

Mixed State Deep Thermalization

Xie-Hang Yu¹, Wen Wei Ho^{2,3}, and Pavel Kos¹

¹Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany

²Department of Physics, National University of Singapore, Singapore 117551

³Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543



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We introduce the notion of the *mixed state projected ensemble (MSPE)*, a collection of mixed states describing a local region of a quantum many-body system, conditioned upon measurements of the complementary region which are incomplete. This constitutes a generalization of the pure state projected ensemble in which measurements are assumed ideal and complete, and which has been shown to tend towards limiting pure state distributions depending only on symmetries of the system, thus representing a new kind of universality in quantum equilibration dubbed *deep thermalization*. We study the MSPE generated by solvable $(1+1)$ d dual-unitary quantum circuit evolution, and identify the limiting mixed state distributions which emerge at late times depending on the size of the incomplete measurement, which we assume to be lossy, finding that they correspond to certain random density matrix ensembles known in the literature. We also derive the rate of the emergence of such universality. Furthermore, we investigate the quantum information properties of the states composing the ensemble, specifically their capacity to teleport quantum information between the ends of the system. The teleportation fidelity is upper bounded by the quantum conditional entropy, which we find exhibits a sharp transition from zero to maximal when the number of measurements lost matches that of the number of degrees of freedom to be teleported. Our results initiate the first investigation of deep thermalization for mixed state ensembles, relevant for present-day quantum simulation experiments wherein measurements are typically not perfect, and also amount to a physical and natural way of sampling from hitherto abstract random density matrix ensembles.

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Introduction—Understanding how isolated quantum many-body systems thermalize is one of the most important questions in quantum dynamics. The standard framework involves studying the state of a local subsystem A (by discarding its complement B , which plays the role of a bath) and asking if over time it converges to the Gibbs state, the maximally entropic equilibrium state [1–4].

Recent advances in controlling and probing modern-day quantum simulator experiments [5–13] have suggested that this framework might be insufficient: in many such platforms, during measurements, it is possible to retain some (classical) information of the bath B , in addition to the subsystem A of interest. This has led to access to quantum states describing different configurations on A , *conditioned* on measurement outcomes α of B . Assuming projective measurements in a fixed local basis, the set of all conditional states $|\psi(\alpha)\rangle_A$ with measurement outcome α

occurring with probability P_α constitutes the so-called *projected ensemble* $\mathcal{E} := \{P_\alpha, |\psi(\alpha)\rangle_A\}$. The projected ensemble probes ergodicity of the wave functions underlying the local state of A , which is tied to the information scrambling nature of the underlying quantum dynamics [14–18].

Interest in the projected ensemble lies in the fact that it tends toward limiting distributions (at late times in generic quantum dynamics), which are universal and depend only on the coarse-grained features of the system, such as symmetries and global conserved charges. This is a novel form of quantum equilibration which has been dubbed *deep thermalization* [19,20]. For example, in the absence (presence) of conservation laws, it has been found that the Haar (generalized Scrooge) ensemble emerges, a uniform (close to uniform but distorted by conserved charges) distribution [17,21]. These have further been argued to be underpinned by generalized maximal entropy principles rooted in quantum information theory, and have been demonstrated to hold across various physical settings, from spins [16,20,22,23], fermions [24], to continuous variable bosonic systems [25], and even in 1D and 2D tensor network states [26]. Importantly, deep thermalization was already experimentally observed in Rydberg quantum simulators [18].

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However, experimental platforms in reality are never perfect. Many kinds of incoherent errors, like erasure and dephasing in dynamics or measurements, inevitably modify the pure states generated in the projected ensemble into mixed ones, raising questions as to how the universality of deep thermalization will be modified under such scenarios. In this work, we initiate the investigation of *mixed state deep thermalization*, focusing on a partial loss of measurement outcomes. We consider the projected ensemble formed under dynamics of dual-unitary quantum circuits [27], a class of recently introduced solvable models which has provided powerful and analytical insights into various aspects of quantum dynamics like entanglement growth and quantum chaos [27–35]. We rigorously derive the limiting universal ensembles emerging at late times and identify them as certain random density matrix ensembles known in the literature [36,37]. Additionally, we examine the quantum information properties of the conditional states of the projected ensemble, namely, their ability to teleport quantum information between distant regions [38,39]. Teleportation fidelity is upper bounded by quantum conditional entropy, which exhibits a sharp transition from zero to maximal, depending on the ratio of teleported and lost degrees of freedom, signifying a fundamental change in the projected ensemble's information-theoretic nature. Our results extend the notion of deep thermalization to systems with imperfections, establishing universality in mixed-state ensembles that arise under physical scenarios.

Mixed state projected ensembles (MSPE)—Consider a quantum state $|\Psi\rangle$ supported on two subsystems A, B with N_A, N_B qudits of local dimension d , respectively. Subsystem B is further divided into three contiguous subparts B_1, B_2, B_3 (Fig. 1) and undergoes rank-1 orthogonal measurements, i.e., complete projective measurements. However, we assume only the measurement results from B_1 and B_3 are kept, while those from m sites in B_2 are lost. These lost measurement outcomes (equivalently, dynamics in the presence of totally depolarizing noise; see discussion and outlook) are the particular type of imperfection we consider, leading to mixed states on subsystem A depending on the measurement results of B_1, B_3 , denoted by the string vector $\alpha = (\alpha_{B_1}, \alpha_{B_3})$. Namely, for each measurement outcome, there is an associated mixed conditional state on A ,

$$\rho_{N_A}(\alpha) = \frac{\text{Tr}_{B_2}[(I_{A+B_2} \otimes \langle \alpha |) |\Psi\rangle \langle \Psi| (I_{A+B_2} \otimes |\alpha\rangle)]}{P_\alpha},$$

occurring with probability $P_\alpha = \langle \Psi | I_{A+B_2} \otimes |\alpha\rangle \langle \alpha| | \Psi \rangle$. Here I_{A+B_2} is the identity matrix acting on subsystems A and B_2 , and $|\alpha\rangle = |\alpha\rangle_{B_1} \otimes |\alpha\rangle_{B_3}$ is the classical state corresponding to outcome α on B_1, B_3 . The MSPE is the set of all possible mixed states on A combined with their probability $\mathcal{E} := \{P_\alpha, \rho_{N_A}(\alpha)\}$. Analogously to the pure-state case, the MSPE can be understood as a distribution,

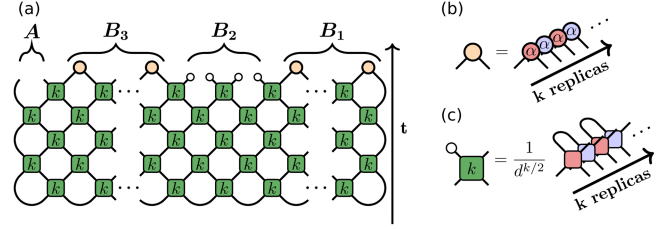


FIG. 1. Quench dynamics leading to a mixed state projected ensemble. (a) The conditional state $\rho_{N_A}^{\otimes k}(\alpha)$ consisting of k pairs of forward and backward evolution (k replica), which together with Born probability P_α determines the moments of the ensemble $\rho_{N_A}^{(k)}$. A product state composed of maximally entangled pairs evolves under a brick-wall quantum circuit with t layers. The system is bipartited into two subsystems, A and B , where B is further split into three subparts. B is measured in 2-qudit basis, with measurement outcomes from B_2 being lost. We consider A 's mixed states conditioned on the measurement outcomes in B_1, B_3 . (b) The orange circle denotes the projective measurement on two qudits with the measurement outcome α in k replicas. (c) The white empty bullet corresponds to lost measurement outcomes and is proportional to the k -copy of the vectorized identity matrix. This effectively implements a partial trace, connecting the forward and backward time evolutions.

but now over the space $\mathcal{V}_A$ of density matrices on A . We are interested in whether this distribution converges to a universal one over $\mathcal{V}_A$, that is, one that does not depend on the microscopic details of the system. If so, we call this *mixed-state deep thermalization*.

To characterize the MSPE, we can analyze its statistical moments. Specifically, the k th moment is captured by

$$\rho_{N_A}^{(k)} = \sum_{\alpha} P_\alpha \rho_{N_A}^{\otimes k}(\alpha), \quad (1)$$

a density matrix acting on the k copy of the subsystem A , see Fig. 1 for a graphical expression. Convergence to a target ensemble $\mathcal{E}'$ entails $\rho_{N_A}^{(k)}$ matching corresponding moments $\rho_{\mathcal{E}'}^{(k)} = \int \rho^{\otimes k} d\mu(\rho)$ for each k , where $d\mu(\rho)$ is the measure induced by $\mathcal{E}'$, quantifiable using the Schatten ξ -norm $\Delta_\xi^{(k)} = [(\|\rho_{N_A}^{(k)} - \rho_{\mathcal{E}'}^{(k)}\|_\xi) / (\|\rho_{\mathcal{E}'}^{(k)}\|_\xi)]$. In this Letter, we focus on $\xi = 1, 2$. For $\forall k$, if $\Delta_\xi^{(k)}$ vanishes (is smaller than ϵ), we say that the MSPE is exactly (ϵ -approximately) deeply thermalized to $\mathcal{E}'$.

Solvable models and results on k moments—We analytically show MSPE arising from a particular treatable protocol, a generalization of the one considered in Ref. [16]. We take the quantum state of $N_A + N_B$ qudits to arise from a 1 + 1D quantum circuit dynamics, to wit: starting from $|\psi\rangle = |\varphi_0\rangle^{\otimes (N_A + N_B)/2}$ composed of products of local maximally entangled pairs $|\varphi_0\rangle = \sum_i^d |ii\rangle / \sqrt{d}$, the system is evolved with a brick-wall quantum circuit with t layers (t is odd in Fig. 1 [40]), with each two-qudit gate

independently sampled from an open set of dual-unitary gates [27], i.e., gates which are unitary when viewed both bottom to top and left to right. The distribution of each gate has a nonzero weight on an open subset of the dual-unitary gates, which allows the spatial evolution to consist of a universal gate set as proved in Ref. [16]. We focus on models without any global conservation laws, and on $t + 1 \geq N_A$.

We assume subsystem B is measured in a basis composed of a tensor product of two-site entangled states. Precisely, the measurement basis is generated from $|\varphi_0\rangle$ acted by the Heisenberg-Weyl operators [41] σ_α as $|\varphi_\alpha\rangle = I \otimes \sigma_\alpha |\varphi_0\rangle$ with $\alpha = \{1, \dots, d^2\}$. These orthogonal measurement bases, generalizing the Bell measurement to the higher dimensions, have maximal bipartite entanglement, and notably satisfy the solvable conditions for dual-unitary circuits [30].

We focus our discussion on the k th moment $\rho_{N_A}^{(k)}$, diagrammatically shown in Fig. 1(a) as a tensor network. The figure is depicted in the vectorized k -replica format, where forward and backward time evolutions are folded on top of each other. Each green gate consists of k copies of forward and backward evolution $\boxed{k} = (U \otimes U^*)^{\otimes k}$, while a projective measurement with outcome α is denoted by an orange circle in Fig. 1(b). We take subsystems B_1, B_3 to be infinitely long, a condition which can be relaxed [42]. In region B_2 , measurement outcomes on m sites are assumed forgotten, which is equivalent to erasing these sites, captured mathematically by tracing out, i.e., the forward and backward time sheets are connected. In Fig. 1(c), this partial trace is represented by m empty bullets. Each empty bullet is proportional to the vectorized identity operator acting on the k replicas $\rho = \frac{1}{d^{k/2}} (\bigwedge)^{\otimes k} = \frac{1}{d^{k/2}} (\sum_{i=1}^d |ii\rangle)^{\otimes k}$.

By an explicit calculation, we prove, for almost all realizations of the circuit model described above, (i) Taking the circuit depth $t \rightarrow \infty$, the MSPE deeply thermalizes to the so-called generalized Hilbert-Schmidt ensemble, Eq. (2). (ii) At large t , the approach is exponentially quick. For the k th moments to be ϵ close in both the two and trace norms, it takes time $t_k \sim \log(k/\epsilon)$.

These two results hold for any ratio of m/N_A . The generalized Hilbert-Schmidt ensemble is defined by tracing out m qudits from Haar random states on $N_A + m$ qudits:

$$\mathcal{E} = \{\text{Tr}_m |\psi\rangle_{N_A+m} \langle \psi| \mid |\psi\rangle \sim \text{Haar}(d^{N_A+m})\}. \quad (2)$$

For $m = N_A$ this ensemble reduces to the so-called Hilbert-Schmidt ensemble [37,45].

The emergence of the generalized Hilbert-Schmidt ensemble as the limiting form of the MSPE—mixed state deep thermalization—constitutes one of our key results. It is further physically reasonable: this ensemble can be viewed as the most ergodic mixed-state ensemble

appearing under the scenario of lost information. Indeed, suppose the number of erased measurement outcomes is zero ($m = 0$), then the MSPE reduces to the maximally-ergodic pure state Haar random ensemble known from deep thermalization [15,16,19,20,22]. When $m > 0$, we expect a loss of purity for each state on A that decreases exponentially with m ; the generalized Hilbert-Schmidt ensemble indeed has this behavior.

We briefly sketch the proof of our results (i) and (ii). This involves a computation of the moments $\rho_{N_A}^{(k)}$. We analyze the evolution of the k -replicated system in the spatial direction from right to left. Assuming that B_1, B_3 are infinitely long, since the gate composed of the circuits form a universal gate set, the circuits there effectively form a unitary k design [16,19] in the space direction. The steady state therefore commutes with $U^{\otimes k}$ for any U [16]. This subspace is spanned by the global permutations of k replicas, so it suffices to track the coefficients of each permutation element. The erased sites in B_2 modify the coefficients of different permutation elements. Lastly, the gates in the subsystem A implement an isometry that preserves these coefficients. Our results follow by combining all these effects and the replica trick, provided in detail in the End Matter and the Supplemental Material [46].

Another interesting limit is to fix t with $t + 1 \geq N_A$ and let $d \rightarrow \infty$, or to take $t, N_A, m \rightarrow \infty$ with $\gamma = m/N_A$ fixed. In the pure-state Haar random ensemble, the k th moment is an equally weighted sum of permutation operators on the k -replicated Hilbert space. Here we find that instead the leading contribution (in d) for $\rho_{N_A}^{(k)}$ of MSPE comes only from the identity permutation, even if only a single measurement outcome is erased. Other permutation contributions are suppressed as $1/d^m$ [50]. It is tempting to argue that in this limit, the MSPE is the totally mixed state ensemble, where every conditional mixed state is the maximally mixed state. However, this is not always the case. The difference can be resolved by, for example, looking at some nonlinear global quantities, such as Rényi- k entropies, which can amplify the subleading contributions, or looking directly at the distribution of the eigenvalues of the mixed states [51].

Generality of mixed state deep thermalization—Here we establish universality of mixed-state deep thermalization beyond the dual-unitary setting. Analytically, we can show that the limiting distribution of the MSPE is also the generalized Hilbert-Schmidt ensemble for global Haar-random circuits with $N_B = \Omega(kN_A)$ and for brick-work random unitary circuits in the large- d limit, see the End Matter and Supplemental Material [46].

We numerically demonstrate this by computing the distance $\Delta_1^{(2)}$ between the MSPE and the generalized Hilbert-Schmidt ensemble for both random unitary circuits and Hamiltonian mixed-field Ising dynamics with local dimension $d = 2$. Figure 2 shows that the saturation value of $\Delta_1^{(2)}$ in all cases decreases exponentially with system

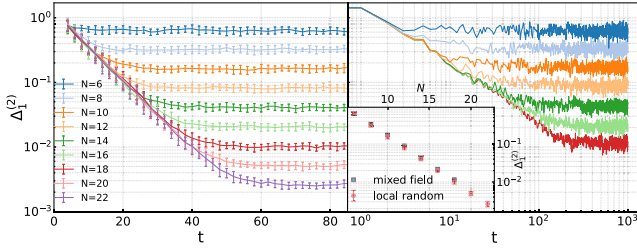
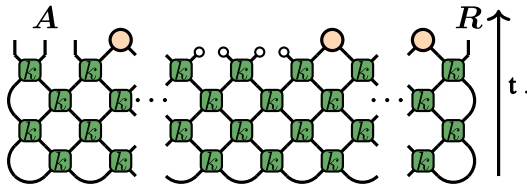


FIG. 2. Distance of $\rho_N^{(k)}$ from the generalized Hilbert-Schmidt ensemble. Left: local Haar-random circuits for system sizes $N = 6$ –22, where error bars indicate statistical fluctuations over different circuit realizations. Right: mixed-field Ising dynamics for $N = 6$ –18, governed by the Hamiltonian $H = \sum_{j=1}^N (h_x \sigma_x^j + h_y \sigma_y^j) + \sum_{j=1}^{N-1} J \sigma_x^j \sigma_x^{j+1}$, with σ_a^j the Pauli matrix on j th qubit and $(h_x, h_y, J) = (0.8090, 0.9045, 1.0)$. Inset: saturated value at late times versus system size.

size, in line with predictions. We further find that the decay rate depends on the dynamics (but not on the system size): for random unitary circuits, $\Delta_1^{(k)}$ decays exponentially before saturation, whereas for Hamiltonian dynamics the decay obeys a power law (this is presumably attributed to energy conservation). In [46] we provide further numerics for the kicked Ising model and dual-unitary circuit. A detailed analysis of finite-time and finite-system effects is left for future work.

Quantum conditional entropy and state teleportation— We return to our analytical setting of dual-unitary circuits. In the limit $d \rightarrow \infty$, the amount of information lost when even only a single measurement outcome is erased, $\log(d)$, diverges. This, as well as the fact that the leading (in d) contribution comes from the identity permutation, suggests that the MSPE should be indistinguishable from the totally mixed ensemble. Nonetheless, the ratio between the lost and the retained information is always fixed to m/N_A . Therefore, how much quantum information can be transported through the region B_2 is not clear. To address this, we examine the average ability of MSPE to *teleport* a quantum state through the bulk [39,52]. Concretely, we consider leaving a single reference (R) qudit at the rightmost boundary of the chain unmeasured, which will act as an input for an arbitrary state in our teleportation thought experiment. We will try to recover quantum information sent into R from region A , mediated by the lossy measurements in B . Diagrammatically, this looks like



Now, the fidelity of teleportation can be upper bounded by the conditional entropy between the subsystem A and the

reference R [53], which matches the coherent information [54], viewing evolution through the bulk as a quantum channel. For simplicity, we compute the annealed average Rényi- k conditional entropy

$$I_{R:A}^{(k)} = S_{AR}^{(k)} - S_A^{(k)} = -\frac{1}{k-1} \log \frac{\overline{\text{Tr} \rho_{AR}^k}}{\overline{\text{Tr} \rho_A^k}}, \quad (3)$$

where the bar indicates the average over the measurement outcomes with associated Born probabilities.

The essence of computing the conditional entropy is to evaluate the Rényi- k purities $\text{Tr}(\rho^k)$, expressible using the k th moment of the density matrix ensemble. Thus, the previously used methods, including the replica trick, also apply. We emphasize that different boundary conditions must be imposed on R depending on whether $S_A^{(k)}$ or $S_{AR}^{(k)}$ is evaluated [46].

We provide the evaluation of Eq. (3) in Supplemental Material [46]. In the limit $t \rightarrow \infty$ and $m, N_A \rightarrow \infty$ while the ratio $\gamma = m/N_A$ is fixed (or equivalently $d \rightarrow \infty$), we can identify two phases for the capacity of quantum state teleportation. If $N_A > m$, $I_{R:A}^{(k)} = -\log(d)$ for any $k \geq 0$, indicating the possibility for perfect quantum state teleportation [53]. The extreme case $m = 0$ reflects that the space evolution is unitary, and no information is lost. For $m > 0$, the erased sites are sparse compared to N_A , and the system still preserves enough quantum information to teleport the state. We call this the teleportation phase.

Conversely, if $N_A < m$, $I_{R:A}^{(k)} = \log(d)$, indicating that the subsystem A and the reference R are decoupled. This can be explained by the monogamy of entanglement [52]: the number of erased sites m is larger than the subsystem size N_A , and all the sites in A are entangled with the environmental degrees of freedom [55]; they thus cannot be entangled with the reference qudit R anymore and so no state teleportation is possible. This corresponds to the decoupled phase.

The above two phases are separated by the transition point $N_A = m$, where the size of subsystem A matches the number of erased sites. There $I_{R:A}^{(k)} = 0$. Since this sharp transition holds for any $k \geq 0$, we have analytically shown a sharp transition from the teleportation phase to the decoupled phase.

Discussion and outlook—In this Letter, we have introduced the mixed state projected ensemble (MSPE), a statistical description of a local subregion of a quantum many-body state, conditioned upon imperfect measurements on the complement. We have identified the universal limiting forms of the MSPE under lossy measurements, yielding a novel form of quantum equilibration which is the conceptual extension of deep thermalization from pure to mixed states. Furthermore, we have investigated the quantum information-theoretic property of the MSPE, and identified a phase transition in terms of its ability to teleport

quantum information between distant regions, which depends on the ratio of the number of erased measurement outcomes to the size of the subsystem of interest. Our findings are particularly relevant for NISQ-era quantum devices, where imperfections and noise are unavoidable, and also provide a practical scheme to generate random states, which have applications in myriad quantum information tasks [56–60]. An experimentally verifiable consequence of mixed-state deep thermalization is the emergence of the so-called Erlang distribution [61–63] governing the probability of probabilities of bit-string outcomes [18,62] upon measuring a projected state of the MSPE, different from the Porter–Thomas distribution known for pure states [17].

While here we have characterized the emergent random density matrix ensembles by their moments [cf. Eq. (1)], in Supplemental Material [46] we analyze the induced eigenvalue-eigenvector distribution of density matrices on the subregion A . The eigenvectors of the MSPE and Hilbert–Schmidt ensembles are distributed uniformly. The eigenvalue distribution for the MSPE in the limit $N_A \rightarrow \infty$ at fixed ratio $\gamma = m/N_A$ is more nontrivial: the variance of the distribution is exponentially suppressed in t , so in this limit it reduces to a delta distribution matching the generalized Hilbert–Schmidt ensemble [45,64]. Remarkably, each instance in this limiting ensemble has a flat spectrum, which supports the intuition that the MSPE is an ergodic ensemble at infinite temperature for given N_A and m .

Our work opens new directions of inquiry. We considered loss of measurement records on a small, nonextensive subregion; it is natural to ask about the universality of the MSPE when erased measurements have finite density. An immediately tractable setting is sparse, randomly distributed erasures, with long regions of retained measurements between. At leading order in d^t , a quick computation yields the same results for deep thermalization and quantum state teleportation, see Supplemental Material [46]. For dense measurement loss, we can prove that global Haar-random circuits still deeply thermalize to the generalized Hilbert–Schmidt ensemble [46], and we conjecture this holds more generally. Notice that in the limit $m, N_A \rightarrow \infty$ while $N_A/m \rightarrow 0$, each instance in the ensemble is similar to the totally mixed state.

We can also go beyond the erasure error model considered here. Our results suggest that discarding measurement outcomes is equivalent to applying *totally* depolarizing channels. What universal ensemble arises if instead these are *partially* depolarizing channels $\rho \rightarrow P\rho + (1-P)I/d$ (in B_2)? For $P = 1(0)$ we recover the pure state Haar ensemble (MSPE), but the case of intermediate P remains open. Similar questions arise for other noisy evolutions and measurements, and to dynamics with symmetries, conservation laws, and beyond spin systems. For example, Ref. [65] considers related questions in conformal field theory with $U(1)$ charges.

More fundamentally, it would be very interesting to understand how the universal mixed-state ensembles unveiled here and those arising in future work discussed above can be argued for in terms of general physical principles like the principle of maximum entropy, see Refs. [17,21,25,66] for the pure-state case. Extending similar notions to mixed state deep thermalization would entail a significant advancement of our understanding of quantum statistical mechanics.

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Data availability—The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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End Matter

Appendix A: k th moment—Here, we describe the calculation of the k th moment $\rho_{N_A}^{(k)}$. It is convenient to consider the spatial evolution progressing from right to left by exchanging the roles of space and time. In this way, the initial state comprises of maximally entangled pairs over $t + 1$ sites in all k replica copies, as indicated in Fig. 3 $\rho_i = (|\varphi_0\rangle^{\otimes(t+1)/2} \langle\varphi_0|^{\otimes(t+1)/2})^{\otimes k}$. The state evolved with N_B layers of brick-wall circuit made of dual-unitary gates, which are also unitary for this evolution in space direction. The original initial state becomes an open boundary condition on the bottom boundary (i.e., an identity matrix on a single site). Projective measurements with outcome α instead imply the action of a single site unitary σ_α on the other (top) boundary. Therefore, if the measurement outcome is recorded, the evolution is unitary.

For any measurement outcome α_1 in B_1 , we denote the spatial unitary in the subpart B_1 as U_{α_1} , whose unitarity is guaranteed by dual-unitary property of the circuit. Because of the assumption that the subpart B_1 is infinitely long, the measurement outcome averaged state ρ_b on the timelike cut thus reaches its steady state before entering B_2 , determined by the eigenvectors of the inhomogeneous spatial transfer matrix with the largest eigenvalues. Since any open set of the dual-unitary gate forms a universal gate set [35], U_{α_1} then explores the whole space of the unitary group [16,44] for different measurement outcomes. According to

Ref. [16], the steady state must commute with $U_{\alpha_1}^{\otimes k}$ for any α_1 , which is equivalent to any unitary. By Schur-Weyl duality, the subspace of the steady states is exclusively given by the representation $\rho_{t+1}(g)$ of a permutation element $g \in \mathbb{S}_k$ on the k replicas, each with $t + 1$ qudits living on the timelike cut. Here $\mathbb{S}_k$ is the permutation group for k replicas. Concretely,

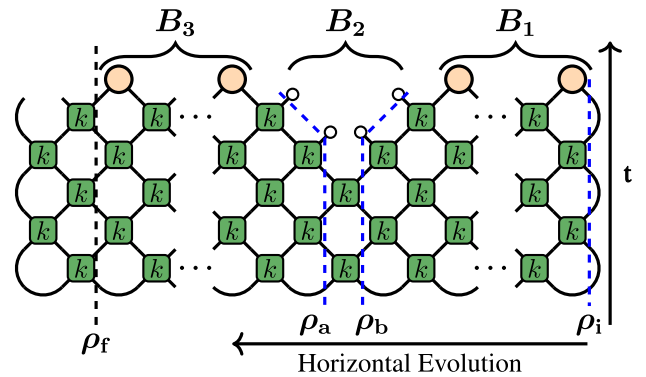


FIG. 3. Quench dynamics leading to MSPE. The symbols have the same meaning as in Fig. 1. Here, we use the unitarity to simplify the region B_2 . We use the dashed line to indicate the states on the timelike cuts, which appear in the derivation of the results.

$$\rho_{t+1}(g) = \sum_{i_1, i_2, \dots, i_k=1}^{d^{t+1}} |i_{g(1)} i_{g(2)} \cdots i_{g(k)}\rangle \langle i_1 i_2 \cdots i_k|. \quad (\text{A1})$$

Note that this is a global state spanning all $t+1$ sites in each replica space, as indicated by the subscript of $\rho(g)$. In the replica picture, the initial state is totally symmetric, thus ρ_b is an equal superposition of all these dominant eigenvectors

$$\rho_b = \sum_g \rho_{t+1}(g). \quad (\text{A2})$$

Here we omit a possible normalization factor, which does not influence the results.

In the region B_2 , measurement outcomes are erased so that the action on the topmost $m/2$ qudits is given by a totally depolarizing quantum channel $(1/d^2) \sum_\alpha \sigma_\alpha(\cdot) \sigma_\alpha^\dagger$. Using the unitarity of the circuit, region B_2 can be simplified as shown in Fig. 3. As a result, the timelike state after the depolarization channel, ρ_a in Fig. 3, is no longer an eigenstate of the spatial transfer matrix.

Since B_3 is also infinitely long, it projects ρ_a back to the corresponding eigenspace. However, due to the replica structure in ρ_a , the final steady state, ρ_f in Fig. 3, does not correspond to the equal superposition of the permutations $\rho_{t+1}(g)$. The overlap between $\rho_{t+1}(g)$ and ρ_f can be determined from the overlap between $\rho_{t+1}(g)$ and ρ_a as $\rho_{t+1}(g)$ remains invariant under the evolution in B_1 : $\text{Tr}(\rho_{t+1}^\dagger(g) \rho_f) = \text{Tr}(\rho_{t+1}^\dagger(g) \rho_a)$. It is important to note that the matrices $\rho_{t+1}(g)$ are not mutually orthogonal with respect to the Frobenius inner product. Taking this into account, we express ρ_f as a linear combination of $\rho_{t+1}(g)$: $\rho_f = \sum_g \alpha(g) \rho_{t+1}(g)$ where $\alpha(g)$ satisfies a series of equations

$$\begin{aligned} & \sum_{g'} \frac{d^{m l(g)/2 + m l(g')/2 + (t+1-m/2) l(g^{-1}g')}}{d^{k(m/2+t+1)}} \\ &= \sum_{g'} \frac{d^{(t+1) l(g^{-1}g')}}{d^{(t+1)k}} \alpha(g') \quad \text{for } \forall g. \end{aligned} \quad (\text{A3})$$

Here $l(g)$ denotes the number of cycles [36] in the permutation element g .

The state ρ_f completely determines the output k th moment $\rho_{N_A}^{(k)}$ in region A. Since the gates in the region A are also dual-unitary, they form an isometry W from the timelike state to the spatial state with $WW^\dagger = I_{N_A}$. This isometry does not mix representations of different permutation elements as $W^{\otimes k} \rho_{t+1}(g) W^{\dagger \otimes k} = \rho_{N_A}(g)$, see Supplemental Material [46] for details. Here $\rho_{N_A}(g)$ is the representation of g defined analogously to Eq. (A1), except that each replica contains N_A sites and thus the sum goes to d^{N_A} . Therefore, $\rho_{N_A}^{(k)}$ has exactly the same weight of

each permutation element as ρ_f . We emphasize that the above calculation has not included the Born probability P_α , which can be included using the replica trick [46]. Taking it into account does not change the result. Consequently, the k th moment of the distribution describing MSPE is expressed as $\rho_{N_A}^{(k)} = \sum_g \alpha(g) \rho_{N_A}(g)$ and entirely determined by Eq. (A3).

To the first order expansion of Eq. (A3) in large t (d^t) limit, we can replace $l(g^{-1}g')$ with a delta function $d^{(t+1) l(g^{-1}g')} \approx d^{(t+1)k} \delta_{g,g'}$. Therefore, Eq. (A3) simplifies to

$$\alpha(g) = \frac{d^{m l(g)}}{d^{mk}}, \quad (\text{A4})$$

which indicates that all the permutation elements contribute to the k th moment. The coefficients $\alpha(g)$ in Eq. (A4) match the ones for the generalized Hilbert-Schmidt ensemble [cf. Eq. (2)]. Thus, in the limit $t \rightarrow \infty$ the MSPE is deeply thermalized to it.

Let us now consider the next order in $1/d^t$, which will give us the deep thermalization time scales. Here, we need to expand the cycle function l up to the next order. For that we have

$$\frac{d^{(t+1) l(g^{-1}g')}}{d^{(t+1)k}} = \delta_{g',g} + \sum_{i < j} \frac{1}{d^{(t+1)}} \delta_{g', g s_{i,j}} + \mathcal{O}\left(\frac{1}{d^{2t}}\right), \quad (\text{A5})$$

where $s_{i,j}$ is the swap operator between the i th and j th replica. Using this expansion, we evaluate the distance to the generalized Hilbert-Schmidt ensemble $\Delta_\xi^{(k)}$ with $\xi = 1, 2$. The full expression can be found in Supplemental Material [46], which simplifies for $k \ll d^m$ to

$$\begin{aligned} \Delta_1^{(k)} &\lesssim \frac{1}{d^{t+1}} \frac{1}{d^m} \left(1 - \frac{1}{d^m}\right) \frac{k(k-1)}{2} + \mathcal{O}\left(\frac{1}{d^{2t}}\right), \\ \Delta_2^{(k)} &\approx \frac{1}{d^{t+1}} \frac{1}{d^m} \left(1 - \frac{1}{d^m}\right) \sqrt{\frac{k(k-1)}{2}} + \mathcal{O}\left(\frac{1}{d^{2t}}\right). \end{aligned} \quad (\text{A6})$$

This indicates that the deep thermalization times scale as $t_k \sim \log(k/\epsilon)$ for an ϵ approximation to the generalized Hilbert-Schmidt ensemble for a given moment k . Note that higher moments require a longer time to achieve ϵ convergence leading to the separation of deep thermalization timescales [16].

On the other hand, if we take $d \rightarrow \infty$ first, the leading contribution in Eq. (A3) is determined by $g = g' = e$ where e is the identity element in the permutation group. This results in $\alpha(g) = \delta_{g,e}$, meaning that the k th moment of the MSPE is always the k -fold tensor product of the identity density matrix for any finite k . This corresponds to an ensemble where a totally mixed state is attributed to all measurement outcomes. The next order expansion in d comes from the swap operator $s_{i,j}$ for arbitrary $i < j$.

We have

$$\alpha(s_{i,j}) = 1/d^m \quad (\text{A7})$$

if $t > 3m/2$, and the number of all the swap operators are $k(k-1)/2$, see Supplemental Material [46].

Appendix B: Local Haar random circuits—We consider the same circuit structure as in Fig. 1, except that each local gate is now drawn Haar randomly. In the limit $d \rightarrow \infty$, $N_{B_1}, N_{B_3} \rightarrow \infty$, and $t > \max(N_A, N_{B_2})$, a typical realization yields an MSPE that is deeply thermalized to the generalized Hilbert-Schmidt ensemble.

The proof of this statement requires an extension of that in Ref. [67]. We show that after measuring subsystems B_1 and B_3 with outcome α , the projected ensemble on the remaining degrees of freedom, $\mathcal{E} = \{P_\alpha, |\psi_\alpha\rangle_{A,B_2}\}$ is equivalent to the Haar-random state ensemble. Tracing out B_2 then yields the generalized Hilbert-Schmidt ensemble on subsystem A following the definition Eq. (2). In the following, we briefly review the argument of Ref. [67] and extend it to show that the Haar-random ensemble emerges once B_1 and B_3 are measured. The difference is that [67] considered a bipartite system, whereas our system is partitioned into four parts.

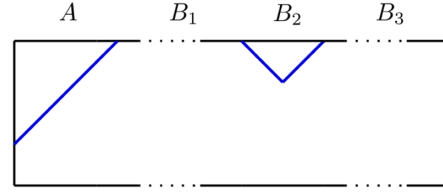


FIG. 4. Membrane-picture illustration of domain-wall evolution (blue lines). The figure shows the trajectory with minimal energy penalty.

In the limit $d \rightarrow \infty$, the circuit dynamics can be described using the membrane picture. Measurements impose distinct boundary conditions at the top of the circuit: one for regions A, B_2 and another for B_1, B_3 . Domain walls form at the interfaces between these regions and evolve backward in time, annihilating either at the boundary or with each other, as shown in Fig. 4. Each domain-wall trajectory incurs an energy penalty d^{-kL} , where L is its length. For small t , the domain walls walk randomly in the circuit. However, for $t > \max(N_A, N_{B_2})$, there exists a unique trajectory which minimizes the energy penalty as shown in Fig. 4. In the $d \rightarrow \infty$ limit, this trajectory dominates the frame potential, leading to the Haar-random state ensemble.