

No Massless Goldstone Bosons in Hamiltonian Time Crystals

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We present a geometric framework for Hamiltonian quantum time crystals, emphasizing the pivotal role of continuous symmetries and their associated conserved charges. In this approach, a time crystal is identified as a minimum-energy ground state that is parallel transported along a family of Fock spaces, with time evolution realized as a symmetry transformation. This parallel transport is implemented via a generalized, time-dependent Bogoliubov transformation, ensuring coherence across the Fock spaces. Revisiting Goldstone's original analysis, we show that in the presence of a time crystal, the Goldstone mode acquires a mass and its direction oscillates periodically in Higgs space. Our formulation circumvents existing no-go theorems that would otherwise forbid quantum time-crystalline order. We place time crystals within the broader context of quantum many-body physics and field theory, drawing analogies to superconductivity and quantum field theory in time-dependent backgrounds.

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A time crystal [1,2] is a phase of matter that exhibits persistent periodic motion even in its lowest energy ground state. Initially proposed for closed Hamiltonian systems, most studies have instead focused on driven nonequilibrium platforms, including Floquet and dissipative systems [3,4]. These realizations display robust subharmonic responses, signaling the breakdown of discrete time-translation symmetry and opening new avenues to study nonequilibrium systems and symmetry-breaking phases of matter [5].

Realizations of classical time crystals have also been presented in a Hamiltonian framework with symmetries [6,7]. These realizations minimize energy while preserving fixed values of the conserved charges, with their mathematical structure based on the Lagrange multiplier theorem [8]. Notable examples include variations [7] of Q ball [9,10], models of closed polymer chains [6], and vortex dynamics in Bose-Einstein condensates [11].

Here we clarify the concept of a quantum Hamiltonian time crystal as an energy-minimizing state whose time evolution is described by symmetry-compatible parallel transport across a family of Fock spaces, realized through time-dependent Bogoliubov transformations between the instantaneous Fock spaces. Analogous generalized

Bogoliubov transformations appear in the theory of superconductivity [12], in many-body systems in connection with the Anderson orthogonality catastrophe [13], and in quantum field theories in time-dependent spacetimes, including the Unruh effect and Hawking radiation [14–16]. By extending the conventional notion of a single Fock space to a family incorporating time-dependent vacua, our framework circumvents no-go theorems [17,18] that prohibit spontaneous time-translation symmetry breaking in a fixed Fock space with a static vacuum.

Our starting point is a phase space parametrized by local coordinates (p_i, q_i) . We introduce a set of general coordinates $\phi^i(p, q)$, with $i = 1, \dots, 2N$, so that the Poisson brackets become $\{\phi^i, \phi^j\} = \omega^{ij}(\phi)$. The Hamiltonian energy function $H(\phi)$ governs the time evolution in terms of Hamilton's equation,

$$\dot{\phi}^i = \{\phi^i, H\} = \omega^{ij} \partial_j H \stackrel{\text{def.}}{=} \mathcal{X}_H^i. \quad (1)$$

If the antisymmetric matrix $\omega^{ij}(\phi)$ is nondegenerate, the energy minimum, as a critical point of $H(\phi)$, must be time independent, precluding the existence of a classical time crystal. However, the presence of conserved charges, which are intimately connected to continuous symmetries and ubiquitous in integrable models, can alter the situation. For this, assume that there are $a = 1, \dots, M < N$ charges $Q_a(\phi)$ that are in involution,

$$\{Q_a, H\} = \{Q_a, Q_b\} = 0. \quad (2)$$

The level sets

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$$C_a(\phi) = Q_a(\phi) - \kappa_a = 0 \quad (3)$$

and fixed constants κ_a define submanifolds of the phase space. In general, the restriction of ω^{ij} on these submanifolds is degenerate, with degeneracies reflecting the underlying symmetry group action [19,20]. Moreover, the minimum of $H(\phi)$ subject to the conditions (3) does not need to be a critical point of $H(\phi)$. Instead, the Lagrange multiplier theorem [8] states that, in that case, the minimum of $H(\phi)$ is a critical point of $H(\phi) - \lambda^a C_a(\phi)$,

$$\partial_i H = \lambda^a \partial_i Q_a \text{ \& } Q_a = \kappa_a, \quad (4)$$

where the λ^a are the Lagrange multipliers. From these $2N + M$ equations we may solve for both the ϕ_{cr}^i and the λ_{cr}^a . If the solution is not unique, the optimal solution is the one with the smallest value of $H(\phi_{cr})$. Whenever any of the λ_{cr}^a is nonzero [7], Hamilton's equation (1), together with (4), imply that the energy minimum evolves in time along a distinct combination of the Hamiltonian vector fields $\mathcal{X}_a$ of the Q_a . Thus, with $\mathcal{L}_a$ denoting the Lie derivative, that is, the derivative along the flow generated by the vector field $\mathcal{X}_a$, and with the initial condition $\phi^i(t=0) = \phi_{cr}^i$ we define a classical Hamiltonian time crystal as a nonstationary solution of

$$\dot{\phi}^i = \omega^{ij} \partial_j H = \lambda_{cr}^a \omega^{ij} \partial_j Q_a = \lambda_{cr}^a \mathcal{X}_a^i = \lambda_{cr}^a \mathcal{L}_a \phi^i. \quad (5)$$

The symmetry group of the time crystal serves as the structure group for this Lie transport, with the solution of (5) evolving along a trajectory generated by the structure group and specified by the initial value ϕ_{cr}^i . The coefficients λ_{cr}^a can be interpreted as the components of the temporal Bravais vector.

Proceeding to quantum time crystals, we recall that the no-go theorem of Watanabe and Oshikawa [18] is formulated within the operator-algebraic framework. Specifically, it addresses quantum systems realized in a single, fixed irreducible Fock representation of the algebra of observables. In such a setting, spontaneous breaking of time-translation symmetry in an equilibrium system is rigorously excluded.

However, our interest lies in situations beyond a single representation. Specifically, we consider families of symmetry-related Fock spaces that arise when the classical phase space supports a continuum of degenerate vacua. In the methodology of [18] this corresponds to a bundle of representations of the C^* algebra of observables, each fiber carrying its own Fock space. While the operator-algebraic approach is effective within a single Fock space, it can obscure geometric and topological structures such as the parallel transport of states along the bundle, e.g., via Bogoliubov transformations, that are essential for understanding systems with multiple vacua. We therefore adopt a geometric approach to quantization [21], as it makes

explicit the interplay between phase space geometry, symmetries, and quantum theory, naturally accommodating quantum time crystals with a bundle structure of Fock spaces.

Following [21], we formulate a quantum version of (5) by replacing classical Lie transport with a symmetry-compatible quantum parallel transport. A putative quantum time crystal is then defined as a time-dependent minimum energy solution to ($\hbar = 1$)

$$\begin{aligned} i\partial_t \psi &= (-i\mathcal{X}_H + \partial_i \mathcal{X}_H^i + H) \psi \stackrel{\text{def.}}{=} (-i\nabla_H + H) \psi \\ &= (-i\lambda_{cr}^a \mathcal{X}_a + \lambda_{cr}^a \partial_i \mathcal{X}_a^i + \lambda_{cr}^a Q_a) \psi \end{aligned} \quad (6)$$

$$\stackrel{\text{def.}}{=} (-i\lambda_{cr}^a \nabla_a + \lambda_{cr}^a Q_a) \psi. \quad (7)$$

Here, the quantum mechanical wave function $\psi(t, \phi)$ is understood as a section of a complex line bundle over phase space, constructed with respect to a chosen *polarization*, that is a set of variables such as positions or momenta in which the wave function depends. The covariant derivatives ∇_H and ∇_a ensure the covariance of time evolution under t -independent phase rotations of $\psi(t, \phi)$, which correspond to potentially polarization altering canonical transformations. The ∂_i are components of a connection one-form $\vartheta = \partial_i d\phi^i$ on the line bundle. It governs the parallel transport of the wave function along phase space trajectories, reflecting the fact that a wave function is only defined up to a phase. The initial wave function ψ_{cr} and the Lagrange multipliers λ_{cr}^a are obtained by solving the appropriate minimization problem; using the λ_{cr}^a values derived from the classical model, we obtain its quantized version. Depending on representation theory of the symmetry group, the κ_a 's in (3) can become subject to quantization conditions.

As a proof of concept for our framework and to demonstrate the existence of quantum time crystals, we analyze the following model, which despite its simplicity encapsulates all the essential features of our general approach:

$$S = \int dt \left[p_i \dot{x}_i - \frac{1}{2} p_i^2 - \frac{1}{2} x_i^2 - \lambda (\epsilon_{ij} x_i p_j - l_z) \right], \quad (8)$$

where $i = 1, 2$ and $Q = \epsilon_{ij} x_i p_j$ is a conserved charge as in (2) and (3). For a fixed constant $\lambda = B$ this is analogous to the classical action of the Landau problem of a charged particle in a constant magnetic field, in the symmetric gauge. However, in the present case, λ is not an externally given quantity; its value must be determined, and the solution to (4) and (5) is the classical Hamiltonian time crystal

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \text{ \& } \begin{cases} \lambda_{cr} = 1 \\ v^2 + w^2 = l_z. \end{cases} \quad (9)$$

From classical to quantum, in the position representation of polarization the wave function is a complex valued function of (x_1, x_2) , and the Eqs. (6) and (7) corresponding to (8) emerge as variational equations of the Schrödinger action

$$S = \int dt d^2x \left\{ \bar{\psi} i \partial_t \psi - \frac{1}{2} \bar{\psi} (-\nabla^2 + |x|^2) \psi \right\} \quad (10)$$

$$- \int dt E \left\{ \int d^2x \bar{\psi} \psi - 1 \right\} \quad (11)$$

$$- \int dt \lambda \left\{ \int d^2x \bar{\psi} (-i \epsilon_{ij} x_i \partial_j) \psi - l_z \right\}. \quad (12)$$

The first integral (10) has a Hamiltonian interpretation with ψ and $\bar{\psi}$ a canonical pair with Poisson bracket

$$\{\bar{\psi}(x), \psi(y)\} = i \delta(x - y). \quad (13)$$

The conserved charge in (11) generates phase rotations of the wavefunction. The normalization of wave function, imposed by the Lagrange multiplier E as a condition (3) on this charge, has no counterpart at the classical level, revealing its distinct quantum origin. The third integral (12) is the quantum representation of the classical conserved charge Q of (8). Variation of the Lagrange multiplier λ imposes the corresponding version of (3) as a condition on the wave function.

We start with (10) and (11). Combining the right-hand side of (6) and (7) we get the time independent two-dimensional Schrödinger equation

$$\frac{1}{2} (-\nabla^2 + |x|^2) \psi = E \psi \text{ \& \; } \int d^2x \bar{\psi} \psi = 1 \quad (14)$$

that we recognize as an example of (4). A textbook result, the harmonic oscillator admits multiple solutions, and the optimal solution is a Gaussian wave function.

The Eq. (6) is the time dependent Schrödinger equation

$$i \partial_t \psi = \frac{1}{2} (-\nabla^2 + |x|^2) \psi = E \psi \Rightarrow \psi(x, t) = \psi_E(x) e^{-i t E}. \quad (15)$$

Technically this exemplifies a quantum time crystal with the wave function normalization as a condition on conserved charge, transforming time-translation invariance into equivariance via the E -dependent phase factor. Since the phase factor has no effect on measurable properties such as the probability density (modulus of wave function) the equivariance is generally ignored, except in cases like holonomy (Berry's phase), energy level transitions, and quantum coherence.

When we include (12) and set $l_z = m$ an integer, $\lambda_{cr} = 0$ and the Eqs. (6) and (7) become the time dependent

Schrödinger equation for the level- m harmonic oscillator. The moduli of the wave functions are rotation invariant and no observable time-crystalline propagation occurs.

When l_z is not an integer the situation changes: the two conditions in (11) and (12) cannot be simultaneously satisfied by any single-valued wave function whose modulus depends solely on the radial coordinate $|\mathbf{x}| = r$. Consequently, the modulus of the energy minimizer $\psi_{cr}(x)$ must depend on both r and the polar angle φ , the multiplier λ_{cr} can not be zero, and from (6) and (7) the time evolution of the energy minimizer is determined by

$$i \partial_t \psi = (E_{cr} + i \lambda_{cr} \epsilon_{ij} x_i \partial_j) \psi \equiv \lambda_{cr} \mathbf{x} \wedge \left(i \nabla + \frac{\Phi}{2\pi} \boldsymbol{\vartheta} \right) \psi$$

$$\psi(x, t=0) = \psi_{cr}(x) \& \; \frac{\Phi}{2\pi} \boldsymbol{\vartheta} = \frac{E_{cr}}{\lambda_{cr}} \nabla \tan^{-1} \left[\frac{x_2}{x_1} \right] \quad (16)$$

with $\boldsymbol{\vartheta}$ a connection on the line bundle and $\Phi = 2\pi E_{cr}/\lambda_{cr}$. The solution describes a genuine, observable time crystal in terms of a rotational motion of the probability density around the origin [22].

As in the case of the harmonic oscillator there are also nonoptimal solutions, each of which evolves according to its corresponding Eq. (16). The Eq. (16) determines an equivariant parallel transport of the wave function, with $\boldsymbol{\vartheta}$ in the role of a Berry connection.

Remarkably, we recognize in (16) the structure of a charge-flux composite, familiar from the theory of Abelian anyons [23] with λ_{cr} as the precession frequency of the wave function's cyclotron motion around the origin, Φ as the statistical parameter, and $\boldsymbol{\vartheta}$ in the role of a Chern-Simons gauge field with curvature localized at the origin. We refer to [11] for numerical solutions in the case of a nonlinear Schrödinger equation.

Since (7) and (16) is a time dependent symmetry transformation of the wave function, we expect Goldstone's theorem [24,25] to have consequences. To investigate this we extend (8) to a field theory, repeat the original argument in [24], and then adapt it to the case of a time crystal: Accordingly, our starting point is the $(D+1)$ dimensional classical action of two real scalar fields with a spontaneously broken potential ($v \neq 0$ & $i = 1, 2$),

$$S = \int \left\{ \pi_i \dot{\phi}_i - \frac{1}{2} \pi_i^2 - \frac{1}{2} |\nabla \phi_i|^2 - \frac{g}{4} (\phi_i^2 - v^2)^2 \right\} \quad (17)$$

The Poisson brackets are $\{\pi_i(x), \phi_j(y)\} = -\delta_{ij}(x - y)$ and, in analogy with (8) we have the conserved charge

$$Q = \int (\pi_1 \phi_2 - \pi_2 \phi_1). \quad (18)$$

By combining Hamilton's equations for (ϕ_i, π_i) we arrive at the relativistic nonlinear equation of motion

$$(\partial_t^2 - \nabla^2)\phi_i = -g(\phi_i^2 - v^2)\phi_i. \quad (19)$$

In [24] the following ansatz of (19) was investigated:

$$\begin{aligned} \phi_1(x) &= v + \sigma_1(x) \\ \phi_2(x) &= \sigma_2(x), \end{aligned} \quad (20)$$

where $\sigma_{1,2}$ are small perturbations around minimum energy. To leading order in powers of σ_i the equation of motion for σ_1 reveals that it has the mass $m^2 = 2gv^2$. Similarly, to leading order the equation for σ_2 shows that it is the massless Goldstone boson.

The choice (20) spontaneously breaks the $SO(2)$ symmetry of the vacuum, and different vacua are related to each other by

$$\begin{bmatrix} v \\ 0 \end{bmatrix} \rightarrow \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix} \xrightarrow{\alpha=\frac{\pi}{2}} \begin{bmatrix} 0 \\ v \end{bmatrix}. \quad (21)$$

Independently of the rotation angle, there is always one perturbation eigenmode with mass-squared $2gv^2$ and one that is massless. Notably for $\alpha = \pi/2$ the original $\alpha = 0$ perturbations change their role, with the one in the ϕ_2 direction now acquiring the mass $2gv^2$ and the one in ϕ_1 direction becoming the massless Goldstone boson.

From classical to quantum, we adopt a second-quantized framework and focus on a single real-valued second-quantized field; this could describe either ϕ_2 or ϕ_1 ;

$$\sigma(x, t) = \int \frac{d^D k}{(2\pi)^{\frac{D}{2}} \sqrt{2\omega_k}} (\mathbf{a}_k e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{x})} + \mathbf{a}_k^\dagger e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{x})}). \quad (22)$$

Here $\mathbf{a}_k^\dagger, \mathbf{a}_k$ are the creation and annihilation operators in the Fock space and $\omega_k = \sqrt{k^2 + m^2}$. We introduce a Bogoliubov transformation which changes the mass in (22) from m to M [14],

$$\begin{aligned} \mathbf{b}_k &= \mathcal{U} \mathbf{a}_k \mathcal{U}^\dagger = \cosh \theta_k \mathbf{a}_k + \sinh \theta_k \mathbf{a}_{-k}^\dagger \\ \mathbf{b}_k^\dagger &= \mathcal{U} \mathbf{a}_k^\dagger \mathcal{U}^\dagger = \cosh \theta_k \mathbf{a}_k^\dagger + \sinh \theta_k \mathbf{a}_{-k}, \end{aligned} \quad (23)$$

where $\mathcal{U}$ is the squeeze operator

$$\begin{aligned} \mathcal{U} &= \exp \left[\sum_k \theta_k (\mathbf{a}_k \mathbf{a}_{-k} - \mathbf{a}_k^\dagger \mathbf{a}_{-k}^\dagger) \right] \\ \cosh \theta_k &= \frac{1}{2} \left(\sqrt{\frac{\omega_k}{\Omega_k}} + \sqrt{\frac{\Omega_k}{\omega_k}} \right) \& \Omega_k = \sqrt{k^2 + M^2}. \end{aligned} \quad (24)$$

In the case of ϕ_2 , from $m = 0$ to $M^2 = 2gv^2$, this mimics a rotation from $\alpha = 0$ to $\alpha = \pi/2$ in (21) demonstrating that different Goldstone boson directions in Higgs space can be related by Bogoliubov transformations. Note that in a thermodynamic limit the overlap between the $\alpha = 0$ and

$\alpha = \pi/2$ vacua goes to zero exponentially in the spatial volume V , in full analogy to the overlap between the Minkowski and the Rindler vacua [15,16]. Accordingly we do not expect the model (17) to support a finite-energy time crystal when $V \rightarrow \infty$.

However, based on [7] we anticipate that the extensions proposed in [9,10] can support finite energy time crystals. Keeping this in mind, we proceed to repeat Goldstone's mean field analysis in the case of (17), in a scenario relevant to time crystals. For this we search for the energy minimum of (17) subject to the condition

$$Q = \int_V (\pi_1 \phi_2 - \pi_2 \phi_1) = \kappa \neq 0. \quad (25)$$

The remaining variational equations in (4) are

$$\delta_{\pi_1}: \pi_1 = \lambda \phi_2 \quad (26)$$

$$\delta \pi_2: \pi_2 = -\lambda \phi_1 \quad (27)$$

$$\delta_{\phi_1}: -\nabla^2 \phi_1 + g(\phi_1^2 + \phi_2^2 - v^2)\phi_1 = -\lambda \pi_2 = -\lambda \dot{\phi}_2 \quad (28)$$

$$\delta_{\phi_2}: -\nabla^2 \phi_2 + g(\phi_1^2 + \phi_2^2 - v^2)\phi_2 = \lambda \pi_1 = \lambda \dot{\phi}_1. \quad (29)$$

Motivated by (9) and by (20) and (21) we start with the following ansatz for the ground state"

$$\begin{pmatrix} \phi_1(x, t) \\ \phi_2(x, t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \begin{pmatrix} \rho \\ 0 \end{pmatrix}. \quad (30)$$

We substitute this in Eqs. (26)–(29). To leading order we get $\alpha^2 = g(\rho^2 - v^2)$ and from (25) we get $\alpha \rho^2 V = \kappa$ so that the ground state is time dependent, reflecting a Hamiltonian time crystal. Next, following (20) we introduce a (small) perturbation around the ground state:

$$\begin{pmatrix} \phi_1(x, t) \\ \phi_2(x, t) \end{pmatrix} = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \begin{pmatrix} \rho \\ 0 \end{pmatrix} + \begin{pmatrix} \sigma_1(x, t) \\ \sigma_2(x, t) \end{pmatrix}. \quad (31)$$

We substitute this into (26)–(29) and obtain the following equation for the small perturbations $\sigma_{1,2}(x, t)$:

$$\begin{aligned} (\partial_t^2 - \nabla^2) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} &= -U_\alpha(t) \begin{pmatrix} \alpha^2 + 2g\rho^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} U_\alpha^{-1}(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \\ &+ \dots, \\ \text{where } U_\alpha(t) &= \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix}. \end{aligned} \quad (32)$$

The Goldstone boson has acquired a mass equal to α and when $\rho \rightarrow v$ ($\kappa \rightarrow 0$) we recover (20). Moreover, the direction of the Goldstone boson in the Higgs space varies

periodically with time. Alternatively, we may remove U_α from the mass term by applying a U_α rotation to the Higgs doublet that sends (32) into

$$(\partial_\mu + U_\alpha^{-1} \partial_\mu U_\alpha)^2 \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = - \begin{pmatrix} \alpha^2 + 2g\rho^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}.$$

Now the σ_2 component of the scalar field remains the Goldstone boson at all times, while a (locally) flat $\text{SO}(2)$ connection akin that in (16) has also emerged. Since we have closely followed [24], we expect that our conclusions will be equally broad in their applicability.

We note that previous studies [26–28] have investigated quantum systems in which time evolution is itself a symmetry transformation. These works showed that, at finite chemical potential, Goldstone modes with symmetry-determined nonzero masses can arise in both field-theoretic and many-body contexts, which is analogous to the present result. While these studies did not directly address time crystals as ground states, they did reveal quasi-time-crystalline dynamics reminiscent of the behavior discussed here.

We arrive at the following framework: Let $\{\mathcal{F}_t\}_{t \in \mathbb{R}/T\mathbb{Z}}$ be a T -periodic bundle with the time circle $\mathbb{R}/T\mathbb{Z} \sim \mathbb{S}^1$ as the base space and the instantaneous Fock spaces $\mathcal{F}_t$ at time t as the fibers. The continuous time evolution of a T -periodic quantum state $|\Psi(x, t)\rangle$ viewed as a section of the bundle takes place in a total space

$$\mathcal{E} = \int_{t \in \mathbb{R}/T\mathbb{Z}}^{\oplus} \mathcal{F}_t d\mu(t)^{\#} \mathbb{R}/T\mathbb{Z} \sim \mathbb{S}^1$$

This is a universal Hilbert space that accommodates representations of the instantaneous Fock spaces. To compare two path-connected Fock spaces $\mathcal{F}_t$ and $\mathcal{F}_{t'}$ as fibers in $\mathcal{E}$, we introduce a connection that governs parallel transport in $\mathcal{E}$. This connection is analogous to θ in (6) and (7), as exemplified in (16); connections on Hilbert space bundles have been extensively studied in the mathematical literature; see, for instance, [29–31]. Furthermore, the transitions between the instantaneous Fock spaces $\mathcal{F}_t$ are governed by time-dependent generalized Bogoliubov transformations, which act as transition functions [32]. Together, the connection and the Bogoliubov transformations establish a gauge framework that ensures a consistent quantum time evolution across different Fock spaces in $\mathcal{E}$.

The equivariant quantum parallel transport of the state $|\Psi(x, t)\rangle$ is governed by the quantum equivariant parallel transport equation

$$i\partial_t |\Psi(x, t)\rangle = \lambda_{cr}^a \mathbf{Q}_a |\Psi(x, t)\rangle, \quad (33)$$

where $\lambda_{cr}^a \mathbf{Q}_a$ is the quantum momentum map associated with the combination $\lambda_{cr}^a Q_a$ of classical conserved charges. It not only incorporates the connection in $\mathcal{E}$ but also

governs the action of these charges as described in (6) and (7), and exemplified by (16). A solution of (33) traces an orbit in $\mathcal{E}$, describing the transport of a quantum state in terms of a time-dependent generalized Bogoliubov transformation. Accordingly we mathematically define a quantum time crystal with Hamiltonian $\mathbf{H}$ as the orbit of $\lambda_{cr}^a \mathbf{Q}_a$ that is represented by the energy minimizer of the corresponding optimization problem (here we keep $\mathcal{F}_t$ fixed),

$$\min_{|\Psi\rangle \in \mathcal{F}_t} \langle \Psi | \mathbf{H} | \Psi \rangle \quad \text{subject to} \quad \langle \Psi | \mathbf{Q}_a | \Psi \rangle = q_a, \quad (34)$$

which we solve using the Lagrange multiplier theorem.

Finally, we set aside the issue of path dependence, noting only that in (16), the connection's curvature is supported solely at the origin. In this case, the connection, flat on $\mathbb{R}^2 \setminus \{0\}$, induces a monodromy that leads to the anyonic statistics of the time crystal.

In summary, we developed a geometric quantization framework for Hamiltonian quantum time crystals, emphasizing the role of symmetries in their formation. Our setup, being more general than the assumptions of no-go theorems [17,18], which preclude quantum time-crystalline behavior in a trivial Fock bundle $\mathbb{R}/T\mathbb{Z} \times \mathcal{F}$, shows that when the bundle is nontrivial, the conclusions can differ. By interpreting second-quantized time-crystal evolution as equivariant parallel transport on a bundle of Fock spaces, we linked it to Bogoliubov transformations, revealing parallels with superconductivity and quantum fields in time-dependent spacetimes. Extending Goldstone's theorem to time-dependent spontaneous symmetry breaking, we find that the Goldstone boson acquires a mass, while its direction in Higgs space exhibits periodic oscillations. For clarity, our examples mimic Goldstone's original argument, but more physically realistic cases can be introduced [6,7,9–11]. We trust that our results will broaden the scope for quantum time crystals in condensed matter and quantum field theory.

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