

Coherence as a Resource for Phase Estimation

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Quantum phase estimation is a core task in quantum technologies ranging from metrology to quantum computing, where it appears as a key subroutine in various algorithms. Here, we quantitatively connect the performance of phase estimation protocols with quantum coherence. To achieve this, we construct and characterize resource theories of quantum networks that cannot generate coherence. Given multiple copies of a unitary encoding an unknown phase and access to a fixed coherent state, we estimate the phase using such networks. For a unified and general approach, we assess the quality of the estimate using a generic cost function that penalizes deviations from the true value. We determine the minimal average cost that can be achieved in this manner and explicitly derive optimal protocols. From this, we construct a family of coherence measures that directly connect a state's coherence with its value for phase estimation, demonstrating that every bit of coherence helps. This establishes coherence as a resource that quantifies the performance of phase estimation, and, thus, of any quantum technology relying on it as a subroutine.

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Introduction—In the past decades, various quantum algorithms promising superpolynomial speedups compared to their classical counterparts have been discovered, most prominently Shor's factorization algorithm [1]. Many of these algorithms [1–3] contain a crucial subroutine, namely *quantum phase estimation*, as detailed in Refs. [4–7]. The goal of phase estimation is to determine an unknown phase ϕ encoded by a unitary as accurately as possible. This also comprises a fundamental problem in metrology itself and has been studied in Ref. [8]. Other applications are found in quantum chemistry and quantum simulation [9]. This makes phase estimation a core problem for quantum technologies.

It is clear that access to coherence, i.e., quantum superposition, is necessary for phase estimation [10,11]. But how much coherence is required for accurate phase estimation or, equivalently, what are the ultimate limitations on the accuracy of any phase estimation protocol that emerge from a lack of coherence? Answering these questions quantitatively is the central goal of this work. For a rigorous approach, we turn to the framework of quantum resource

theories [12], and in particular to the resource theory of coherence [13,14].

The quality of a phase estimate depends on the context. For example, to solve integer factorization via Shor's algorithm, one requires estimates that are sufficiently close to the true value because this ensures that the classical post-processing succeeds with sufficiently high probability. In contrast, in metrology, one tries to optimize the mean squared error. For a unified treatment of the diverse applications of phase estimation, we adopt the approach of Ref. [6] and consider generic cost functions that penalize deviations of our estimates $\hat{\phi}$ from the true value ϕ .

We evaluate the minimal average cost that an optimal phase estimation protocol consisting of a fixed coherent state and quantum networks that cannot generate coherence can achieve and show that this cost is fully determined by a family of coherence monotones. This establishes a quantitative relationship between coherence and the quality of phase estimation. Furthermore, we show that any amount of coherence provides a direct operational advantage in phase estimation, and give an explicit construction of optimal protocols achieving this advantage. For other examples that relate resource-theoretic quantities with the efficiency of specific quantum algorithms, see, e.g., Refs. [15–21]. Moreover, Refs. [22,23] derived *necessary* entanglement-based conditions for superpolynomial speedups in quantum algorithms and Ref. [24] linked contextuality and non-stabilizerness to quantum computational advantages. The relevance of such results lies in the fact that they provide a deeper understanding of the properties that underpin the advantages of both specific and general quantum algorithms.

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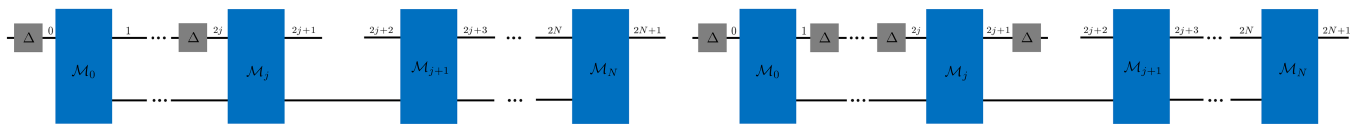


FIG. 1. A MIO-compatible network satisfies the condition that incoherent inputs at the first j inputs, i.e., systems $0, 2, \dots, 2j$, imply incoherent outputs at the output systems $1, 3, \dots, 2j+1$ for all $0 \leq j \leq N$.

Moreover, our work includes an alternative proof for the results of Ref. [6] concerning optimal phase estimation without resource constraints and characterizes the structure of coherence non-generating networks and supermaps, which we will introduce in the next sections.

Coherence as a resource—Resource theories of coherence [13,14] quantify the value of superposition with respect to a fixed orthonormal basis $\{|i\rangle\}_i$ referred to as the incoherent basis. A quantum state σ is called incoherent (or free) if it is diagonal in the incoherent basis, i.e., iff $\Delta(\sigma) = \sigma$, where $\Delta(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|$ denotes total dephasing in the incoherent basis $\{|i\rangle\}_i$. We denote the set of incoherent states by $\mathcal{I}$. States not contained in $\mathcal{I}$ are called coherent and considered resourceful. Since eigenstates of the unitary that encodes the phase we intend to estimate are invariant under its action, they cannot be used to infer any information about the phase. In the context of phase estimation, it is thus natural to choose this eigenbasis as the incoherent one, which we do. To use coherence optimally for phase estimation (or any other application), we need to be able to manipulate it, which is done with the help of free channels. The choice of free channels is not unique, and there are several relevant classes of free channels considered in the literature [13,14,25–31] emerging, for example, from practical or conceptual considerations. A minimal requirement for a meaningful resource theory is that free channels cannot create coherence [12,32]. The largest set of channels that satisfies this requirement is called the set of *maximally incoherent* operations, denoted by MIO [25,33–35], which consists of all channels $\mathcal{M}$ that satisfy $\mathcal{M}\Delta = \Delta\mathcal{M}\Delta$. To determine the ultimate limitations that restricted access to coherence imposes on phase estimation, we choose MIO as the set of free channels, since this allows for the most advantageous manipulation and thus usage of coherence.

Free quantum networks—The most general way to use a set of N channels $\mathcal{N}_1, \dots, \mathcal{N}_N$ in a laboratory is to insert them into a quantum network [36,37], i.e., to interlace them with a sequence of channels $\mathcal{M}_0, \dots, \mathcal{M}_N$, transforming them to $\mathcal{S}_N[\mathcal{N}_1, \dots, \mathcal{N}_N] = \mathcal{M}_N(\text{id} \otimes \mathcal{N}_N)\mathcal{M}_{N-1}(\text{id} \otimes \mathcal{N}_{N-1})\mathcal{M}_{N-2} \dots \mathcal{M}_1(\text{id} \otimes \mathcal{N}_1)\mathcal{M}_0$, where id refers to identity channels on auxiliary systems. Different networks can lead to the same effective transformation $\mathcal{S}_N$, which is called a supermap. Using a generalization of the Choi-Jamiołkowski isomorphism [38,39], such supermaps can be represented by quantum combs [36,37] (likewise, a quantum channel can be represented by a Choi state and implemented with different Stinespring dilations).

Quantum supermaps will be our main tool for the study of optimal phase estimation protocols with limited coherence, which is only meaningful if the supermaps under consideration have well-defined restrictions on their ability to create coherence. To this end, we divide the set of quantum supermaps into free and nonfree ones (as we did in the previous section for states and channels). Since quantum supermaps are implemented by quantum networks, it is intuitive and operational to consider the supermaps that can be implemented via a network of free channels as free.

Definition 1—A quantum supermap $\mathcal{S}_N$ corresponds to a MIO network and is considered free if its action can be decomposed into a sequence of MIO channels $\mathcal{M}_0, \dots, \mathcal{M}_N \in \text{MIO}$, i.e., $\mathcal{S}_N[\mathcal{N}_1, \dots, \mathcal{N}_N] = \mathcal{M}_N(\text{id} \otimes \mathcal{N}_N)\mathcal{M}_{N-1} \dots \mathcal{M}_1(\text{id} \otimes \mathcal{N}_1)\mathcal{M}_0$ for all channels $\mathcal{N}_1, \dots, \mathcal{N}_N$.

It is clear (see also Supplemental Material (SM) [40]) that the above definition is also solid from a structural perspective in the sense that the composition of MIO networks leads to another MIO network. In particular, this implies that a MIO network cannot transform free states and channels into resourceful ones, and thus cannot generate resources for free. For further details on how to build resource theories of quantum supermaps in general, see SM, and also Refs. [43–49]. However, since it seems that the set of MIO networks is hard to characterize, as a technical tool, we will utilize the following relaxation.

Definition 2—A quantum supermap $\mathcal{S}_N$ is called MIO-compatible if it satisfies the conditions depicted in Fig. 1 for all $0 \leq j \leq N$; i.e., if the first j inputs are incoherent, then the first j outputs are incoherent too.

The intuition behind this definition is that a free network should not be able to output coherence before coherence is fed into it. In SM [40], we show that the set of MIO-compatible supermaps is the largest set of superchannels that is consistent with MIO channels from a resource-theoretical perspective. This implies that a superchannel (i.e., $\mathcal{S}_N = \mathcal{S}_1$) is completely MIO preserving (see also, e.g., Refs. [45,46]); i.e., it maps every MIO channel to a MIO channel in a complete sense, iff it is MIO-compatible. Importantly, a supermap $\mathcal{S}_N$ is MIO-compatible iff its comb $J_{\mathcal{S}_N}$ satisfies

$$\Delta_{0,2,\dots,2j} J_{\mathcal{S}_N} = \Delta_{0,2,\dots,2j} \Delta_{1,3,\dots,2j+1} J_{\mathcal{S}_N} \quad \forall j: 0 \leq j \leq N, \quad (1)$$

where $\Delta_{x_1,\dots,x_n}$ denotes the application of a total dephasing to subsystems $x_1, \dots, x_n$. This is a semidefinite constraint which allows the use of the powerful methods of the comb representation [36,37] and semidefinite

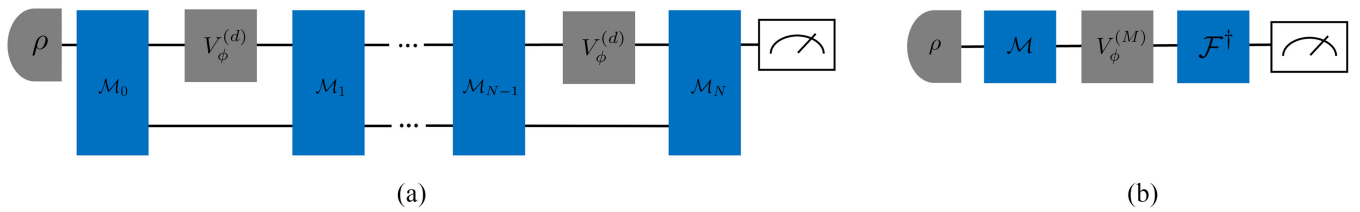


FIG. 2. (a) Phase estimation protocol composed of a network of MIO operations probing N copies of $V_\phi^{(d)}$ and a (resourceful) input state ρ . (b) Protocol that achieves the minimal average cost in Theorem 3, where $\mathcal{F}$ denotes the quantum Fourier transform, $\mathcal{M} \in \text{MIO}$, $M = (d-1)N + 1$, and the measurement is a projective measurement in the computational basis.

programming [50,51]. In SM, we show that MIO networks are a subset of MIO-compatible supermaps. We will see later that the two sets are equally powerful for phase estimation.

Quantum phase estimation—Phase estimation is a central subroutine in many quantum algorithms. Given N copies of the black-box unitary $V_\phi^{(d)} = \sum_{n=0}^{d-1} e^{i\phi n} |n\rangle\langle n|$, the goal is to estimate the unknown phase ϕ as accurately as possible. As shown in Ref. [4], many quantum algorithms can be phrased in this manner. For example, Shor’s integer factorization algorithm, in which we try to estimate a phase containing information that allows us to find a prime factor, is one of them (see Ref. [4] and SM for further information). For a unified description of different applications of phase estimation, we therefore evaluate the performance of a phase estimation protocol using a generic cost function $C(\phi - \hat{\phi})$ that only depends on the difference between ϕ and our estimate $\hat{\phi}$ and satisfies the following mild assumptions: (i) $C(\phi) \geq 0$, (ii) $C(\phi + 2\pi) = C(\phi)$, (iii) $\int_0^{2\pi} d\phi C(\phi) < \infty$.

To investigate limitations that restricted access to coherence imposes on phase estimation, we supply a fixed state ρ and N copies of the unitary $V_\phi^{(d)}$, which we insert into a MIO network as in Definition 1 followed by a measurement; see Fig. 2(a). Based on the measurement outcomes x with probability distribution $p^{(d,N)}(x|\phi, \rho)$ we assign phase estimates $\hat{\phi}_x$. Assuming a uniform prior distribution of the phase, our goal is to determine the minimal average cost

$$C_{\min}^{(d,N)}(\rho) = \inf \sum_x \int_0^{2\pi} \frac{d\phi}{2\pi} C(\phi - \hat{\phi}_x) p^{(d,N)}(x|\phi, \rho). \quad (2)$$

The infimum is taken over all MIO networks, all measurements (with a potentially unbounded number of outcomes) leading to the probability distribution $p^{(d,N)}(x|\phi, \rho)$, and all possible assignment rules for the phase estimate $x \mapsto \hat{\phi}_x$. A MIO network is considered optimal if it achieves $C_{\min}^{(d,N)}(\rho)$.

Since MIO is the largest set of free channels, no other set of free channels allows for better utilization of the coherence contained in ρ . If we choose any subset of MIO as free, the correspondingly defined minimal average cost cannot be lower than the one defined via MIO. Moreover,

the minimal average cost in Eq. (2) is not limited to a specific protocol because we optimize over all MIO networks, measurements, and assignment rules—and it is in this sense that Eq. (2) determines the ultimate limitations that restricted access to coherence imposes on phase estimation. Due to the generality of our approach, we are modeling neither a specific phase estimation protocol nor hardware-dependent experimental restrictions. Nonetheless, any cost that can be obtained with a protocol that corresponds to a MIO network is lower bounded by Eq. (2). By choosing MIO as free, we consider an idealization in the sense that, in a specific experiment, not all MIO channels are necessarily easy to implement. On the other hand, in such a specific experiment, it might be reasonably easy to implement operations outside of MIO. By allowing for such operations, we do, however, no longer consider coherence as a restricted resource but are modeling other constraints instead (which are experiment dependent). Since we are interested in a general approach, we do not pursue this direction, although it can be highly relevant for the specific experiment under consideration. This idealization allows us to single out and rigorously investigate the role of coherence in phase estimation. A similar approach has been pursued in the investigation of the role of entanglement in quantum information theory [52–54], where local operations and classical communication are often considered free, although many local operations are difficult to implement in practice but entangling two qubits close to each other is often feasible.

Main results—We now show that the optimal achievable average cost using an arbitrary protocol as depicted in Fig. 2(a) is achieved by the surprisingly simple phase estimation protocol depicted in Fig. 2(b).

Theorem 3—Let $Y^{(M)} \in \mathbb{C}^{M \times M}$ be the Toeplitz matrix defined by

$$Y^{(M)} = \sum_{n,m=0}^{M-1} \int \frac{d\phi}{2\pi} C(\phi) e^{i\phi(n-m)} |n\rangle\langle m|. \quad (3)$$

Given N copies of the unitary $V_\phi^{(d)}$, the minimal average cost in Eq. (2) is

$$C_{\min}^{(d,N)}(\rho) = C_{\min}^{(M,1)}(\rho) = \min_{\mathcal{M} \in \text{MIO}} \text{Tr}[Y^{(M)} \mathcal{M}(\rho)], \quad (4)$$

where $M = (d - 1)N + 1$ and the input and output dimensions of the channel $\mathcal{M}$ are fixed by ρ and $Y^{(M)}$, respectively. The minimal average cost is achieved by the protocol shown in Fig. 2(b): the N copies of the black-box unitary $V_\phi^{(d)}$ are combined with MIO operations to implement the unitary $V_\phi^{(M)}$. This unitary is probed with a state $\mathcal{M}(\rho)$, where $\mathcal{M}$ is a MIO channel that optimizes Eq. (4). The output is subsequently measured in the Fourier basis, and upon outcome x , a phase estimate $\hat{\phi}_x = (2\pi x/M)$ is assigned.

We emphasize that the optimal phase estimation protocol depicted in Fig. 2(b) depends on the cost function C only via the channel $\mathcal{M}$. The implementation of $V_\phi^{(M)}$, the measurement in the Fourier basis, and the phase estimates are optimal for any cost function. Moreover, using a memory system in the quantum network is never necessary. We also want to mention that the quantum Fourier transform in Fig. 2(b) is not an MIO operation. This does not contradict our premise to consider only MIO networks since we can always absorb it into the subsequent measurement, and the resulting measurement can never create coherence. We choose to depict the protocol including the Fourier transform to highlight the similarity to the usual depiction of phase estimation protocols. The right-hand side of Eq. (4) is solvable via a semidefinite program.

Our results generalize Ref. [6], where optimal phase estimation protocols without coherence constraints have been studied. The authors of Ref. [6] showed that the protocol in Fig. 2(b), where $\mathcal{M}$ is replaced by an optimal unconstrained channel, is optimal (for $d = 2$). Our results include their findings, as providing enough coherence allows us to implement any channel [13]. However, their proof methods relying on dilations of the channels in the network cannot be extended to the scenario that we consider because MIO channels do not possess a free dilation [28–30]. Our method of showing Theorem 3 is to relax the optimization over the set of all MIO networks in Eq. (2) to the set of MIO-compatible supermaps introduced in Definition 2. Using symmetry arguments similar to Ref. [55, Chap. 4.4] to reexpress the optimization problem, we prove that it is sufficient to consider M measurement outcomes and phase estimates $\hat{\phi}_x = (2\pi x/M)$. This allows us to show that the resulting optimization problem can be solved using an SDP. Via the dual problem, we then prove that the right-hand side of Eq. (4) is a lower bound on the minimal average cost. To conclude the proof, we show that the protocol depicted in Fig. 2(b) achieves this lower bound. This also implies that Theorem 3 holds if we define $C_{\min}^{(d,N)}(\rho)$ via an optimization over MIO-compatible supermaps.

To understand the relevance of coherence in phase estimation, we first consider two extremal cases supplying either an incoherent or a maximally coherent state of

unbounded dimension. By providing an incoherent input state, according to Theorem 3, the optimal average cost reduces to the average of the cost function; i.e., for any $\sigma \in I$, we have $C_{\min}^{(M,1)}(\sigma) = \int (d\phi/2\pi) C(\phi) =: C_0$, where C_0 is the cost associated with a guess solely based on the uniform prior distribution. Intuitively, this is expected as without coherence, no information about the phase is encoded, which could improve our guesses. If, on the other hand, we supply a maximally coherent state of sufficient dimension, we can prepare any possible quantum state, and in particular, the optimal input states of Refs. [6,56], such that the minimal average cost is given by the minimal eigenvalue $\lambda_{\min}$ of $Y^{(M)}$. For any coherent state $\rho \notin I$ between these two extremal cases, we expect to be able to infer some information about the phase and lower the average cost below C_0 . To quantify this, for an arbitrary but fixed cost function, we introduce the functional

$$\mathcal{A}^{(M)}(\rho) := \max_{\mathcal{M} \in \text{MIO}} \text{Tr}[(\Delta(Y^{(M)}) - Y^{(M)})\mathcal{M}(\rho)], \quad (5)$$

where $Y^{(M)}$ is defined as in Eq. (3). This directly determines the decrease in average cost compared to random guesses with a cost of C_0 since

$$C_{\min}^{(M,1)}(\rho) = C_0 - \mathcal{A}^{(M)}(\rho). \quad (6)$$

We now show that the functionals in Eq. (5) define a family of coherence monotones [14].

Theorem 4—The functionals $\mathcal{A}^{(M)}$ are convex coherence monotones, i.e., $\mathcal{A}^{(M)}(\mathcal{N}(\rho)) \leq \mathcal{A}^{(M)}(\rho)$ for all $\mathcal{N} \in \text{MIO}$ and all states ρ . For any cost matrix $Y^{(M)}$ with $Y^{(M)} \neq \Delta Y^{(M)}$, the monotones are faithful; i.e., $\mathcal{A}^{(M)}(\rho) \geq 0$ with equality iff $\rho \in I$.

This, together with Eq. (6), shows that the optimal performance of phase estimation is intrinsically linked to the available coherence. Unless the cost function is constant and thus trivial, there always exists a sufficiently large M such that the corresponding cost matrix $Y^{(M)}$ is not diagonal. This implies that every amount of coherence is useful for phase estimation problems described by a non-trivial cost function, and the operational advantage it provides is directly quantified by the task-tailored coherence measures in Eq. (5). Therefore, coherence is a crucial resource for phase estimation protocols and, thus, for algorithms that use it as a subroutine.

Linking operational advantages directly to employed resources in real applications is a primary motivation for resource theories in general and coherence theory in particular. This has been studied in recent years for various forms of discrimination, exclusion, and detection games [57–69]. For example, trying to discriminate between unitaries applying different phases from a discrete set is such a discrimination game, in which coherence emerges as a crucial resource [57,58]. Our work generalizes this to the

continuous setting. Moreover, in various applications in computation (see, e.g., Refs. [19,20,70–73]), the pivotal role of coherence has been studied. Showing that the available coherence directly quantifies the advantage in optimal phase estimation underscores its fundamental role in quantum computation, making the intuition that coherence is a necessary resource quantitative.

Naturally, the measures in Eq. (5) are highly tailored to a specific cost function, respectively, the task at hand, and it would be interesting to separate the advantage attributed to coherence into a problem-specific and a coherence-dependent part. If $\dim(\rho) = 2$ and $M = 2$, this is indeed possible and $\mathcal{A}^{(2)}(\rho) = (C_0 - \lambda_{\min}(Y^{(2)}))C_R(\rho)$, where $C_R(\rho)$ is the (generalized) robustness of coherence [57,58]. For the general case, we provide the following upper bound on $\mathcal{A}^{(M)}$, which holds for any cost function.

Proposition 5—The advantage $\mathcal{A}^{(M)}$ is bounded by

$$\mathcal{A}^{(M)}(\rho) \leq (C_0 - \lambda_{\min}(Y^{(M)}))W(\rho), \quad (7)$$

where the weight of coherence W [74,75] is given by

$$W(\rho) = \min_{w \geq 0} \{\rho = (1-w)\sigma + w\tau, \sigma \in \mathcal{I}, \tau \text{ a state}\}. \quad (8)$$

Recall that for the optimal phase estimation protocol with unrestricted access to coherence, the best achievable advantage over classical strategies is given by $C_0 - \lambda_{\min}(Y^{(M)})$ [6]. Since $0 \leq W(\rho) \leq 1$ [75], the bound in Eq. (7) interpolates between the optimal advantage without coherence and with arbitrarily much coherence. As we demonstrate in SM [40], this implies that for commonly used cost functions $\lim_{M \rightarrow \infty} C_{\min}^{(M,1)}(\rho) \geq C_0(1 - W(\rho))$, which can be strictly larger than zero, while in the unrestricted case [56], the limit converges to zero. As such, restricted access to coherence imposes a fundamental bound to the performance of phase estimation even in the limit of unboundedly many copies of the unitary.

Discussion—In this work, we established a direct quantitative connection between the optimal performance of phase estimation protocols and the coherence employed in the form of a resourceful input state. This provides valuable insight into phase estimation that is twofold: on one hand, lack of coherence poses quantifiable limitations to the accuracy of phase estimation. On the other hand, every state containing coherence also provides a quantifiable advantage. Thus, coherence is a key resource for phase estimation. Together with Refs. [19,20,70–73], this establishes a quantitative connection between coherence and advantages in quantum computation. Moreover, our work is an example for the application of resource theories of supermaps, see also Refs. [43–49], to problems of interest.

Coherence in the context of phase estimation has previously been studied [10,11,19]. Reference [10] investigated the optimal average cost with respect to a fixed cost

function, namely, the periodized variance, that can be achieved given an arbitrary state with a fixed amount of coherence measured by the robustness of coherence to which a single copy of the phase gate is applied. This allowed Ref. [10] to obtain asymptotic expressions. In contrast, in our work, we take an operational approach. Optimizing the physical usage of the employed coherence via MIO-compatible supermaps gives rise to tailored families of coherence measures that exactly quantify the performance of phase estimation protocols. This is not only operationally well-defined but also allows us to show that every amount of coherence serves as a resource for phase estimation. Moreover, in Ref. [11], quantum parameter estimation in Bayesian metrology and its connection to coherence was studied. The coherence of state ensembles, defined as the ensemble relative entropy of coherence [11], was identified to directly quantify the difference between Holevo quantity [76,77] and mutual information. On the one hand, the results of Ref. [11] are broader in scope than ours in the sense that phase estimation is a special case of parameter estimation. On the other hand, mutual information provides a lower bound on the average mean-squared error [78] and thus a specific cost function, while we consider general cost functions. In Ref. [19], the role of coherence in a specific implementation of Shor’s algorithm was investigated. This specificity allowed bounding its success probability by dynamical coherence measures [19,35] of the employed channels (which include static coherence as studied here as a special case). In SM [40], we discuss why here, in contrast, we decided to present our results in terms of static coherence and to which extent this encompasses the dynamical case. This allows us to determine the role of coherence in general phase estimation problems that are relevant not only for other implementations of Shor’s algorithm but also for tasks beyond factoring. In summary, our work unifies and expands key aspects from Refs. [10,11,19]: We consider arbitrary tasks for which phase estimation is relevant in the form of generic cost functions. Our operational approach of considering the optimal achievable protocol (with respect to the average cost) unveils the fundamental limitations that access to restricted coherence imposes, and is thus not limited to hardware- or implementation specific protocols.

We want to emphasize that, while coherence is a resource that quantifies the performance of phase estimation protocols, this does not imply that it is the only relevant resource. Importantly, MIO operations can convert coherence to entanglement [79,80], which is a valuable resource not only for quantum metrology [81–84], but also for quantum computation [22,85,86], and cryptography [87]. From the proof of Theorem 3 in SM [40], it is apparent that the optimal protocol in Fig. 2(b) uses the coherence provided to entangle the input states of the different copies of the unitary encoding the phase. Generalizing our results to non-uniform prior distributions, dynamical resources (including

measurements [35]), or resourceful networks, as well as parameter estimation beyond phase estimation, could provide further insights into the role of (dynamical) coherence for quantum advantages and is left for future work.

Note added—Shortly after this work appeared on the arXiv, the independent work Ref. [88] investigated the limitations that energy constraints impose on parameter estimation (and specifically phase estimation) using methods similar to ours.

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Data availability—No data were created or analyzed in this study.

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