dagger\}$ with the action $S = \int dt L$. Above all, we can figure out the Lagrange multipliers $\lambda_{\mathbf{k}\alpha} = \sum_\beta [\mathcal{P}_{\alpha\beta}(\mathbf{k}) - \delta_{\alpha\beta}] (\delta H_1 / \delta c_{\mathbf{k}\beta}^\dagger)$, which does not vanish when both interaction and nonconstant kernel $\mathcal{P}_{\alpha\beta}$ are present. Then we can reach the EL equation, with details in Supplemental Material [64],

$$i \partial_t c_{\mathbf{k}\alpha} = \sum_\beta h_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\beta} + \sum_\beta \mathcal{P}_{\alpha\beta}(\mathbf{k}) \frac{\delta H_1}{\delta c_{\mathbf{k}\beta}^\dagger}, \quad (8)$$

while the dynamics of $i \partial_t c_{\mathbf{k}\alpha}^\dagger$ is obtained by taking the Hermitian conjugate of Eq. (8). The kernel $\mathcal{M}_{\alpha\beta}(\mathbf{k})$ appears in the interaction term, which can contribute to the continuity equation,

$$\begin{aligned} -i \partial_t \rho(\mathbf{q}) = & \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}) f_{\mathbf{k}+\mathbf{q}}^\dagger f_{\mathbf{k}} \\ & - \sum_{\mathbf{k}\alpha\beta} \left[c_{\mathbf{k}+\mathbf{q}\alpha}^\dagger \mathcal{P}_{\alpha\beta}(\mathbf{k}) \frac{\delta H_1}{\delta c_{\mathbf{k}\beta}^\dagger} \right. \\ & \left. - \frac{\delta H_1}{\delta c_{\mathbf{k}+\mathbf{q}\alpha}} \mathcal{P}_{\alpha\beta}(\mathbf{k} + \mathbf{q}) c_{\mathbf{k}\beta} \right]. \end{aligned} \quad (9)$$

Here $\rho(\mathbf{q}) = \sum_{\mathbf{k}} \langle u_{\mathbf{k}+\mathbf{q}} | u_{\mathbf{k}} \rangle f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{q}}$ is the projected density operator. There are no terms from H_1 in Eq. (9) when the kernel $\mathcal{P}_{\alpha\beta}(\mathbf{k})$ is a constant function of momentum $\mathbf{k}$. The Noether current $\mathbf{j}_{\mathbf{q}}$ can be obtained by expanding Eq. (9) to the form $\partial_t \rho(\mathbf{q}) = -i \mathbf{q} \cdot \mathbf{j}_{\mathbf{q}}$.

From the continuity equation in Eq. (9), we can determine two contributions of currents: band dispersion and interactions. The current that depends on band dispersion can be obtained by directly projecting the original current operator. In contrast, the interaction-driven current is not achievable through this method, and we call it

the QG current. What kind of H_1 can contribute to a nonvanishing QG current? We require that H_1 cannot be expressed in the form of the free Hamiltonian H_0 . Specifically, it should not take the form of terms like $\sum_{\mathbf{k}\alpha\beta} V_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\beta}$ or $\sum_{\mathbf{k}\alpha\beta} V_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\alpha}^\dagger c_{-\mathbf{k}\beta}^\dagger + \text{H.c.}$ We will show that there are QG currents from both the interaction and disorder terms in the examples provided. The Noether current is nonlocal as seen by expanding L_λ in Eq. (5) in real space, $L_\lambda = \sum_{\mathbf{r}\alpha} \{ \lambda_{\mathbf{r}\alpha}^\dagger c_{\mathbf{r}\alpha} - \sum_{\mathbf{r}'n} (1/n!) [(\mathbf{r}' - \mathbf{r}) \cdot \nabla_{\mathbf{r}}]^n \lambda_{\mathbf{r}\alpha}^\dagger \phi_\alpha^*(\mathbf{r}' - \mathbf{r}) f_{\mathbf{r}} \} + \text{H.c.}$: the Noether current can involve high derivatives of fields f and $f^\dagger$ [57].

Generalized Peierls substitution—We now generalize the Peierls substitution on bands of Wannier obstructions to derive the current operators. In a lattice model, the Peierls substitution is an approximation of the continuum minimal coupling in the response to a slowly varying gauge potential $A_{\mathbf{q}}(t)$ by attaching U(1) phase factors to hopping terms. For Wannier-obstructed bands, the effective Lagrangian in Eq. (7) contains L_λ which realizes the band projection. The term L_λ is invariant under global U(1) transformations, but not under local U(1) transformations. Thus, the U(1) symmetry in Eq. (6) operates by involving the $f_{\mathbf{k}\alpha}$, $c_{\mathbf{k}\alpha}$, $\lambda_{\mathbf{k}\alpha}$ and their conjugates.

According to the Peierls substitution, we can introduce the gauge coupling to the kinetic term $H_0[A]$. For a slowly varying gauge potential, we can keep the leading order of A ,

$$\begin{aligned} H_0[A] = & \sum_{\mathbf{k}\alpha\beta} \left[h_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\beta} \right. \\ & - \sum_{\mathbf{q}\mu} A_{\mathbf{q}}^\mu \partial_\mu h_{\alpha\beta} \left(\mathbf{k} + \frac{\mathbf{q}}{2} \right) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}+\mathbf{q}\beta} \\ & \left. + \frac{1}{2} \sum_{\mathbf{q}\mu\nu} A_{\mathbf{q}}^\mu A_{-\mathbf{q}}^\nu \partial_\mu \partial_\nu h_{\alpha\beta}(\mathbf{k}) c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\beta} \right], \end{aligned} \quad (10)$$

with the abbreviation $\partial_\mu \equiv \partial / \partial k_\mu$. The terms linear and quadratic in A represent the paramagnetic and diamagnetic response, respectively. The key step in the generalization is to introduce a gauge coupling to L_λ , which gives rise to

$$L_\lambda[A] = L_\lambda + L_\lambda^1 + L_\lambda^2, \quad (11)$$

with

$$L_\lambda^1 = \sum_{\mathbf{k}\mathbf{q}\alpha\mu} \lambda_{\mathbf{k}\alpha}^\dagger A_{\mathbf{q}}^\mu \partial_\mu u_{\mathbf{k}+\frac{\mathbf{q}}{2}\alpha}^* f_{\mathbf{k}+\mathbf{q}} + \text{H.c.}, \quad (12)$$

$$L_\lambda^2 = \frac{1}{2} \sum_{\mathbf{k}\mathbf{q}\alpha\mu\nu} \lambda_{\mathbf{k}\alpha}^\dagger A_{\mathbf{q}}^\mu A_{-\mathbf{q}}^\nu \partial_\mu \partial_\nu u_{\mathbf{k}\alpha}^* f_{\mathbf{k}} + \text{H.c.} \quad (13)$$

It is essential to keep both terms L_λ^1 and L_λ^2 to obey the U(1) gauge symmetry. Therefore, the Lagrangian $L[A]$ after the

Peierls substitution has the form

$$L[A] = \sum_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}^\dagger i \partial_t c_{\mathbf{k}\alpha} - H_0[A] - H_1 - L_\lambda[A]. \quad (14)$$

We can again solve the modified constraint which, to leading order of A , becomes

$$c_{\mathbf{k}\alpha} = u_\alpha^*(\mathbf{k}) f_{\mathbf{k}} - \sum_{\mathbf{q}\mu} A_{\mathbf{q}}^\mu \mathcal{D}_\mu u_{\mathbf{k}+\frac{\mathbf{q}}{2}}^* f_{\mathbf{k}+\mathbf{q}} - \frac{1}{2} \sum_{\mathbf{q}\mu\nu} A_{\mathbf{q}}^\mu A_{-\mathbf{q}}^\nu G_{\mu\nu}(\mathbf{k}) u_{\mathbf{k}\alpha}^* f_{\mathbf{k}}. \quad (15)$$

Here $\mathcal{D}_\mu = \partial_\mu + iA_\mu$ is the covariant derivative with Berry connection $A_\mu(\mathbf{k}) = i\langle u_{\mathbf{k}} | \partial_\mu u_{\mathbf{k}} \rangle$, and $G_{\mu\nu}(\mathbf{k})$ is the quantum metric, $G_{\mu\nu}(\mathbf{k}) = \text{Re} \langle \partial_\mu u_{\mathbf{k}} | (1 - |u_{\mathbf{k}}\rangle\langle u_{\mathbf{k}}|) | \partial_\nu u_{\mathbf{k}} \rangle$. The covariant derivative $\mathcal{D}_\mu$ appears due to the gauge redundancy $u_{\mathbf{k}\alpha} \rightarrow u_{\mathbf{k}\alpha} e^{i\phi(\mathbf{k})}$. The solution in Eq. (15) captures the response of a Bloch wave to a gauge potential $A_{\mathbf{q}}(t)$. An effective theory can be derived by integrating out the fields $\lambda_{\mathbf{k}\alpha}$ and $c_{\mathbf{k}\alpha}$, with $Z[A] = \int Df_{\mathbf{k}} Df_{\mathbf{k}}^\dagger e^{i \int dt L[f,A]}$ where the Lagrangian $L[f,A]$ becomes

$$L[f,A] = \sum_{\mathbf{k}} f_{\mathbf{k}}^\dagger i \partial_t f_{\mathbf{k}} - H_0[f,A] - H_1[f,A], \quad (16)$$

with $H_0[f,A] = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}} - \sum_{\mathbf{k}\mu} A_{\mathbf{q}}^\mu \partial_\mu \epsilon_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{q}} + \frac{1}{2} \sum_{\mathbf{k}\mu\nu} A_{\mathbf{q}}^\mu A_{-\mathbf{q}}^\nu \partial_\mu \partial_\nu \epsilon_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}}$. The interaction term $H_1[f,A]$ is obtained by substituting the solution in Eq. (15) into the original H_1 . The effective theory in Eq. (16) encodes band dispersion and Bloch wave changes by the gauge potential for Wannier-obstructed bands. We will derive the current operator based on the effective theory.

Quantum geometric current from interactions—We first examine currents from the free term H_0 . From Eq. (16), it is easy to find the paramagnetic current and diamagnetic current tensor from H_0 , respectively, $j_{\mathbf{p},\mathbf{q}}^\mu = \sum_{\mathbf{k}} \partial_\mu \epsilon_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{q}}$ and $K_{\mu\nu} = \sum_{\mathbf{k}} \partial_\mu \partial_\nu \epsilon_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}}$ with the projected density operator $\rho_{\mathbf{q}} = \sum_{\mathbf{k}} f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{q}}$. The interaction term $H_1[f,A]$ can also contribute to the paramagnetic current $j_{\text{QG},\mathbf{q}}^\mu$ and diamagnetic current tensor $K_{\text{QG},\mu\nu}$,

$$j_{\text{QG},\mathbf{q}}^\mu = \frac{\delta H_1[f,A]}{\delta A_{-\mathbf{q}}^\mu} \Big|_{A \rightarrow 0}, \quad K_{\text{QG},\mu\nu} = \frac{\delta^2 H_1[f,A]}{\delta A_{\mathbf{q}}^\mu \delta A_{-\mathbf{q}}^\nu} \Big|_{A \rightarrow 0}, \quad (17)$$

even though H_1 does not contribute to the current operator in the full Hilbert space. Obviously, both $j_{\text{QG},\mathbf{q}}^\mu$ and $K_{\text{QG},\mu\nu}$ in Eq. (17), which we call QG currents, are proportional to the interaction strength. The QG current encodes the dynamics from the interactions in the subspace $\mathbb{H}^{\text{sub}}$. This is similar to the lowest Landau level [57,58]. For a

specific system, such as a flat-band superconductor, one can simplify the QG current expression by employing the ground state ansatz, as illustrated in the examples below.

One may question whether the Peierls substitution can be applied to a band with a topological obstruction, where long-range hopping terms exist in the tight-binding models. There are two key observations. First, the free Hamiltonian H_0 does not contribute to QG current, and the related current only depends on band dispersion. Second, the projector in real space $P_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} P_{\alpha\beta}(\mathbf{k})$ decreases exponentially for a large distance $|\mathbf{r}-\mathbf{r}'|$ in any dimensions [65–69], leading to a short-range effective interaction [70]. One typical example is the lowest Landau level, where the projected Coulomb interaction gets suppressed by a Gaussian decay factor.

Example I: Diamagnetic response in a flat-band SC—We revisit the SC in a flat-band system which possesses finite quantum geometry [32,33,50] from the perspective of the electromagnetic response. We consider the interaction term: $H_1 = -(U/N) \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} \Delta_{\mathbf{k}}^\dagger(\mathbf{p}) \Delta_{\mathbf{k}'}(\mathbf{p})$ with the pairing operator $\Delta_{\mathbf{k}}(\mathbf{p}) = \sum_{\alpha} c_{-\mathbf{k}\alpha\downarrow} c_{\mathbf{k}+\mathbf{p}\alpha\uparrow}$. We examine its gauge coupling

$$\Delta_{\mathbf{k}}(\mathbf{p}) \rightarrow \Delta_{\mathbf{k}}(\mathbf{p}) + \sum_{\mathbf{q}\mu} A_{\mathbf{q}}^\mu \mathcal{D}_{\mathbf{q}}^\mu(\gamma_{\mathbf{k}}^*(\mathbf{p})) \bar{\Delta}_{\mathbf{k}}(\mathbf{q}+\mathbf{p}) - \sum_{\mathbf{q}\mu\nu} A_{\mathbf{q}}^\mu A_{-\mathbf{q}}^\nu G_{\mu\nu}(\mathbf{k}) \bar{\Delta}_{\mathbf{k}}(\mathbf{p}), \quad (18)$$

with $\bar{\Delta}_{\mathbf{k}}(\mathbf{p}) \equiv f_{-\mathbf{k}\downarrow} f_{\mathbf{k}+\mathbf{p}\uparrow}$ and $\gamma_{\mathbf{k}}(\mathbf{p}) \equiv \langle u_{\mathbf{k}} | u_{\mathbf{k}+\mathbf{p}} \rangle$. For convenience, we abbreviate $\mathcal{D}_{\mathbf{q}}^\mu(\langle u_{\mathbf{k}} | u_{\mathbf{k}'} \rangle) \equiv \langle \mathcal{D}_\mu u_{\mathbf{k}+(\mathbf{q}/2)} | u_{\mathbf{k}'} \rangle + \langle u_{\mathbf{k}} | \mathcal{D}_\mu u_{\mathbf{k}'+(\mathbf{q}/2)} \rangle$. Then the paramagnetic current and diamagnetic current tensor are both proportional to the interaction,

$$j_{\mathbf{p},\mathbf{q}}^\mu = -\frac{U}{N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} \gamma_{\mathbf{k}'}(\mathbf{p}) \mathcal{D}_{\mathbf{q}}^\mu(\gamma_{\mathbf{k}}^*(\mathbf{p})) \bar{\Delta}_{\mathbf{k}'}^\dagger(\mathbf{p}) \bar{\Delta}_{\mathbf{k}}(\mathbf{p}+\mathbf{q}) + (\mathbf{k} \leftrightarrow \mathbf{k}'), \quad (19)$$

$$K_{\mu\nu} = \frac{U}{N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} [G_{\mu\nu}(\mathbf{k}) \gamma_{\mathbf{k}'}(\mathbf{p}) + G_{\mu\nu}(\mathbf{k}') \gamma_{\mathbf{k}}^*(\mathbf{p})] \bar{\Delta}_{\mathbf{k}'}^\dagger(\mathbf{p}) \bar{\Delta}_{\mathbf{k}}(\mathbf{p}). \quad (20)$$

The Meissner effect can be inferred from the total current in linear response $\langle j_{\mathbf{q}}^\mu \rangle = -\sum_{\nu} D_s^{\mu\nu} A_{\mathbf{q}}^\nu$, where the superfluid weight is given by $D_s^{\mu\nu} = R_{\mu\nu}(\mathbf{q},t) + \langle K_{\mu\nu} \rangle$ with the paramagnetic current response tensor $R_{\mu\nu}(\mathbf{q},t) = -\langle [j_{\mathbf{p},\mathbf{q}}^\mu(t), j_{\mathbf{p},-\mathbf{q}}^\nu(0)] \rangle$. The response can be evaluated based on the mean-field order parameter $\Delta = 1/N \sum_{\mathbf{k}} \langle f_{-\mathbf{k}\downarrow} f_{\mathbf{k}\uparrow} \rangle$, which determines a BCS ground state [64]. At leading order in Δ , we can decompose the paramagnetic current, $j_{\mathbf{p},\mathbf{q}}^\mu = 2U\Delta \sum_{\mathbf{k}} \mathcal{D}_{\mathbf{q}}^\mu(\langle u_{\mathbf{k}} | u_{\mathbf{k}} \rangle) [f_{-\mathbf{k}\downarrow} f_{\mathbf{k}+\mathbf{q}\uparrow} - f_{\mathbf{k}\uparrow}^\dagger f_{-\mathbf{k}-\mathbf{q}\downarrow}^\dagger]$, which vanishes when $\mathbf{q} \rightarrow 0$.

In Supplemental Material [64], we show that $R_{\mu\nu}(\mathbf{q}, t)$ vanishes even at finite temperatures where the quasiparticle state populations are nonzero. Thus, the current response at finite temperature is only determined by the diamagnetic response $\langle K_{\mu\nu} \rangle = 2U\Delta^2 \bar{G}_{\mu\nu}$ with $\bar{G}_{\mu\nu}$ being the averaged quantum metric over the first Brillouin zone. This is consistent with Refs. [32,33,50] from different approaches. Therefore, the decrease of superfluid weight can be attributed to the shrinking of the pairing gap at finite temperature [64]. More importantly, the absence of paramagnetic current response at finite temperature indicates that the quasiparticle excitations do not carry dissipative currents, which can provide a mechanism behind the suppression of nonequilibrium quasiparticle transport observed in Ref. [71]. By contrast, in the two-fluid model of a conventional superconductor, dissipation is always present at finite temperatures due to the quasiparticle current [72].

Concerns may arise regarding the validity of linear response theory in a flat-band SC state, where only one energy scale exists. We note that the amplitude of the form factor $\gamma_{\mathbf{k}}^*(\mathbf{p})\gamma_{\mathbf{k}'}(\mathbf{p})$, which appears in the projected interaction term $H_I \rightarrow -(U/N) \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} \gamma_{\mathbf{k}}^*(\mathbf{p})\gamma_{\mathbf{k}'}(\mathbf{p})\bar{\Delta}_{\mathbf{k}}^\dagger(\mathbf{p})\bar{\Delta}_{\mathbf{k}'}(\mathbf{p})$, is smaller than 1. Therefore, it is likely still reasonable to conduct the linear response analysis.

Example II: Drude weight in a flat-band metal—As the second example, we investigate the delocalization effect in a flat-band metal. Reference [73] showed (through a diagrammatic expansion in the full Hilbert space) that there is a finite Drude weight in a flat-band metal with a nonzero quantum metric when the temperature is much larger than the interaction strength. We consider an interaction $H_I = (1/N) \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} V_{\mathbf{p}} \rho_{\mathbf{k}'}^\dagger(\mathbf{p}) \rho_{\mathbf{k}}(\mathbf{p})$ with $\rho_{\mathbf{k}}(\mathbf{p}) = \sum_{\alpha} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}+\mathbf{p}\alpha}$. With the generalized Peierls substitution, we have the paramagnetic current in response to a gauge potential A_t^μ of an electric field,

$$j_{\mathbf{p},t}^\mu = \frac{1}{N} \sum_{\mathbf{k}\mathbf{k}'\mathbf{p}} V_{\mathbf{p}} \mathcal{J}_{\mathbf{k}\mathbf{k}'}^\mu(\mathbf{p}) \bar{\rho}_{\mathbf{k}'}^\dagger(\mathbf{p}) \bar{\rho}_{\mathbf{k}}(\mathbf{p}), \quad (21)$$

with $\mathcal{J}_{\mathbf{k}\mathbf{k}'}^\mu(\mathbf{p}) = \mathcal{D}_0^\mu(\gamma_{\mathbf{k}'}^*(\mathbf{p}))\gamma_{\mathbf{k}}(\mathbf{p}) + \gamma_{\mathbf{k}'}^*(\mathbf{p})\mathcal{D}_0^\mu(\gamma_{\mathbf{k}}(\mathbf{p}))$. When the temperature is much larger than the bandwidth and the interaction strength, we can ignore the finite band dispersion from the Hartree-Fock decomposition of the interactions. In linear response [64], we can evaluate the Drude weight

$$D_{\mu\nu}(T) = \frac{\pi[\bar{\nu}(1-\bar{\nu})]^2}{T} \left[\int_{\mathbf{k}\mathbf{k}'\mathbf{p}} |V_{\mathbf{p}}|^2 \mathcal{J}_{\mathbf{k}\mathbf{k}'}^\mu(\mathbf{p}) \mathcal{J}_{\mathbf{k}\mathbf{k}'}^\nu(-\mathbf{p}) - \int_{\mathbf{k}\mathbf{p}\mathbf{p}'} V_{\mathbf{p}} V_{-\mathbf{p}'} \mathcal{J}_{\mathbf{k}\mathbf{k}}^\mu(\mathbf{p}) \mathcal{J}_{\mathbf{k}\mathbf{k}}^\nu(-\mathbf{p}') \right], \quad (22)$$

with $\bar{\nu}$ being the filling factor. In the special case of the potential $V_{\mathbf{p}} = V\delta_{\mathbf{p},0}$, the Drude weight is related to the

Berry curvature variance, which is consistent with Ref. [73]. Physically, the current $j_{\mathbf{p},t}^\mu$ describes a cooperative hopping process of two particles, which does not have a one-body operator correspondence. At low temperatures, the dispersion induced by the interaction becomes relevant, a problem we leave for future studies. Moreover, it will be interesting to consider the conductivity in the twisted bilayer graphene system where the narrow band possesses a finite quantum geometry and the QG current can be induced by interactions as in Eq. (21).

Example III: Delocalization from disorder—We further show that QG currents from disorder can induce a tendency toward delocalization in a flat-band system, in the absence of any conductivity in the clean limit [74]. For this purpose, we can consider an on-site disorder in two spatial dimensions with $H_I = \sum_{\mathbf{r}\alpha} w_{\mathbf{r}} c_{\mathbf{r}\alpha}^\dagger c_{\mathbf{r}\alpha}$, where the disorder satisfies $\overline{w_{\mathbf{r}} w_{\mathbf{r}'}} = \gamma^2 \delta_{\mathbf{r}\mathbf{r}'}$ with $\overline{\cdot}$ denoting the disorder averaging. Our generalized Peierls substitution to couple to A_t^μ of an electric field yields

$$L = \sum_{\mathbf{k}} i f_{\mathbf{k}}^\dagger \partial_t f_{\mathbf{k}} - \sum_{\mathbf{k}\mathbf{p}} w_{\mathbf{p}} \langle u_{\mathbf{k}+\mathbf{p}} | u_{\mathbf{k}} \rangle f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{p}} - A_t^\mu j_{\mathbf{p},t}^\mu, \quad (23)$$

with $w_{\mathbf{p}} = 1/N \sum_{\mathbf{r}} w_{\mathbf{r}} e^{i\mathbf{p}\cdot\mathbf{r}}$. Here the paramagnetic current

$$j_{\mathbf{p},t}^\mu = \sum_{\mathbf{k}\mathbf{p}} w_{\mathbf{p}} \mathcal{J}_{\mathbf{k}}^\mu(\mathbf{p}) f_{\mathbf{k}}^\dagger f_{\mathbf{k}+\mathbf{p}}, \quad (24)$$

with $\mathcal{J}_{\mathbf{k}}^\mu(\mathbf{p}) = \langle u_{\mathbf{k}+\mathbf{p}} | \mathcal{D}_\mu u_{\mathbf{k}} \rangle + \langle \mathcal{D}_\mu u_{\mathbf{k}+\mathbf{p}} | u_{\mathbf{k}} \rangle$. This paramagnetic current $j_{\mathbf{p},t}^\mu$ depends on the disorder configuration.

In a flat-band system, introducing disorder can destroy the destructive quantum interference, underpinning the vanishing group velocity. We note two effects of disorder in a flat-band system. First, disorder can lead to a finite bandwidth in the disorder-averaged Green's function $\bar{G}_{\mathbf{k}} = (1/\omega/2 - i\gamma)$ for $|\omega| \ll \gamma$ from the Dyson equation [64]. Second, the disorder drives the QG current, Eq. (24), which is absent in the clean limit. Ignoring the quantum fluctuations from Diffuson and Cooperon, we estimate the finite static conductivity when the disorder is much weaker than the band gap, $\sigma_{\mu\nu} \sim \int [d^2\mathbf{k} d^2\mathbf{p}/(2\pi)^4] \mathcal{J}_{\mathbf{k}}^\mu(\mathbf{p}) \mathcal{J}_{\mathbf{k}}^{\nu*}(\mathbf{p})$. Such delocalization effect can potentially be linked to inverse Anderson localization [75,76]. The above discussion describes a regime at scales below the Anderson localization length, which in two dimensions can be large for weak disorder [77,78], as well as energy dependent when band topology protects a delocalized state.

The resulting detailed phenomenology, possibly including a “finite-size metal-insulator transition,” is an interesting object that we leave to future study. Experimentally, the Lieb lattice, already realized in cold atomic systems 15 years ago [79–86], could provide a platform for experimental investigations.

Concluding remarks—It is essential to consider changes in Bloch waves when studying the response of bands of

Wannier obstructions to general perturbations, which are not limited to electromagnetic fields. We note that our projection method can be extended to cases without translational symmetry where one can enforce projection onto the descendants of the flat band through $L_\lambda = \int_{\mathbf{r}} \lambda_{\mathbf{r}}^\dagger (c_{\mathbf{r}} - \sum_n \psi_n^*(\mathbf{r}) f_n) + \text{H.c.}$, with n labeling the eigenstate of the bands and $\psi_n(\mathbf{r})$ the corresponding wave functions. The projection restricts electrons to be spatially correlated, such that $c_{\mathbf{r}} = \int_{\mathbf{r}'} G(\mathbf{r}, \mathbf{r}') c_{\mathbf{r}'}$ with $G(\mathbf{r}, \mathbf{r}') = \sum_n \psi_n^*(\mathbf{r}) \psi_n(\mathbf{r}')$.

Finally, based on the examples above, one may view the interaction-induced QG currents as integral to Fermi liquid behavior [13,87]. This immediately raises the question of how to extend Landau's Fermi liquid theory to incorporate QG currents.

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- [64] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/xkxw-1134> for further details of results and calculation. Sec. I: Review on the Peierls

- substitution; Sec. II: Euler-Lagrange equation; Sec. III: Example: Flat-band superconductor; Sec. IV: Example: Drude weight in flat-band metal from interaction; and Sec. V: Example: Delocalization by disorder in a flat system.
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