

Spreading of Magic Resource under Unitary Clifford Dynamics

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Nonstabilizerness, or quantum magic resource, presents a valuable resource in quantum error correction and computation. We study the dynamics of locally injected nonstabilizerness in unitary Clifford circuits, where the total nonstabilizerness is conserved. However, the absence of physical observables quantifying nonstabilizerness precludes a direct microscopic or hydrodynamic description of its local distribution and dynamics. Using insights from stabilizer quantum error correcting codes, we rigorously show that the spatial distribution of nonstabilizerness can be inferred from a canonical representation of low-magic states, dubbed the bipartite magic gauge. Moreover, we propose two operationally relevant magic length scales. We numerically establish that, at early times, both length scales grow ballistically at distinct velocities set by the entanglement velocity, after which nonstabilizerness delocalizes. Our Letter sheds light on the spatiotemporal structure of quantum resources and complexity in many-body dynamics, opening up avenues for investigating their transport properties and further connections with quantum error correction.

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Building scalable quantum computers hinges on the ability to protect logical information from being corrupted. To protect this information, one encodes it via a quantum error correcting (QEC) code [1–7]; however, using certain QEC codes limits the logical operations implementable transversally, i.e., with physically local operations [8–10]. To perform universal quantum computation, one needs to augment QEC codes with special states. For example, magic states are required for stabilizer codes [11]. These states contain a quantum resource, dubbed nonstabilizerness (NS) or “magic,” that can make the classical simulation of quantum systems hard [12–15].

Magic and other quantum resources such as entanglement, non-Gaussianity, etc., not only play a crucial role in building practical quantum devices [13–23], but also have profound implications on quantum many-body dynamics. Signatures of quantum chaos emerge when injecting NS into Clifford dynamics [24,25], which alters the entanglement spectral distribution from Poisson to Wigner-Dyson (commonly associated with integrable and chaotic dynamics, respectively) [24]. Generalized notions of quantum

thermalization require NS [26–36] because ensembles of magic-free (stabilizer) states form quantum state k designs only with $k \leq 3$ [35–38]. Magic resource has also been used in studying a variety of aspects of quantum many-body systems, including (symmetry-protected) topological order [39–41], ground and critical states [42–48], conformal field theories [49–52], quantum scars [53–55], monitored and noisy dynamics [56–66], and quantum chaos [67–71].

Unlike the accumulation of entanglement [72–75] or of NS [56,76–78] in ergodic quantum systems, the *spreading* of NS under dynamics that conserves it remains largely unexplored, raising an important question: how does locally injected magic resource spread? Answering this may be relevant practically for the optimal allocation of classical and quantum simulation resources, and physically for unveiling the local mechanisms behind quantum phenomena such as generalized thermalization.

Here we study how magic resource, injected via a single local unitary in a short-range entangled state, spreads under random local Clifford unitary evolution—a model of scrambling quantum many-body dynamics. While Clifford dynamics conserves the total amount of NS, it creates entanglement and spreads operators, thereby spreading the initially localized magic resource. Our focus is precisely on the spatiotemporal distribution of this resource.

We identify two magic length scales (MLS), which are the sizes of (i) the smallest contiguous region from which magic resource can be extracted or destroyed, and (ii) the region outside of which operations cannot reduce the NS (Fig. 1). A key perspective in identifying these length scales

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is that the time-evolved state encodes a single logical qubit through a dynamically generated stabilizer QEC code. This enables a mapping of the spatial structure of magic resource to that of the code's logical operators.

We rigorously prove in Theorem 1 the existence of a gauge of logical operators and stabilizers which reveals the spatial structure of magic resource, for any bipartition of the system. This provides an efficient classical algorithm to *exactly* compute the NS in any subsystem, regardless of the amount of entanglement present. Conversely, exactly computing NS in typical (especially volume-law entangled) states would be exponentially costly in (sub)system size. Using our approach, we numerically find that, at early times, the two MLS spread ballistically at rates given by the entanglement velocity (v_E) and twice thereof ($2v_E$), respectively. Once these MLS reach their maximal values, magic resource delocalizes, i.e., it becomes extractable from any $\approx L/2$ qubits, while the QEC code approaches a random $[[L, 1]]$ stabilizer code.

Quantum circuit setup—We consider the dynamics of magic resource in unitary quantum circuits with L qubits and depth t , cf. Fig. 1. Each Clifford gate is chosen either as the identity gate or from a uniform distribution over the two-qubit Clifford group [79] with probability $p \in [0, 1]$ and $1 - p$, respectively. Such identity-doped circuits admit, upon averaging, parametrically tunable butterfly $v_B(p)$ and entanglement $v_E(p)$ velocities [80,81],

$$v_B(p) = \frac{3}{5} \frac{1-p}{1+p}, \quad v_E(p) \approx \frac{\log(1-v_B) + \log(1+v_B)}{\log(1-v_B) - \log(1+v_B)}. \quad (1)$$

Applying a magic-injecting non-Clifford gate on a stabilizer state dynamically generates a stabilizer quantum error correcting code [56]. For simplicity, we consider a pure short-range entangled stabilizer state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and apply a T_j gate on qubit j , localizing the NS at this site. Because $T_j = \alpha\mathbb{1} + \beta Z_j$, this adds NS if Z_j anticommutes with at least one stabilizer of ρ_0 . We can always choose the generators of the stabilizer group $\mathcal{S}(|\psi_0\rangle)$ such that $\mathcal{S}(|\psi_0\rangle) = \langle s_1, \mathcal{G} \rangle$, where $\mathcal{G} = \langle g_2, \dots, g_L \rangle$, with $[T, g_i] = 0 \ \forall \ i$ and $\{Z_j, s_1\} = 0$ [79]. Thus, $\rho = T\rho_0 T^\dagger$ is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2^L} \left(\mathbb{1} + \frac{\bar{Z} - \bar{Y}}{\sqrt{2}} \right) \prod_{i=2}^L (\mathbb{1} + g_i), \quad (2)$$

where we used $Ts_1T^\dagger = (s_1 + is_1Z_j)/\sqrt{2}$ and defined $\bar{X} := Z_j$, $\bar{Z} := s_1$ and $\bar{Y} := i\bar{X}\bar{Z}$. $|\psi\rangle$ encodes one logical qubit in L physical qubits via a stabilizer QEC code [4,88] with stabilizer group $\mathcal{G}$ and logical operators $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ [56]. For concreteness, we here take $|\psi_0\rangle = \bigotimes_{i=1}^{L/2} |\Phi^+\rangle_{2i-1,2i}$ defined from Bell states $|\Phi^+\rangle_{ij} = (|0_i0_j\rangle + |1_i1_j\rangle)/\sqrt{2}$, locally inject NS by applying a non-Clifford gate

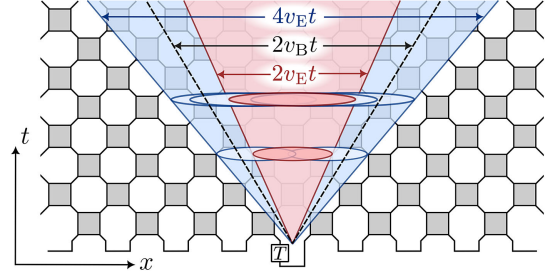


FIG. 1. Magic resource spreading in a T -doped random Clifford circuit with L qubits, time t , and a brickwall of random two-qubit Clifford gates (gray). Magic resource is locally injected by a T gate acting on a short-range entangled stabilizer state with the two central qubits in a Bell pair. The typical size of the smallest contiguous region where nonstabilizerness can be extracted (red ellipse) spreads at the entanglement velocity v_E (red cone). The union of the smallest mutually independent such intervals (blue and red ellipses) indicates the region outside of which operations cannot reduce the nonstabilizerness (blue cone), which spreads at $2v_E$. For comparison, operators spread at the butterfly velocity v_B (black dashed cone).

$T \sim e^{-i\pi Z/8}$ on qubit $L/2$, and evolve the state using an identity-doped Clifford circuit.

Quantifying magic resource—There exist multiple quantifiers of nonstabilizerness in a state ρ . In our setup, the presence of magic resource is essentially encoded in the structure of Eq. (2) [see also Eq. (5)]. For illustration purposes, we focus on two commonly used quantifiers: the second stabilizer Rényi entropy (SRE) [89] and the robustness of magic (ROM) [90]. While SRE is not a faithful magic measure for mixed states, since it can be nonzero for classical mixture of stabilizer states [e.g., $\rho_\lambda = \lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|$ for any $\lambda \in (0, 1)$, $\lambda \neq 1/2$], we include it for comparison convenience. Conversely, ROM is a faithful magic measure even for mixed states. Since ROM is also a magic monotone, it is particularly well suited to quantifying NS in subsystems in our setup or, more generally, in setups with mixed initial states.

The second SRE for pure states is $\mathcal{M}_2(\rho) := S_2(\Xi(\rho)) - L$, with Pauli spectrum $\Xi(\rho) := \{\text{Tr}[\rho P]^2/2^L\}$ for a basis of L -qubit Pauli operators $\{P\}$ [91] and Rényi-2 entropy S_2 . For mixed states, the Pauli spectrum needs to be normalized [89], corresponding to subtracting the Rényi-2 entropy of ρ , which leads to the mixed-state SRE

$$\tilde{\mathcal{M}}_2(\rho) := \mathcal{M}_2(\rho) - S_2(\rho) = -\log_2 \frac{\sum_P \text{Tr}[\rho P]^4}{\sum_P \text{Tr}[\rho P]^2}. \quad (3)$$

The ROM of an L -qubit state ρ is [90]

$$\mathcal{R}(\rho) := \min_{\{x_i\}} \left\{ \sum_i |x_i| \left| \rho = \sum_i x_i \sigma_i, \sum_i x_i = 1 \right. \right\}, \quad (4)$$

where the decomposition is over *all* pure L -qubit stabilizer states σ_i . For pure, mixed, and convex combinations (i.e.,

$x_i > 0$ in the optimal decomposition) of stabilizer states, the ROM is $\mathcal{R}(\rho) = 1$, whereas for nonstabilizer states it satisfies $\mathcal{R}(\rho) > 1$ [90]. Conversely, while $\tilde{\mathcal{M}}_2(\rho) = 0$ for pure stabilizer states, $\tilde{\mathcal{M}}_2(\rho) > 0$ does not imply that ρ is a nonstabilizer state if ρ is mixed.

To quantify how magic resource is distributed across the system, we compute the SRE and ROM for various subsystems by tracing out their respective complements. If such a subsystem has the same amount of magic resource as the full system, we say that these are “contained” within the extent of the subsystem; otherwise, the size of the subsystem needs to be increased until it holds the full amount of nonstabilizerness.

Operational definitions of magic resource—We next define two operational notions of nonstabilizerness linked to locality. We focus on a subsystem $A \subset [L] := \{1, \dots, L\}$ and pure states with a single global unit of nonstabilizerness; see Supplemental Material [81] for generalizations.

Our first operational definition is *unitarily extractable magic resource*. Considering an ancilla qubit C initialized as $|0\rangle_C$, we say magic resource is unitarily extractable from A if there exists a Clifford unitary $U = \mathbb{1}_B \otimes U_{AC}$ such that $U|\psi\rangle \otimes |0\rangle_C = |\psi'\rangle_{AB} \otimes |\psi'\rangle_C$, with magic state $\mathcal{R}(|\psi'\rangle_C) > 1$. In our setup, it suffices to focus on $|\psi'\rangle_C = |T\rangle_C$ with magic state $|T\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$. As we show in [81], NS can be extracted from A iff ρ_A has a full magic unit, i.e., $\mathcal{R}(\rho_A) > 1$. Moreover, by Theorem 1, this is equivalent to all logicals being reducible to A [81]. Altogether, these equivalent notions of extractable magic resource correspond to NS (and logical information) being concentratable in A . Here we excluded extracting NS to C via measurements since these can teleport it—e.g., by measuring Pauli operators, which are not logicals, in a region B that contains NS, we can move this NS to its complement A , even if A contained no NS.

Instead, measurements lead to our second operational definition, dubbed *projectively destroyable magic resource*. We say magic resource is destroyable in a subsystem A if there exists a Pauli operator fully supported in A which, if measured, collapses $|\psi\rangle$ to a stabilizer state [92]. All possible Paulis that can destroy these are the logicals, and one can efficiently [in $\mathcal{O}(L^3)$ time] determine if there is any logical whose support can be reduced to A [93]. Nonetheless, in this Letter, we mainly focus on unitary extraction to avoid subtleties stemming from the teleportation of magic resource via measurements.

Computing the precise spatial structure of magic resource requires analyzing all possible subsystems A . This is generically hard because there are 2^L choices, requiring $\exp(L)$ runs. This challenge arises regardless of how hard it is to determine if a single subsystem contains nonstabilizerness. To quantify the key features of this spatial structure, we hence focus on subsystems that are contiguous intervals [94]. This choice enables us to resolve the magic structure efficiently and points toward two

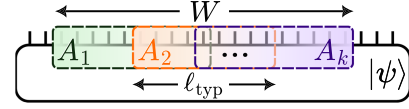


FIG. 2. Magic length scales: typical magic length ℓ_{typ} and full linear extent of magic W , defined in terms of mutually irreducible and minimal-length contiguous subsystems $\{A_i\}_{i=1}^k$ where nonstabilizerness is extractable from $|\psi\rangle$.

relevant magic length scales, sharpening our view on the spatiotemporal structure of magic resource besides the broad structure already captured by the causal set and “causal cone of magic” proposed in Ref. [56].

Magic length scales—The spatial structure of magic resource is captured by the set of mutually irreducible intervals where this can be extracted. Multiple intervals may arise since the logical information of a QECC may be altered by physically acting on distinct qubit sets, as reflected in the freedom of choosing the generators [97]. Consider two magic-extractable subsystems A_1 and A'_1 with A_1 contained in A'_1 (i.e., $A_1 \subset A'_1$). Since NS is extractable from A_1 alone, A'_1 is redundant. Let another magic-extractable subsystem A_2 be contained in A'_1 ($A_2 \subset A'_1$) but independent from A_1 ($A_1 \not\subset A_2$ and $A_2 \not\subset A_1$): by the same logic, A_2 is more precise than A'_1 ; however, it is equally relevant as A_1 as they are mutually irreducible.

We collect these *minimal linear magic intervals* (MLMI) in the set $\mathcal{I}(\rho) = \{A_1, \dots, A_k\}$ (Fig. 2). MLMIs must mutually overlap: suppose there exist two nonoverlapping MLMI A_1 and A_2 , where NS is extractable. By definition, one can extract the nonstabilizerness with local operations on A_2 only; however, by the same definition, any operations on the complement of A_1 , which includes A_2 , cannot reduce the magic resource in A_1 . Hence, we arrive at a contradiction. To obtain these subsystems in practice, one first finds all magic-extractable contiguous intervals, then applies the above logic recursively.

The first magic length scale is the typical length ℓ_{typ} of the MLMIs (Fig. 2). To quantify the spreading of magic resource, we consider how ℓ_{typ} grows. For illustration purposes, we introduce a related quantity: we define the *linear magic length* (LML) $\ell(t)$ as the length of the shortest left-right symmetric and centered (on the injection site) interval where NS is extractable. The LML grows within the geometric light cone of the non-Clifford gate, unlike ℓ_{typ} , which can inherit subtleties from entanglement already present in the state [81]. Moreover, the growth rate of $\ell(t)$ is consistent with that of $\ell_{\text{typ}}(t)$ [81]. The second length scale is the width of the union of the MLMIs (Fig. 2), which we call the *full linear extent of magic* (FLEOM) and denote by $W := \text{width}(\cup_{A \in \mathcal{I}} A)$. Physically, local operations such as measurements, unitaries, decoherence, etc., applied outside of $\cup_{A \in \mathcal{I}} A$ cannot reduce the total NS in the system.

We can compute these two MLS efficiently, i.e., with $\text{poly}(L)$ classical resources. For contiguous subsystems, the

computational hardness comes from assessing a subsystem's nonstabilizerness. In Supplemental Material [81], we provide efficient algorithms for computing both the subsystem magic $\tilde{\mathcal{M}}_2(\rho_A)$, $\mathcal{R}(\rho_A)$, and Pauli spectrum $\Xi(\rho_A)$ exactly. While the latter might naïvely have an exponential $\sim 4^{|A|}$ cost [89], the following construction significantly reduces the cost by exposing the local structure of operators in Eq. (2).

Bipartite magic gauge—We introduce a gauge of stabilizer generators and logical operators of ρ for any bipartition, which enables the classical computation of the ROM and Pauli spectrum, and thus SRE, in $\mathcal{O}(L^3)$ time. We dub this the *bipartite magic gauge* (BMG). We rigorously show its existence and provide an algorithm for its construction in [81], assuming one knows the stabilizers of ρ . This is our first key analytical result.

Theorem 1—Let $|\psi\rangle$ be a pure L -qubit state with one unit of nonstabilizerness (i.e., one unitarily extractable magic state), and encoding one logical qubit, where $\rho = |\psi\rangle\langle\psi| = (1/2^L)[\mathbb{1} + (\bar{Z} - \bar{Y}/\sqrt{2}) \prod_{j=2}^L (\mathbb{1} + g_j)]$ with logical operators $\bar{\sigma}$ and stabilizer group $\mathcal{G} = \langle g_2, \dots, g_L \rangle$. For any bipartition into subsystem A and its complement B , there exists a bipartite magic gauge, i.e., a choice of logical operators and stabilizer generators such that

$$\rho = \left(\frac{\mathbb{1}}{2} + \frac{\bar{Z} - \bar{Y}}{2\sqrt{2}}\right) \prod_{i=1}^{n_A} \frac{\mathbb{1} + a_i}{2} \prod_{j=1}^{n_B} \frac{\mathbb{1} + b_j}{2} \prod_{k=1}^m \frac{\mathbb{1} + h_k}{2} \quad (5)$$

with $L - 1 = n_A + n_B + m$, where the support of stabilizers a_i , b_j , and h_k can(not) be reduced to $A(B)$, $B(A)$, and neither A nor B , respectively. Furthermore, using the reducibility of the logicals $\bar{Z}$ and $\bar{Y}$, we can compute the magic resource of subsystem A , as summarized in Table I.

To guide our intuition for the various cases in Table I, it is useful to consider our second rigorous result—Lemma 1, proven in the End Matter—on the spatial structure of logicals. The magic resource of a subsystem A depends on whether logical operators, which enter in ρ , survive the tracing $\text{Tr}_B[\cdot]$; how many stabilizers survive is unimportant

since their contribution to $\tilde{\mathcal{M}}_2$ from $\mathcal{M}_2$ is canceled by that of the entanglement 2-Rényi entropy S_2 [cf. Eq. (3)], and also for $\mathcal{R}(\rho_A)$. It is crucial to determine if these logicals can be chosen to be fully supported on A , and this is where Lemma 1 comes into play.

Lemma 1—For any stabilizer code encoding one logical qubit in the pure state of L qubits, and a bipartition into subsystems A and $B = [L] \setminus A$, it is true that either (i) all three logicals $\bar{X}, \bar{Y}, \bar{Z}$ are reducible to A , or (ii) all three are reducible to B , or (iii) only one logical can be supported only on A or B , while the other two are supported on both A and B .

A few comments are in order. First, the states $|\psi\rangle$ considered in Theorem 1 extend beyond unitary circuits [Fig. 1] to monitored Clifford + T circuits, provided measurements do not remove the magic resource. Second, the logicals $\bar{Z}, \bar{Y}$ and prefactors $a_{\pm} = \pm 1/\sqrt{2}$ are generic [98]; the phenomenology holds for other local non-Clifford gates, with $\mathcal{R}(\rho_A) > 1$ indicating the presence of nonstabilizerness. Third, while Theorem 1 considers $|\psi\rangle$ with a single magic unit, or $[[L, 1]]$ stabilizer codes, we anticipate that it can be extended to states with multiple ($k \geq 1$) magic units, corresponding to $[[L, k]]$ stabilizer codes.

Phenomenology of magic resource spreading—Here we report numerical results on the growth of LML and FLEOM and heuristically argue their early time growth rates.

Focusing on random Clifford circuits with open boundary conditions (BC) at doping rate p and extractable NS for FLEOM and LML, we find that, for early times, both $W(t)$ and $\ell(t)$ grow ballistically at rates v_W and v_{ℓ} , respectively (Fig. 3). While LML saturates to $\ell(t \rightarrow \infty) \simeq L/2$, the ballistic growth of FLEOM slows down at intermediate times [99], only to later saturate at $W(t \rightarrow \infty) = L$. Throughout the entire p range, the p dependence of the early-time velocities agrees remarkably well with that of the entanglement velocity $v_E(p)$ [Eq. (1)], suggesting that $v_W(p) \simeq 2v_E(p)$ and $v_{\ell}(p) \simeq v_E(p)$ [Fig. 3 inset]. This agreement is further established in [81] using results on doped dual-unitary circuits and periodic BC.

Magic resource spreading at $\partial_t W/2 \simeq 2v_E$ can be faster than the geometric light cone with $v_{LC} = 1$, but does this violate causality? No. Magic resource spreading

TABLE I. Reducibility of logicals $\bar{Z}, \bar{Y}$ to a subsystem and the subsystem magic resource. All possible cases (i)–(v) are discussed in the proof of Theorem 1. The columns labeled “X red.” indicate whether the support of a logical $\bar{\sigma}$ can be reduced to be fully contained in subsystem X ; ✓ (✗) indicates yes (no). $\tilde{\mathcal{M}}_2(\rho_A)$ and $\mathcal{R}(\rho_A)$ are the stabilizer two-Rényi entropy and robustness of magic of ρ_A [Eqs. (3) and (4)], respectively.

Case	(i)		(ii)		(iii)		(iv)		(v)	
	A red.	B red.	A red.	B red.	A red.	B red.	A red.	B red.	A red.	B red.
Support of $\bar{Z}$	✓	✗	✗	✓	✗	✗	✓	✓	✗	✗
Support of $\bar{Y}$	✓	✗	✗	✓	✗	✗	✗	✗	✓	✓
SRE $\tilde{\mathcal{M}}_2(\rho_A)$	$\log_2(4/3)$		0		0		$\log_2(6/5)$		$\log_2(6/5)$	
ROM $\mathcal{R}(\rho_A)$	$\sqrt{2}$		1		1		1		1	

is influenced by both entanglement growth and operator spreading, giving rise to an upper bound $[W(t) - W(0)]/2t \leq 3v_{\text{LC}}$ [81]. To shed light on this interplay (see Supplemental Material [81] for details), it is instructive to consider $UT|\psi\rangle = (UTU^\dagger)U|\psi\rangle$, for Clifford unitary U and short-range entangled $|\psi\rangle$. While $UTU^\dagger$ spreads at the butterfly velocity v_B , entanglement builds up in $U|\psi\rangle$ at rate v_E , and both contribute to nonstabilizerness spreading. What is nontrivial is how the structure of the operator $UTU^\dagger$ interacts with that of the entanglement in $U|\psi\rangle$.

That $v_\ell \approx v_E$ can be understood through the dynamics of purity. To link purity to magic resource, a key observation is that if $\rho_A = (\mathbb{1}_A/d_A)$ is maximally mixed, then subsystem A cannot host NS. For random circuits where entanglement grows ballistically, v_E corresponds to the decay rate of the circuit-averaged subsystem purity $\overline{\mathcal{P}(\rho_A(t))} = \overline{\text{Tr}[\rho_A^2(t)]}$ [73,74]. The decay $\overline{\mathcal{P}(\rho_A(t))} \propto e^{-v_E t}$ implies that, starting from a product state, at time t any ρ_A with A contiguous and $|A| \lesssim 2v_E t$ ($|A| \gtrsim 2v_E t$) is (not) maximally mixed with high probability; hence, $\ell(t) \gtrsim 2v_E t$ with high probability. Conversely, for A to host NS, by counting the logicals in $\Xi(\rho_A)$, A must be in the causal cone set by entanglement and operator spreading and satisfy $|A| \gtrsim 2v_E t$ [81]. Focusing on symmetric-centered intervals, there hence exists with high probability a subsystem A_ϵ supporting at least one logical, with $|A_\epsilon| = (1 + \epsilon)2v_E t$ for some $0 < \epsilon \ll 1$, and thus, $\ell(t) = |A_\epsilon| \Rightarrow \partial_t \ell = 2v_E(1 + \epsilon) \simeq 2v_E$. Beyond the scope of this heuristic argument, an interesting open problem is to find the scaling of ϵ with system size L . We lastly remark on a similarity between the growth of LML and that of entanglement in a stabilizer state. There, v_E characterizes the rate at which each stabilizer generator grows when imposing a collective constraint—the “clipped gauge” [73]. In our setup, v_ℓ characterizes the rate at which both logical generators grow under the constraint that the interval containing them is both *minimal* and *centered*. Remarkably, despite these different constraints, the growth rates v_ℓ and v_E agree exceptionally well.

A way to understand the growth rate $v_W \simeq 2v_E$ at early times is as follows. Consider the left- and rightmost MLMIs. Assuming these two MLMIs have typical sizes $\ell_{\text{typ}}(t)$, then $W(t) \lesssim 2\ell_{\text{typ}}(t)$ by the mutual-overlap condition. Because the circuit dynamics is otherwise structureless, it is reasonable to expect that there is no other mechanism constraining the spreading of MLMIs. One might naively anticipate an $\mathcal{O}(1)$ overlap between the left- and rightmost MLMIs; however, due to operator fronts broadening (see Supplemental Material [81] for a depiction) [74,100], we instead expect an $\mathcal{O}(\sqrt{t})$ overlap. We numerically confirm, as detailed in [81], that the left- and rightmost MLMIs indeed overlap on $\sim t^{0.45}$ qubits at early times. Altogether, since $\ell_{\text{typ}}(t) \simeq 2v_E t$, we thus expect to leading order $W(t) \simeq 2\ell_{\text{typ}}(t)$, which implies $W(t) \simeq 4v_E t$.

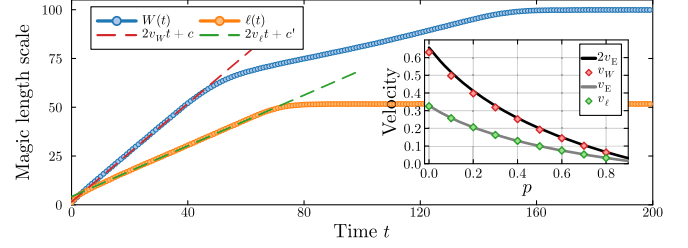


FIG. 3. Growth of magic length scales over time in random Clifford circuits for $L = 102$ and identity doping rate $p = 0$. Blue (orange) circles show FLEOM (LML). At early times, both MLS grow ballistically at fitted rates $2v_W$ and $2v_\ell$, respectively (dashed lines). Inset: p dependence of v_W and v_ℓ compared to the entanglement velocity $v_E(p)$. Data averaged over 2×10^3 circuit realizations, negligible error bars not shown.

Magic resource delocalizes at late times—After a saturation time $t_{\text{sat}} \sim \mathcal{O}(L)$, the FLEOM reaches the full system size, indicating that nonstabilizerness delocalizes (Fig. 3). Physically, in this regime, the full amount of nonstabilizerness can be unitarily extracted from *any* (i.e., not necessarily contiguous) subsystem with $\gtrsim L/2$ qubits [see Fig. 5(a)]. In other words, nonstabilizerness no longer has a notion of locality since the spatial proximity of qubits becomes irrelevant, and an extensive number of qubits $\gtrsim L/2$ is needed to distill a magic state. In contrast, to unitarily extract a magic state at earlier times, subsystems with at least $\approx \ell_{\text{typ}}(t)$ qubits are needed, and these subsystems typically overlap with or are contained within the FLEOM’s light cone. Note that the LML saturates before the FLEOM and approaches $\ell(t \rightarrow \infty) \simeq L/2$ [Fig. 3]. In the End Matter, by considering the dynamical QECC, we argue that, at late times, this code approaches a random $[[L, 1]]$ stabilizer code, which is known to saturate the so-called quantum Singleton bound [4,101], thereby explaining the saturation value $\ell(t \rightarrow \infty) \simeq L/2$. By using a proxy for the quantum channel capacity of a channel that erases a finite fraction of randomly chosen qubits [102–104], we also numerically show that nonstabilizerness can be unitarily extracted from any $\gtrsim L/2$ qubits.

Discussion and outlook—We have shown, for pure states with one magic unit and using insights from QECC, how to find the spatiotemporal structure of magic resource. Specifically, we proved the existence of a bipartite magic gauge and, based on it, developed algorithms to exactly compute subsystem nonstabilizerness efficiently. We proposed two length scales that are operationally relevant to magic distillation protocols. We reported numerical evidence for a ballistic growth of these length scales in random Clifford circuits, followed by magic delocalization at late times.

Moving forward, while we provided analytical bounds and heuristic arguments for the typical magic velocities, it would be desirable to understand more rigorously the underlying physical mechanisms, e.g., through coarse-grained models

and hydrodynamical descriptions of nonstabilizerness spreading. It would also be interesting to generalize Theorem 1 to states with multiple units of magic, and extend the associated algorithm (see also Refs. [105,106]). To this end, we note that our operational definitions extend to such states, cf. Supplemental Material [81]. Using these to examine “interaction effects” between spatially separated magic sources in monitored and open quantum dynamics would be appealing [56,57,107]. More generally, exploring how quantum resources (including NS, non-Gaussianity, and coherence) spread may clarify how complexity builds up in quantum many-body systems [108–110], how these systems deeply thermalize [26–32,34–36], and how logical information is causally organized in the spacetime of dynamical QEC codes [111].

Note added—While finalizing this manuscript, we became aware of an independent work by Maity and Hamazaki [112], which studies the spreading of local magic resource under Clifford dynamics. While they use a different approach and focus on distinct spatiotemporal structures, our results are consistent where they overlap: certain magic length scales grow ballistically until nonstabilizerness delocalizes at late times.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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- that $\tilde{\mathcal{M}}_2(\rho_A) > 0$ and $\mathcal{R}(\rho_A) > 1$ for a subsystem A where at least one logical can be reduced. The specific values of $|a_{\pm}|$ set the values of “full” [case (i)] and “partial” [cases (iv), (v)] NS. See also Supplemental Material [81].
- [99] For intermediate times, the data from Fig. 3 is consistent with a growth rate of FLEOM given by $\partial_t W \sim 4v_E/3$. We attribute the distinct intermediate-time growth of $W(t)$ to a combination of effects due to boundary conditions and shrinking of large logicals, as detailed in [81].
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End Matter

Appendix A: Proof of Lemma 1—Here we prove one of our key technical results, Lemma 1, which exposes the spatial structure of both logicals and stabilizer generators of $|\psi\rangle$. The proof of our other key technical result, Theorem 1, follows from this lemma and is found in [81].

Let the stabilizer group of our code be $\mathcal{G} = \langle g_2, \dots, g_L \rangle$, where g_i are the stabilizer generators. The logical operators of this code are nontrivial Hermitian Pauli operators $\bar{\sigma}$ that commute with all generators $[\bar{\sigma}, g] = 0 \ \forall \ g \in \mathcal{G}$ and are not elements of $\pm\mathcal{G}$. Thus, a logical operator $\bar{\sigma}$ is in one of the three cosets $\bar{X}\mathcal{G}$, $\bar{Y}\mathcal{G}$, and $\bar{Z}\mathcal{G}$ (up to a sign).

It is useful to represent the code and its logical operators through a pure stabilizer state $|\Phi_{ABR}\rangle$ on an enlarged system with an additional reference qubit R [113]. We depict this in Fig. 4, where for illustration purposes we take subsystems A and B as contiguous regions; however, our proof holds for any bipartition with $B = [L] \setminus A$. The stabilizer group of $|\Phi_{ABR}\rangle$ is defined as $\mathcal{S} := \langle \bar{X}X_R, \bar{Z}Z_R, g_2, \dots, g_L \rangle$, which also contains $-\bar{Y}Y_R$.

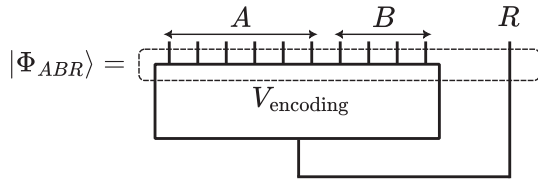


FIG. 4. Representation of a $[[n, k]]$ stabilizer code with n qubits and k logical qubits as a pure stabilizer state $|\Phi_{ABR}\rangle$ on an enlarged system ABR , with $n = |AB|$ and $k = |R|$. $|\Phi_{ABR}\rangle$ is the Choi state of the isometry V_{encoding} . The spatial structure of the logical operators, for any subsystem A and its complement B , can be inferred from the spatial structure of the stabilizers of $|\Phi_{ABR}\rangle$, as discussed in the proof of Lemma 1.

Using this mapping, we shall determine whether logical operators $\bar{\sigma}$ of the code can be fully supported on a subsystem (A or B). To illustrate this, note that any logical $\bar{\sigma}$ corresponds to a stabilizer of $|\Phi_{ABR}\rangle$ which has nontrivial support on the reference qubit R , whereas any stabilizer $g \in \mathcal{G}$ of $|\psi\rangle$ is represented by a stabilizer of $|\Phi_{ABR}\rangle$ with trivial support on R . Hence, to determine the spatial structure of the logicals, it suffices to work with the corresponding stabilizers of $|\Phi_{ABR}\rangle$.

$|\Phi_{ABR}\rangle$ is locally Clifford unitarily (LCU) equivalent, i.e., with Clifford $U \equiv U_A \otimes U_B \otimes U_R$, to a tensor product of tripartite GHZ states, EPR (Bell) pairs, and single-qubit states [82]. Here, since $k = |R| = 1$, there can be at most one GHZ state $|\text{GHZ}_{abR}\rangle = \frac{1}{\sqrt{2}}(|0_a 0_b 0_R\rangle + |1_a 1_b 1_R\rangle)$ for some qubits $a \in A$, $b \in B$; $|\text{GHZ}_{abR}\rangle$ has stabilizers in $\langle X_a X_b X_R, Z_a Z_b, Z_b Z_R \rangle$. The EPR pairs can be between any two subsystems out of A , B , and R , and are defined as $|\Phi_{ij}^+\rangle = \frac{1}{\sqrt{2}}(|0_i 0_j\rangle + |1_i 1_j\rangle)$ for qubits i and j , with stabilizers in $\langle X_i X_j, Z_i Z_j \rangle$. Since the reference qubit R is maximally entangled with the subsystem AB for $|\Phi_{ABR}\rangle$ and U is local, these remain entangled, and thus, after the LCU, subsystem R must be involved in either a GHZ state or an EPR pair. Hence, we can choose exactly two generators of $U|\Phi_{ABR}\rangle$ that are supported on R with the operators X_R and Z_R ; these are not necessarily $U\bar{X}X_RU^\dagger$ and $U\bar{Z}Z_RU^\dagger$ since, when choosing these two generators, we might need to multiply $U\bar{\sigma}\sigma_RU^\dagger$ by some $UgU^\dagger$ for $g \in \mathcal{G}$ and we can also choose $\sigma = Y$.

Next, focusing on $U|\Phi_{ABR}\rangle$, we proceed into three cases depending on which entangled state R belongs to: (i) $|\Phi_{aR}^+\rangle$ for qubit $a \in A$, (ii) $|\Phi_{bR}^+\rangle$ for $b \in B$, or (iii) $|\text{GHZ}_{abR}\rangle$ for $a \in A$ and $b \in B$. We take $U_R = \mathbb{1}_R$ for simplicity, but our

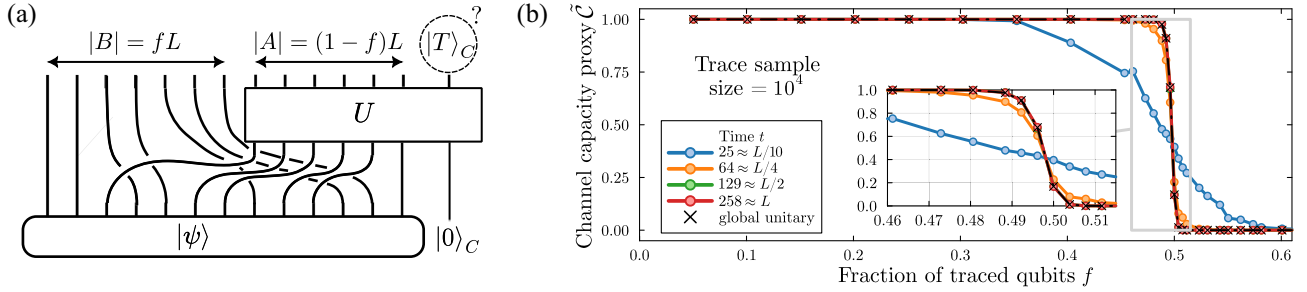


FIG. 5. (a) Quantum circuit setup for unitarily extracting the magic resource from subsystem A of state $|\psi\rangle$ onto an ancilla qubit C into a magic state $|T\rangle_C$ using a Clifford unitary U . The complementary subsystem B contains a finite fraction f of randomly chosen qubits. At late times, the extraction succeeds with high probability for any A with $|A| \gtrsim L/2$. (b) Channel capacity proxy $\tilde{C}$ of the channel that traces out $fL = |B|$ randomly chosen qubits after evolution up to time t under a random brickwork Clifford circuit with identity doping $p = 0.1$ (circles) and a global random Clifford (crosses). For $t \gtrsim t_{\text{sat}}$, $\tilde{C} = \mathcal{O}(1)$ for $f \leq 0.5$ indicating the preservation of magic resource; for $f \gtrsim 0.5$, $\tilde{C}$ quickly drops to zero and magic resource is lost (see inset). Data averaged over 10^4 realizations of B for a single random circuit instance, negligible error bars not shown.

conclusions hold for any Clifford gate U_R (this can only permute how we label our logicals).

Case (i). Since the stabilizers of $|\Phi_{AR}^+\rangle$ are in $\langle X_a X_R, Z_a Z_R \rangle$, we can choose the representatives of the logical operators (encoded in $U|\Phi_{ABR}\rangle$) to be $\bar{X}' = X_a$ and $\bar{Z}' = Z_a$, thus, $\bar{Y}' = Y_a$. Reverting the LCU transformation $U|\Phi_{ABR}\rangle \mapsto U^\dagger U|\Phi_{ABR}\rangle = |\Phi_{ABR}\rangle$, we find that the representatives of the logical operators of our initial stabilizer code (encoded in $|\Phi_{ABR}\rangle$) can be taken $\bar{\sigma} = U^\dagger \bar{\sigma}' U$ for $\bar{\sigma}' \in \{X_a, Y_a, Z_a\}$. Hence, since U is local, for each coset $\bar{\sigma}\mathcal{G}$, there exist representative logical operators $\bar{\sigma}$ that are simultaneously fully supported on subsystem A , i.e., $\bar{\sigma}|_B = \mathbb{1}_B$.

Case (ii). This is analogous to case (i), with A replaced by B , and we find that there exist representative logical operators $\bar{\sigma}$ that are fully supported on subsystem B .

Case (iii). Here, the two logical generators representatives $\bar{X}'X_R$ and $\bar{Z}'Z_R$, i.e., the generators of $U|\Phi_{ABR}\rangle$ supported on R , must be chosen from the stabilizer group of the GHZ state $\langle X_a X_b X_R, Z_a Z_b, Z_b Z_R \rangle$. Thus, we can take without loss of generality $\bar{X}' = X_a X_b$ and $\bar{Z}' = Z_a$ (or $\bar{Z}' = Z_b$), leading to $\bar{Y}' = Y_a X_b$ (or $\bar{Y}' = X_a Y_b$). Then, because U is local, reverting the LCU does not change whether a logical $\bar{\sigma} = U^\dagger \bar{\sigma}' U$ is supported on subsystem A , or B , or both. Hence, two logicals (here, $\bar{X}$ and $\bar{Y}$) are supported on both A and B , while the third (here, $\bar{Z}$) can be chosen to be supported only on either A or B ; moving the third logical from A to B and vice versa can be done by multiplying it by a stabilizer $g \in \mathcal{G}$ (here, $Z_a Z_b$). This concludes our proof. ■

Appendix B: Details on magic resource delocalization at late times—By focusing on QEC aspects of the dynamically generated stabilizer code, we here show that this code approaches a random $[[L, 1]]$ stabilizer code at late times, which in turn implies that one can unitarily

distill a magic state from *any* subsystem with $\gtrsim L/2$ qubits, and thus that magic resource delocalizes.

To show this, we consider a subsystem B with a finite fraction $f = |B|/L$ of qubits chosen at random, and ask if magic can be unitarily extracted from its complement $A := [L] \setminus B$, as depicted in Fig. 5(a). By Lemma 1, this question is equivalent to asking if the logical information of the code can be transmitted through a quantum channel $\mathcal{N}_B = \mathcal{E}_B \circ U$, with a Clifford unitary U and an error channel $\mathcal{E}_B$, chosen as either tracing out or measuring in a complete stabilizer basis of B . To answer the latter question, we use a proxy $\tilde{C}(\mathcal{N}_B[\rho])$ for the single-copy channel capacity [102–104], with $\tilde{C} = 1(0)$ if the logical qubit is (not) preserved.

We can evaluate the single-copy channel capacity proxy $\tilde{C}$ through the coherent information $I_c(\Psi_{AR})$ by using the construction from the proof of Lemma 1 [Fig. 4]. Let Φ be the stabilizer state on ABR that encodes $U\rho U^\dagger$. Acting with the error channel $\mathcal{E}_B$ yields $\Psi = \mathcal{E}_B \otimes \mathbb{1}_{AR}[\Phi]$. The logical information (or equivalently, magic resource) is preserved if the mutual information between subsystem A and the ancilla R satisfies $I_\Psi(A, R) = 2$ [113], where $I_\Psi(X, Y) = S(\Psi_X) + S(\Psi_Y) - S(\Psi_{XY})$ with von Neumann entanglement entropy S . By a simple calculation, we can recast this condition in terms of the coherent information $I_c(\Psi_{AR}) = S(\Psi_A) - S(\Psi_{AR})$ as $I_c(\Psi_{AR}) = 1$; conversely, $I_c(\Psi_{AR}) = 0$ if the logical information is lost.

Numerically, we focus on a single circuit (code) realization with fixed f , sample N sets B of fL qubits, and thus estimate $\tilde{C}(\mathcal{N}_B)$ at different evolution times. (We checked multiple code realizations, finding similar results.) Random stabilizer codes are known to saturate the quantum Singleton bound [4, 101], which is a bound on the code distance: for $f \lesssim 0.5$, their logical information is preserved asymptotically ($L \rightarrow \infty$), whereas for $f > 0.5$, it is highly likely lost due to the No Cloning theorem [4]. We find that,

at $t > t_{\text{sat}}$, the channel capacity proxy $\tilde{\mathcal{C}}(\mathcal{N}_B) = 1$ for $f \lesssim 0.48$, finite up to $f \simeq 0.5$, and then drops to zero for $f > 0.5$, as expected from the quantum Singleton bound [Fig. 5(b)]. Comparing our results with global random

Clifford gates, which generate random stabilizer codes, we find excellent agreement, further supporting the approach of the dynamical code to a random $[[L, 1]]$ stabilizer code and the delocalization of magic resource at late times.