Self-Propulsion Symmetries Determine Entropy Production of Active Particles with Hidden States

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Entropy production distinguishes equilibrium from nonequilibrium. Calculating the entropy production rate (EPR) is challenging in systems where some degrees of freedom cannot be observed. Here we introduce a perturbative framework to calculate the time irreversibility of an active particle with hidden self-propulsion, termed the "partial EPR," which provides a lower bound to the full, physical EPR. We find that the parity symmetry, $\mathcal{P}$, and time reversibility, $\mathcal{T}$, of the hidden variable determine the partial EPR. Nontrivial partial EPR appears at least at *sixth order* in the self-propulsion velocity. We apply our framework to two processes that break $\mathcal{P}$- and $\mathcal{T}$ symmetries, respectively: an asymmetric telegraph process and diffusion with stochastic resetting.

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Violation of time reversal symmetry (TRS) is a fundamental feature of nonequilibrium systems and essential for the existence of life [1–3]. The extent of TRS-breaking places thermodynamic constraints on biological processes such as reproduction, sensing, and signaling [4–7]. A pressing theoretical challenge is to estimate or bound the time irreversibility of systems that cannot be fully observed, since visible degrees of freedom may break TRS to a lesser extent than the complete underlying system [8], and some degrees of freedom, in particular internal states, are difficult to observe experimentally. When *all* microscopic details are visible, time irreversibility is measured by the full, physical entropy production rate (EPR), which may intuitively be thought of as a "distance" of the system from equilibrium [9]. Zero EPR corresponds to a time-reversible (i.e., equilibrium) system. A number of approaches have been proposed for partially hidden systems: for discrete state systems, lower bounds to the full entropy production rate have been derived from waiting time distributions and transition frequencies [10–16] and estimates proposed based on inference of hidden system parameters [17]. In continuous systems, approaches to quantifying irreversibility include "milestoning" continuous trajectories into discrete states [18,19], inferring entropy production from measured moments [20] and explicit integration for particles with hidden Gaussian (Ornstein-Uhlenbeck) activity [9,21,22].

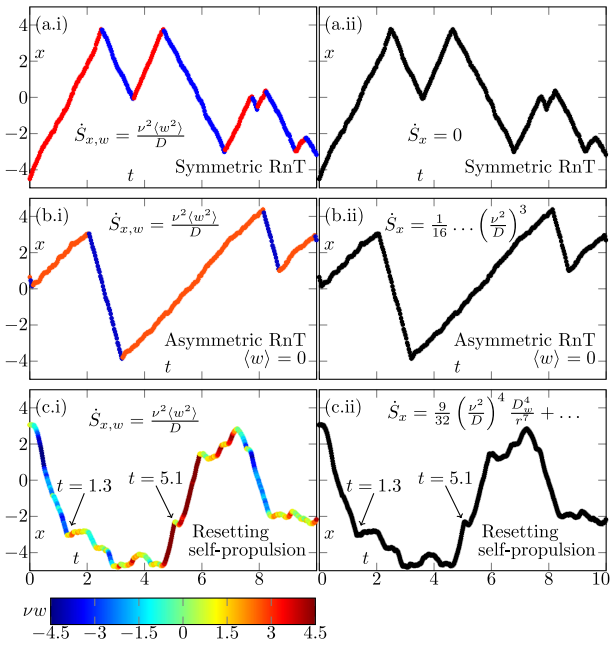


FIG. 1. Three processes $x(t)$ with self-propulsion velocity $\nu w(t)$ [left column, (i)] fully resolved (color-coded), and [right column, (ii)] hidden. Self-propulsion $w(t)$ is (a) a symmetric telegraph process (symmetric RnT), (b) an asymmetric telegraph process with vanishing net drift $\langle w\rangle = 0$ (asymmetric RnT), Eq. (15), (c) diffusion with stochastic resetting (resetting self-propulsion), Eq. (16). Red and blue coloring (color bar) correspond to positive and negative self-propulsion velocity $\nu w(t)$, respectively. In (a) trajectories break TRS when fully observed (left, since time reversal switches red and blue) but obey TRS when the self-propulsion state is hidden (right). In (b) and (c) trajectories break TRS even with hidden $w(t)$. In (c) two resetting events are marked. All processes are discussed below, also see Table I.

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In this Letter we calculate a "partial EPR" that quantifies the TRS-breaking of a canonical partially observable system: an overdamped active particle with hidden stochastic self-propulsion [21–24]. This partial EPR captures the time irreversibility of spatial trajectories only and is a lower bound to the full, physical EPR of the fully observable system [25,26]. Our perturbative framework allows us to characterize *generic* active particles, going beyond the previously studied case of active Ornstein-Uhlenbeck particles [9,21,22]. Intriguingly, we find that the *symmetries* of the *hidden* degree of freedom determine whether or not, and the extent to which, the TRS of spatial trajectories is broken. We demonstrate this by calculating the leading-order partial EPR of two processes that minimally break the required symmetries to observe TRS breaking in their spatial trajectories.

Intuition is provided by considering a symmetric run-and-tumble (RnT) particle [27,28] moving in one dimension, Fig. 1(a) When both its spatial position and self-propulsion velocity can be observed, trajectories can easily be distinguished from their time-reversed counterparts, meaning that the process has a nonzero EPR. However, when *only* its position can be observed, trajectories are time reversal symmetric and thus the process has zero *partial* EPR. Figures 1(b) and 1(c) show processes where the partial EPR is nonzero when the self-propulsion velocity is hidden.

Key derivation—In the following, the Langevin equation [27,28]

$$\dot{x}(t) = \nu w(t) + \xi(t), \tag{1a}$$

$$\langle\!\langle \xi(t)\xi(t')\rangle\!\rangle = 2D\delta(t-t'), \tag{1b}$$

acts as a phenomenological description of an overdamped active particle at position $x(t) \in [0, L)$ with periodic boundary conditions, as to enforce a steady state. It is self-propelled with velocity $\nu w(t)$, where ν has dimensions of velocity and $w(t)$ is a steady-state dimensionless stochastic process, to be specified further, but evolving independently of $x(t)$. The particle is further subject to independent Gaussian white noise $\xi(t)$ with diffusivity D and vanishing mean, $\langle\!\langle \xi \rangle\!\rangle = 0$, where $\langle\!\langle \bullet \rangle\!\rangle$ denotes the average over the noise. The full, physical EPR of Eq. (1) depends on the ratio of the path probabilities of forward and reversed trajectories of $\{x(t)\}$ and $\{w(t)\}$ and is given by [29–34]

$$\dot{S}_{x,w} = \lim_{T\to\infty} \frac{1}{T} \int \mathcal{D}x \int \mathcal{D}w\, \mathbb{P}[\{x(t)\},\{w(t)\}] \times \ln\left(\frac{\mathbb{P}[\{x(t)\},\{w(t)\}]}{\mathbb{P}[\{x(T-t)\},\{w(T-t)\}]}\right), \tag{2}$$

where T is the trajectory duration. Trajectories are assumed to be initialized according to steady-state distributions, negating transient contributions arising from initial conditions (which vanish anyway due to the limit $T \to \infty$). $\mathbb{P}[\{x(t)\},\{w(t)\}]$ denotes a normalized measure for the steady-state probability density of a joint path $[\{x(t)\},\{w(t)\}]$ for $t \in [0, T]$ to occur in process Eq. (1). Correspondingly, $\mathbb{P}[\{x(T-t)\},\{w(T-t)\}]$ is the probability density for the reverse path to occur. Such path probabilities are easy to construct when they derive from a known distribution, e.g., using an Onsager-Machlup functional based on Eq. (1) for the path probability [[35,36], Ch. 5.2.2.3]

$$\mathbb{P}[\{x(t)\}|\{w(t)\}] \propto \exp\left(-\frac{1}{4D}\int_0^T \mathrm{d}t[\dot{x}(t)-\nu w(t)]^2\right), \tag{3}$$

of $\{x(t)\}$ *conditioned* on $\{w(t)\}$, derived from the density of the noise $\{\xi(t)\}$. To avoid ambiguity, we use the Stratonovich convention throughout. Here and in the following, $\dot{x}(s)$ shall denote the derivative of $x(t)$ with respect to t, evaluated at s, so that

$$\mathbb{P}[\{x(T-t)\}|\{w(T-t)\}] \propto \exp\left(-\frac{1}{4D}\int_0^T \mathrm{d}t[-\dot{x}(T-t)-\nu w(T-t)]^2\right). \tag{4}$$

Multiplying Eq. (3) by $\mathbb{P}[\{w(t)\}]$, which generally is difficult to obtain, finally produces $\mathbb{P}[\{x(t)\},\{w(t)\}]$. Instead of using such an explicit expression in Eq. (2), one may use that $x(t)$, $w(t)$ is Markov, so that $\mathbb{P}[\{x(t)\},\{w(t)\}]$ factorizes in time slices [29] and the limit of large T does not need to be taken, reducing the ratio of path probabilities to that of transition rates [30,37].

Such a bypass is no longer available if we consider the scenario where the position $x(t)$ of the particle can be observed, but *not* its internal self-propulsion state $w(t)$, as it is *hidden*. The position $x(t)$ without accompanying $w(t)$ leaves the system non-Markovian. This description of the system has a corresponding *partial* EPR

$$\dot{S}_x = \lim_{T\to\infty} \frac{1}{T} \int \mathcal{D}x\, \mathbb{P}[\{x(t)\}] \ln\left(\frac{\mathbb{P}[\{x(t)\}]}{\mathbb{P}[\{x(T-t)\}]}\right), \tag{5}$$

which by Jensen's inequality can be shown to be bounded from above by $\dot{S}_{x,w} \geq \dot{S}_x$. The word *partial* is used to highlight that $\dot{S}_x$ represents the time irreversibility of the partially hidden system, and that $\dot{S}_x$ does not correspond to the rate of physical entropy production, unlike the full EPR $\dot{S}_{x,w}$. Our approach to calculate Eq. (5) is to express the path probability as

$$\mathbb{P}[\{x(t)\}] = \int \mathcal{D}w\, \mathbb{P}[\{x(t)\}|\{w(t)\}]\mathbb{P}[\{w(t)\}], \tag{6}$$

where the dummy variable $w(t)$ can equally be replaced by $w(T-t)$.

Using Eqs. (3), (4), and (6) in Eq. (5) allows the term quadratic in $\dot{x}(t)$ to be canceled across the fraction as $x(t)$ is a dummy variable of the outer path integral. Using that $\mathrm{d}x(t)/\mathrm{d}t = -\dot{x}(T-t)$, the partial EPR takes the form

$$\dot{S}_x = \lim_{T\to\infty}\frac{1}{T}\int \mathcal{D}x\,\mathbb{P}[\{x(t)\}] \times \ln\left(\frac{\langle \exp(\frac{\nu}{2D}\int_0^T \mathrm{d}t\, w(t)\dot{x}(t))\rangle_*}{\langle \exp(-\frac{\nu}{2D}\int_0^T \mathrm{d}t\, w(T-t)\dot{x}(t))\rangle_*}\right) \quad (7)$$

$$\text{with}\quad \langle\bullet\rangle_* = \frac{1}{\mathcal{N}_*(\nu)}\int \mathcal{D}w\,\bullet\exp\left(-\frac{\nu^2}{4D}\int_0^T \mathrm{d}t\, w(t)^2\right)$$

$$\mathbb{P}[\{w(t)\}] = \int \mathcal{D}w\,\bullet\,\mathbb{P}_*[\{w(t)\}], \quad (8)$$

where $\mathcal{N}_*(\nu)$ is the normalization of the modified path probability distribution $\mathbb{P}_*[\{w(t)\}]$ defined above. To leading order ν^0 of the exponential, $\langle\bullet\rangle_*$ is the expectation $\langle\bullet\rangle$ under $\mathbb{P}[\{w(t)\}]$, Supplemental Material [38]. The logarithm of the expectation of an exponential in Eq. (7) can be recognized as the cumulant-generating function of $\mathbb{P}_*$ with $\dot{x}(t)$ as the conjugate variable. Writing these expectations as

$$\overline{\dot{x}(t_1)\ldots\dot{x}(t_n)} = \int \mathcal{D}x(\dot{x}(t_1)\ldots\dot{x}(t_n))\mathbb{P}[\{x(t)\}] \quad (9)$$

Eq. (7) can be rewritten in terms of cumulants and correlators,

$$\dot{S}_x = \lim_{T\to\infty}\frac{1}{T}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\nu}{2D}\right)^n\int_0^T \mathrm{d}t_1\ldots\mathrm{d}t_n\overline{\dot{x}(t_1)\ldots\dot{x}(t_n)} \times\Big(\langle w(t_1)\ldots w(t_n)\rangle_*^{(\mathrm{c})} - (-1)^n\langle w(T-t_1) \times\ldots w(T-t_n)\rangle_*^{(\mathrm{c})}\Big), \quad (10)$$

where the superscripted (c) indicates the cumulants. Equation (10) is central to the results derived in the following.

Results—Analyzing Eq. (10), we observe that the partial EPR of processes of the form Eq. (1) depends on the *symmetries* obeyed by the hidden self-propulsion $w(t)$. The relevant symmetries are parity symmetry $\mathcal{P}$, which is obeyed by $\mathbb{P}[\{w(t)\}]$ iff

$$\mathbb{P}[\{w(t)\}] = \mathbb{P}[\{-w(t)\}] = \mathbb{P}[\mathcal{P}\{w(t)\}] \quad (11)$$

and time reversal symmetry $\mathcal{T}$

$$\mathbb{P}[\{w(t)\}] = \mathbb{P}[\{w(T-t)\}] = \mathbb{P}[\mathcal{T}\{w(t)\}]. \quad (12)$$

By construction of Eqs. (2) and (5), the EPR vanishes if $\mathbb{P}[\{x(t)\},\{w(t)\}]$ or $\mathbb{P}[\{x(t)\}]$, respectively, are $\mathcal{T}$ symmetric. As $(x-1)\ln(x)\geq 0$ for $x\in\mathbb{R}^+$ the converse holds as well.

The $\mathcal{PT}$ symmetry of $\int_0^T \mathrm{d}t\, w^2(t)$ in Eq. (8) implies that the relevant symmetries of $\mathbb{P}_*[\{w(t)\}]$ are identical to those of $\mathbb{P}[\{w(t)\}]$, so we focus on the latter. By inspection of Eq. (7), if $\mathbb{P}[\{w(t)\}]$ is invariant under replacing $\{w(t)\}$ by $\{-w(T-t)\}$, then the logarithm vanishes, so that $\dot{S}_x = 0$ under $\mathcal{PT}$-symmetry of $w(t)$ [23]. Given $(x-1)\ln(x)\geq 0$, the converse follows from Eq. (7), i.e., a process Eq. (1) with vanishing partial EPR must be driven by $\mathcal{PT}$-symmetric $w(t)$.

Further, the expansion Eq. (10) reveals separate cancellation of odd and even terms under $\mathcal{P}$ and $\mathcal{T}$ symmetry, respectively. If $\mathbb{P}[\{w(t)\}]$ is $\mathcal{P}$ symmetric, then any odd moments and cumulants of $w(t)$ vanish, i.e., odd n vanish in Eq. (10). If $\mathbb{P}[\{w(t)\}]$ is $\mathcal{T}$ symmetric, $\langle w(t_1)\ldots w(t_n)\rangle_* = \langle w(T-t_1)\ldots w(T-t_n)\rangle_*$, then even n cancel.

The lowest order $n=1$ in Eq. (10) indicates "trivial" partial EPR as $x(t)$ displays a net drift, $\langle w\rangle_* = \langle w\rangle_*^{(\mathrm{c})} \neq 0$, as we will illustrate below. The second lowest order, $n=2$, in one dimension always vanishes in the steady state, as $\langle w(t_1)w(t_2)\rangle_*^{(\mathrm{c})}$ can be a function only of $|t_1-t_2|$ and therefore equals $\langle w(T-t_1)w(T-t_2)\rangle_*^{(\mathrm{c})}$. This immediately explains that free active Ornstein-Uhlenbeck particles have vanishing partial EPR, $\dot{S}_x = 0$, as all cumulants beyond $n=2$ of $\mathbb{P}[\{w(t)\}]$ and $\mathbb{P}_*[\{w(t)\}]$ vanish. More terms need to be considered in the presence of an external potential [50], resulting in vanishing partial EPR in harmonic, but generally nonzero partial EPR in anharmonic potentials [24]. External potentials may capture space-dependent self-propulsion as observed in colloidal and living systems.

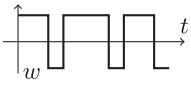
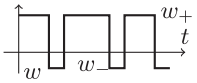
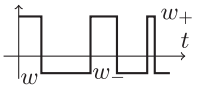
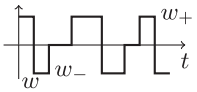
After symmetrizing Eq. (10), the leading order contribution to $\dot{S}_x$ can be written as (Supplemental Material [38])

$$\dot{S}_x = \lim_{T\to\infty}\frac{1}{2T}\frac{1}{n^*!}\left(\frac{\nu^2}{2D}\right)^{n^*}\int_0^T \mathrm{d}t_1\ldots\mathrm{d}t_{n^*} \times\left(\langle w(t_1)\ldots w(t_{n^*})\rangle - (-1)^{n^*}\langle w(T-t_1) \times\ldots w(T-t_{n^*})\rangle\right)^2 + \mathcal{O}\left(\frac{\nu^{2n^*+2}}{D^{n^*+1}}\right), \quad (13)$$

where n^* denotes the lowest order for which the squared bracket does not vanish. Inspecting Eq. (13), the leading order n^* is even when $w(t)$ obeys $\mathcal{P}$ symmetry and odd when $w(t)$ obeys $\mathcal{T}$ symmetry.

Applications—In the following we consider several processes to illustrate the calculation of their partial EPR on the basis of Eqs. (10) and (13). The sole mathematical challenge is to determine higher order correlation functions and cumulants of $w(t)$, relegated to Supplemental Material [38]. The results are summarized in Table I. The first three

TABLE I. Different $w(t)$ processes give rise to different trajectories $x(t)$, Eq. (1), and therefore different EPR, as discussed in the main text. The first three rows produce conventional run-and-tumble behavior with different parameters. The resetting noise in the fourth row is more akin to active Brownian particles, as $w(t)$ evolves piecewise continuously in time. The last row is a variation of the first. Symmetries of the w process are indicated in the third column. If broken, they are struck through.

$w(t)$ process	Sample graph	Symmetries	$\dot{S}_{x,w}$	$\dot{S}_w$	$\dot{S}_x$
Symmetric telegraph, $\langle w\rangle = 0$		$\mathcal{P}$, $\mathcal{T}$, $\mathcal{PT}$	$\langle w^2\rangle(\nu^2/D)$	0	0
Asymmetric telegraph, $\langle w\rangle \neq 0$		$\mathcal{T}$, $\not{\mathcal{P}}$	$\langle w^2\rangle(\nu^2/D)$	0	$\langle w\rangle_\nu^2(\nu^2/D) + \cdots$ Eq. (14)
Asymmetric telegraph, $\langle w\rangle = 0$		$\mathcal{T}$, $\not{\mathcal{P}}$	$\langle w^2\rangle(\nu^2/D)$	0	$(-1/16)(\nu^2/D)^3(w_+^3 w_-^3/(\alpha_+\alpha_-))$ $[(\alpha_+ - \alpha_-)/(\alpha_+ + \alpha_-)]^2 + \cdots$ Eq. (15)
Resetting, $\langle w\rangle = 0$		$\not{\mathcal{P}}$, $\not{\mathcal{T}}$	∞	∞	$(9/32)(\nu^2/D)^4(D_w^4/r^7) + \cdots$ Eq. (16)
Three-state cyclic, $\langle w\rangle = 0$		$\not{\mathcal{P}}$, $\not{\mathcal{T}}$, $\not{\mathcal{PT}}$	$\langle w^2\rangle(\nu^2/D) + \dot{S}_w$	$(\alpha - \beta)\ln(\alpha/\beta)$	0

are based on $w(t) \in \{w_+, w_-\}$ determined by a telegraph process, which switches from w_+ to w_- with rate α_- and back with rate α_+. From first principles [30,51] $\dot{S}_{x,w} = \nu^2\langle w^2\rangle/D$ with $\langle w^2\rangle \geq \langle w\rangle^2$ and $\langle w\rangle = (w_+\alpha_- + w_-\alpha_+)/(\alpha_+ + \alpha_-)$. This $w(t)$ process is bound to be $\mathcal{T}$ symmetric, even when $\alpha_+ \neq \alpha_-$, as these rates merely determine occupation times in the two states. The only terms that contribute in Eq. (10) are therefore those with odd n.

Iff $w_+ = -w_-$ and $\alpha_+ = \alpha_-$, the process is also $\mathcal{P}$ symmetric, so that $\dot{S}_x = 0$, which is expected as $x(t)$ is indistinguishable from its reverse $x(T-t)$, Fig. 1(a). If $\langle w\rangle \neq 0$, then Eq. (10) gives

$$\dot{S}_x = \frac{\nu^2}{D}\langle w\rangle^2 + \mathcal{O}\left(\frac{\nu^4}{D^2}\right), \tag{14}$$

which holds for any process with net current $\nu\langle w\rangle$. If the net drift vanishes, $\langle w\rangle = 0$, Fig. 1(b), the leading order becomes, Supplemental Material [38],

$$\dot{S}_x = -\frac{1}{16}\left(\frac{\nu^2}{D}\right)^3 \frac{w_+^3 w_-^3}{\alpha_+\alpha_-}\left(\frac{\alpha_+ - \alpha_-}{\alpha_+ + \alpha_-}\right)^2 + \mathcal{O}\left(\frac{\nu^8}{D^4}\right), \tag{15}$$

as confirmed by a numerical sanity check in Supplemental Material [38]. When $\mathcal{P}$ symmetry is restored by taking $\alpha_+ = \alpha_-$ and $w_+ = -w_-$, the right-hand sides of Eqs. (14) and (15) vanish as expected.

The entropy production rate $\dot{S}_w$ vanishes in all three telegraphic processes, while $\dot{S}_x$ may not, implying that $x(t)$ producing entropy does not require $w(t)$ to produce entropy.

As a $w(t)$ process that is $\mathcal{P}$ symmetric but not $\mathcal{T}$ symmetric, we choose the case of diffusion with diffusivity D_w and stochastic resetting to $w(t) = 0$ with Poissonian rate r [52,53], Fig. 1(c). As there is no bias, the statistics of $w(t)$ are independent of its sign. $\mathcal{T}$ symmetry is broken by the instantaneous resetting to the origin, causing both $\dot{S}_{x,w}$ and $\dot{S}_w$ to diverge. As derived in Supplemental Material [38] using [54],

$$\dot{S}_x = \frac{9}{32}\left(\frac{\nu^2}{D}\right)^4 \frac{D_w^4}{r^7} + \mathcal{O}\left(\frac{\nu^{10}}{D^5}\right). \tag{16}$$

We finally consider a $w(t)$ process that is neither $\mathcal{P}$ nor $\mathcal{T}$ symmetric, but $\mathcal{PT}$ symmetric, given by a cyclic change from w_+ to $w_- = -w_+$ to 0 with rate α and backward with rate a rate $\beta < \alpha$ [23]. The EPR solely of the w process is $\dot{S}_w = (\alpha - \beta)\ln(\alpha/\beta)$ [30]. The full EPR is, equally from first principles, $\dot{S}_{x,w} = \dot{S}_w + \langle w^2\rangle/D$ with $\langle w^2\rangle = 2w_+^2/3$, while the partial EPR of the $x(t)$ process by itself vanishes, $\dot{S}_x = 0$.

Discussion and outlook—Spatial trajectories of particles with nonzero net drift can trivially be distinguished from their time reverse due to the swapped drift direction, and have leading order partial EPR $\propto \nu^2/D$, i.e., $n^* = 1$, Eq. (14) and Table I. Trajectories of particles with zero net drift differ more subtly from their time reverse. Steady state stochastic processes $w(t)$ obey $\langle w(t_1)w(t_2)\rangle = \langle w(T-t_1)w(T-t_2)\rangle$, implying that two-time correlation functions are time

reversible, so that n^* can never be 2 in one dimension. We thus find that such current-free particles with steady state self-propulsion processes reveal their time-asymmetry only at the level of three-time correlators or above, as observed in [26]. When the self-propulsion $w(t)$ breaks $\mathcal{P}$ symmetry, a leading order contribution appears at $n^* = 3$, Eq. (15). When self-propulsion obeys $\mathcal{P}$ symmetry but breaks $\mathcal{T}$ symmetry, as in the case of stochastic resetting of $w(t)$, the four-time correlator is the lowest order that reveals time asymmetry of $x(t)$, resulting in a leading order contribution at $n^* = 4$, Eq. (16). As a rule of thumb at small self-propulsion, the harder it is to qualitatively distinguish forward and reversed spatial trajectories of a process, the higher the order n^* of the leading term in Eq. (10). As shown in Fig. 1(c) time irreversibility is very difficult to see with the naked eye but reveals itself through a sudden jump to zero velocity, e.g., at $t = 1.3$, 5.1 (marked by arrows).

Entropy production measures "how broken" $\mathcal{T}$ symmetry is [9,29], Eqs. (2) and (5). How come the $\mathcal{PT}$ symmetry and not just the $\mathcal{T}$ symmetry of $w(t)$ enters into the partial EPR of $x(t)$ as given by Eq. (1)? The answer is found by writing the equation of motion of $x^R(t) = x(T-t)$, which produces $\dot{x}^R(t) = -\dot{x}(T-t) = -\nu w(T-t) - \xi(T-t)$, so that the statistics of $x^R(t)$ is identical to that of $x(t)$ if the statistics of $-w(T-t)$ and $-\xi(T-t)$ are identical to that of $w(t)$ and $\xi(t)$, respectively, i.e., when their statistics is $\mathcal{PT}$ symmetric. In other words, the reason why the $\mathcal{P}$ symmetry of $w(t)$ matters for the EPR of $x(t)$ is that w features on the right of x's time derivative. One might think of extensions to Eq. (1) that make other symmetries of $w(t)$ visible in the partial EPR of $x(t)$.

To use the above formalism, the mathematical challenge is to calculate correlators and higher order moments, Supplemental Material [38]. This is still much easier than calculating full path integrals, Eq. (5), but can be challenging in some cases. The formalism is readily extended to incorporate potentials [50], in which case even the simplest run-and-tumble setup produces entropy. The required correlators for a harmonic potential are available from [55], but are quite unwieldy.

Our Letter is of particular relevance to experimental systems, where the position of particles can be observed, but not so easily their internal state. Such experimental systems typically "live" in dimensions greater than one. In higher dimensions, expressions like Eq. (10) are more cumbersome as $w(t)\dot{x}(t)$ in Eq. (7) becomes $\mathbf{w}(t)\cdot\dot{\mathbf{x}}(t)$. The symmetry arguments carry through, as odd correlators of $\mathbf{w}$ still vanish in the presence of $\mathcal{P}$ symmetry and even terms still cancel in the presence of $\mathcal{T}$ symmetry. However, for two components $\alpha \neq \beta$ steady state does not necessarily imply equality of $\langle w_\alpha(t_1) w_\beta(t_2)\rangle$ and $\langle w_\alpha(T-t_1) w_\beta(T-t_2)\rangle = \langle w_\beta(t_1) w_\alpha(t_2)\rangle$, so that the lowest order can be $n^* = 2$ in the presence of, say, a "chiral" $\mathbf{w}$. There might be further symmetries to consider in higher dimensions. An experimentalist can deduce symmetries of the self-propulsion velocity by measuring the dependence of the partial EPR on the temperature and thus D, possibly helped by tools like those of [56].

A thermodynamic interpretation of the partial EPR beyond a lower bound to the full EPR is an important topic of future work. The level of description of a system determines how it can be manipulated [[57], Ch. 5]. Connection could be made between partial EPR and maximum extractable power in partially hidden systems [58]. The framework could also be extended to capture the (partial) EPR of a system with an *inferred* internal state [23,59,60]. Extending this formalism to spatially discrete systems and particles in potentials [50] would allow applications to a wider array of models of biological systems including molecular motors and hair cell bundles [26,61,62].

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Data availability—The data that support the findings of this article are available upon request to the corresponding authors.

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