

Superradiant Phase Is a Finite Size Effect in Two-photon Processes

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Two-photon light-matter interactions exhibit distinctive features such as spectral collapse. The two-photon Dicke model has been reported to exhibit a superradiant phase which could be useful in quantum applications. Here we show that this superradiant phase is not a genuine thermodynamic phase but a finite-size effect. Combining analytical and numerical analyses, we demonstrate that the superradiant region shrinks with increasing system size and disappears in the thermodynamic limit, while spectral collapse remains. Our results clarify the nature of superradiant conditions in two-photon systems and constrain its realization in quantum platforms.

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Understanding the interaction between light and matter is a central problem in quantum optics [1] and underpins a wide range of quantum technologies, including quantum metrology [2–6], simulation [7,8], and communication [9–12]. Cavity and circuit quantum electrodynamics provide controlled settings in which collective atom-field effects emerge [9,13], often leading to enhanced coherence and cooperative phenomena that have no single-particle counterpart.

Among the paradigmatic models describing collective light-matter interactions is the Dicke model [14–18], which captures the coupling between an ensemble of two-level systems (qubits) and a single bosonic mode. The Dicke model exhibits a well-established superradiant phase transition in the thermodynamic limit [19–23], characterized by macroscopic occupation of the field mode and collective atomic excitation. This transition has been extensively studied both theoretically [24–35] and experimentally [36–40] and forms a cornerstone of the understanding of collective quantum behavior.

Recent interest has shifted toward multiphoton light-matter interactions [41,42], motivated by advances in experimental platforms such as trapped ions, superconducting circuits, Rydberg atoms, quantum dots, and quantum batteries [43–54]. In particular, two-photon processes,

where atomic excitations are accompanied by the simultaneous creation or annihilation of two bosonic excitations, lead to qualitatively new physics [41,42,55–59]. The two-photon Rabi and Dicke models display rich spectral properties, including spectral collapse [60–63], in which discrete energy levels merge into a continuous band at a critical coupling strength. This phenomenon has no analogue in single-photon models.

The two-photon Dicke model [17,60,63–68] has attracted special attention because it has been reported to exhibit a superradiant phase [69,70] based on analytical descriptions using Holstein-Primakoff and squeezed-state mean-field approaches. This superradiant phase has been found even in the presence of dissipation [71,72]. It has motivated proposals for exploiting two-photon interactions in quantum technologies expected to operate under superradiant conditions [53,54,73]. However, the nature of this superradiant phase and its stability in the thermodynamic limit have remained insufficiently scrutinized.

In this Letter, we show that the superradiant phase of the two-photon Dicke model is not a genuine thermodynamic phase but rather a finite-size effect. By combining analytical mean-field results with numerical simulations, we demonstrate that the region associated with superradiant behavior systematically shrinks as the system size increases and vanishes entirely in the thermodynamic limit. Employing a mean-field description based on squeezed vacuum states, we show that the solutions of the superradiant phase are unphysical in the thermodynamic limit.

Our results clarify the relationship between superradiance and spectral collapse in two-photon light-matter systems. We find that spectral collapse survives the thermodynamic limit and admits a clear interpretation in bosonic phase space, whereas superradiance does not.

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This distinction resolves an apparent inconsistency in previous analyses and imposes fundamental constraints on the realization of superradiant behavior in two-photon platforms.

Two-photon Dicke model—The two-photon Dicke model describes the collective interaction between $\mathcal{N}$ two-level systems and a single-mode bosonic field of frequency ω , where atomic excitations are accompanied by the simultaneous creation or annihilation of two bosonic quanta. Setting $\hbar = 1$, the Hamiltonian reads as

$$\hat{H}_D = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{\gamma}{\mathcal{N}} (\hat{a}^{\dagger 2} + \hat{a}^2) (\hat{J}_+ + \hat{J}_-), \quad (1)$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators satisfying $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, respectively. The collective atomic operators $\hat{J}_z$ and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ obey the standard angular-momentum commutation relations $[\hat{J}_z, \hat{J}_\pm] = \pm \hat{J}_\pm$ and $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$ and are defined as $\hat{J}_{x,y,z} = (1/2) \times \sum_{k=1}^{\mathcal{N}} \hat{\sigma}_{x,y,z}^k$. In the above Hamiltonian, ω_0 denotes the atomic transition frequency, and γ is the atom-field coupling strength. The coupling is bounded by the critical value $\gamma_{sc} = \omega/2$, at which spectral collapse occurs [60–62].

The Hamiltonian $\hat{H}_D$ possesses a discrete parity symmetry, $[\hat{H}_D, \hat{\Pi}] = 0$, where the parity operator is defined as $\hat{\Pi} = \exp(i\pi\hat{\Lambda})$, with $\hat{\Lambda} = \hat{a}^\dagger \hat{a}/2 + \hat{J}_z + \hat{J}_\pm$. This symmetry partitions the Hilbert space into four invariant subspaces labeled by the eigenvalues $p = \pm 1, \pm i$ of $\hat{\Pi}$, corresponding to the simultaneous transformations $\hat{a} \rightarrow i\hat{a}$ and $\hat{J}_x \rightarrow -\hat{J}_x$.

In addition, the total collective spin $\hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ is conserved, $[\hat{H}_D, \hat{\mathbf{J}}^2] = 0$, allowing the Hilbert space to be decomposed into sectors characterized by the eigenvalue $j(j+1)$. Throughout this Letter, we focus on the fully symmetric subspace with $j = \mathcal{N}/2$, which governs the collective behavior of the model and is relevant for the thermodynamic limit.

Ground state, classical limit, and superradiant phase—The ground state of the two-photon Dicke model can be obtained at the mean-field level by applying a Holstein-Primakoff transformation to the collective spin operators, followed by a Bogoliubov transformation of the bosonic mode [69]. This procedure yields a squeezed-vacuum ground state of the form $|E_0\rangle = |r_b\rangle = e^{(r_b^* \hat{a}^2 - r_b \hat{a}^{\dagger 2})/2} |0\rangle$, where $r_b \in \mathbb{R}$ is a real squeezing parameter and $|0\rangle$ denotes the bosonic vacuum. Minimization of the ground-state energy E_0 identifies two regimes separated by the critical coupling $\gamma_c = \sqrt{\omega\omega_0 j/2}$. For $\gamma < \gamma_c$ the system is in a normal phase (NP), while for $\gamma > \gamma_c$ a superradiant phase (SP) emerges. Details of this derivation are given in Ref. [69].

The previous mean-field description does not allow for analyzing the scaling of observables with the number of atoms, nor does it yield the thermodynamic limit. Thus, we introduce a similar mean-field approximation based on the

tensor product $|\mathbf{x}\rangle \equiv |\xi\rangle \otimes |\beta\rangle$, where $|\xi\rangle = e^{(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})/2} |0\rangle$ is a squeezed vacuum state and $|\beta\rangle = (1 + |\beta|^2)^{-j} e^{\beta \hat{J}_+} |j, -j\rangle$ is a Bloch coherent state for the collective spin. The squeezing parameter is a complex number $\xi = r e^{i\theta} \in \mathbb{C}$ with magnitude r and phase θ . We introduce the canonical variables $\mathbf{x} = (q, p; Q, P)$ describing the bosonic and atomic degrees of freedom. The atomic variables (Q, P) are related to the Bloch parameter $\beta = (Q + iP)/\sqrt{4 - Q^2 - P^2}$, while the bosonic variables (q, p) are

$$(q, p) = \sqrt{\frac{2}{j}} \sinh(r) [\cos(\theta), \sin(\theta)]. \quad (2)$$

The scaling factor $\sqrt{2/j}$ in Eq. (2) allows us to define a classical Hamiltonian,

$$\begin{aligned} h_D(\mathbf{x}) &= \frac{1}{j} \langle \mathbf{x} | \hat{H}_D | \mathbf{x} \rangle \\ &= \Omega(\mathbf{x}) - 2\gamma q Q \sqrt{\left(\frac{1}{2j} + \frac{q^2 + p^2}{4}\right) \left(1 - \frac{Q^2 + P^2}{4}\right)}, \end{aligned} \quad (3)$$

with

$$\Omega(\mathbf{x}) = \frac{\omega}{2} (q^2 + p^2) + \frac{\omega_0}{2} (Q^2 + P^2) - \omega_0, \quad (4)$$

where the energy is scaled with the number of atoms $j = \mathcal{N}/2$, namely $h_D(\mathbf{x}) = E/j$, and its thermodynamic limit provides an intensive classical energy.

The critical points from Hamilton's equations of motion of the Hamiltonian $h_D(\mathbf{x})$ are $\mathbf{x}_m$ [74]. One of these extrema $\mathbf{x}_0 \in \mathbb{R}$ is related to the normal phase, while the remaining extrema $\mathbf{x}_\pm \in \mathbb{R}$ are related to the superradiant phase. Using these extrema, we can obtain analytical expressions [74] for the ground-state energy,

$$\begin{aligned} E_0 &= \langle \mathbf{x}_m | \hat{H}_D | \mathbf{x}_m \rangle \\ &= - \begin{cases} j|\omega_0| & \text{for (NP)} \\ \frac{i\omega}{2} \left(\frac{1}{j} - \sqrt{\left(1 - \frac{4\gamma^2}{\omega^2}\right) \left(\frac{1}{j^2} - \frac{\omega_0^2}{\gamma^2}\right)} \right) & \text{for (SP)} \end{cases}, \end{aligned} \quad (5)$$

the number of photons

$$\langle \mathbf{x}_m | \hat{a}^\dagger \hat{a} | \mathbf{x}_m \rangle = \begin{cases} 0 & \text{for (NP)} \\ \frac{j}{2} \left(\sqrt{\frac{\frac{\omega_0^2}{j^2} - \frac{1}{j}}{1 - \frac{4\gamma^2}{\omega^2}}} - \frac{1}{j} \right) & \text{for (SP)} \end{cases}, \quad (6)$$

and the atomic inversion

$$\frac{\langle \mathbf{x}_m | \hat{J}_z | \mathbf{x}_m \rangle}{j} = - \begin{cases} \text{sgn}(\omega_0) & \text{for (NP)} \\ \frac{\omega_0 \omega}{2\gamma^2} \sqrt{\frac{1 - \frac{4\gamma^2}{\omega^2}}{\frac{\omega^2}{j^2} - \frac{\omega_0^2}{\gamma^2}}} & \text{for (SP)} \end{cases} \quad (7)$$

Thermodynamic limit and disappearance of the superradiant phase—The thermodynamic limit of the two-photon Dicke model is defined by $\mathcal{N} \rightarrow \infty$ or, equivalently, when $j = \mathcal{N}/2 \rightarrow \infty$. In this limit, collective atom-field systems are expected to display macroscopic behavior and well-defined phase transitions. The analytical expressions derived in Eqs. (5)–(7) allow us to track explicitly how the ground-state properties evolve with increasing system size.

Figures 1(a1)–1(a3) present phase diagrams of the normalized atomic excitation $1 + \langle \hat{J}_z \rangle / j$ [Eq. (7)] as functions of the scaled atomic frequency ω_0/ω and the scaled coupling strength γ/ω for increasing values of j . For finite system sizes, two distinct regions corresponding to the normal and superradiant phases are separated by the critical coupling $\gamma_c = \sqrt{\omega\omega_0 j/2}$. However, as the system size increases, the superradiant region shrinks progressively. In the limit $j \rightarrow \infty$, this region collapses entirely, leaving only the normal phase. The numerical results in Figs. 1(b1)–1(b3), obtained with the ground state from the diagonalization of $\hat{H}_D$ [Eq. (1)], are in complete agreement with the analytical predictions. We present similar phase diagrams for the number of photons $\langle \hat{a}^\dagger \hat{a} \rangle$ [Eq. (6)] and the

ground-state energy E_0 [Eq. (5)] in Supplemental Material [74].

The previous behavior is further illustrated in Figs. 2(a1)–2(c1), which show projections of the ground-state energy, photon number, and atomic inversion as functions of ω_0/ω at fixed coupling $\gamma/\omega < 1/2$, respectively. For finite j , these observables display the smooth crossover associated with the analytically defined superradiant phase. As j increases, the crossover region narrows and converges to a singular structure in the thermodynamic limit. At $\omega_0 = 0$, the ground-state energy develops a finite discontinuity, the photon number approaches a deltalike peak, and the atomic inversion becomes a step function. These limiting behaviors are incompatible with the existence of a stable superradiant phase. The numerical results in Figs. 2(a2)–2(c2) again show full agreement with the analytical results.

The disappearance of the superradiant phase can be understood directly from the analytical expressions. In Eqs. (5)–(7), the critical points associated with the superradiant phase require $j^2 \omega_0^2 < \gamma^2$, a condition that cannot be satisfied as $j \rightarrow \infty$ at fixed γ and ω_0 . Consequently, the corresponding mean-field extrema $\mathbf{x}_\pm$ become complex and unphysical in the thermodynamic limit, signaling the breakdown of the superradiant-phase solutions. This analysis can also be confirmed by taking the thermodynamic limit in Eqs. (5)–(7), leading to complex expressions for the three scaled observables [74].

From the previous evidence, we conclude that the superradiant phase does not survive the thermodynamic limit in the two-photon Dicke model. The phase identified

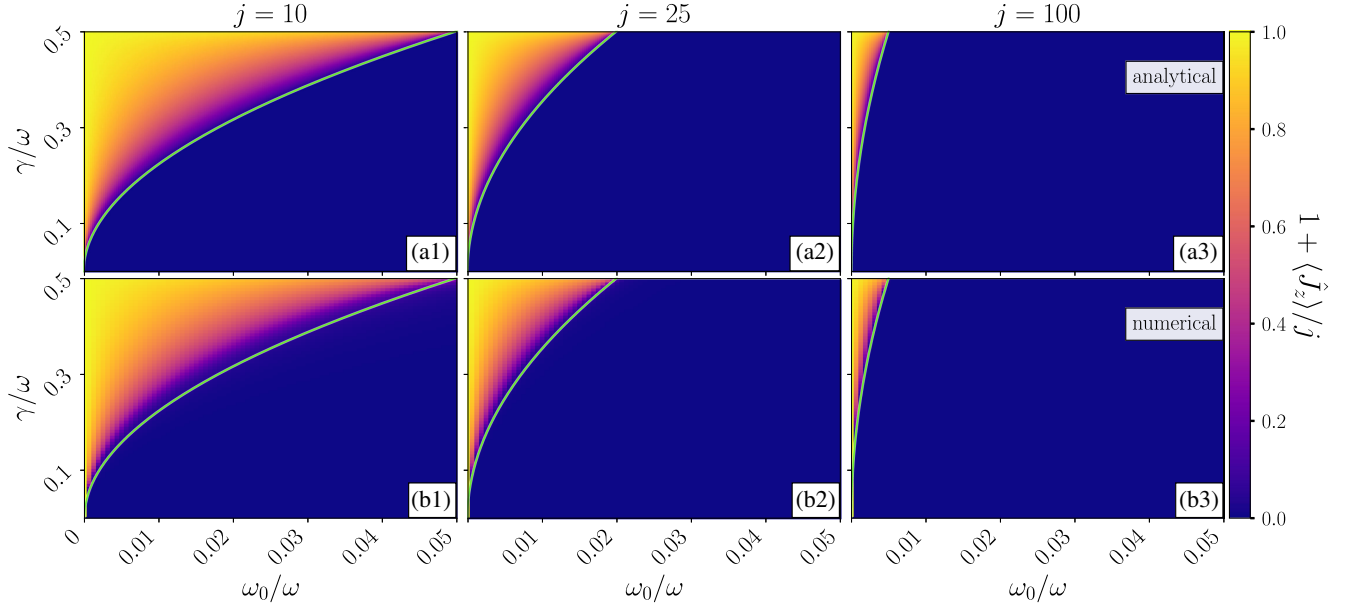


FIG. 1. (a1)–(a3) Phase diagrams of the normalized atomic excitation $1 + \langle \hat{J}_z \rangle / j$ [Eq. (7)] as a function of the scaled atomic frequency ω_0/ω and the scaled coupling strength γ/ω . The color scale represents the numerical value of the normalized atomic excitation. The green solid line represents the scaled critical coupling $\gamma_c/\omega = \sqrt{\omega_0 j / (2\omega)}$. (b1)–(b3) Numerical implementations of panels (a1), (a3). Each column identifies a different system size: (a1), (b1) $j = 10$, (a2), (b2) $j = 25$, and (a3), (b3) $j = 100$.

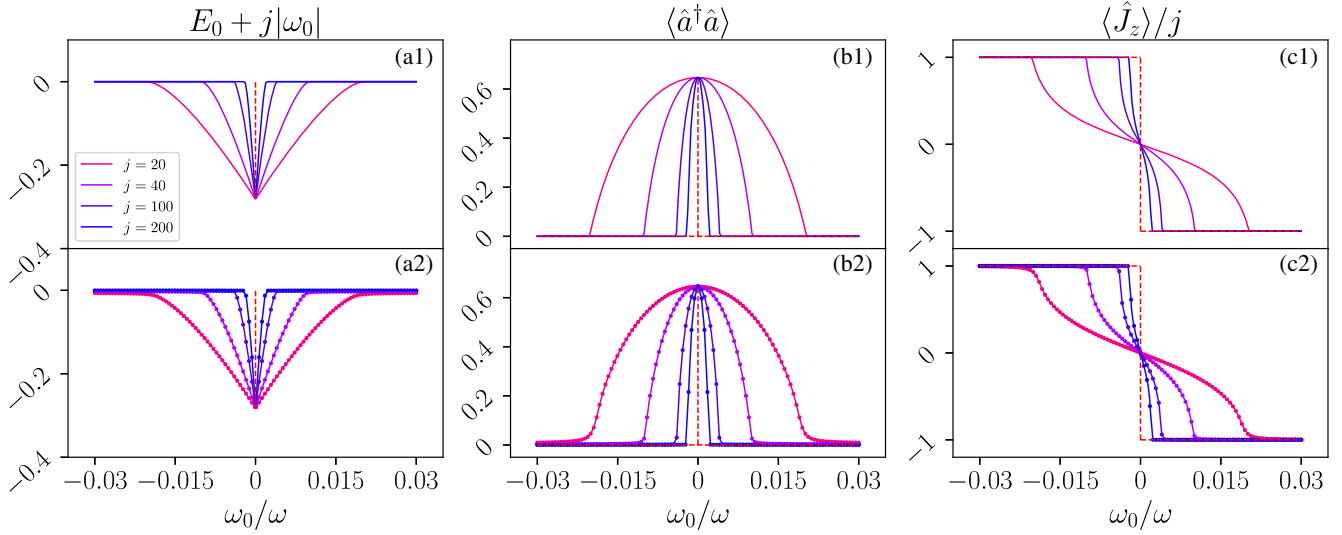


FIG. 2. Projections of (a1) displaced ground-state energy $E_0 + j|\omega_0|$ [Eq. (5)], (b1) number of photons $\langle \hat{a}^\dagger \hat{a} \rangle$ [Eq. (6)], and (c1) atomic operator $\langle \hat{J}_z \rangle / j$ [Eq. (7)], as a function of the scaled atomic frequency ω_0 / ω . Each color identifies a different system size $j = 20, 40, 100, 200$. The red dashed line represents the thermodynamic limit $j \rightarrow \infty$. (a2)–(c2) Numerical implementations of panels (a1)–(c1). The coupling strength is $\gamma / \omega = 0.45$.

at finite system sizes represents a finite-size crossover rather than a genuine quantum phase transition.

Spectral collapse in bosonic phase space—Despite the fact that the superradiant phase does not survive in the thermodynamic limit of the two-photon Dicke model, the classical signature of the spectral collapse can be identified in the same limit. In the limit $j \rightarrow \infty$, the term $1/(2j)$ disappears in Eq. (3) and a new Hamiltonian, which is independent of the number of atoms $j = \mathcal{N}/2$, can be defined as

$$h_D^\infty(\mathbf{x}) = \lim_{j \rightarrow \infty} h_D(\mathbf{x}) = \Omega(\mathbf{x}) - \gamma q Q \sqrt{(q^2 + p^2) \left(1 - \frac{Q^2 + P^2}{4}\right)}, \quad (8)$$

where $\Omega(\mathbf{x})$ is given in Eq. (4).

The critical points from Hamilton's equations of motion of the Hamiltonian $h_D^\infty(\mathbf{x})$ are $\mathbf{x}_0 \in \mathbb{R}$ (normal phase) and $\mathbf{x}_\pm^\infty \in \mathbb{C}$ (superradiant phase) [74], where the only real solution $\mathbf{x}_0$ reproduces the ground-state energy of the normal phase, $h_D^\infty(\mathbf{x}_0) = E_0/j = -\omega_0$. The previous critical points can also be obtained by taking the thermodynamic limit of the critical points that come from the Hamiltonian $h_D(\mathbf{x})$ [Eq. (3)], namely, $\lim_{j \rightarrow \infty} \mathbf{x}_0 = \mathbf{x}_0 \in \mathbb{R}$ and $\lim_{j \rightarrow \infty} \mathbf{x}_\pm = \mathbf{x}_\pm^\infty \in \mathbb{C}$.

The Hamiltonian $h_D^\infty(\mathbf{x})$ allows us to visualize the spectral collapse in the bosonic phase space of the two-photon Dicke model. Eliminating the atomic variables by solving $\partial_Q h_D^\infty(\mathbf{x}) = \partial_P h_D^\infty(\mathbf{x}) = 0$, we obtain the solutions $Q_m = Q_m(q, p)$ and $P_m = 0$, leading to an effective energy surface projected onto the bosonic variables

$$h_D^\infty(q, p) = h_D^\infty(q, p; Q_m, 0) = \frac{\omega}{2} (q^2 + p^2) - \omega_0 \sqrt{1 + \frac{\gamma^2}{\omega_0^2} q^2 (q^2 + p^2)}. \quad (9)$$

Figures 3(a1)–3(a4) show the resulting energy surface for increasing values of the scaled coupling strength γ/ω at fixed resonance $\omega_0 = 2\omega$. In Figs. 3(b1)–3(b4), we show the orthogonal projection of the energy surfaces over the bosonic plane $q - p$. For weak coupling, the surface is bounded and displays a single global minimum, characteristic of the normal phase. As γ/ω approaches the collapse value $\gamma_{sc}/\omega = 1/2$, the surface progressively flattens and eventually becomes unbounded, developing extended regions at the ground-state energy across the p axis [Figs. 3(a4) and 3(b4)]. This transition from a bounded to an unbounded energy landscape provides a clear classical signature of spectral collapse in bosonic phase space.

A complementary description using a mean-field description with coherent states is presented in Supplemental Material [74]. In this case the ground state is always in the normal phase. There are, however, two relevant aspects which are missed in the coherent state description: the presence of the squeezed superradiant phase for finite number of atoms and the squeezed ground state in the limit $\omega_0 \rightarrow 0$.

Conclusion—We demonstrated that the superradiant phase of the two-photon Dicke model is not a genuine thermodynamic phase but a finite-size effect. Although a superradiant region emerges at finite system size, it collapses systematically as the number of atoms increases and disappears in the thermodynamic limit. In this limit, the

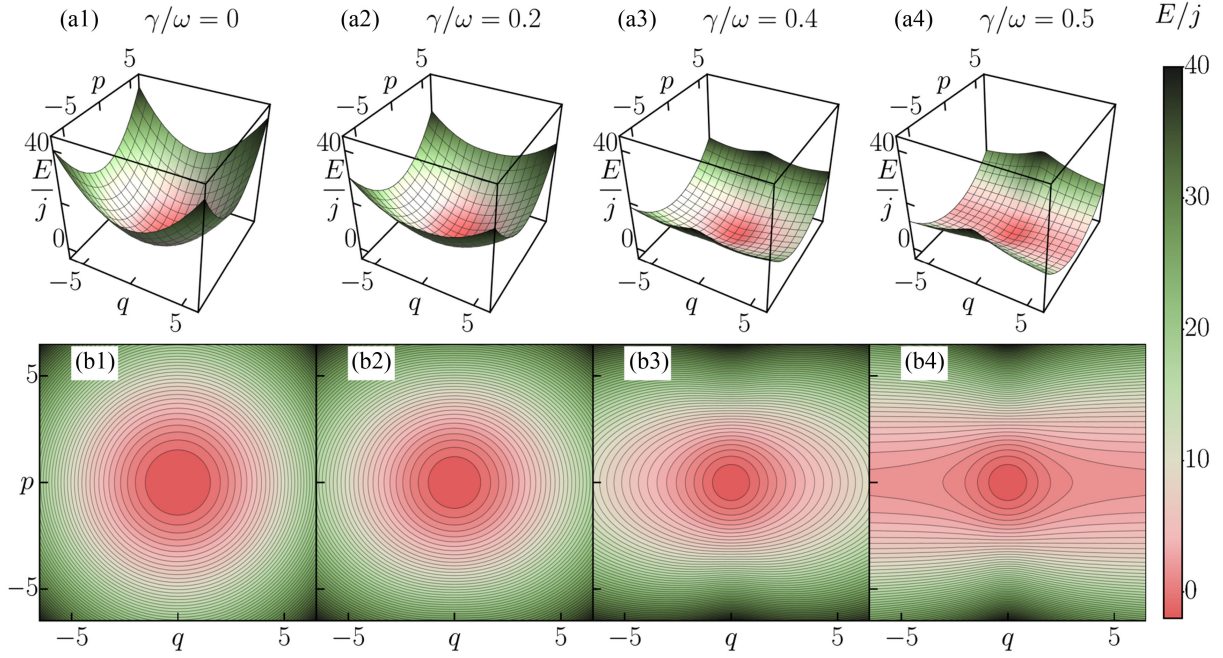


FIG. 3. (a1)–(a4) Energy surface $h_D^\infty(q, p)$ [Eq. (9)] as a function of the bosonic coordinates (q, p) for increasing values of the scaled coupling strength $\gamma/\omega = 0.1, 0.2, 0.4, 0.5$. (b1)–(b4) Orthogonal projections of panels (a1)–(a4) over the bosonic plane $q - p$. The color scale shows values of the scaled classical energy E/j that begins at the scaled ground-state energy $E_0/j = -\omega_0$. System parameters: $\omega = 1$ and $\omega_0 = 2\omega$.

critical points associated with superradiance $\mathbf{x}_\pm^\infty \in \mathbb{C}$ become unphysical, and the classical Hamiltonian $h_D^\infty(\mathbf{x})$ admits only the normal phase and correctly captures the onset of spectral collapse.

Our results demonstrate that superradiance and spectral collapse are fundamentally distinct phenomena in two-photon light-matter systems. In particular, spectral collapse survives the thermodynamic limit, whereas superradiance does not. This resolves an apparent contradiction in previous analyses based on squeezed-state mean-field descriptions and places strong constraints on the realization of superradiant behavior in two-photon platforms. Whether dissipation [71,72] can stabilize a genuine superradiant phase remains an open and nontrivial question.

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Data availability—The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be

prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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