

Maximally Nonprojective Measurements Are Not Always Symmetric Informationally Complete

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Standard quantum measurements are projective. However, the full scope of quantum measurements is represented by positive operator-valued measures (POVMs) and many of these break the limitations of projective measurements as resources in quantum information. It is therefore natural to consider how accurately an experimenter with access only to projective measurements and classical processing can simulate POVMs. The most well-known class of nonprojective measurements is called symmetric informationally complete (SIC). Such measurements are both ubiquitous in the broader scope of quantum information theory and known to be the most strongly nonprojective measurements in qubit systems. Here, we show that beyond qubit systems, the SIC property is in general not associated with the most nonprojective measurement. For this, we put forward a semidefinite programming criterion for detecting genuinely nonprojective measurements. This method allows us to determine quantitative simulability thresholds for generic POVMs and to put forward a conjecture on which qutrit and ququart measurements are most strongly nonprojective.

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Introduction—Standard measurements in quantum theory are complete sets of orthogonal projectors. However, it is well-established that the full scope of quantum measurements is represented by the more general notion of positive operator-valued measures (POVMs). On a given Hilbert space, there exist nonprojective POVMs that cannot be reduced to projective measurements [1]. The seminal example of such a measurement is the symmetric informationally complete (SIC) POVM [2,3]. A d -dimensional SIC-POVM has the largest possible number of irreducible outcomes and all the angles between its measurement operators are identical. These measurements are broadly important in quantum theory. For instance, they are optimal for state tomography [4] and randomness generation [5], play a key role in some approaches to the foundations of quantum theory [6], and serve as tools for entanglement witnessing [7,8], contextuality [9], non-locality [10,11], self-testing [12,13], etc. A plethora of experimental platforms developed for implementing nonprojective measurements have primarily focused on realizing SIC-POVMs [14–20]. Their existence in all dimensions is a famous open-ended conjecture [2,21,22].

In view of this, it may be expected that SIC-POVMs are, in the appropriate sense, the POVMs that most strongly defy the limitations of standard projective measurements. This was conjectured in Ref. [23]. The same work also formalized the study of simulating POVMs using only projective measurements and classical resources. This is not only conceptually motivated but it is also practically relevant since nonprojective measurements are considerably more expensive to implement than projective measurements; they need ancilla systems and entangling gates. As in most resource theories, there are infinitely many mathematically legitimate ways of quantifying projective simulability. However, for projective simulability the standard benchmark employed in the literature is based on a depolarizing channel (see, e.g., Refs. [23–28]). This is because the depolarization is unbiased with respect to the POVM under consideration, experimentally meaningful, and mathematically elegant. The extent to which a given POVM is genuinely nonprojective is then the smallest amount of depolarization noise that it can be exposed to before admitting a projective simulation.

For qubit systems, it has been proven that SIC-POVMs indeed are the most nonprojective measurements [24]. Here, however, we show that the same is not in general true for dimensions higher than two. For this, we put forward a necessary condition for projective simulability which admits a formulation in terms of semidefinite program (SDP), a standard optimization tool used in quantum information [29]. We first deploy it to investigate the projective simulability of SIC-POVMs in dimension

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three and show that the SIC property is insufficient to determine the simulability thresholds, i.e., some SIC-POVMs are more nonprojective than others. Next, we consider SIC-POVMs in dimension four and show that none of them correspond to the most nonprojective measurement. Remarkably, we find that all four-dimensional SIC-POVMs are less robust than their lower-dimensional counterparts. Perhaps counterintuitively, this shows that SIC-POVMs do not always become harder to simulate as we increase the dimension.

In view of these results, it is natural to ask the following question: if not SIC-POVMs, then which measurements are the most nonprojective? Toward answering this, we use our SDP criterion as an oracle in a systematic numerical search for the most strongly nonprojective POVM. For qutrits, we find that it returns one single SIC-POVM (among infinitely many possible). For ququarts, it returns a ten-outcome POVM which is not a SIC-POVM but bears a clear conceptual resemblance to it. This leads us to investigate this new type of “flagged” measurement and its capability of defying projective simulations.

Projective simulation of POVMs—Consider that we are given an n -outcome POVM $\mathbf{E} = \{E_a\}_{a=1}^n$ on a d -dimensional Hilbert space, where a denotes the outcome. Our goal is to simulate it by using only the following restricted class of resources. (i) Classical randomness, represented by a stochastic variable λ subject to a probability distribution $\{q_\lambda\}$. (ii) A set of projective measurements, $\{P_{k|\lambda}\}$, where k represents the outcome. The variable λ is interpreted as the measurement choice. (iii) Classical postprocessing of k and λ , through a probabilistic rule $p(a|k, \lambda)$, which produces the final output a . If these resources can reproduce the POVM, it takes the form $E_a = \sum_\lambda q_\lambda \sum_k p(a|k, \lambda) P_{k|\lambda}$. If such a model exists, we say that $\mathbf{E}$ admits a projective simulation [23]. However, it turns out that one can without loss of generality omit the postprocessing, i.e., we can always select it as $p(a|k, \lambda) = \delta_{a,k}$. This leads to the simpler form

$$E_a = \sum_\lambda q_\lambda P_{a|\lambda}, \quad (1)$$

where the projective measurements $P_{a|\lambda}$ in the simulation can also be null matrices. In Supplemental Material [30] (see also Refs. [31–34]), we provide a constructive proof in which it is shown how to explicitly eliminate the postprocessing. Notably, such simulable POVMs can be realized without access to additional degrees of freedom, even though they are not necessarily projective. This shows that other (nonconvex) measures of nonprojectiveness, like unsharpness, are insufficient to capture this feature [35].

We denote the set of projectively simulable POVMs by $\mathcal{P}$. Assume that $\mathbf{E}$ is not projectively simulable ($\mathbf{E} \notin \mathcal{P}$). Then, a natural way to quantify its magnitude of nonprojectivity is through applying the depolarization channel,

$\Phi_v(X) = vX + [(1-v)/d]\text{Tr}(X)\mathbb{1}$, to each of the POVM elements and determine the largest visibility, $v \in [0, 1]$, for which a projective simulation is possible [36]. The projective simulability threshold (sometimes called critical visibility) for $\mathbf{E}$ becomes

$$v^*(\mathbf{E}) = \max\{v | \{\Phi_v(E_a)\}_a \in \mathcal{P}\}. \quad (2)$$

This measure can be interpreted as white noise tolerance of genuine nonprojective features.

Necessary condition for projective simulability—How can we determine if a given POVM $\mathbf{E}$ admits a projective simulation? A complete solution is not known. We identify a necessary condition in the form of an SDP. Such programs are known to be efficiently computable [29]. The main idea is to partition the space of projective measurements over d -dimensional Hilbert space by the ranks of the measurement operators. Specifically, consider a vector $\vec{r} = (r_1, \dots, r_n)$ such that r_a are non-negative integers and $\sum_{a=1}^n r_a = d$. Every such vector can be associated with a distinct class of projective measurements, $\{P_a\}_a$, in which $\text{rank}(P_a) = r_a$. The union of all these classes is the set of all projective measurements. Therefore, we call $\vec{r}$ a rank-vector and use it as an index for partitioning the space of projective measurement. Hence, we can substitute the projective measurement index λ in Eq. (1) with the pair $(\vec{r}, \mu)$, where $\vec{r}$ specifies the rank-vector and μ indexes the projective measurement within that class. Inserted into Eq. (1), this leads to $E_a = \sum_{\vec{r}} F_{a|\vec{r}}$, where $F_{a|\vec{r}} = \sum_\mu q_{\vec{r}, \mu} P_{a|\vec{r}, \mu}$. By construction, we have that $F_{a|\vec{r}} \geq 0$. Furthermore, using conditional probability we can write $q_{\vec{r}, \mu} = q_{\vec{r}} q_{\mu|\vec{r}}$. Then, it follows from measurement completeness and normalization that $\sum_a F_{a|\vec{r}} = q_{\vec{r}} \mathbb{1}$. Finally, since by definition $\text{rank}(P_{a|\vec{r}, \mu}) = r_a$ and P_a is projective, it follows that $\text{tr}(P_{a|\vec{r}, \mu}) = r_a$. Hence, $\text{tr}(F_{a|\vec{r}}) = q_{\vec{r}} r_a$. Putting this together, our necessary condition for $\mathbf{E} \in \mathcal{P}$ becomes

$$\begin{aligned} & \text{find } \{F_{a|\vec{r}}\}_{a, \vec{r}} \\ & \text{such that } \sum_{\vec{r}} F_{a|\vec{r}} = E_a, \quad \text{tr}(F_{a|\vec{r}}) = q_{\vec{r}} r_a, \\ & \sum_a F_{a|\vec{r}} = q_{\vec{r}} \mathbb{1}, \quad \text{and } F_{a|\vec{r}} \geq 0. \end{aligned} \quad (3)$$

which is an SDP. Note that the second constraint implies that the distribution over the rank-vectors is given by $q_{\vec{r}} = (1/d) \sum_a \text{tr}(F_{a|\vec{r}})$. In addition, the SDP can be adapted to compute upper bounds on the projective simulability threshold visibility $v^*(\mathbf{E})$. For this, we need only to introduce the visibility, v , as the objective of the SDP, replace E_a with $\Phi_v(E_a)$ in the constraints, and run a maximization of the objective. For qubits and qutrits Eq. (3) reduces to the one presented in Ref. [23]. Our implementation is available at Ref. [37].

Moreover, sometimes the necessary condition Eq. (3) is also sufficient for projective simulation. This is the case for any qubit or qutrit POVM. To see this, recall that a projective measurement cannot have more than d nonzero outcomes. First, all extremal dichotomic measurements are projective [38]. Second, d -outcome measurements are a strict superset of the measurements defined via the criterion Eq. (3). This implies that it is exact for qubits. Moreover, it was shown in Ref. [23] that qutrit measurements with unit-trace measurement operators only have projective extremals. Hence, optimality follows also for qutrits.

Another relevant use of the SDP criterion is to determine witnesses for $\mathbf{E} \notin \mathcal{P}$. These witnesses come in the form of linear inequalities, satisfied by all $\mathbf{E} \in \mathcal{P}$, in the probabilities obtained from probing the measurement with a few selected preparations. These inequalities can in general be extracted from the dual formulation of the SDP; see the Supplemental Material [30]. However, we also propose a simple witness ansatz. Consider a rank-one nonprojective measurement $\tilde{\mathbf{E}}$ where $\tilde{E}_a = c_a |\psi_a\rangle\langle\psi_a|$. To show that an arbitrary POVM $\mathbf{E} \notin \mathcal{P}$, we consider a witness inspired by quantum state discrimination,

$$W_{\tilde{\mathbf{E}}}(\mathbf{E}) = \frac{1}{d} \sum_{a=1}^n \langle \psi_a | E_a | \psi_a \rangle, \quad (4)$$

i.e., we probe the measurement with the eigenstates of the target POVM with a uniform prior ($1/d$). Note that the witness is upper bounded by 1 for any given $\mathbf{E}$, $\max(1/d) \sum_a \text{tr}(\psi_a E_a) \leq (1/d) \sum_a \lambda_{\max}(E_a) \leq (1/d) \sum_a \text{tr}(E_a) = 1$. Moreover, for $\mathbf{E} = \tilde{\mathbf{E}}$ we have $W_{\tilde{\mathbf{E}}}(\tilde{\mathbf{E}}) = 1$. Thus, we would expect that $\mathbf{E}$ attains a value close to unit. To show that this is sufficient to elude projective simulation we need to determine the largest value of $W_{\tilde{\mathbf{E}}}$ over all projective measurements. This can be achieved by SDP, by using Eq. (4) as the objective over the constraints in Eq. (3). In the Supplemental Material [30], we discuss how to leverage the linearity of $W_{\tilde{\mathbf{E}}}$ to evaluate the SDP more efficiently.

Main problem—For a given dimension, d , it is natural to ask which POVM most strongly defies a projective simulation. This corresponds to finding the specific $\mathbf{E}$ that solves the problem

$$v_d \equiv \min_{\mathbf{E}} v^*(\mathbf{E}). \quad (5)$$

Here, v_d is the visibility associated with the most non-projective measurement in the given dimension.

A natural candidate for solving this problem is SIC-POVMs. A POVM over d -dimensional Hilbert space is SIC if and only if $E_a = (1/d) |\psi_a\rangle\langle\psi_a|$ where $|\langle\psi_a|\psi_b\rangle|^2 = [1/(d+1)]$ for $a \neq b$. Such a measurement has $n = d^2$ outcomes, which is the largest possible number for an extremal POVM [1]. All known SIC-POVMs (with

one exception for $d = 8$) are generated by the Weyl-Heisenberg group [21]. Define the group's generators $X = \sum_{j=0}^{d-1} |j \oplus 1\rangle\langle j|$ and $Z = \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j|$, where $\omega = e^{2\pi i/d}$. Then, a SIC-POVM is constructed from the orbit

$$|\psi_a\rangle = X^{a_0} Z^{a_1} |\phi\rangle, \quad (6)$$

where $|\phi\rangle$ is a suitably chosen vector called a SIC-fiducial and where $a = (a_0, a_1) \in \{0, \dots, d-1\}^2$.

It was conjectured in Ref. [23] and proven in Ref. [24] that for $d = 2$, the unique SIC-POVM $\mathbf{E}_{\text{SIC2}}$ is the solution to Eq. (5). It achieves the visibility $v_2 = \sqrt{2/3} \approx 81.6\%$. To achieve this simulation, one requires six rank-one projective measurements corresponding to rank-vectors that are permutations of $(1, 1, 0, 0)$. The corresponding Bloch vectors form a cuboctahedron on the Bloch sphere. It was further conjectured in Ref. [23] that SIC-POVMs also are the solution for $d > 2$.

Qutrit SIC-POVMs—For qutrits, it is known that all SIC-POVMs admit the form Eq. (6) [39,40]. Also, up to a unitary or antiunitary transformation, they can all be obtained from the one-parameter family of SIC-fiducials

$$|\phi^{(\theta)}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\theta}|2\rangle) \quad (7)$$

for $\theta \in [0, \pi/9]$ [41]. Denoting these SIC-POVMs by $\mathbf{E}_\theta$, we use the SDP criterion to compute $v^*(\mathbf{E}_\theta)$. The results are plotted in Fig. 1. Since the curve is not flat, it means that the SIC property is insufficient to determine the degree of projective simulability of the POVMs. This stems from the fact that while the SIC property determines all the magnitudes of the overlaps between the POVM elements, it does not determine their relative phases. Nevertheless, from the plot we see that every qutrit SIC-POVM has a projective simulability threshold visibility lower than that of the qubit SIC-POVM. In particular, the specific SIC-POVM that attains the smallest visibility ($\theta = 0$) is unique and known as the Hesse SIC [42]. Its nine (unnormalized) vectors $\{|\psi_a\rangle\}_a$ are the columns of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & -\omega^2 & -\omega & 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 0 & 0 & 0 & -1 & -\omega^2 & -\omega \\ -1 & -\omega^2 & -\omega & 1 & \omega & \omega^2 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

and it leads to

$$v_3 \leq v^*(\mathbf{E}_{\text{Hesse}}) = \frac{1}{6} \left[1 + 4 \cos\left(\frac{\pi}{9}\right) \right] \approx 79.3\%. \quad (9)$$

To better understand the projective simulation of the Hesse SIC-POVM, we have found an analytical model that gives the closed expression above for the decimal solution returned by our SDP criterion. Our simulation strategy requires 72 equiprobable projective measurements. The main idea is to consider the projective Clifford group, which has 216 elements, and compute the orbit of a suitably chosen fiducial vector. The resulting 216 vectors can be grouped into 72 sets of 3 vectors, such that each set forms an orthonormal basis, i.e., a projective measurement. The model is detailed in Supplemental Material [30]. In contrast, the SIC-POVM with highest visibility threshold ($\theta = \pi/9$) is known in the literature as Norrell SIC and is related to resourceful states in the resource theory of stabilizer computation [43].

Ququart SIC-POVMs—In contrast to three dimensions, there exists up to unitary and antiunitary transformations only a single SIC-POVM in dimension four [44]. This POVM, $\mathbf{E}_{\text{SIC}_4}$, is obtained from Eq. (6) by selecting the appropriate SIC-fiducial vector [45]. In contrast to the lower-dimensional cases, the value obtained by the SDP yields only an upper bound on the projective simulability threshold visibility. Remarkably, however, the bound is exact for the SIC-POVM and we obtain $v^*(\mathbf{E}_{\text{SIC}_4}) \approx 82.6\%$. It follows from the fact that the solution returned by our SDP has a particular structure: all the nonzero operators take the form $F_{a|\vec{r}} = q_{\vec{r}} O_{a|\vec{r}}$ for some projective operator $O_{a|\vec{r}}$. From Eq. (3) we have that $\sum_a F_{a|\vec{r}} = q_{\vec{r}} \mathbb{1}$ which then implies that $\sum_a O_{a|\vec{r}} = \mathbb{1}$. Thus, $\{O_{a|\vec{r}}\}_a$ is a projective measurement for every relevant $\vec{r}$. Hence, we have an explicit projective strategy that saturates our upper bound, thus implying that it is optimal. Our simulation relies on 132 projective measurements.

The most nonprojective POVM—We now set out to investigate which are the most nonprojective measurements beyond qubit systems. To study this problem, we have used

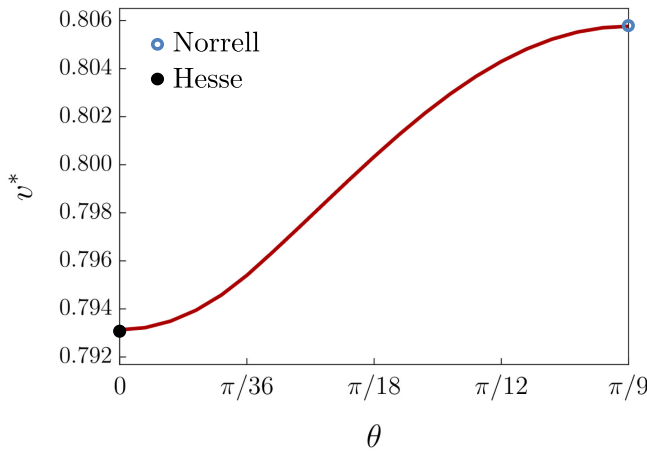


FIG. 1. The projective simulability threshold visibility $v^*(\mathbf{E}_\theta)$ defined in Eq. (2), for projective simulation of depolarized qutrit SIC-POVMs constructed from fiducial vector $|\varphi^{(\theta)}\rangle$ in Eq. (7).

a numerical search algorithm. This algorithm uses the SDP criterion for projective simulability as an oracle to iteratively explore the space of nonprojective measurements. Specifically, its steps are the following. (i) Select a nonprojective measurement from which to begin the search and use the SDP criterion to compute a bound on its visibility. (ii) From the dual SDP (see Supplemental Material [30]), extract a witness for genuine nonprojectivity. (iii) Compute the measurement that gives the largest violation of this witness. This can also be done by an SDP. (iv) Return to step (i) and use the measurement from (iii) as the input. This process is repeated until desired convergence is found. Our implementation is available at Ref. [37]. As a basic check, we have run this process first for qubits and find that it essentially always converges to the SIC-POVM, which we know is the correct answer.

For qutrits, we have evaluated this search with many different initial POVMs and found that it has, essentially always, two possible fates: either it gets stuck in a specific local extremum or it bypasses it to converge to the Hesse SIC-POVM (8). This provides significant evidence that Eq. (9) actually is an equality and that the Hesse SIC-POVM is the most nonprojective qutrit measurement. Furthermore, also the POVM found in the local extremum is interesting, because it has only five (out of nine possible) outcomes. In fact, it turns out to closely resemble the qubit SIC-POVM, $\mathbf{E}_{\text{SIC}_2}$. Its POVM elements are

$$\{E_1, E_2, E_3, E_4\} = \mathbf{E}_{\text{SIC}_2} \oplus 0, \quad E_5 = |2\rangle\langle 2|. \quad (10)$$

Thus, the first four outcomes are just an embedding of the qubit SIC-POVM and the fifth outcome is a vector orthogonal to the qubit sub-space. Therefore, we call this a flagged SIC-POVM, $\mathbf{E}_{\text{fSIC}_2}$. Its visibility is $v^*(\mathbf{E}_{\text{fSIC}_2}) \approx 79.6\%$, which is higher than for the Hesse SIC-POVM but lower than for the qubit SIC-POVM. Hence, simply by appending an orthogonal projection (“flag”), in contrast to intuition, the nonprojective features of the qubit SIC-POVM are amplified. In Supplemental Material [30] we construct an analytical projective simulation of the noisy $\mathbf{E}_{\text{fSIC}_2}$, as well as its generalizations to higher dimension, which shows that it exhibits a monotonically decreasing simulability threshold with increasing dimension. For example, the qubit SIC flagged in $d = 4$ yields a threshold visibility of 78.6%, which is lower than $v^*(\mathbf{E}_{\text{SIC}_4})$. This shows that $\mathbf{E}_{\text{SIC}_4}$ cannot be the most nonprojective measurement in $d = 4$.

Therefore, we have repeated the numerical search also for ququarts. The best search result returns that $v_4 \lesssim 78.2\%$. Interestingly, the corresponding POVM does not use all the possible 16 outcomes, but only ten of them. Indeed, this one too has a flagged structure: it is an embedding of the Hesse SIC-POVM appended with an orthogonal projection,

$$\{E_1, \dots, E_9\} = \mathbf{E}_{\text{Hesse}} \oplus 0, \quad E_{10} = |3\rangle\langle 3|. \quad (11)$$

We call this POVM $\mathbf{E}_{\text{fSIC3}}$. In analogy with the qutrit case, we construct a projective simulation of the noisy $\mathbf{E}_{\text{fSIC3}}$ (and its higher-dimensional generalizations) in Supplemental Material [30] which confirms $v^*(\mathbf{E}_{\text{fSIC3}}) \approx 78.2\%$. The visibility of this POVM is lower than that of the SIC-POVM embedded into it and we conjecture that this is the most nonprojective ququart measurement.

Discussion—We have shown that SIC-POVMs in general are not the POVMs with the largest depolarization noise robustness with respect to projective simulations. For qutrits, we have shown that not all SIC-POVMs have the same robustness and for ququarts we have shown that SIC-POVMs are far from the most nonprojective measurements. Nevertheless, our conjectures on the qutrit and ququart measurements that are the most nonprojective clearly relate to SIC-inspired features.

In particular, contrary to what we initially expected, we have found that the robustness of SIC-POVMs does not always decrease as the Hilbert space dimension gets larger. In Fig. 2 we showcase this feature, while the quantitative results are listed in Supplemental Material [30]. In this figure, we have also used the witness ansatz (4) to compute upper bounds on the projective simulability threshold of SIC-POVMs in dimensions five and six. Notably, this method returns optimal results for $d = 2$ and the Hesse SIC but somewhat suboptimal results for $d = 4$. In $d = 5$, by explicit construction we have shown that a projective simulation is possible up to $v \approx 82.7\%$ (see Ref. [37]). Thus, the fact that SIC-POVMs are not the most nonprojective measurements does not seem to be related to whether the dimension is prime or not. Moreover, we observe increasing simulability thresholds from dimensions $d = 3$ to $d = 5$. Whether this truly reflects a decaying

nonprojective property of SIC-POVMs with increasing dimension is an interesting problem beyond the scope of this Letter.

So far, we have limited our attention to the case of depolarizing noise. A complementary model that is often used in more mathematically motivated literature on resource theories of measurements is derived from considering maximally detrimental measurements [46]. This worst-case noise has well-behaved mathematical properties [47]. It corresponds to the situation in which the target POVM, $\mathbf{E}$, is mixed with another arbitrary POVM, $\mathbf{E}_{\text{noise}}$. By this measure, the projective simulation threshold visibility of $\mathbf{E}$ is the largest v such that one can find some $\mathbf{E}_{\text{noise}}$ that satisfies $v\mathbf{E} + (1-v)\mathbf{E}_{\text{noise}} \in \mathcal{P}$ (see Supplemental Material [30]); we refer to this as the “worst-case” noise. For example, if $\mathbf{E} = \mathbf{E}_{\text{SIC2}}$ then this worst-case noise will be another SIC-POVM to which $\mathbf{E}_{\text{SIC2}}$ is elementwise orthogonal. As shown in Supplemental Material [30], the corresponding threshold visibility for any fixed POVM cannot decrease when adding flag dimensions to it, thus, it does not suffer from the unintuitive behavior that we observe for $\mathbf{E}_{\text{fSIC2}}$ and $\mathbf{E}_{\text{fSIC3}}$, namely, that they get more robust to noise if additional trivial flags are added to it.

For the sake of completeness, we have used our SDP criterion to analyze the problem of finding the most nonprojective measurement also under this global measure. The results are qualitatively analogous to those for depolarization noise. For qubits and qutrits we find that the same SIC-POVMs are optimal. What is more, for ququarts, we again find that the SIC-POVM is not the most nonprojective measurement and the flagged Hesse SIC-POVM (11) performs better. See Supplemental Material [30] for quantitative results. This indicates that the main results reported here are not artefacts unique to the choice of depolarization as the quantifier of nonprojective properties.

Our findings open several paths for future investigations. First, the question arises whether the results can be extended to the case of quantum instruments, for which a similar concept of simulability was introduced recently [48]. Second, although our Letter provides crucial numerical evidence that the flagged Hesse SIC is the maximally robust POVM in four dimensions, this conjecture remains to be proven. Moreover, while the maximally nonprojective POVM for qubits (i.e., the SIC-POVM) provides advantages over other POVMs in several tasks, it would be interesting to show the same for maximally nonprojective POVMs in other dimensions.

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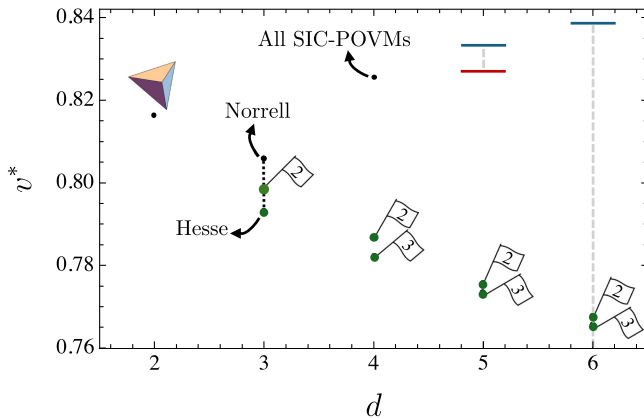


FIG. 2. The threshold visibility v^* from Eq. (2) obtained for various POVMs in $2 \leq d \leq 6$. Black dots represent SIC-POVMs, while green ones fSIC-POVMs (the number in the flag corresponds to the dimension of the embedded SIC-POVM), for which the visibility thresholds are calculated in Ref. [30]. The blue lines represent upper bounds on the visibility obtained for SIC-POVMs in $d = 5, 6$ using the witness method, while the red line denotes a lower bound in $d = 5$ found by explicit construction.

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Data availability—The data that support the findings of this article are openly available in the Supplemental Material [30].

- [1] G. M. D'Ariano, P. Lo Presti, and P. Perinotti, Classical randomness in quantum measurements, *J. Phys. A* **38**, 5979 (2005).
- [2] G. Zauner, Grundzüge einer nichtkommutativen Designtheorie, Ph.D. dissertation, University of Vienna, 1999.
- [3] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, *J. Math. Phys. (N.Y.)* **45**, 2171 (2004).
- [4] A. J. Scott, Tight informationally complete quantum measurements, *J. Phys. A* **39**, 13507 (2006).
- [5] A. Acín, S. Pironio, T. Vértesi, and P. Wittek, Optimal randomness certification from one entangled bit, *Phys. Rev. A* **93**, 040102(R) (2016).
- [6] C. A. Fuchs and R. Schack, Quantum-Bayesian coherence, *Rev. Mod. Phys.* **85**, 1693 (2013).
- [7] J. Shang, A. Asadian, H. Zhu, and O. Gühne, Enhanced entanglement criterion via symmetric informationally complete measurements, *Phys. Rev. A* **98**, 022309 (2018).
- [8] A. Tavakoli and S. Morelli, Enhanced Schmidt-number criteria based on correlation trace norms, *Phys. Rev. A* **110**, 062417 (2024).
- [9] I. Bengtsson, K. Blanchfield, and A. Cabello, A Kochen–Specker inequality from a SIC, *Phys. Lett. A* **376**, 374 (2012).
- [10] N. Gisin, Entanglement 25 years after quantum teleportation: Testing joint measurements in quantum networks, *Entropy* **21**, 325 (2019).
- [11] C.-J. Huang, G.-Y. Xiang, Y. Guo, K.-D. Wu, B.-H. Liu, C.-F. Li, G.-C. Guo, and A. Tavakoli, Nonlocality, steering, and quantum state tomography in a single experiment, *Phys. Rev. Lett.* **127**, 020401 (2021).
- [12] A. Tavakoli, D. Rosset, and M.-O. Renou, Enabling computation of correlation bounds for finite-dimensional quantum systems via symmetrization, *Phys. Rev. Lett.* **122**, 070501 (2019).
- [13] P. Mironowicz and M. Pawłowski, Experimentally feasible semi-device-independent certification of four-outcome positive-operator-valued measurements, *Phys. Rev. A* **100**, 030301(R) (2019).
- [14] Y.-y. Zhao, N.-k. Yu, P. Kurzyński, G.-y. Xiang, C.-F. Li, and G.-C. Guo, Experimental realization of generalized qubit measurements based on quantum walks, *Phys. Rev. A* **91**, 042101 (2015).
- [15] Z. Bian, J. Li, H. Qin, X. Zhan, R. Zhang, B. C. Sanders, and P. Xue, Realization of single-qubit positive-operator-valued measurement via a one-dimensional photonic quantum walk, *Phys. Rev. Lett.* **114**, 203602 (2015).
- [16] Z. Hou, J.-F. Tang, J. Shang, H. Zhu, J. Li, Y. Yuan, K.-D. Wu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, Deterministic realization of collective measurements via photonic quantum walks, *Nat. Commun.* **9**, 1414 (2018).
- [17] A. Tavakoli, M. Smania, T. Vértesi, N. Brunner, and M. Bourennane, Self-testing nonprojective quantum measurements in prepare-and-measure experiments, *Sci. Adv.* **6**, eaaw6664 (2020).
- [18] R. Stricker, M. Meth, L. Postler, C. Edmunds, C. Ferrie, R. Blatt, P. Schindler, T. Monz, R. Kueng, and M. Ringbauer, Experimental single-setting quantum state tomography, *PRX Quantum* **3**, 040310 (2022).
- [19] X. Wang, X. Zhan, Y. Li, L. Xiao, G. Zhu, D. Qu, Q. Lin, Y. Yu, and P. Xue, Generalized quantum measurements on a higher-dimensional system via quantum walks, *Phys. Rev. Lett.* **131**, 150803 (2023).
- [20] L.-T. Feng, X.-M. Hu, M. Zhang, Y.-J. Cheng, C. Zhang, Y. Guo, Y.-Y. Ding, Z. Hou, F.-W. Sun, G.-C. Guo, D.-X. Dai, A. Tavakoli, X.-F. Ren, and B.-H. Liu, Higher-dimensional symmetric informationally complete measurement via programmable photonic integrated optics, *Optica* **12**, 1014 (2025).
- [21] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, The SIC question: History and state of play, *Axioms* **6**, 21 (2017).
- [22] P. Horodecki, L. Rudnicki, and K. Życzkowski, Five open problems in quantum information theory, *PRX Quantum* **3**, 010101 (2022).
- [23] M. Oszmaniec, L. Guerini, P. Wittek, and A. Acín, Simulating positive-operator-valued measures with projective measurements, *Phys. Rev. Lett.* **119**, 190501 (2017).
- [24] F. Hirsch, M. T. Quintino, T. Vértesi, M. Navascués, and N. Brunner, Better local hidden variable models for two-qubit Werner states and an upper bound on the Grothendieck constant $K_G(3)$, *Quantum* **1**, 3 (2017).
- [25] L. Guerini, J. Bavaresco, M. Terra Cunha, and A. Acín, Operational framework for quantum measurement simulability, *J. Math. Phys. (N.Y.)* **58**, 092102 (2017).
- [26] T. Singal, F. B. Maciejewski, and M. Oszmaniec, Implementation of quantum measurements using classical resources and only a single ancillary qubit, *npj Quantum Inf.* **8**, 82 (2022).
- [27] D. Martínez, E. S. Gómez, J. Cariñe, L. Pereira, A. Delgado, S. P. Walborn, A. Tavakoli, and G. Lima, Certification of a non-projective qudit measurement using multiport beam-splitters, *Nat. Phys.* **19**, 190 (2023).
- [28] M. Kotowski and M. Oszmaniec, Pretty-good simulation of all quantum measurements by projective measurements, *arXiv:2501.09339*.
- [29] A. Tavakoli, A. Pozas-Kerstjens, P. Brown, and M. Araújo, Semidefinite programming relaxations for quantum correlations, *Rev. Mod. Phys.* **96**, 045006 (2024).
- [30] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/slt1-gfv6> for details on the proofs, explicit constructions of the simulations and numerical results.

- [31] P. Skrzypczyk and D. Cavalcanti, *Semidefinite Programming in Quantum Information Science* (IOP Publishing, Bristol (England), 2023).
- [32] D.M. Appleby, Symmetric informationally complete positive operator valued measures and the extended Clifford group, *J. Math. Phys. (N.Y.)* **46**, 052107 (2005).
- [33] A.J. Scott and M. Grassl, Symmetric informationally complete positive-operator-valued measures: A new computer study, *J. Math. Phys. (N.Y.)* **51**, 042203 (2010).
- [34] Z. Webb, The Clifford group forms a unitary 3-design, *Quantum Inf. Comput.* **16**, 1379 (2016).
- [35] Y. Liu and S. Luo, Quantifying unsharpness of measurements via uncertainty, *Phys. Rev. A* **104**, 052227 (2021).
- [36] Since the depolarization channel is self-dual, this is equivalent to performing the noise-free measurement on depolarised input states.
- [37] Code for PVM-simulability and numerical search, https://github.com/GabrieleCobucci/POVM_simulation (2025).
- [38] E. B. Davies and J. T. Lewis, An operational approach to quantum probability, *Commun. Math. Phys.* **17**, 239 (1970).
- [39] F. Szöllösi, All complex equiangular tight frames in dimension 3, [arXiv:1402.6429](https://arxiv.org/abs/1402.6429).
- [40] L. P. Hughston and S. M. Salamon, Surveying points in the complex projective plane, *Adv. Math.* **286**, 1017 (2016).
- [41] H. Zhu, SIC POVMs and Clifford groups in prime dimensions, *J. Phys. A* **43**, 305305 (2010).
- [42] O. Hesse, Über die Elimination der Variablen aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variablen, *J. Reine Angew. Math.* **1844**, 68 (1844).
- [43] V. Veitch, S. H. Mousavian, D. Gottesman, and J. Emerson, The resource theory of stabilizer quantum computation, *New J. Phys.* **16**, 013009 (2014).
- [44] H. Zhu, Y. S. Teo, and B.-G. Englert, Two-qubit symmetric informationally complete positive-operator-valued measures, *Phys. Rev. A* **82**, 042308 (2010).
- [45] The fiducial is $|\varphi_4\rangle \propto \sqrt{2}|0\rangle + [z(1-z)(\phi^{3/2} + \bar{z})]|1\rangle + (2 - \sqrt{2})i|2\rangle + [z(1-z)(\phi^{3/2} - \bar{z})]|3\rangle$ with $z = e^{i\pi/4}$ and the golden ratio $\phi = (1 + \sqrt{5})/2$.
- [46] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [47] J. Schluck, G. Murta, H. Kampermann, D. Bruß, and N. Wyderka, Continuity of robustness measures in quantum resource theories, *J. Phys. A* **56**, 255303 (2023).
- [48] S. Khandelwal and A. Tavakoli, Simulating quantum instruments with projective measurements and quantum postprocessing, *Phys. Rev. Lett.* **135**, 040202 (2025).