

Kaluza-Klein AdS Virasoro-Shapiro Amplitude near Flat Space

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We bootstrap the first-order correction in the curvature expansion of the Virasoro-Shapiro amplitude in AdS spacetime, for arbitrary Kaluza-Klein charges of external operators. By constructing a universal ansatz based on single-valued multiple polylogarithms as well as an AdS $\times$ S formalism, and matching it with the low-lying result, we derive a unified formula in terms of world-sheet integrals. Our result predicts an infinite number of Wilson coefficients that were not available in previous literature.

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Introduction—The application of analytic bootstrap methods to correlators in holographic conformal field theory models has recently uncovered many hidden structures in these observables in the supergravity limit. In particular, in the prototypical model of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM), contrary to the naive expectation from complicated bulk interaction vertices, these new developments reveal that these correlators enjoy universal features that are guided by simple principles such as the hidden higher dimensional conformal symmetry [1]. This offers a useful probe into the structure of particle interactions in anti-de Sitter (AdS). When taking into consideration the full type IIB superstring dual to SYM, one may wonder whether the above simplicity extends to the entire string scattering [2]. At least, when we perform curvature expansion on (the Mellin representations of) the four-point correlator of half-BPS operators, i.e., with respect to $\lambda^{-\frac{1}{2}} = \alpha'/R^2$, the leading piece in the genus-0 scattering is well captured by the renowned Virasoro-Shapiro amplitude

$$\mathcal{A}^{(0)}(S, T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}, \quad (1)$$

where S, T, U are the Mandelstam variables, satisfying $S + T + U = 0$. It should be ideal that a similar compact formula exists for the complete genus-0 correlator.

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As a promising step toward this goal, recent studies of the stress-tensor correlator $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ as well as its higher Kaluza-Klein (KK) charge counterparts $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$ show that the corrections in the above curvature expansion can also be expressed as a Riemann sphere integral with insertions [8–10]. It is then natural to inspect whether these developments can be made systematic for arbitrary KK charge configurations. On the one hand, such an all KK result will definitely help us ease the analysis of the operator spectrum as well as provide new insights on universal structures of the correlators. On the other hand, computation in these general situations is typically not a straightforward task, since existing methods (such as dispersive sum rules [11,12]) struggle with superconformal block expansion among different KK modes.

In this Letter, we resolve the above challenge through a novel AdS $\times$ S formalism [3], establishing the world-sheet formula for the first curvature correction to the AdS Virasoro-Shapiro amplitude of arbitrary KK modes. The bootstrap computation leading to this result starts with an ansatz where the world-sheet integrand is built using single-valued multiple polylogarithms (SVMPLs). After constraining the ansatz by crossing symmetry, the remaining degrees of freedom turn out to be uniquely fixed by matching the known $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$ result [10]. This circumvents the computational complexity in dispersive sum rules. We validate the universality of our result via low-energy expansions [3–7], where consistency with localization/integrated predictions [13,14] is confirmed by precise agreement on Wilson coefficients. This world-sheet formula further provides an infinite set of new predictions for the Wilson coefficients.

The correlators—In this Letter we consider the supergravitons in AdS₅ $\times$ S⁵. On the boundary they are half-BPS operators $\mathcal{O}_{p \geq 2}(x, y)$ in $\mathcal{N} = 4$ SYM theory, with protected dimension $\Delta = p$ and transforming as $[0, p, 0]$ under the SU(4) $\simeq$ SO(6) R-symmetry group. Here, x denotes the

space-time dependence while y represents the $\text{SO}(6)$ null vector. We can derive the reduced correlator $\mathcal{H}(u, v)$ from four-point correlation function $\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle$ by solving the superconformal Ward identity [15,16] [17]. The reduced Mellin amplitude is defined by

$$\mathcal{H}(u, v; y_{ij}) = \int \frac{ds dt}{(2\pi i)^2} u^{\frac{s+4}{2}} v^{\frac{t-p_{23}}{2}} M(s, t; y_{ij}) \Gamma_{p_i}, \quad (2)$$

where we abbreviate $p_{ij} = p_i + p_j$ and $y_{ij} = y_i \cdot y_j$. The function Γ_{p_i} reads,

$$\Gamma_{p_i} = \Gamma\left(\frac{p_{12}-s}{2}\right) \Gamma\left(\frac{p_{34}-s}{2}\right) \Gamma\left(\frac{p_{14}-t}{2}\right) \\ \times \Gamma\left(\frac{p_{23}-t}{2}\right) \Gamma\left(\frac{p_{13}-\tilde{u}}{2}\right) \Gamma\left(\frac{p_{24}-\tilde{u}}{2}\right),$$

and

$$s + t + \tilde{u} = p_1 + p_2 + p_3 + p_4 - 4 = 2\Sigma - 4.$$

Note that s , t , and $\tilde{u}$ are the Mellin analogues of the usual Mandelstam variables for four-point amplitudes in flat space.

The genus zero contributions of $M(s, t)$ can be decomposed into [18]

$$M^{\text{genus-0}} = M^{\text{SG}} + \lambda^{-\frac{3}{2}} M^3 + \lambda^{-\frac{5}{2}} M^5 + \dots, \quad (3)$$

where M^{SG} originates from the supergravity contributions [19] while M^i corresponds to higher-derivative stringy effective corrections. The divergences in the asymptotic expansion Eq. (3) are removed via a Borel transform [12], which resums the perturbative series into a finite amplitude

$$A(S, T; y_{ij}) = \lambda^{\frac{3}{2}} \Gamma(\Sigma - 1) \int \frac{d\alpha}{2\pi i} \frac{e^\alpha}{\alpha^{\Sigma+2}} \\ \times M\left(\frac{2\sqrt{\lambda}S}{\alpha} + \frac{2\Sigma-4}{3}, \frac{2\sqrt{\lambda}T}{\alpha} + \frac{2\Sigma-4}{3}; y_{ij}\right). \quad (4)$$

This transformation analytically continues the original divergent series into a well-defined amplitude, reorganizing the curvature expansion in $\lambda^{-\frac{1}{2}}$ around flat space,

$$A^{\text{genus-0}}(S, T; y_{ij}) = \sum_{k=0}^{\infty} \lambda^{-\frac{k}{2}} A^{(k)}(S, T; y_{ij}). \quad (5)$$

The leading term $A^{(0)}$ is proportional to Virasoro-Shapiro amplitude with an extra factor fixed by known supergravity terms on S^5 [19,20]. $A^{(0)}$ exhibits a low-energy expansion whose coefficients lie in the ring of single-valued multiple

zeta values (MZVs) [21–24]. For example, in the $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2 \rangle$ case,

$$A_{2222}^{(0)} = \frac{1}{STU} + 2\zeta_3 + (S^2 + T^2 + U^2)\zeta_5 + STU\zeta_3^2 + \dots$$

We use ζ_n to represent the Riemann ζ function. The appearance of single-valued MZVs indicates the structure of world-sheet integrals. Building on this observation one can systematically construct curvature corrections $A^{(k)}(S, T)$ by generalizing the integrand on a compact Riemann sphere [11,12].

Our goal is to ascertain the first curvature correction $A^{(1)}(S, T)$ for *arbitrary* KK modes. However, for the scattering involving generic KK modes, the dispersive sum rule method necessitates applying superconformal block decomposition and carefully handling the intricate internal $\text{SU}(4)$ symmetry. These difficulties underscore the necessity to enhance the current framework.

The AdS $\times$ S formalism—It is convenient to introduce the AdS $\times$ S Mellin formalism [3] when one deals with the KK levels. This S transform sends n_{ij} to y_{ij} similar to the Mellin transformation, which sends $s/t/u$ to x_{ij}^2 ,

$$M(s, t; y_{ij}) = \sum_{n_{ij}} \mathcal{M}(s, t; n_{ij}) \times \prod_{i < j} \frac{y_{ij}^{n_{ij}}}{\Gamma(n_{ij} + 1)}, \quad (6)$$

where n_{ij} can be understood as the Mellin variables on the sphere S^5 . To obtain the usual Mellin amplitude, one sums over all integers n_{ij} under the following constraints:

$$\sum_j n_{ij} = 0, \quad n_{ij} = n_{ji}, \quad n_{ii} = -(p_i - 2). \quad (7)$$

The γ functions in the denominator naturally set a cutoff for the range of n_{ij} , leading to a finite sum. This formalism simplifies the structures that depend on internal symmetry into problems with finite parameters. This makes it highly effective in diverse scenarios, including the bootstrap process for low-energy expansion [7], computations of 10d effective actions [6], and higher-point calculations [25].

Within this framework, the tree-level supergravity amplitude for arbitrary KK charges [19] has a remarkable simplification [3],

$$\mathcal{M}^{\text{SG}} = -\frac{1}{(\mathbf{s} + 1)(\mathbf{t} + 1)(\mathbf{u} + 1)}, \quad (8)$$

here $\mathbf{s}$ and $\mathbf{t}$ are associated with our notation as

$$\mathbf{s} = \frac{1}{2}(s - p_{12}) + n_{12}, \quad \mathbf{t} = \frac{1}{2}(t - p_{14}) + n_{14},$$

and satisfy the on-shell relation $\mathbf{s} + \mathbf{t} + \mathbf{u} = -4$. Regarding the stringy corrections, this formalism can also recast them

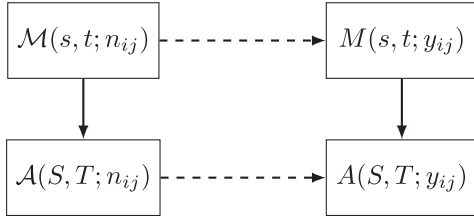


FIG. 1. We use solid arrows to represent the Borel transformations Eq. (4) from $\mathcal{M}$ to $\mathcal{A}$ and from M to A , and dashed arrows to represent the S transformations Eq. (6) from $\mathcal{M}$ to M and from $\mathcal{A}$ to A . Since these two transformations act on different variables, we conclude that they are commutative.

into a simpler form. For instance, in the case of the $\lambda^{-\frac{3}{2}}$ term, the result for the most general KK charges [4,5] can be reorganized as a constant [3]

$$\mathcal{M}^3 = 2(\Sigma - 1)_3 \zeta_3, \quad (9)$$

where $(a)_b \equiv \Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol.

Having introduced the Borel transform Eq. (4) and the S transform Eq. (6), a useful insight is their commutativity, as illustrated in Fig. 1. Previous discussions only introduced the Borel transform for $M(s, t; y_{ij})$ [10–12], but we suggest considering the transformation from $\mathcal{M}(s, t; n_{ij})$ to $\mathcal{A}(S, T; n_{ij})$ through the Borel transform directly. This framework provides a practical approach for deriving arbitrary KK results while maintaining explicit crossing symmetry.

For instance, the tree-level supergravity contribution to $\mathcal{A}^{(1)}(S, T)$ derived from Eq. (8) yields

$$\mathcal{A}^{(1),\text{SG}} = \frac{2 - \Sigma}{6S^2 T^2 U^2} \times (3(n_s S^2 + n_t T^2 + n_u U^2) - (n_s + n_t + n_u - 1)(S^2 + T^2 + U^2)), \quad (10)$$

where the crossing symmetric variables are defined as

$$n_s = n_{12} + n_{34}, \quad n_t = n_{14} + n_{23}, \quad n_u = n_{13} + n_{24}$$

and satisfy the on-shell relation $n_s + n_t + n_u = \Sigma - 4$. Furthermore, taking into account the contribution $\mathcal{A}^{(1),i}$ of each string correction $\mathcal{M}^i$ under the Borel transform, we can write the full low-energy expansion of $\mathcal{A}^{(1)}(S, T)$ as

$$\mathcal{A}^{(1)}(S, T; n_{ij}) = \mathcal{A}^{(1),\text{SG}} + \mathcal{A}^{(1),3} + \mathcal{A}^{(1),5} + \dots \quad (11)$$

Notably, deriving $\mathcal{A}^{(1),i}$ from $\mathcal{M}^i$ is straightforward. For example, Eq. (9) predicts $\mathcal{A}^{(1),3} = 0$. Conversely, reconstructing $\mathcal{M}(s, t)$ from $\mathcal{A}(S, T)$ is uniquely ascertained due to the polynomiality of string correction [7]. Toward the

end of this Letter, we will revisit the discussion on Wilson coefficients of $\mathcal{A}^{(1)}$ and provide more precise results.

The results for the low-lying KK configurations can be reformulated as an integral over the Riemann sphere, suggesting the feasibility of constructing analogous structures in $\text{AdS} \times S$ space. The supergravity contribution $\mathcal{A}^{(1),\text{SG}}$, as part of the low-energy expansion of $\mathcal{A}^{(1)}$, already delineates a pathway for incorporating n_{ij} dependence into the amplitude. Next, we will present a compact formalism that systematically unifies the treatment of world-sheet integrals and internal symmetry dependencies.

World-sheet correlator—Let us construct our ansatz. On the one hand, we choose the single-valued multiple polylogarithms (SVMPLs) as basis, which ensures that the amplitude contains only single-valued MZVs. On the other hand, the coefficients should contain the internal symmetry dependencies. We assume the curvature correction for the world-sheet correlator as

$$\mathcal{A}^{(1)}(S, T) = \mathcal{B}^{(1)}(S, T) + \mathcal{B}^{(1)}(S, U) + \mathcal{B}^{(1)}(U, T), \quad (12)$$

which respects the crossing symmetry. $\mathcal{B}^{(1)}(S, T)$ is an integral over the Riemann sphere

$$\mathcal{B}^{(1)}(S, T) = \int d^2 z |z|^{-2S-2} |1-z|^{-2T-2} \mathcal{G}(z, \bar{z}). \quad (13)$$

In the above $|z|^{-2S-2} |1-z|^{-2T-2}$ is the Koba-Nielsen factor, z the complex cross ratio on the Riemann sphere, and the integration measure reads $d^2 z = dz d\bar{z}/(-2\pi i)$.

Our ansatz generalizes the structure of $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$, with the integrand

$$\mathcal{G}(z, \bar{z}) = \sum_{i=1}^4 \mathcal{L}_i^s \mathcal{R}_i^s + \sum_{j=1}^3 \mathcal{L}_j^a \mathcal{R}_j^a. \quad (14)$$

Here $\mathcal{R}_i^{a/s}(S, T)$ are homogeneous rational functions in the Mandelstam variables S, T, U and $\mathcal{L}_i^{s/a}$ denote combinations of pure transcendental weight 3 SVMPLs $\mathcal{L}$. Both of them are carefully chosen to ensure consistency with the pole structure and low-energy behavior of the AdS Virasoro-Shapiro amplitude. The symmetric components $\mathcal{L}_i^s$ and antisymmetric components $\mathcal{L}_j^a$ under the exchange $z \leftrightarrow 1-z$ [8,9] are defined as,

$$\mathcal{L}_i^s = (\mathcal{L}_{000}^s(z), \mathcal{L}_{001}^s(z), \mathcal{L}_{010}^s(z), \zeta(3)), \quad (15a)$$

$$\mathcal{L}_i^a = (\mathcal{L}_{000}^a(z), \mathcal{L}_{001}^a(z), \mathcal{L}_{010}^a(z)), \quad (15b)$$

with explicit expressions provided in Supplemental Material [26]. The prefactors $\mathcal{R}_i^{a/s}(S, T)$ are designed

with a universal denominator U^2 and enjoyed the general structure

$$\mathcal{R}_i^{a/s}(S, T) = \frac{f_1^{a/s} S^2 + f_2^{a/s} ST + f_3^{a/s} T^2}{(S + T)^2}, \quad (16)$$

where $f_i^{a/s}$ are linear combinations of

$$\{\Sigma n_s, \Sigma n_t, \Sigma n_u, \Sigma, n_s, n_t, n_u\}$$

with unknown coefficients. This parametrization is motivated by two observations. First, supergravity contributions $\mathcal{A}^{(1), \text{SG}}$ already encode these n_{ij} dependencies. Second, the known $A_{22pp}^{(1)}$ exhibit at most quadratic dependence on p , ensuring the sufficiency of the ansatz. Notice that the on-shell relation $n_s + n_t + n_u = \Sigma - 4$ reduces the number of basis. This ansatz systematically encapsulates the SU(4) information for arbitrary KK charges.

We establish key consistency conditions that constrain the ansatz Eq. (14). The ansatz must satisfy the following fundamental constraints of crossing symmetry and single valuedness.

Crossing symmetry: invariance under the exchange $z \leftrightarrow 1 - z$ translates to a symmetry in the Mandelstam variables $S \leftrightarrow T$ and $n_s \leftrightarrow n_t$ enforcing

$$\mathcal{B}^{(1)}(S, T; n_s, n_t) = \mathcal{B}^{(1)}(T, S; n_t, n_s).$$

The crossing symmetry impose nontrivial relationships among the coefficients of $f_i^{a/s}$, enabling partial determination of their values.

Single-valuedness: The use of SVMPLs guarantees that the amplitude only contains the single-valued MZVs, a critical requirement for closed-string amplitudes.

After imposing the crossing relation, 75 unknown coefficients remain in Eq. (14). All of these coefficients can be uniquely determined by comparing to the known $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$ result presented in [10]. The complete bootstrap procedure is presented in Supplemental Material [26]. We find that the final result exhibits a striking simplification and offers several insightful physical perspectives.

Results—We now present the first curvature correction to the Kaluza-Klein AdS Virasoro-Shapiro amplitude. Although the ansatz was constructed with the full basis of weight-3 SVMPLs, by properly reorganizing the bootstrap result, we observed that it only depends on five linearly independent SVMPLs,

$$\begin{aligned} \mathcal{L}_{1/4} &= \mathcal{L}_{000}^{s/a}(z), & \mathcal{L}_2 &= -\mathcal{L}_{010}^s(z) + 4\zeta_3, \\ \mathcal{L}_3 &= \mathcal{L}_{000}^s(z) - \mathcal{L}_{001}^s(z) - \mathcal{L}_{010}^s(z), \\ \mathcal{L}_5 &= \mathcal{L}_{000}^a(z) - \mathcal{L}_{001}^a(z) - \mathcal{L}_{010}^a(z), \end{aligned} \quad (17)$$

where $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ are symmetric, while $\mathcal{L}_4$ and $\mathcal{L}_5$ are antisymmetric. This basis automatically eliminates divergences of weight three or lower in the Riemann surface integral Eq. (13). Specifically, none of these basis elements generate logarithmic singularities of the form $\log^i(0)$ for $i \leq 3$. This is consistent with the world-sheet toy model proposed by [8], where all such divergences cancel due to the intrinsic properties of closed-string amplitudes. We anticipate analogous cancellations in higher-order corrections (e.g., $\mathcal{A}^{(2)}$), and we leave this analysis for future work. In terms of this basis, the curvature correction takes the elegantly simple form

$$\mathcal{G}(z, \bar{z}) = (\Sigma - 2) \times \sum_{i=1}^5 \mathcal{L}_i \mathcal{R}_i, \quad (18)$$

where the rational prefactors $\mathcal{R}_i(S, T)$ are explicitly

$$\begin{aligned} \mathcal{R}_1 &= \frac{(n_s - n_t)(S - T) + 2U}{12(S + T)}, & \mathcal{R}_2 &= \frac{1}{2} \frac{1}{\Sigma - 2}, \\ \mathcal{R}_3 &= \frac{n_s S + n_t T + n_u U}{12(S + T)}, & \mathcal{R}_4 &= \frac{n_s - n_t}{12}, \\ \mathcal{R}_5 &= \frac{n_s S - n_t T + (n_u + 2)(T - S)}{12(S + T)}. \end{aligned} \quad (19)$$

This result yields several interesting insights. First, an overall prefactor $\Sigma - 2$ appears in the integrand. In fact, this factor emanates from the supergravity contribution Eq. (8). All basis elements $\mathcal{L}_i$ involve the polar contributions such as $1/S^4$ or $1/T^4$ except $\mathcal{L}_2$. These polar contributions are related to Eq. (8). This illustrates why $\mathcal{R}_2$ looks different. Second, our ansatz reproduces the supergravity result automatically without explicitly imposing additional conditions. This alignment arises from the hidden 10d conformal symmetry of supergravity [1], which is preserved when extended to AdS $\times$ S framework.

Low-energy expansion—To derive the low-energy expansion of $\mathcal{A}^{(1)}$ from the world-sheet correlator, we first need to compute the expansion of $\mathcal{B}^{(1)}(S, T)$ around $S = T = 0$. This requires evaluating the integrals over basis

$$I_w(S, T) = \int d^2 z |z|^{-2S-2} |1 - z|^{-2T-2} \mathcal{L}_w(z). \quad (20)$$

These integrals were computed in [8] using the method developed in [23], yielding the following result:

$$I_w = P_w + \sum_{i,j=0} (-S)^i (-T)^j \sum_{W \in 0^i \sqcup 1^j \sqcup w} (\mathcal{L}_{0W}(1) - \mathcal{L}_{1W}(1)), \quad (21)$$

where P_w represents the polar term on S or T , arising from logarithmic divergences of the integrand either around

$z = 0$ for $w = 0^n$ or around $z = 1$ for $w = 1^n$. In our case we have

$$P_{0^n} = -\frac{1}{S^{n+1}}, \quad P_{1^n} = -\frac{1}{T^{n+1}}.$$

$\sqcup$ denotes the shuffle product. In our case, $0^i \sqcup 1^j \sqcup w$ represents all possible ways to merge the sequences 0^i , 1^j , and w , while maintaining their individual orderings. An example of the integral is

$$I_{000} = -\frac{1}{S^4} - (8S + 6T)\zeta_5 - (8ST + 6T^2)\zeta_3^2 + \dots$$

Now we would like to systematically analyze the low-energy behavior of the Mellin amplitude $\mathcal{M}$ and its Borel-transformed counterpart $\mathcal{A}$. The behavior of $\mathcal{M}$ under Borel transform is dominated by the power of s , t , and $\tilde{u}$. Critically, the large- p limit [3] imposes an upper bound on the power of $\mathbf{s}$ while the perturbative order we considered constrains the lower bound of $\mathbf{s}$. This dual constraint ensures that only a small part in $\mathcal{M}$ contributes to $\mathcal{A}^{(1)}$, allowing us to categorize contributions as leading or subleading based on the degree of $\mathbf{s}/\mathbf{t}/\mathbf{u}$ in λ^{-1} expansion. The leading terms in $\lambda^{-(i/2)}\mathcal{M}^i$ correspond to homogeneous symmetric polynomials in $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ of degree $i - 3$. These terms are fully determined by the flat-space Virasoro-Shapiro amplitude $\mathcal{A}^{(0)}$ through the Borel transform, contributing dominantly to $\mathcal{A}^{(0)}$ and partially to $\mathcal{A}^{(1)}$. Subleading terms, however, involve mixed polynomials of lower degree in $\mathbf{s}/\mathbf{t}/\mathbf{u}$ and linear dependence on $n_s/n_t/n_u/\Sigma$,

$$(\mathbf{s}^j n_s + \mathbf{t}^j n_t + \mathbf{u}^j n_u) \mathcal{S}^k, \quad \mathcal{S}^{j+k}, \quad \Sigma \mathcal{S}^{j+k},$$

where $j + k = i - 4$ and $\mathcal{S}^m$ denote a homogeneous symmetric polynomial in $\mathbf{s}$, $\mathbf{t}$, and $\mathbf{u}$ of degree m . Constructing a complete basis for these terms requires systematic enumeration of symmetric invariants, as detailed in [7].

By plugging in our result Eq. (18) into the integral, in principle we can predict subleading Wilson coefficients in $\mathcal{M}^i$ for arbitrary i . The first few terms are worked out as

$$\mathcal{M}^5 = (\sigma_{2,0} - 5\sigma_{1,1} + \dots)\zeta_5, \quad (22a)$$

$$\mathcal{M}^6 = \frac{2}{3}(\sigma_{3,0} - 9\sigma_{2,1} + (6\Sigma - 3)\sigma_{2,0} + \dots)\zeta_3^2, \quad (22b)$$

$$\mathcal{M}^7 = \left(\sigma_{4,0} - 14\sigma_{3,1} + \left(8\Sigma - \frac{51}{8}\right)\sigma_{3,0} + \dots\right)\zeta_7, \quad (22c)$$

$$\mathcal{M}^8 = \frac{4}{5}\left(\sigma_{5,0} - 20\sigma_{4,1} + \left(10\Sigma - \frac{55}{4}\right)\sigma_{4,0} + \dots\right)\zeta_3\zeta_5, \quad (22d)$$

the “...” represents the contributions from $\mathcal{A}^{(k \geq 2)}$ and we use a shorthand

$$\sigma_{a,b} = (\Sigma - 1)_{a+3}(\mathbf{s}^a n_s^b + \mathbf{t}^a n_t^b + \mathbf{u}^a n_u^b). \quad (23)$$

These expansions perfectly align with the analysis above. Our results $\mathcal{M}^5$, $\mathcal{M}^6$, and $\mathcal{M}^7$ precisely agree with previous bootstrap result [6,7], supersymmetric localization [13], and integrated constraints [14]. These perfect alignments verify the validity of our approach. The subleading terms of $\mathcal{M}^8$ are not fixed in previous studies [6,7]. It will be ideal to validate these results further from different perspectives, i.e., the OPE data [31–33] and some special limits [34,35].

Discussion—This work establishes a universal framework for computing curvature corrections to the AdS Virasoro-Shapiro amplitude with arbitrary KK modes. By utilizing the AdS $\times$ S formalism and constructing an ansatz based on weight-3 single-valued multiple polylogarithms, we circumvented the complicated superconformal block expansion, which is a long-standing challenge in correlators of nonidentical KK modes. The derived amplitude not only reproduces the known results but also predicts an infinite tower of subleading Wilson coefficients. These coefficients encode both bulk string corrections and effective higher-dimensional interactions, offering a systematic tool to probe AdS geometry beyond the supergravity regime.

The prefactor $\Sigma - 2$ in Eq. (18) hints that, an underlying 10-dimensional conformal symmetry inherited from the supergravity limit may persist in the world-sheet framework, which ensures consistency between stringy corrections and the low-energy effective action. A more in-depth exploration of this symmetry in AdS $\times$ S space is desirable.

Our results also provide a pathway to extracting CFT data for massive stringy operators, such as the dimensions of single-trace operators. These data probe the fine structure of $\mathcal{N} = 4$ SYM theory and can be directly compared with predictions from integrability methods.

It is interesting to generalize our method to higher-order corrections such as $\mathcal{A}^{(2)}$ [36], where we need single-valued multiple polylogarithms of weight 6. The cancellation mechanism of divergences observed for weight-3 terms (e.g., $\log^3(0)$ terms) needs rigorous validation at higher weights. The generalization to other AdS backgrounds [37–39] or AdS Veneziano amplitude [40,41] could reveal universal features of AdS string amplitudes. Finally, the higher genus expansion encodes the non-planar effects, we can consider world-sheet correlators on genus-one Riemann surface, while the one-loop supergravity terms were provided in [42–49]. These advancements aim to bridge perturbative string theory with nonperturbative holography, ultimately forging a comprehensive dictionary for quantum gravity in AdS.

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Data availability—No data were created or analyzed in this study.

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